

# ANALYSIS AND DESIGN OF MINIMAX-OPTIMAL INTERPOLATORS \*

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## Abstract

We consider a class of interpolation algorithms, including the least-squares optimal Yen interpolator, and we derive a closed-form expression for the interpolation error for interpolators of this type. The error depends on the eigenvalue distribution of a matrix which is specified for each set of sampling points. The error expression can be used to prove that the Yen interpolator is optimal. The implementation of the Yen algorithm suffers from numerical ill-conditioning, forcing the use of a regularized, approximate solution. We suggest a new, approximate solution, consisting of a sinc-kernel interpolator with specially chosen weighting coefficients. The newly designed sinc-kernel interpolator is compared with the usual sinc interpolator using Jacobian (area) weighting, through numerical simulations. We show that the sinc interpolator with Jacobian weighting works well only when the sampling is nearly uniform. The newly designed sinc-kernel interpolator is shown to perform better than the sinc interpolator with Jacobian weighting.

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# 1 Introduction

The problem of signal reconstruction from nonuniformly sampled data can be found in various contexts, such as in the design of irregularly-spaced antenna arrays [1, 2] and the reconstruction of signals for cases with missing samples. Generalizing to two dimensions, interpolation from non-Cartesian data grids is an important problem arising in various Fourier imaging problems, such as tomography [3], synthetic aperture radar [4], and radio astronomy [5, 6].

In most signal processing applications, the signal to be reconstructed from its nonuniform samples is modeled as bandlimited. Many types of interpolation algorithms have been devised for the reconstruction of bandlimited signals from nonuniform samples. Recent overviews of a number of reconstruction techniques for nonuniform sampling can be found in [7, 8]. Among the available algorithms, the sinc kernel figures prominently. A common interpolation formula, using the sinc kernel, is

$$x_L(t) = \sum_{i=1}^L b_i x(t_i) \frac{\sin(\sigma(t - t_i))}{\pi(t - t_i)}, \quad (1)$$

where the  $b_i$ 's are chosen to be the sample spacings (Jacobian,  $b_i = t_{i+1} - t_i$ ) around the nonuniform sample locations  $t_i$ , and  $\sigma$  is the bandlimit of  $x(t)$ . This choice of  $\{b_i\}$  follows from a Riemann sum approximation to the integral identity

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \frac{\sin(\sigma(t - \tau))}{\pi(t - \tau)} d\tau. \quad (2)$$

This algorithm is not optimal in any usual sense. An optimal (in the least-squares sense) bandlimited interpolation algorithm was first derived by Yen [9]. The computation in the Yen interpolator involves the inversion of an  $L \times L$  matrix where  $L$  is the number of samples. When  $L$  is large, the matrix often becomes highly ill-conditioned, so that regularization must

be used to reduce the numerical errors in inversion. Regularization results in smoothing of the signal spectrum at high frequencies [10]. Although Yen interpolation is best in theory, there are extreme problems in computing the interpolated values numerically, which prevent realization of the expected performance. Hence, the Yen interpolator is of modest use in practice.

The Yen interpolation algorithm has been derived using several different approaches [2, 11, 12, 13, 14, 15, 17]. Brown [2] and Levi [11] rediscovered Yen's interpolator. Oetken et al. [12] posed the interpolation problem as a finite-duration impulse response (FIR) filter design when the signal spectrum is known. They obtained the same interpolator by minimizing a squared-error measure. Kolba and Parks [13] solved the same problem for the  $\infty$  norm case and their solution was the same as the Yen interpolator. In [14], the algorithm was derived by assuming data matching at the sample points and finding the minimum-norm solution. The Yen algorithm was also derived in this same paper by optimization in the frequency domain, using a linear time-varying system model. In [15], the algorithm was derived by the so-called optimal recovery approach, which was originally derived by Micchelli and Rivlin [16]. It was shown that the pseudo inverse of the sampling operator is an optimal algorithm. In [17], the algorithm was derived by minimizing the least-squares error, assuming a special form of the interpolation algorithm. Later, in this paper, we suggest a modified minimax optimality criterion that we feel is better suited for measuring the performance of interpolation algorithms.

The primary contribution of this paper is the development of an approximate form of the Yen interpolator, which is easier to implement than the Yen interpolator, and less sensitive

to noise. In our approach, we first select a measure of interpolator error and then define the optimal interpolator to be that which minimizes the worst-case error. It is shown that the Yen interpolator is optimal when there is no restriction on the form of the interpolator. The framework in this paper differs from previous research in that we do not assume either data matching or linearity of the optimal interpolator for the derivation of the Yen algorithm. Furthermore, we use a modified version of the optimality criterion stated in [15] and [17]. Our derivation of the optimal interpolator is based on this modified definition of minimax optimality. We then seek an interpolator having a restricted form, which can be computed more easily and stably than the Yen interpolator. The optimal choice of the parameters in the approximate interpolator is given by clustering the eigenvalues of a matrix product, around 1. We suggest a simple method that achieves this clustering, and we quantify the performance of this scheme through numerical simulation.

Section 2 states the bandlimited interpolation problem and devises a suitable optimality criterion. In Section 3, we derive an explicit expression for the interpolation error and identify an eigenvalue property that characterizes the optimal interpolator. In Section 4, we suggest a simple, nearly optimal method of determining the interpolator coefficients. Section 5 presents simulation results which demonstrate the performance of the newly proposed interpolator by comparing it with the sinc-kernel interpolator using usual Jacobian weighting and with the optimal Yen interpolator. Section 6 summarizes the paper and suggests some directions for future research.

## 2 Bandlimited Interpolation and Optimality Criterion

The problem of bandlimited signal interpolation can be stated as follows. Given samples (possibly nonuniform) of a bandlimited signal  $x$ , find a signal that best approximates  $x$ . That is, given  $\{x(t_1), x(t_2), \dots, x(t_L)\}$ ;  $t_i$  distinct, find a bandlimited signal  $x_L$  such that  $\|x - x_L\|$  is minimized, where  $\|\cdot\|$  is an appropriate norm that measures the distance between two signals. We define  $\sigma$  to be the bandlimit of  $x$ , so that  $X(\omega) = 0$  for  $|\omega| > \sigma$ , where  $X(\omega)$  is the Fourier transform of  $x(t)$ .

