Reliable Minimum Finding Comparator Networks

Piotr Denejko¹ Krzysztof Diks^{1,2} * Andrzej Pelc² ** Marek Piotrów³ ***

 ¹ Instytut Informatyki, Uniwersytet Warszawski, Banacha 2, 02-097 Warszawa, Poland.
² Département d'Informatique, Université du Québec à Hull, Hull, Québec J8X 3X7, Canada.
³ Heinz Nixdorf Institut, Universität-GH-Paderborn, Warburger Str. 100, 33098 Paderborn, Germany.

Abstract. We consider the problem of constructing reliable minimum finding networks built from unreliable comparators. In case of a faulty comparator inputs are directly output without comparison. Our main result is the first nontrivial lower bound on depths of networks computing minimum among n > 2 items in the presence of k > 0 faulty comparators. We prove that the depth of any such network is at least max($\lceil \log n \rceil + 2k, \log n + k \log \frac{\log n}{k+1}$). We also describe a network whose depth nearly matches the lower bound. The lower bounds should be compared with the first nontrivial upper bound $O(\log n + k \log \frac{\log n}{\log k})$ on the depth of k-fault tolerant sorting networks that was recently derived by Leighton and Ma [6].

1 Introduction

Networks built from comparators are commonly used to perform such tasks as selection, sorting and merging. A comparator is a 2 input-2 output device which sorts two items. Networks of minimum size, i.e. using the minimum number of comparators for a given task, have been studied e.g. in [1, 3, 5]. In particular Ajtai, Komlos and Szemeredi [1] showed an *n*-input sorting network that uses $O(n \log n)$ comparators. Another measure of performance of a network built from comparators is its depth, i.e. the time in which it performs its task assuming that nonoverlapping comparators (those which do not have common inputs) can act simultaneously and one comparison takes a unit of time. The network from [1] is asymptotically optimal from this point of view: it has depth $O(\log n)$.

Yao and Yao [9] originated a new approach to the study of such networks. They supposed that some comparators can be faulty and a faulty comparator does not work at all: inputs are output directly without comparison. In [9] networks for sorting, merging and minimum selection using a small number of comparators were built under two alternative fault models. In the stochastic model comparators fail

** Research supported in part by NSERC grant OGP 0008136. Em: pelc@uqah.uquebec.ca

^{*} Research supported in part by NSERC International Fellowship and by grant KBN 2-2043-92-03. Email: diks@mimuw.edu.pl

^{***} Research supported in part by Alexander von Humboldt-Stiftung, Volkswagen Stiftung and the ESPRIT Basic Research Action No. 7141 (ALCOM II). Permanent address: Instytut Informatyki, Przesmyckiego 20, Wrocław, Poland. Em: marekp@uni-paderborn.de

independently with fixed probability δ and the goal is to construct (ϵ, δ) -stochastic networks which work correctly with probability at least $1 - \epsilon$ under this assumption. In the k-fault model the goal is to build k-tolerant networks, that is networks which work correctly if any set of at most k comparators are faulty.

Most attention in literature has been devoted to fault-tolerant networks for sorting. Yao and Yao [9] constructed such a k-tolerant network of minimum size. Leighton and Ma [6] derived the first nontrivial upper bound $O(\log n + k \log \frac{\log n}{\log k})$ on the depth of a k-tolerant sorting network. In their construction the constant in the O-notation depends on the expander used to build the network. The probabilistic model as well as other types of faulty comparators were also studied in [2, 6, 7, 9] in this context.

In this paper we consider networks finding the minimum term of a vector of real numbers under the k-fault model. Yao and Yao constructed a k-tolerant minimum finding network of minimum size. We are interested in building such networks with small depths. We construct a k-tolerant n-input network using the minimum number of comparators, of depth at most

$$\min\left(\left\lceil \log n \right\rceil + k\left(\left\lceil \log (\left\lceil \log n \right\rceil + 1\right)\right\rceil, \ 1.5 \left\lceil \log n \right\rceil + 3k + 1, \ \left(\left\lceil \frac{n}{2} \right\rceil + 2k\right)\right).$$

We also establish the corresponding lower bound

$$\max\left(\left\lceil \log n \right\rceil + 2k, \ \log n + k \log\left(1.28\frac{\left\lceil \log n \right\rceil}{k+1} + 1.92\right)\right)$$

which shows that the depth of our network is asymptotically optimal both for fixed n and arbitrary k and for fixed k and arbitrary n. No such tight bounds were known previously.

