

Computads for Finitary Monads on Globular Sets

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ABSTRACT. A finitary monad A on the category of globular sets provides basic algebraic operations from which more involved 'pasting' operations can be derived. To make this rigorous, we define A -computads and construct a monad on the category of A -computads whose algebras are A -algebras; an action of the new monad encapsulates the pasting operations. When A is the monad whose algebras are n -categories, an A -computad is an n -computad in the sense of R.Street. When A is associated to a higher operad (in the sense of the author), we obtain pasting in weak n -categories. This is intended as a first step towards proving the equivalence of the various definitions of weak n -category now in the literature.

Introduction

This work arose as a reflection on the foundation of higher dimensional category theory. One of the main ingredients of any proposed definition of weak n -category is the shape of diagrams (pasting scheme) we accept to be composable. In a globular approach [3] each k -cell has a source and target $(k-1)$ -cell. In the opetopic approach of Baez and Dolan [1] and the multitopic approach of Hermida, Makkai and Power [7] each k -cell has a unique $(k-1)$ -cell as target and a whole $(k-1)$ -dimensional pasting diagram as source. In the theory of strict n -categories both source and target may be a general pasting diagram [9, 14, 15].

The globular approach being the simplest one seems too restrictive to describe the combinatorics of higher dimensional compositions. Yet, we argue that this is a false impression. Moreover, we prove that this approach is a basic one from which the other type of composable diagrams may be derived. One theorem proved here asserts that the category of algebras of a finitary monad on the category of n -globular sets is **equivalent** to the category of algebras of an appropriate monad on the special category (of computads) constructed from the data of the original

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monad. In the case of the monad derived from the universal contractible operad [3] this result may be interpreted as the equivalence of the definitions of weak n -categories (in the sense of [3]) based on the ‘globular’ and general pasting diagrams. It may be also considered as the first step toward the proof of equivalence of the different definitions of weak n -category.

We also develop a general theory of computads and investigate some properties of the category of generalized computads. It turned out, that in a good situation this category is a topos (and even a presheaf topos under some not very restrictive conditions, the property firstly observed by S.Schanuel for 2-computads in the sense of Street and proved in [4] for arbitrary n).

1. Preliminary discussion

Computads were defined by R.Street [12, 14] for the purposes of the presentation of (strict) n -categories. The definition is inductive. Let us give it here.

A 0-computad is a set, a free 0-category on it is this set itself.

A 1-computad is a directed graph, a free 1-category on it is the usual category of directed paths in that graph.

Suppose we know what an $(n-1)$ -computad \mathcal{C} is and have a construction of a free (strict) $(n-1)$ -category $\mathcal{F}_{n-1}\mathcal{C}$ on it. Then an n -computad consists of a set C_n of n -cells, an $(n-1)$ -computad \mathcal{C} and two functions (source and target)

$$s_{n-1}, t_{n-1} : C_n \longrightarrow (\mathcal{F}_{n-1}\mathcal{C})_{n-1}$$

where $(\mathcal{F}_{n-1}\mathcal{C})_{n-1}$ is the set of $(n-1)$ -cells in $\mathcal{F}_{n-1}\mathcal{C}$. These functions must satisfy the equations

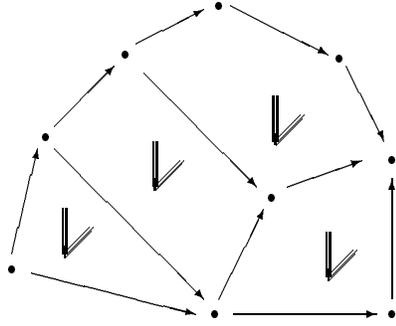
$$(1.1) \quad s_{n-2} \cdot s_{n-1} = s_{n-2} \cdot t_{n-1}, \quad t_{n-2} \cdot s_{n-1} = t_{n-2} \cdot t_{n-1},$$

where s_{n-2}, t_{n-2} are source and target functions in $\mathcal{F}_{n-1}\mathcal{C}$.

The n -computads form a category, where a morphism of n -computads consists of a morphism between corresponding $(n-1)$ -computads and a map between the sets of n -cells which agree one to another via the source and target functions.