Consider an interpolation algorithm  $A$ . Let  $S : X \rightarrow Y$  be the sampling operator, where  $X$  is the domain and  $Y$  is the range, respectively.  $S$  is assumed to be linear and have full rank. In our discussion,  $X$  is the set of all signals bandlimited to  $|\omega| < \sigma$ , and  $Y = \mathcal{R}^L$ . We define the sampling operator  $Sx = \{x(t_1), \dots, x(t_L)\}$ ;  $t_i$  distinct. Let us assume that  $X$  is a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ , and let  $\|\cdot\|$  be the natural norm associated with the inner product. The interpolation error can be defined as

$$\|x - ASx\|. \quad (3)$$

If the range space of  $S$  is closed in  $Y$ ,  $X$  admits the orthogonal decomposition

$$X = \mathcal{N}(S) \oplus \mathcal{N}^-(S), \quad (4)$$

where  $\mathcal{N}(S)$  is the null space of  $S$  and  $\mathcal{N}^-(S)$  is its orthogonal complement. Any  $x \in X$  can be decomposed uniquely as  $x = x_N + x_S$ , where  $x_N \in \mathcal{N}(S)$  and  $x_S \in \mathcal{N}^-(S)$ .

In the previous literature [15, 17], the minimax-optimal interpolator  $A_{opt}$  has been defined to be the one satisfying

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = \sup_{x \in X_1} \|x - A_{opt}Sx\|, \quad (5)$$

where  $\mathcal{A}$  is the class of algorithms under consideration, and  $X_1$  is the subset of  $X$  having elements with unit energy. If the null space of  $S$ ,  $\mathcal{N}(S)$ , is empty, then the sampling operator (which is linear) is invertible, so that exact reconstruction is possible, i.e.<sup>1</sup>

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = 0. \quad (6)$$

If  $\mathcal{N}(S) \neq \emptyset$ , let  $A(0) = x^A$  for each  $A \in \mathcal{A}$ . We can write  $x^A = x_N^A + x_S^A$ , where  $x_N^A \in \mathcal{N}(S)$  and  $x_S^A \in \mathcal{N}^-(S)$ . If  $x_N^A = 0$ , we choose any  $x \in \mathcal{N}(S) \cap X_1$ . Then, we have  $\|x - ASx\|^2 = \|x - x_S^A\|^2 = \|x\|^2 + \|x_S^A\|^2 \geq 1$ . If  $x_N^A \neq 0$ , for  $x = -x_N^A / \|x_N^A\| \in \mathcal{N}(S) \cap X_1$ ,  $\|x - ASx\|^2 = \|x_S^A\|^2 + (1 + 1/\|x_N^A\|)^2 \|x_N^A\|^2 \geq 1$ . So, we have

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| \begin{cases} = 0 & \text{if } \mathcal{N}(S) = \emptyset \\ \geq 1 & \text{otherwise} \end{cases}. \quad (7)$$

Setting the interpolator output to be identically zero, i.e., choosing  $ASx = 0$  for all  $x \in X_1$ , would give  $\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = 1$ . Thus, according to the optimality criterion in (5), this trivial algorithm is as good as the Yen algorithm. A reasonable minimax algorithm should satisfy  $\sup_{x \in X_1} \|x - ASx\| = 1$  if  $\mathcal{N}(S) \neq \emptyset$  (in which case  $x^A = 0$  or  $A(0) = 0$ .) This suggests that the optimality criterion defined by (5) is not useful for measuring the performance of interpolation algorithms, because the minimax error in (5) is completely specified by  $S$  and is independent of  $A$  for reasonable  $A$ .

In this section, we slightly modify the definition of minimax optimality, to provide a useful measure of the performance of interpolation algorithms. The optimal algorithm  $A_{opt}$  should satisfy

$$\inf_{A \in \mathcal{A}} \sup_{x_S \in \mathcal{N}^\perp(S) \cap X_1} \|x_S - ASx_S\| = \sup_{x_S \in \mathcal{N}^\perp(S) \cap X_1} \|x_S - A_{opt}Sx_S\|, \quad (8)$$

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<sup>1</sup>When  $L$  is finite, the null space of  $S$  is not empty, and exact reconstruction is not possible.

where, comparing with (5), we have restricted the domain of the supremum to  $\mathcal{N}^-(S) \cap X_1$ .

The analysis below justifies this modification of the error criterion.

Since  $S$  is linear,

$$ASx = A(Sx_N + Sx_S) = ASx_S. \quad (9)$$

Write  $ASx_S = \tilde{x}_N + \tilde{x}_S$  where  $\tilde{x}_N \in \mathcal{N}(S)$  and  $\tilde{x}_S \in \mathcal{N}^-(S)$ . Then, we have

$$\begin{aligned} \|x - ASx\|^2 &= \|x_N + x_S - ASx_S\|^2 \\ &= \|x_N - \tilde{x}_N\|^2 + \|x_S - \tilde{x}_S\|^2. \end{aligned} \quad (10)$$

$x_S$  can be exactly recovered from the available samples, since  $S$  is invertible when restricted to  $\mathcal{N}^-(S)$ . Let  $y = Sx$ . Then  $x_S$  can be found using the minimum-norm inverse of  $S$  as

$$x_S = S^*(SS^*)^{-1}y, \quad (11)$$

where  $S^*$  is the adjoint operator of  $S$ . Since  $x$  is assumed to have unit energy, the energy of  $x_N$  is constrained to be

$$\|x_N\|^2 = 1 - \|x_S\|^2. \quad (12)$$

Since we do not have any other information on  $x_N$ ,  $x_N$  can be any signal in  $\mathcal{N}(S)$  with the energy given by (12). To minimize the worst-case value of (10), we note that we must set  $\tilde{x}_S = x_S$ , and we choose  $\tilde{x}_N$  so that

$$\sup_{\|x_N\|^2=1-\|x_S\|^2} \|x_N - \tilde{x}_N\| \quad (13)$$

is minimized. This is achieved by setting  $\tilde{x}_N \equiv 0$ , since  $x_N$  lies in a balanced set. This choice of  $\tilde{x}_S$  and  $\tilde{x}_N$  corresponds to the interpolator originated by Yen [9]. Given samples of  $x(t)$