The paper is organized as follows. In Section 2 we present our terminology and establish basic facts used in the paper. Section 3 is devoted to establish the lower bound on the depth of k-tolerant minimum finding networks and in Section 4 we construct and analyze a network whose depth nearly matches the lower bound from Section 3.

2 Preliminaries

Let $n \geq 2$ be an integer and \mathcal{R}^n – the set of *n*-element vectors of reals. For every $\overline{x} \in \mathcal{R}^n$, $\overline{x}[i]$ denotes the *i*-th term of \overline{x} . For $1 \leq i < j \leq n$, the comparator [i:j] is a mapping from \mathcal{R}^n to \mathcal{R}^n which transforms a vector \overline{x} into vector $\overline{x}' = \overline{x}[i:j]$ defined as follows:

$$\overline{x}'[k] = \begin{cases} \overline{x}[k] & k \neq i, j \\ \min(\overline{x}[i], \overline{x}[j]) & k = i \\ \max(\overline{x}[i], \overline{x}[j]) & k = j. \end{cases}$$

Thus [i : j] compares $\overline{x}[i]$ with $\overline{x}[j]$ and places the smaller of them in position i and the larger in position j.

Let α be a finite sequence of comparators $[i_1 : j_1], \ldots, [i_r : j_r]$. α transforms each vector $\overline{x} \in \mathcal{R}^n$ into $\overline{y} = \overline{x}\alpha$ defined as follows:

$$\overline{x}^{(0)} = \overline{x}; \overline{x}^{(k)} = \overline{x}^{(k-1)}[i_k : j_k], \text{ for } 1 \le k \le r; \overline{x}\alpha = \overline{x}^{(r)}.$$

Two comparators $[i_1 : i_2], [i_3 : i_4]$ are called nonoverlapping if $i_1 \neq i_3, i_4$ and $i_2 \neq i_3, i_4$.

Proposition 1. Let C be a set of pairwise nonoverlapping comparators and $\phi_1(C)$, $\phi_2(C)$ arbitrary permutations of all elements from C. Then $\overline{x}\phi_1(C) = \overline{x}\phi_2(C)$, for any $\overline{x} \in \mathcal{R}^n$.

An *n*-input network α is any sequence C_1, C_2, \ldots, C_r of nonempty sets of comparators on \mathcal{R}^n such that in each set C_i comparators are pairwise nonoverlapping. For any $i = 1, \ldots, r$ let $\phi_i(C_i)$ be any permutation of C_i . The *n*-input network α transforms any $\overline{x} \in \mathcal{R}^n$ into $\overline{x}\alpha = \overline{x}\phi_1(C_1)\ldots\phi_r(C_r)$. The sets C_1, \ldots, C_r are called phases of the network α and r is said to be its depth. We denote $r = |\alpha|$.

We say that β is a *j*-fault subnetwork of α if β can be obtained from α by deleting exactly *j* comparators. By definition of a faulty comparator, instead of deleting comparators it is equivalent to say that comparators in question fail.

An *n*-input network α is called an mf-network (minimum finding network) if for every $\overline{x} \in \mathcal{R}^n$, $\overline{x}\alpha[1] = \min(\overline{x}[1], \ldots, \overline{x}[n])$.

An *n*-input mf-network α is said to be *k*-tolerant if every *j*-fault subnetwork of α , $j \leq k$, is also an mf-network. We denote by $T_k(n)$ the minimum depth of a *k*-tolerant *n*-input mf-network. In the sequel we assume n > 2. Observe that $T_k(2) = k + 1$, for any $k \geq 0$.

Throughout the paper $\log x$ is used for $\log_2 x$ and |A| denotes the size of a set A.

3 Lower Bounds

In this section we give two nontrivial lower bounds on $T_k(n)$. The first theorem establishes a lower bound which is good in the case when n is fixed and k can be arbitrarily large.