Computads turned out to be an excellent device for working with presentations of n -categories and pasting operations (see [14] for many examples), especially in the dimension 2 where R.Street proved that 2-categories may be characterised in terms of an abstract pasting operation on an appropriate 2-computad. More precisely, the forgetful functor \mathcal{W}_2 from 2-computads to 2-categories is monadic [12].

The 2-computads appear also in the calculus of string diagrams [8]. To explain this connection look at a typical 2-cell in a free 2-category generated by a terminal 2-computad:



If we take a planar dual of this diagram we obtain a topological graph that is usually called string diagram [14]. A deep theorem of A.Joyal and R.Street [8] asserts that the free 2-category on the terminal 2-computad is isomorphic to the 2-category of deformation classes of progressive plane graphs.

However, if we try to go on one dimension up and develop the calculus of surface diagrams in 3-dimensional space we find that 3-computads are not sufficient. The right notion for this is Gray-computad in the sense of [11]. Again we have a theorem that a free Gray-category on a terminal Gray-computad is isomorphic to the Gray-category of deformation classes of surface diagrams [11].

This suggests that higher dimensional analogues of Gray-computad might be very important as they, probably, connect higher category theory to the geometry of real space.

Furthermore, the notion of computad appears in connection with the appropriate notion of higher-dimensional category. According to [3] these categories are algebras of an appropriate operad (in the sense of [3]) on the category of globular sets. By a globular set we mean a sequence of sets

$$X_0, X_1, \dots, X_r, \dots$$

together with source and target maps

$$s_{n-1}, t_{n-1} : X_n \longrightarrow X_{n-1}$$

satisfying the equation (1.1). Every operad on globular sets generates a finitary monad whose algebras are algebras of the operad. In the case of the terminal operad this monad assigns to a globular set X the underlying globular set of the free ω -category generated by X (see [3] or [16] for the description of this monad in terms of the combinatorics of trees). Thus we expect more generally, that every finitary monad on globular sets generates its own notion of computad. In the case of terminal operad we should get Street's n -computads. We prove this in the next sections.

2. Definition of computad

Let $\mathcal{G}l_n$ be the subcategory of Set which has as objects the sets $\{0, k\}$, $0 \leq k \leq n$ and only the constant function $\{0, k\} \rightarrow \{0, l\}$, $k > l$, as nonidentity morphisms. The category $Glob_n$ of n -globular sets (the category $Glob$ of globular sets for $n = \infty$) is the category of functors

$$\mathcal{G}l_n \rightarrow Set.$$

For an n -globular set C we write C_k for $C\{0, k\}$ and s_l, t_l for the image under C of the functions $\{0, k\} \rightarrow \{0, l\}$, $k > l$, constant at $0, l$, respectively.

For every $k \leq n$, we have a functor

$$L_k : Glob_k \rightarrow Glob_n$$

which assigns to every k -globular set X a globular set $L_k X$ which coincides with X up to dimension k and has empty sets in all dimensions greater than k . This functor has a right adjoint tr_k which assigns the k -truncation to each globular set. The counit of this adjunction will be denoted by \wp_k .

Let $A_n = (A_n, \mu_n, \epsilon_n)$ be a finitary monad on $Glob_n$ (recall that monad is called finitary if its functor part preserves filtered colimits [10]). For every $k \leq n$, we have a monad $A_k = (A_k, \mu_k, \epsilon_k)$ on $Glob_k$ with underlying functor $tr_k A_n L_k$, multiplication

$$tr_k A_n L_k tr_k A_n L_k \xrightarrow{\wp_k} tr_k A_n^2 L_k \xrightarrow{\mu_n} tr_k A_n L_k$$

and unit

$$I \xrightarrow{id} tr_k L_k \xrightarrow{\epsilon_k} tr_k A_n L_k.$$

We denote by Alg_k the category of algebras of A_k , by $W_k : Alg_k \rightarrow Glob_k$ the corresponding forgetful functor and by $F_k : Glob_k \rightarrow Alg_k$ the left adjoint of W_k . The functor tr_k obviously induces a functor

$$Alg_n \longrightarrow Alg_k$$

which will be denoted by tr_k as well.