at  $t_1, \dots, t_L$ , the interpolated signal is given by

$$y(t) = \sum_{m=1}^L \sum_{n=1}^L \gamma_{mn} x(t_n) \varphi(t, t_m), \quad (14)$$

where  $\gamma_{mn}$  is the  $(m, n)^{th}$  element of the inverse of the matrix  $\Phi$ , where  $\Phi$  is the matrix whose  $(i, j)^{th}$  element is  $\varphi(t_i, t_j)$ , with

$$\varphi(s, t) = \frac{\sin(\sigma(s - t))}{\pi(s - t)}. \quad (15)$$

An interpolation algorithm has no hope of estimating  $x_N$ ; interpolation should be thought of as a method of finding an estimate of  $x_S$ . Thus, the performance of an interpolation algorithm should be measured by its ability to estimate  $x_S$  in  $\mathcal{N}^-(S)$ . Our definition of  $A_{opt}$  in (8) follows from this consideration. Although the optimal algorithm under the criterion in (8) is the widely known Yen algorithm, the new criterion is more suitable for evaluating the performances of non-optimal algorithms. In (8) we assumed  $x_S$  has unit energy, without loss of generality. Further, since we did not specify any particular inner product for the Hilbert space  $X$  in our development, the optimality criterion in (8) is suitable for a natural norm associated with any inner product.

Although the Yen interpolator is optimal, it is difficult to implement since it involves inversion of an  $L \times L$  matrix ( $\Phi$ ), which is often highly ill-conditioned, especially when  $L$  is large and the sampling is very nonuniform. Many regularization schemes have been devised for solving ill-posed least-squares problems. However, most of these methods are cumbersome to implement and generally require an enormous amount of computation to obtain a satisfactory result, especially when the number of samples is large. Hence, we suggest restricting the class of interpolation algorithms to a set that is more easily computed,

and less sensitive to noise. The optimal interpolator among this set will approximate the Yen interpolator. As a first step in designing simpler interpolation algorithms, we analyze the error in interpolation, in the next section. We derive an explicit expression for the worst-case interpolation error for arbitrary coefficients,  $\gamma_{mn}$ , in Eq. (14). Given the error expression, we then formulate the interpolator design as a constrained optimization problem in a finite-dimensional Hilbert space. We next seek a simple approximate solution to this optimization problem, which leads to a simple approximation to the optimal Yen interpolator.

### 3 Analysis of Interpolation Error

Given a bandlimited signal  $x(t)$ , consider an interpolation algorithm of the form

$$x_L(t) = \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_n) \varphi(t, t_m), \quad (16)$$

where  $t_l$ ,  $l = 1, \dots, L$ , are the sampling points and samples of  $x(t)$  are available at locations  $\{t_l\}$ . The  $\{b_{mn}\}$  are parameters of the interpolator to be chosen according to some optimality criterion. In our discussion, we assume that the parameters  $\{b_{mn}\}$  are symmetric, that is,  $b_{mn} = b_{nm}$ , because this symmetric choice of  $\{b_{mn}\}$  includes the Yen interpolator and the analysis of the interpolation error simplifies. More importantly, it can be shown that an asymmetric choice of  $\{b_{mn}\}$  cannot reduce interpolation error over a symmetric choice.

The parameters  $\{b_{mn}\}$  should be chosen to be optimal for a given sampling point set. But, as noted in the last section, the exact evaluation of the optimal  $\{b_{mn}\}$  suffers from numerical ill-conditioning. Hence, it is of interest to consider using non-optimal values for  $\{b_{mn}\}$ . As preparation for this, we derive an expression for the interpolation error in terms of

the parameters  $\{b_{mn}\}$ . Later we will see how this formula assists us in selecting the  $\{b_{mn}\}$  in a suboptimal, but well conditioned manner that can still produce small interpolation error.

We now find the expression for the error between the original signal  $x$  and the reconstructed signal  $x_L$ . Define the inner product between two  $\sigma$ -bandlimited signals  $f(t)$  and  $g(t)$  as follows:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt. \quad (17)$$

The norm induced by this inner product is given as

$$\|f\| = \sqrt{\langle f, f \rangle} = \left[ \int_{-\infty}^{\infty} |f(t)|^2 dt \right]^{1/2}. \quad (18)$$

In our interpolation problem, the signals we consider belong to a reproducing kernel Hilbert space of functions which are bandlimited to  $\sigma$ , with inner product  $\langle f, g \rangle$  and reproducing kernel

$$\varphi(s, t) = \frac{\sin(\sigma(s - t))}{\pi(s - t)}. \quad (19)$$

By the reproducing property, we have

$$x(t) = \langle x(s), \varphi(s, t) \rangle. \quad (20)$$

For a signal  $x$  with unit energy, we have

$$\begin{aligned} \|x - x_L\|^2 &= \langle x(t) - x_L(t), x(t) - x_L(t) \rangle \\ &= \langle x(t), x(t) \rangle - 2\langle x(t), x_L(t) \rangle + \langle x_L(t), x_L(t) \rangle. \end{aligned} \quad (21)$$

Using the reproducing property, we obtain

$$\langle x(t), x_L(t) \rangle = \langle x(t), \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_n) \varphi(t, t_m) \rangle$$

$$\begin{aligned}
&= \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_n) \langle x(t), \varphi(t, t_m) \rangle \\
&= \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_m) x(t_n).
\end{aligned} \tag{22}$$

Similarly, we obtain

$$\begin{aligned}
\langle x_L(t), x_L(t) \rangle &= \left\langle \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_n) \varphi(t, t_m), \sum_{p=1}^L \sum_{q=1}^L b_{pq} x(t_q) \varphi(t, t_p) \right\rangle \\
&= \sum_{m=1}^L \sum_{n=1}^L \sum_{p=1}^L \sum_{q=1}^L b_{mn} b_{pq} x(t_n) x(t_q) \langle \varphi(t, t_m), \varphi(t, t_p) \rangle \\
&= \sum_{m=1}^L \sum_{n=1}^L \sum_{p=1}^L \sum_{q=1}^L b_{mn} b_{pq} x(t_n) x(t_q) \varphi(t_m, t_p).
\end{aligned} \tag{23}$$