Theorem 2. $T_k(n) \ge \lfloor \log n \rfloor + 2k$.

Proof. Induction on k.

k = 0

This is the well-known fact that any *n*-input network computing minimum has depth at least $\lceil \log n \rceil$.

k > 0

Assume that the theorem holds for networks with less than k faulty comparators. Suppose that there is a k-tolerant n-input mf-network with depth less than $\lceil \log n \rceil + 2k$. Let $\alpha = A_1, \ldots, A_d$ be such a network with the smallest depth d. By the inductive hypothesis $d \ge \lceil \log n \rceil + 2(k-1)$, since any k-tolerant mf-network is also (k-1)-tolerant. Consider the last phase of α . This phase must contain a comparator [1:f], for some $2 \le f \le n$, otherwise it would be superfluous (but α is a shortest k-tolerant n-input mf-network). W.l.o.g. assume that [1:f] is the only comparator in A_d (the others are useless). Since α is k-tolerant, the network $\alpha' = A_1, \ldots, A_{d-1}$ is (k-1)-tolerant. This and the inductive hypothesis imply $d = \lceil \log n \rceil + 2(k-1) + 1$. Since α' is a (k-1)-tolerant n-input mf-network with the smallest depth (by the inductive hypothesis), the phase A_{d-1} must contain a comparator [1:g], for some $1 < g \le n$.

We show f = q. Suppose $f \neq q$. Since α' is a shortest (k-1)-tolerant mf-network, the comparator [1:g] in the phase A_{d-1} is essential — the minimum can be placed on the line g after the execution of the phases A_1, \ldots, A_{d-2} in the presence of at most k-1 faulty comparators. Such a minimum is never moved to the line 1 in the network α if the comparator [1:q] is faulty. It contradicts the assumption of ktolerance of α . Hence f = g. Let s be the index of the latest phase in α not containing a comparator [1:f]. Since n > 2 and α is an mf-network, such an index exists and $1 \leq s < \lceil \log n \rceil + 2(k-1)$. Suppose that A_s contains a comparator [1:h], for some $1 < h < n, h \neq f$. We show that the comparator [1:h] is unessential with respect to the network α' , i.e. the minimum can be never located on the line h after the execution of phases A_1, \ldots, A_{s-1} , in the presence of at most k-1 faults. Otherwise such a minimum is never moved to the line 1 in the network α if the comparator [1:h] is faulty. This contradicts k-tolerance of α . Similarly one can prove that if A_s contains a comparator [f:m], for some f < m < n, then this comparator is also unessential with respect to the (k-1)-tolerant network α' (the minimum can be never located on the line m after the execution of the first s-1 phases). This implies that the network $\alpha'' = A_1, \ldots, A_{s-1}$ always places the minimum on one of the lines 1 and f in the presence of at most k-1 faults. Now consider two cases: (1) In the presence of at most k-2 faults α'' always places minimum on the line 1.

In this case the network $A_1, \ldots, A_{s-1}, \{[1:f]\}$ is a (k-1)-tolerant mf-network with depth less than $\lceil \log n \rceil + 2(k-1) - a$ contradiction.

(2) There are input data such that α'' places minimum on the line f in the presence of at most k-l faults, for some $2 \leq l \leq k$. Consider the largest such l. In this case s < d-l, because we need l+1 more comparators [1:f] in α to move such a minimum from line f to line 1 in the presence of l additional faults. This implies that the network

$$A_1, \dots, A_{s-1}, \underbrace{\{[1:f]\}, \dots, \{[1:f]\}}_{l-\text{times}}$$

is a (k-1)-tolerant mf-network with depth less than $\lceil \log n \rceil + 2(k-1)$ - a contradiction.

The next Theorem establishes another lower bound which is nontrivial when k is fixed and n can be arbitrarily large.