Combining these results, we have

$$\|x - x_L\|^2 = 1 - 2 \sum_{m=1}^L \sum_{n=1}^L b_{mn} x(t_m) x(t_n) + \sum_{m=1}^L \sum_{n=1}^L \sum_{p=1}^L \sum_{q=1}^L b_{mn} b_{pq} x(t_n) x(t_q) \varphi(t_m, t_p). \tag{24}$$

Letting  $\underline{x} = [x(t_1) \cdots x(t_L)]^T$ ,  $B = [b_{ij}]$ , and  $\Phi = [\varphi(t_i, t_j)]$ , gives an expression for the interpolation error in matrix notation:

$$\begin{aligned}
\|x - x_L\|^2 &= 1 - 2\underline{x}^T B \underline{x} + \underline{x}^T B \Phi B \underline{x} \\
&= 1 - \underline{x}^T (2B - B \Phi B) \underline{x}.
\end{aligned} \tag{25}$$

Let us proceed to find  $x \in \mathcal{N}^-(S) \cap X_1$  giving the worst-case interpolation error. For  $x \in \mathcal{N}^-(S)$ ,  $x$  is exactly reconstructable from its samples, and it is given by

$$x(t) = \sum_{m=1}^L \sum_{n=1}^L x(t_n) \gamma_{mn} \varphi(t, t_m), \tag{26}$$

where  $\gamma_{mn}$  is  $(m, n)^{th}$  element of  $\Phi^{-1}$ . The energy of  $x$  can be written as

$$\|x\|^2 = \left\langle \sum_{m=1}^L \sum_{n=1}^L x(t_m) \gamma_{mn} \varphi(t, t_n), \sum_{i=1}^L \sum_{j=1}^L x(t_i) \gamma_{ij} \varphi(t, t_j) \right\rangle$$

$$\begin{aligned}
&= \sum_{i=1}^L \sum_{j=1}^L \left[ \sum_{m=1}^L x(t_m) \gamma_{im} \right] \left[ \sum_{n=1}^L x(t_n) \gamma_{jn} \right] \varphi(t_i, t_j) \\
&= \underline{x}^T \Phi^{-1} \Phi \Phi^{-1} \underline{x} \\
&= \underline{x}^T \Phi^{-1} \underline{x}.
\end{aligned} \tag{27}$$

Since  $x \in \mathcal{N}^-(S) \cap X_1$ , we have the following constraint on  $x$ :

$$x^T \Phi^{-1} x = 1. \tag{28}$$

Hence, the worst-case error occurs at the solution of the following constrained maximization problem:

$$\begin{aligned}
&\text{maximize} && 1 - \underline{x}^T (2B - B\Phi B) \underline{x}, \\
&\text{subject to} && \underline{x}^T \Phi^{-1} \underline{x} = 1.
\end{aligned}$$

This constrained optimization problem can be solved using the Lagrange multiplier method.

At the solution of the problem, there exists a  $\lambda$  such that

$$-(B\Phi B - 2B) \underline{x} + \lambda \Phi^{-1} \underline{x} = 0. \tag{29}$$

Multiplying by  $\Phi$  and rearranging terms, we obtain

$$(2\Phi B - (\Phi B)^2 - \lambda I) \underline{x} = 0. \tag{30}$$

This implies that  $\lambda$  is an eigenvalue of the matrix  $2\Phi B - (\Phi B)^2$ , and  $\underline{x}$  is the eigenvector corresponding to that eigenvalue. And, in this case, the interpolation error can be written as follows:

$$\begin{aligned}
\sup \|x - x_L\|^2 &= 1 - \underline{x}^T (2B - B\Phi B) \underline{x} \\
&= 1 - \lambda \underline{x}^T \Phi^{-1} \underline{x} \\
&= 1 - \lambda.
\end{aligned} \tag{31}$$

The worst-case error occurs when  $\underline{x}$  is the eigenvector corresponding to the minimum eigenvalue of  $2\Phi B - (\Phi B)^2$ , and the maximum error is  $\sqrt{1 - \lambda_{min}}$ .

The above derivation gives a criterion for the choice of interpolator parameters  $b_{mn}$ . Let  $\alpha$  be an eigenvalue of  $\Phi B$ . Then,  $2\alpha - \alpha^2$  is an eigenvalue of  $2\Phi B - (\Phi B)^2$ . Since  $2\alpha - \alpha^2 \geq \lambda_{min}$ , we have

$$1 - \sqrt{1 - \lambda_{min}} \leq \alpha \leq 1 + \sqrt{1 - \lambda_{min}}. \quad (32)$$

Hence, minimizing the worst-case error  $\sqrt{1 - \lambda_{min}}$  is equivalent to clustering the eigenvalues of  $\Phi B$  near 1. Any suboptimal choice of the parameters  $b_{mn}$  should try for this eigenvalue clustering. Also, notice that if  $2\Phi B - (\Phi B)^2$  is positive definite, then the interpolation error is always less than the signal energy, and in this case we can say we have a meaningful reconstruction of the signal. But, if  $2\Phi B - (\Phi B)^2$  is not positive definite, there exists a signal for which the reconstruction error is greater than the signal energy. In this case, the interpolation is useless in the minimax sense.

## 4 Generalization of the Sinc-Kernel Interpolation

The model of the commonly used sinc-kernel interpolator was given in Eq. (1). As mentioned earlier, the parameters  $\{b_l\}$  are weighting coefficients which compensate for the irregularity of sampling and usually are chosen to be the sample spacing (Jacobian) of the sampling grid. However, this choice of  $\{b_l\}$  has little theoretical basis.

The interpolator in Eq. (1) falls within the class of the interpolators described in the previous sections and is a special case of Eq. (16) with  $b_{mn} = 0$  for  $m \neq n$ . This choice of

$\{b_{mn}\}$  corresponds to using a diagonal matrix for  $B$ , which is reasonable since  $\Phi$  frequently has large elements on its diagonal, with off-diagonal elements that are relatively small and quickly decreasing with distance from diagonal. Thus, we expect that its inverse is also well approximated by a diagonal matrix. This implies that the interpolated value at a particular point depends heavily on the samples near that point, and less on those samples far from it.