Theorem 3. $T_k(n) \ge \log n + k \log \left(1.28 \frac{\lceil \log n \rceil}{k+1} + 1.92 \right).$

Proof. Let $\alpha = C_r C_{r-1} \dots C_1$ be a k-tolerant mf-network of depth r. For $i = r, r - 1, \dots, 0$ we define a partition of the set of line numbers $\{1, 2, \dots, n\}$ into pairwise disjoint sets (A_0^i, A_1^i, \dots) which classify line numbers after r - i initial phases of α and then, following the ideas of Berlekamp (cf. [4, 8]), we assign a weight w_i to each partition. Next we prove that this weight cannot decrease too much during one phase. This will give a lower bound on r.

Let A_j^i (i = r, r - 1, ..., 0, j = 0, 1, ...) consist of all numbers m such that: (i) there is a *j*-fault tolerant subnetwork α' of $C_r C_{r-1} ... C_{i+1}$ and an input vector $\overline{x} \in \mathcal{R}^n$ such that $(\overline{x}\alpha')[m] = \min(\overline{x})$, and

(ii) if j > 0 then for each (j-1)-fault subnetwork α' of $C_r C_{r-1} \dots C_{i+1}$ and for each input vector $\overline{x} \in \mathcal{R}^n$ of pairwise distinct numbers $(\overline{x}\alpha')[m] \neq \min(\overline{x})$.

One can observe that $A_0^r = \{1, 2, ..., n\}$, $A_1^r = A_2^r = ... = \emptyset$ and $A_0^0 = \{1\}$, $A_1^0 = A_2^0 = ... = A_k^0 = \emptyset$, $\bigcup_{j > k} A_j^0 = \{2, ..., n\}$, since α is a k-tolerant mf-network. Let

$$w_i = \sum_{j=0}^k \left(\binom{i}{k-j} \right) \left| A_j^i \right|,$$

where $\binom{i}{j} = \binom{i}{0} + \binom{i}{1} + \cdots + \binom{j}{j}$ is a sum of binomial coefficients. Asumme that $\binom{i}{j} = 0$ for j < 0 and $\binom{i}{j} = 0$ for j > i. Observe that $\binom{i+1}{j} = \binom{i}{j} + \binom{i}{j-1}$. Equivalently, instead of associating the weight w_i with a partition, we can assign an individual weight $v_m^i = \binom{i}{k-j}$ to each line $m \in A_j^i$ and consider w_i as $v_1^i + v_2^i + \cdots + v_n^i$.

Thus $w_r = \binom{r}{k}n$ and $w_0 = 1$. In order to finish the proof of the theorem we need the following lemma.

Lemma 4. For i = r, ..., 1,

$$w_{i-1} \ge \begin{cases} (1/2)w_i & i \ge 2k, \\ (2/5)w_i & 1 < i < 2k, \\ (1/3)w_i & i = 1. \end{cases}$$

Proof. Due to space limitations, it will appear in the full version of the paper.

It follows from Lemma 4 that $w_r \leq 2^r \frac{3}{2} (\frac{5}{4})^{2k-2} w_0$. Hence $\binom{r}{k} n \leq 2^r \frac{3}{2} (\frac{5}{4})^{2k-2}$ and consequently, using the result of Theorem 2, $T_k(n) \geq \min\{r \geq \lceil \log n \rceil + 2k : (\binom{r}{k})n \leq 2^r \frac{3}{2} (\frac{5}{4})^{2k-2}\}$. While $\binom{r}{k} \geq \binom{2r-k+3}{k+1}^k \geq (\frac{2\lceil \log n \rceil + 3k+3}{k+1})^k$, a few transformations of the above inequality yield

$$T_k(n) \ge \log n + k \log \left(1.28 \frac{\lceil \log n \rceil}{k+1} + 1.92 \right),$$

which concludes the proof of the theorem.

4 Upper Bound

In this section we construct a k-tolerant mf-network whose depth is nearly optimal. Unfortunately we are not able to compute this depth precisely but we will give a good estimate.

In what follows the term "network" means an *n*-input comparator network with fixed n > 4. It is easy to verify that $T_k(m) = \lceil \log m \rceil + 2k$, for m = 3, 4 and arbitrary $k \ge 0$.