In view of the optimality criterion described in Sections 2,  $\{b_l\}$  should be chosen so that the matrix product  $\Phi B$ , where  $B = \text{diag}(b_1, \dots, b_L)$ , has eigenvalues clustered around 1. However, although we can evaluate the eigenvalues of  $\Phi B$  for a particular choice of  $B$ , it is difficult to find a best  $B$ . The optimal  $B$  for an interpolator of this form is a diagonal matrix minimizing

$$\|\Phi B - I\|_2, \quad (33)$$

where  $\|\cdot\|_2$  is the spectral norm of the matrix. In other words, the problem is to choose  $B$  so that  $\Phi B$  is close to the identity in the sense of spectral norm. Since the minimization of the spectral norm is intractable, we explore using the Frobenius norm. Although the minimization of the Frobenius norm does not guarantee the minimum value of the spectral norm, it will provide an approximate solution to the problem. The minimum of  $\|\Phi B - I\|_F$  is achieved when  $B$  is designed with

$$\begin{aligned} b_i &= \frac{\varphi(t_i, t_i)}{\sum_{j=1}^L \varphi^2(t_j, t_i)} \\ &= \frac{\pi}{\sigma} \left[ \sum_{j=1}^L \text{sinc}^2(\sigma(t_j - t_i)) \right]^{-1}. \end{aligned} \quad (34)$$

In the next section, we compare the performance of this choice of  $B$  with the interpolator using the usual Jacobian weighting.

Rather than restricting  $B$  to be diagonal, we might consider other forms for  $B$  such as tridiagonal, general banded or block diagonal. But, restricting  $B$  to be diagonal makes the computation of  $B$  very simple and is numerically attractive. We expect that the performance of our interpolator could be improved by incorporating nonzero off-diagonal elements in  $B$ , at the expense of increased computation and numerical instability. The choice of a block diagonal  $B$  may be good if the sample points are grouped in a few clusters.

## 5 Simulation Results

We conducted simulations to demonstrate the performance of the new, suboptimal interpolator suggested in Section 4. In the first set of simulations, we compared this interpolator with the sinc-kernel interpolator using constant weighting and with the sinc-kernel interpolator using Jacobian weighting. For this simulation, a set of  $L = 16$  samples was used with average sampling interval  $T = 1$  sec, giving a nominal bandwidth  $\sigma = \pi$  rad/sec. The sampling instants were obtained by beginning with uniformly distributed points  $\{t_n = nT, n = 1, 2, \dots, L\}$  and adding random noise, distributed on the range  $[-u, u]$ , to these initial positions. Sampling instants falling outside the interval  $[0, LT]$  were wrapped inside. We constructed the  $B$  matrix according to Eq. (34) for the new sinc interpolator and computed the eigenvalues of  $\Phi B$ . The worst-case error  $\mathcal{E}_{wc} = \sqrt{1 - \lambda_{min}}$  was recorded. The same procedure was performed for the other sinc-kernel interpolators except that  $b_i$  was chosen to be either constant or the value of the sample spacing around the  $i^{th}$  sampling point. For the Yen interpolator, we know that  $\mathcal{E}_{wc} = 0$  for any distribution of sample points, because we restricted our error measure to the set of signals belonging to  $\mathcal{N}^-(S)$  when we

defined optimality in Section 2.

Figures 1 through 5 show the squared worst-case error  $\mathcal{E}_{wc}$  for various nonuniform sample point sets. The horizontal axis corresponds to the trial number, with 100 different sample point sets in each experiment. The irregularity of sampling was controlled by varying the value of parameter  $u$ , which is the maximum possible offset of sample points from a uniformly spaced grid. The solid line represents the result with  $B = I$ , and the dotted line represents  $\mathcal{E}_{wc}$  for the sinc kernel interpolator with Jacobian weighting. The dashed line represents the result for the sinc kernel interpolator with  $B$  matrix chosen as in Eq. (34). In this set of simulations, we did not include the Yen interpolator because we know that the worst-case error  $\mathcal{E}_{wc}$  is zero for the Yen interpolator. In cases where a plotted value is greater than 1, the matrix  $2B - B\Phi B$  is not positive definite and the interpolator can be useless (for that particular sampling grid) for some signals, which means that it is useless in the minimax sense.

When the sampling was almost uniform ( $u = 0.1$ ), we note that the three choices of  $B$  gave almost the same worst-case error. And, note that in all cases,  $\mathcal{E}_{wc}$  was relatively small and thus,  $\Phi B$  had eigenvalues clustered around 1. In this case, the interpolation error is expected to be small. As the irregularity of sampling increased,  $\mathcal{E}_{wc}$  for the sinc kernel interpolator without weighting ( $B = I$ ) became large, and in many cases, the matrix  $2B - B\Phi B$  was not positive definite; hence the interpolation may be useless. When  $u = 0.5$ , we note that the interpolator with Jacobian weighting still had  $\mathcal{E}_{wc} \leq 1$  for most of the cases, and the newly designed interpolator behaved similarly. As the irregularity of sampling increased further, the worst-case error of the sinc interpolator with Jacobian weighting degraded badly. When

$u \geq 1$ ,  $2B - B\Phi B$  lost positive definiteness for the usual Jacobian weighting. But, we note that the newly designed interpolator maintained  $\mathcal{E}_{wc} \leq 1$  even when the sampling was very nonuniform.

In summary, we notice that the sinc kernel interpolator with Jacobian weighting worked well only when the sampling was nearly uniform. For very nonuniform sampling, the interpolator designed in the previous section performed much better.

A second set of simulations compared the performances of the Yen interpolator without regularization (Yen-1), Yen with  $\epsilon$ -regularization (Yen-2), sinc interpolator with uniform weighting (Sinc-1), sinc interpolator with the Jacobian weighting (Sinc-2), and the newly designed sinc interpolator with weightings given by Eq. (34) (Sinc-3). The Yen algorithm with  $\epsilon$ -regularization was designed by adding a small positive constant,  $\epsilon$ , to the diagonal elements of  $\Phi$  before inverting it. In all simulations for the  $\epsilon$ -regularized Yen interpolator, the optimal value of  $\epsilon$  was used, which was determined according to the SNR of the data samples [18]. We point out, however, that when the SNR is not known exactly, the performance will be poorer than observed in these simulations. The sampling instants were generated in the same way as for the first simulations, and the interpolator input was taken to be a noisy, sampled version of the superposition of 50 sinc functions having cutoff frequency  $\sigma = \pi$  rad/s, having random amplitudes on the range  $[0,1]$ , and randomly centered on the interval  $[LT, 2LT]$  with  $L = 16$ . The noise was additive, white, and Gaussian. For each interpolator, a uniformly-sampled version of the output signal, with sampling interval  $T/16$ , was reconstructed as an approximation of the continuous-time signal. To evaluate the reconstruction quality of the five interpolators, the signal-to-error (S/E) ratio was computed in dB for five different values

of the maximum displacement  $u$ . The S/E ratio was computed as the energy of the signal divided by the energy of the reconstruction error. The means and standard deviations of the S/E ratio, averaged over 100 trials (with different sampling instants and different signals), are reported in Tables 1 through 5 as a function of interpolator type and signal-to-noise ratio (SNR) of the data. For all simulations, double-precision arithmetic was used to minimize the numerical errors.

It is seen that all the interpolators behaved in a very similar way when the maximum displacement  $u$  was small ( $u = 0.1$ ), that is, when the sampling was almost uniform. Note that even the Yen algorithm cannot provide a perfect reconstruction in the noiseless case because, given a finite number of samples, the null space of the sampling operator is not empty, and we cannot hope to reconstruct the signal component lying in the null space of the sampling operator. As the distribution of the sampling points became more nonuniform, the performance of each interpolator worsened. As the level of the noise which corrupted the data samples became higher, the performance of the Yen-1 interpolator degraded very quickly, as expected, especially when the sample distribution was very nonuniform. The Yen-2 interpolator performed well, even in the presence of noise. The newly proposed Sinc-3 interpolator performed similarly. Notice, however, that the Sinc-3 interpolator generally had lower standard deviations for the error than the Yen-2 interpolator. Also, the Sinc-3 interpolator required no knowledge of the SNR. If the regularization parameter in the Yen-2 interpolator had been fixed at a constant, independent of SNR, then the Yen-2 performance would have been much worse than that of the Sinc-3 interpolator, as we confirmed in other simulations. Among the three sinc interpolators, Sinc-3 interpolator showed the best perfor-

mance by far.

## 6 Conclusion

In this paper, we formulated the problem of signal interpolation from nonuniform samples when the original signal is bandlimited. It was pointed out that the minimax error measure defined in previous literature should be restricted to the orthogonal complement of the null space of the sampling operator. In terms of this modified optimality measure, an explicit expression for the interpolation error was obtained. It was shown that the design of a minimax optimal interpolator corresponds to the design of the eigenvalue distribution of a particular matrix product. An approximate design strategy was proposed and the performance of this newly designed interpolation algorithm was examined through numerical simulations.

From the simulation results, we saw that the performance of a simple sinc-kernel interpolator with appropriate weighting can be as good as the Yen interpolator with proper regularization. The ill-posedness of the problem, when the sampling is very nonuniform and the noise level is high, results in large error for the unregularized Yen interpolator. The excellent performance of the new sinc-kernel interpolator is most welcome, since the regularized Yen interpolator is cumbersome to implement, and the SNR may not be known to properly set the regularization parameter.

Although we demonstrated the performance of the new sinc interpolator through numerical simulations, a theoretical analysis of the new algorithm and a theoretical comparison with other algorithms are still needed. (Some recent work in this direction can be found in [21, 22, 23, 24] for a number of reconstruction algorithms.) Also, other methods of clustering

the eigenvalues may produce even better interpolators.

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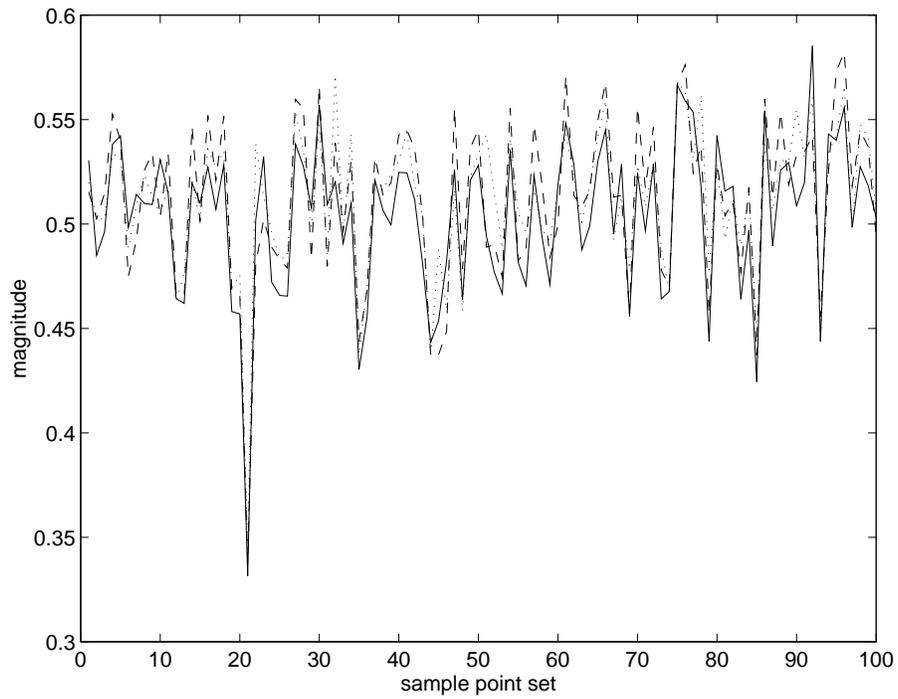