Let α be a network of depth l with phases A_1, \ldots, A_l . We say that numbers l_i, r_i are the left and the right bounds of the phase A_i , respectively, if $l_i = \min(\{x : [x : y] \in A_i\})$ and $r_i = \max(\{y : [x : y] \in A_i\})$, for $i = 1, \ldots, l$. For two networks $\alpha = A_1, \ldots, A_a$ and $\beta = B_1, \ldots, B_b$ we define the network $\Gamma(\alpha, \beta)$ as follows:

Let *i* be the smallest non-negative integer $\geq a - b$ such that for each j > i, either j > a or $j \leq a$ and the left bound of the (j-i)-th phase in β is larger than the right bound of the *j*-th phase in α . Then

$$\Gamma(\alpha,\beta) = A_1, \ldots, A_i, A_{i+1} \cup B_1, \ldots, A_a \cup B_{a-i}, B_{a-i+1}, \ldots, B_b.$$

The depth of $\Gamma(\alpha, \beta)$ is i + b.⁴

Given $k \ge 1$ and networks $\alpha_1, \ldots, \alpha_k$ we define the k-run network $\Gamma_k(\alpha_1, \ldots, \alpha_k)$ with runs $\alpha_1, \ldots, \alpha_k$ as follows:

$$\Gamma_k(\alpha_1,\ldots,\alpha_k) = \begin{cases} \alpha_1 & k = 1\\ \Gamma(\Gamma_{k-1}(\alpha_1,\ldots,\alpha_{k-1}),\alpha_k) & k > 1. \end{cases}$$

Let $\gamma = \Gamma_k(\alpha_1, \ldots, \alpha_k)$ be a k-run network. For each $1 < i \leq k$, the number

$$|\Gamma_i(\alpha_1,\ldots,\alpha_i)| - |\Gamma_{i-1}(\alpha_1,\ldots,\alpha_{i-1})|$$

is called the delay of the run α_i with respect to γ . The delay of α_1 is defined to be its depth $|\alpha_1|$.

Proposition 5. Let $\gamma = \Gamma_k(\alpha_1, \ldots, \alpha_k)$ be a k-run network and let D_i , for $i \leq k$, be the delay of the run α_i with respect to γ . Then the depth of γ equals $D_1 + \cdots + D_k$.

Proposition6. Let $\gamma = \Gamma_{k+1}(\alpha_1, \ldots, \alpha_{k+1})$ be a (k+1)-run network whose runs are mf-networks. Then γ is a k-tolerant mf-network.

Proof. Deleting at most k comparators from γ leaves at least one minimum finding run α_i intact.

An *n*-input mf-network $\alpha = A_1, \ldots, A_a$ is called normal iff the following constraints are satisfied:

(1) α contains exactly n-1 comparators.⁵

(2) For every $1 \leq j < a$, if $[i_x : x] \in A_j$ and $[i_y : y] \in A_{j+1}$ then x > y.

(3) For every $1 \leq j \leq a$, if $[i_x : x] \in A_j$, $[i_y : y] \in A_j$ and $x \neq y$ then x > y iff $i_x < i_y$.

For every interval of lines $1 \le x, x+1, \ldots, y \le n$ we define the set of comparators COMP(x, y) as follows: Let $s = \lfloor \frac{x+y-1}{2} \rfloor$. Then

$$COMP(x, y) = \{ [x: y], [x+1: y-1], \dots, [s: y-(s-x)] \}.$$

We now describe the k-tolerant mf-network MIN_{k+1} whose depth is close to optimal. To this end we define the infinite sequence min_1, min_2, \ldots of normal minimumfinding networks and then MIN_{k+1} will be defined to be the (k + 1)-run network $\Gamma_{k+1}(min_1, \ldots, min_{k+1})$. The networks $min_i = M_1^i, \ldots, M_{d_i}^i$ are defined inductively on *i*.

i = 1

In this case $d_1 = \lceil \log n \rceil$ and $M_j^1 = COMP(1, \lceil \frac{n}{2^{j-1}} \rceil)$, for $j = 1, \ldots, \lceil \log n \rceil$. i > 1

Suppose that the network min_{i-1} is constructed. Let r_j^{i-1} be the right bound of the *j*-th phase in min_{i-1} , for $1 \leq j \leq d_{i-1}$. Set $r_j^{i-1} = 0$, for all $j > d_{i-1}$. Denote by s

⁴ It is important for further considerations that the last phase of β is not earlier than the last phase of α in the network $\Gamma(\alpha, \beta)$.