Figure 1: Worst-case interpolation error :  $u = 0.1$

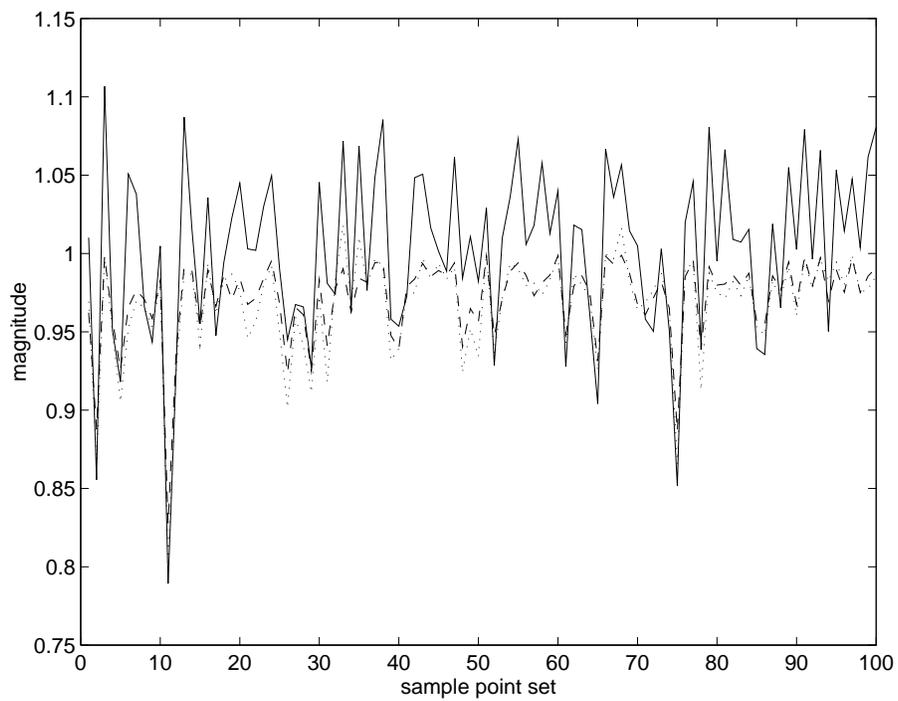


Figure 2: Worst-case interpolation error :  $u = 0.5$

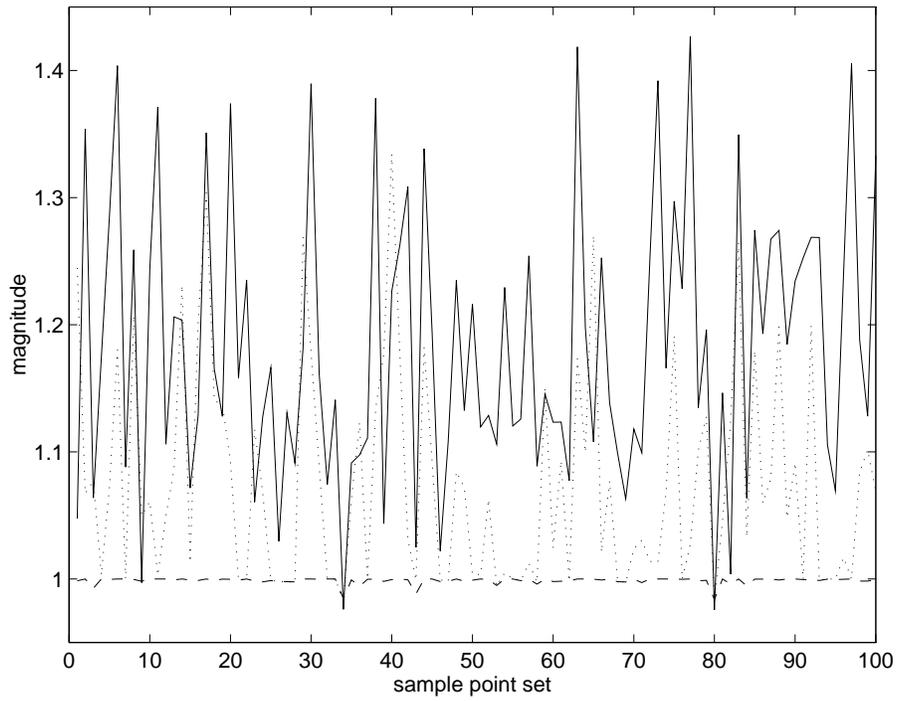


Figure 3: Worst-case interpolation error :  $u = 1.0$

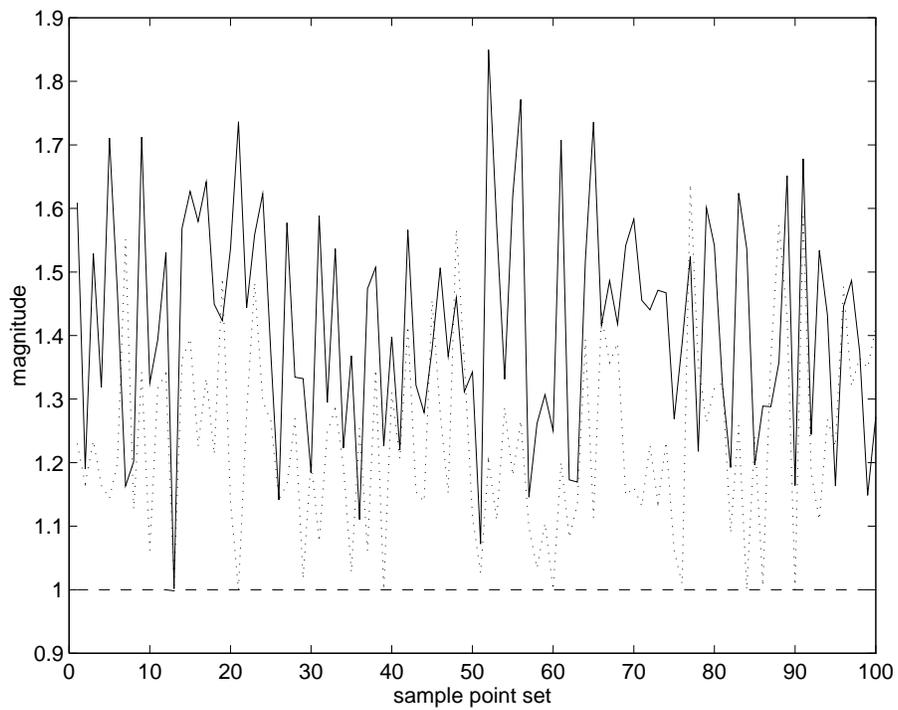


Figure 4: Worst-case interpolation error :  $u = 5.0$

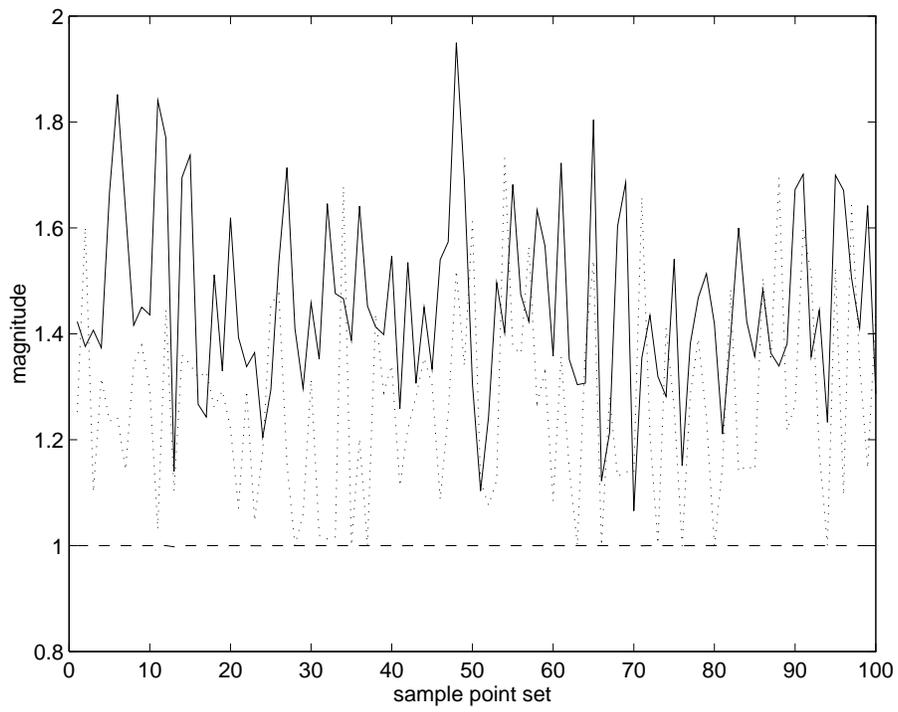


Figure 5: Worst-case interpolation error :  $u = 16.0$

Interpolation Method		SNR of data samples				
		no noise	40 dB	30 dB	20 dB	10 dB
Yen-1	Mean	17.46	17.38	16.71	14.78	9.09
	S. D.	6.06	5.92	5.39	3.72	1.99
Yen-2	Mean	17.46	17.42	16.45	14.18	8.99
	S. D.	6.06	6.01	5.14	3.36	1.86
Sinc-1	Mean	14.05	13.96	13.96	13.04	8.80
	S. D.	3.37	3.24	3.38	2.87	1.91
Sinc-2	Mean	14.26	14.20	14.16	13.17	8.85
	S. D.	3.48	3.42	3.42	2.92	1.92
Sinc-3	Mean	14.15	14.06	14.00	13.14	8.91
	S. D.	3.44	3.32	3.41	2.91	1.92

Table 1: Mean and standard deviation of S/E for different interpolators :  $u = 0.1$

Interpolation Method		SNR of data samples				
		no noise	40 dB	30 dB	20 dB	10 dB
Yen-1	Mean	15.44	15.87	15.21	12.71	5.56
	S. D.	5.69	5.06	4.78	3.85	3.69
Yen-2	Mean	15.44	15.61	14.53	11.77	7.59
	S. D.	5.69	4.94	4.65	3.09	2.00
Sinc-1	Mean	5.69	5.80	6.15	5.44	4.45
	S. D.	2.21	2.28	2.32	2.02	1.77
Sinc-2	Mean	6.42	6.68	7.14	6.30	4.92
	S. D.	2.28	2.39	2.49	1.94	1.69
Sinc-3	Mean	7.18	7.70	7.95	7.21	6.12
	S. D.	2.64	2.57	2.56	2.19	1.74

Table 2: Mean and standard deviation of S/E for different interpolators :  $u = 0.5$

Interpolation Method		SNR of data samples				
		no noise	40 dB	30 dB	20 dB	10 dB
Yen-1	Mean	13.47	12.91	8.18	3.40	-4.83
	S. D.	5.56	6.24	7.36	8.05	9.64
Yen-2	Mean	13.47	9.23	7.30	6.09	4.96
	S. D.	5.56	4.32	3.93	3.29	2.00
Sinc-1	Mean	2.94	2.88	2.91	2.61	2.46
	S. D.	2.27	1.88	1.78	1.86	1.82
Sinc-2	Mean	3.91	3.79	3.69	3.42	3.06
	S. D.	2.27	1.88	1.83	2.06	1.79
Sinc-3	Mean	5.75	5.61	5.32	5.46	4.98
	S. D.	2.46	2.01	2.35	2.20	1.77

Table 3: Mean and standard deviation of S/E for different interpolators :  $u = 1.0$

Interpolation Method		SNR of data samples				
		no noise	40 dB	30 dB	20 dB	10 dB
Yen-1	Mean	11.64	-4.09	-11.25	-19.85	-29.46
	S. D.	4.76	19.63	21.78	23.15	23.59
Yen-2	Mean	11.64	6.14	5.42	4.45	3.83
	S. D.	4.76	3.71	3.25	2.66	1.97
Sinc-1	Mean	1.40	1.40	1.40	1.37	0.92
	S. D.	2.20	2.20	2.21	2.20	2.12
Sinc-2	Mean	2.04	2.03	2.01	1.91	1.18
	S. D.	2.27	2.28	2.28	2.28	2.27
Sinc-3	Mean	4.65	4.65	4.64	4.59	4.08
	S. D.	1.98	1.98	1.98	1.96	1.77

Table 4: Mean and standard deviation of S/E for different interpolators :  $u = 5.0$

Interpolation Method		SNR of data samples				
		no noise	40 dB	30 dB	20 dB	10 dB
Yen-1	Mean	10.45	-14.40	-18.24	-25.31	-35.95
	S. D.	5.45	27.29	24.87	25.22	23.46
Yen-2	Mean	10.45	5.08	5.09	4.61	3.56
	S. D.	5.45	3.77	3.33	3.04	2.18
Sinc-1	Mean	0.66	0.94	1.02	0.75	0.51
	S. D.	2.33	2.74	2.38	2.55	2.48
Sinc-2	Mean	1.94	1.71	1.97	2.06	1.26
	S. D.	2.26	2.27	2.08	2.16	2.06
Sinc-3	Mean	4.50	4.35	4.58	4.84	3.95
	S. D.	2.16	2.18	2.19	2.39	2.10

Table 5: Mean and standard deviation of S/E for different interpolators :  $u = 16.0$