⁵ Observe that for each $1 < x \le n$, α must contain exactly one comparator of the form $[i_x : x]$.

the largest index such that $r_s^{i-1} \ge n-1$. For every $p \ge 1$ let l_p^i and r_p^i be defined as follows:

Let d_i be the smallest p such that $r_p^i = 2$. Then

$$min_i = COMP(l_1, r_1), COMP(l_2, r_2), \dots, COMP(l_{d_i}, r_{d_i}).$$

Easy induction on *i* shows that the networks min_i , $i \ge 1$, are normal mf-networks. This implies the following theorem:

Theorem 7. MIN_{k+1} is a (k + 1)-run network whose runs min_1, \ldots, min_{k+1} are normal mf-networks.

The network MIN_3 for n = 16 is illustrated in Fig. 1.

We will estimate the depth of network MIN_{k+1} from above. The following theorem will be helpful in this task.

Theorem 8. The depth of the network MIN_{k+1} is the minimum depth of all (k+1)run networks $\Gamma_{k+1}(\alpha_1, \ldots, \alpha_{k+1})$ with normal, minimum finding runs $\alpha_1, \ldots, \alpha_{k+1}$.

Proof. Let γ be a (k + 1)-run network $\Gamma_{k+1}(\alpha_1, \ldots, \alpha_{k+1})$ with normal, minimum finding runs $\alpha_1, \ldots, \alpha_{k+1}$. Denote by $I_{\gamma}(x, l)$ the index of the phase in γ containing a comparator $[i_x^l : x]$ from the run α_l , for $x = 2, \ldots, n$ and $l = 1, \ldots, k+1$. The theorem is an immediate consequence of the following lemma. Due to space limitations its proof will appear in the full version of the paper.

Lemma 9. For all x, l such that $1 < x \leq n$ and $1 \leq l \leq k + 1$, $I_{MIN_{k+1}}(x, l) \leq I_{\gamma}(x, l)$.



Fig. 1 The network MIN_3 for n = 16.

In order to give upper bounds on the depth of the network MIN_{k+1} we estimate it now at $1.5 \lceil \log n \rceil + 3k + 1$ and then construct two (k+1)-run networks Net_1 and Net_2 with normal minimum finding runs and of depths $\lceil \log n \rceil + k \lceil (\lceil \log n \rceil + 1) \rceil$ and $\lceil \frac{n}{2} \rceil + 2k$, respectively. By Theorem 8 we get

Theorem 10.

$$T_k(n) \le |MIN_{k+1}| \le \min \begin{cases} \lceil \log n \rceil + k \lceil \log(\lceil \log n \rceil + 1) \rceil, \\ 1.5 \lceil \log n \rceil + 3k + 1, \\ \lceil \frac{n}{2} \rceil + 2k. \end{cases}$$

The network MIN_{k+1} , whose depth is bounded by the depth of networks Net_1 and Net_2 , contains exactly (k+1)(n-1) comparators. It should be noted that this number is optimal (cf. [9]).

Estimation of the depth of MIN_{k+1}

We would like to estimate rights bounds of all k + 1 runs of MIN_{k+1} . To this end we define R_j^i , the right bound of the *i*-th run in the *j*-th phase, assuming that $R_j^i = n$ before the start of the run and $R_j^i = 0$ after its end. According to (1) we can state

$$R_{j}^{i} = \begin{cases} 0 & \text{if } i = 0 \text{ or } R_{j-1}^{i} = 2, \\ n & \text{if } 0 < i \le k+1 \text{ and } j = 0, \\ \lfloor \frac{1}{2}(R_{j-1}^{i-1} + R_{j-1}^{i} + 1) \rfloor & \text{if } 0 < j \text{ and } 0 < i \le k+1 \text{ and } R_{j-1}^{i} \neq 2. \end{cases}$$

Let us notice that for each $j \ge 0$, $R_j^0 \le R_j^1 \le \ldots \le R_j^{k+1}$, moreover, the depth of MIN_{k+1} is equal to the minimum j such that $R_j^1 = R_j^2 = \ldots = R_j^{k+1} = 0$. We can rewrite the recurrence for R_j^i using the following definition. Let $f(a_1, a_2, \ldots, a_{k+1}) = (g(0, a_1), g(a_1, a_2), g(a_2, a_3), \ldots, g(a_k, a_{k+1}))$, where $g(x, y) = \lfloor \frac{x+y+1}{2} \rfloor$ if x + y > 2 and 0 otherwise. One can easily observe that

$$f(R_{j-1}^1, R_{j-1}^2, \dots, R_{j-1}^{k+1}) = (R_j^1, R_j^2, \dots, R_j^{k+1}).$$

Let $f^{(j)}$ denote the *j*-th iteration of *f*. Thus $f^{(j)}(n, n, \ldots, n) = (R_j^1, R_j^2, \ldots, R_j^{k+1})$. Let an inequality of sequences of terms means inequalities of respective terms and let $\binom{n}{k} = \sum_{i=0}^{\min(n,k)} \binom{n}{i}$. The following lemmas give an estimation of $R_{\lceil \log n \rceil}^i$, $1 \le i \le k+1$.

Lemma 11. For $j = 0, 1, ..., \lceil \log n \rceil$,

$$f^{(j)}(n,n,\ldots,n) \leq \left(\lceil \frac{n}{2^j} \rceil \binom{j}{0}, \lceil \frac{n}{2^j} \rceil \binom{j}{1}, \ldots, \lceil \frac{n}{2^j} \rceil \binom{j}{k} \right).$$

Proof. Induction on j using the inequality $\lfloor \frac{\lceil x \rceil m+1}{2} \rfloor \leq \lceil \frac{x}{2} \rceil m$, where $x \geq 0$ and $m \geq 1$ is an integer. Notice that f is nondecreasing with respect to each argument.

Lemma 12.
$$f^{(m)}(n, n, ..., n) \le (0, 2^{m/3+2}, 2^{m/3+4}, ..., 2^{m/3+2k}), \text{ for } m = \lceil \log n \rceil$$

Proof. By induction on m one can prove that $\binom{m}{k} < 2^{m/3+2k}$ and get the result by substituting this in the inequality from Lemma 11.

To get the upper bound $1.5 \lceil \log n \rceil + 3k + 1$ we need the following lemma.

Lemma 13. $f^{(3\lfloor m/6 \rfloor + 3k+1)}(0, 2^{\lceil m/3 \rceil + 2}, 2^{\lceil m/3 \rceil + 4}, \dots, 2^{\lceil m/3 \rceil + 2k}) = (0, 0, \dots, 0).$

Proof. Let a > 1 be an integer. An easy calculation gives the following relations:

$$\begin{split} & f^{(3)}(0, a \cdot 2^3, a \cdot 2^5, a \cdot 2^7, \dots, a \cdot 2^{2k+1}) \leq (0, a, a \cdot 2^3, a \cdot 2^5, \dots, a \cdot 2^{2k-1}) \\ & f^{(4)}(0, 2^4, 2^6, 2^8, \dots, 2^{2k+2}) = (0, 0, 8, 38, 156, 625 \cdot 2^0, 625 \cdot 2^2 \dots, 625 \cdot 2^{2(k-5)}) \\ & f^{(3)}(\underbrace{0, \dots, 0}_{i-\text{times}}, \underbrace{8, 38, 156, 625, 625 \cdot 2^2 \dots, 625 \cdot 2^{2(k-i-4)}}_{\bar{x}}, 625 \cdot 2^{2(k-i-3)}) \leq (\underbrace{0, \dots, 0}_{i+1-\text{times}}, \bar{x}) \end{split}$$

We start with the sequence $(0, 2^{\lceil m/3 \rceil + 2}, 2^{\lceil m/3 \rceil + 4}, \dots, 2^{\lceil m/3 \rceil + 2k})$ and using $\lfloor m/6 \rfloor$ -times the first inequality we get the upper-bound sequence $(0, 2^4, 2^6, 2^8, \dots, 2^{2k+2})$, then we use second inequality and end up using (k-1)-times the last one.

Construction of the Network Net_1

The network Net_1 is a (k + 1)-run network whose first run is a network α and the remaining runs are copies of a network β . The networks α and β are defined below.

Let $H = \lceil \log n \rceil$ and $L = \lceil \log(\lceil \log n \rceil + 1) \rceil$. The network α is a sequence A_H , A_{H-1}, \ldots, A_1 of H phases (i.e. A_1 is the last phase in α), where

$$A_i = \begin{cases} COMP(1, 2^i) & 1 \le i < H\\ COMP(n - 2(n - 2^{H-1}) + 1, n) & i = H. \end{cases}$$

It is easy to observe that α is a normal mf-network (after performing phases A_H, \ldots, A_i , the minimum is always on one of the lines $1, 2, \ldots, 2^{i-1}$). Denote by r_i^{α} the right bound of the phase A_i in α . Then r_i^{α} equals 2^i , for $i = 1, \ldots, H - 1$, and n if i = H.

Now we describe the network β . Consider a sequence of integers $(v_i)_{i\geq 1}$, defined as follows:

$$v_i = \begin{cases} 2^i & 1 \le i \le L \\ 2v_{i-1} - 2^{i-L} & i > L. \end{cases}$$

Observe that

$$v_{L+H-1} = 2^{L+H-1} - (H-1)2^{H-1} = 2^{H-1}(2^L - H + 1) \ge 2^{H-1}(H + 1 - H + 1) = 2^H \ge n$$

Let S be the smallest integer such that $v_{L+S} \ge n$. For $i = 1, \ldots, L + S$ define:

$$r_i^{\beta} = \begin{cases} v_i & i < L + S - 1, \\ n & i = L + S; \end{cases} \qquad l_i^{\beta} = \begin{cases} 1 & i \le L, \\ 2^{i-L} + 1 & L < i \le L + S - 1, \\ n - 2(n - r_{L+S-1}^{\beta}) + 1 & i = L + S. \end{cases}$$

Observe that $l_i^{\beta} < r_i^{\beta}$, for each $1 \le i \le L + S$. The network β consists of L + S phases $B_{L+S}, B_{L+S-1}, \ldots, B_1$, where $B_i = COMP(l_i^{\beta}, r_i^{\beta})$. Note that for each $1 < i \le L + S$, $\lfloor \frac{l_i^{\beta} + r_i^{\beta}}{2} \rfloor = r_{i-1}^{\beta}$. This implies that β is a normal mf-network. Define

$$Net_1 = \Gamma_{k+1}(\alpha, \underbrace{\beta, \dots, \beta}_{k-\text{times}}).$$

Observe that $l_j^{\beta} = 1$, for $j = 1, \ldots, L$, but $r_i^{\beta} \leq r_i^{\alpha} = 2^i < 2^i + 1 = l_{L+i}^{\beta}$, for each $i = 1, \ldots, S$. This implies that the delay of each copy of β in Net_1 is L. Since $|\alpha| = H$ we get (by Proposition 6) that the depth of Net_1 equals H + kL.

Construction of the Network Net_2

The network Net_2 consists of k + 1 identical runs α . Now the network α is defined as follows:

Let $S = \lfloor \frac{n}{2} \rfloor$. For $i = 1, \dots, S$ define:

$$r_i^{\alpha} = \begin{cases} 2i & i < S - 1, \\ n & i = S; \end{cases} \qquad l_i^{\alpha} = \begin{cases} 1 & i = 1, 2, \\ 2(i-2) + 1 & 2 < i < S, \\ n - 2(n - 2(S - 1)) + 1 & i = S. \end{cases}$$

Obviously $l_i^{\alpha} < r_i^{\alpha}$, for $i \leq S$. The network α consists of S phases $A_S, A_{S-1}, \ldots, A_1$, where $A_i = COMP(l_i^{\alpha}, r_i^{\alpha})$. Now define

$$Net_2 = \Gamma_{k+1}(\underbrace{\alpha, \dots, \alpha}_{(k+1)-\text{times}}).$$

The same argument as before shows that Net_2 is a k-tolerant mf-network of depth S + 2k.

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