Let us give the following inductive definition:

The category $Comp_0$ of A_0 -computads is $Glob_0$. The functors

$$\mathcal{W}_0 = W_0 : Alg_0 \rightarrow Comp_0$$

$$\mathcal{F}_0 = F_0 : Comp_0 \rightarrow Alg_0$$

are forgetful and free A_0 -algebra functors, respectively.

Let us suppose now that the category $Comp_{n-1}$ of A_{n-1} -computads is already defined together with two functors:

$$\mathcal{W}_{n-1} : Alg_{n-1} \rightarrow Comp_{n-1}$$

$$\mathcal{F}_{n-1} : Comp_{n-1} \rightarrow Alg_{n-1}$$

such that \mathcal{F}_{n-1} is left adjoint to \mathcal{W}_{n-1} .

DEFINITION 2.1. An A_n -computad \mathcal{C} is a triple (C, ϕ, \mathcal{C}') consisting of an n -globular set C , an A_{n-1} -computad \mathcal{C}' and an isomorphism ϕ in $Glob_{n-1}$

$$\phi : W_{n-1}(\mathcal{F}_{n-1}\mathcal{C}') \rightarrow tr_{n-1}C.$$

A morphism

$$F : (C, \phi, \mathcal{C}') \rightarrow (D, \psi, \mathcal{D}')$$

of A_n -computads is a pair (f, Φ) where $f : C \rightarrow D$ is a morphism of n -globular sets and $\Phi : \mathcal{C}' \rightarrow \mathcal{D}'$ is a morphism of A_{n-1} -computads such that the following diagram commutes

$$\begin{array}{ccc}
 W_{n-1}(\mathcal{F}_{n-1}\mathcal{C}') & \xrightarrow{W_{n-1}(\mathcal{F}_{n-1}\Phi)} & W_{n-1}(\mathcal{F}_{n-1}\mathcal{D}') \\
 \phi \downarrow \swarrow & & \psi \downarrow \swarrow \\
 tr_{n-1}C & \xrightarrow{tr_{n-1}f} & tr_{n-1}D
 \end{array}$$

The most important case for us are the monads coming from higher operads in the sense of [3]. We refer the reader to [3, 16] for the definition. The brief explanation is given in the example 2 of section 6 of this paper.

DEFINITION 2.2. For an n -operad \mathcal{A}_n in $Span_n$, the category of \mathcal{A}_n -computads is the category of A_n -computads, where A_n is the monad on $Glob_n$ generated by \mathcal{A}_n .

3. Existence

To give sense to this definition we have to construct an adjoint pair of functors

$$\mathcal{F}_n \dashv \mathcal{W}_n.$$

Let G be an object of Alg_n . The counit of the adjunction $\mathcal{F}_{n-1} \dashv \mathcal{W}_{n-1}$ gives a morphism

$$r_{n-1} : \mathcal{F}_{n-1}\mathcal{W}_{n-1}tr_{n-1}G \rightarrow tr_{n-1}G.$$

Define an n -globular set \mathcal{G} in the following way. The $(n-1)$ -skeleton of \mathcal{G} coincides with $W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}tr_{n-1}G$ and

$$\begin{aligned}
 \mathcal{G}_n = \{ & (\xi, a, \eta) \in \mathcal{G}_{n-1} \times G_n \times \mathcal{G}_{n-1} \mid s_{n-2}\xi = s_{n-2}\eta, t_{n-2}\xi = t_{n-2}\eta, \\
 & s_{n-1}a = r_{n-1}(\xi), t_{n-1}a = r_{n-1}(\eta) \}.
 \end{aligned}$$

Define

$$s_{n-1}(\xi, a, \eta) = \xi, \quad t_{n-1}(\xi, a, \eta) = \eta.$$

Then put

$$\mathcal{W}_n G = (\mathcal{G}, id, \mathcal{W}_{n-1}tr_{n-1}G).$$

Now our goal is to construct a left adjoint \mathcal{F}_n for \mathcal{W}_n .

For an A_n -computad $\mathcal{C} = (C, \phi, \mathcal{C}')$, define

$$V_n(\mathcal{C}) = C$$

and $V_0 = id$ for $n = 0$. Define also the truncation functor

$$tr_{n-1} : Comp_n \longrightarrow Comp_{n-1}$$

by

$$tr_{n-1}\mathcal{C} = \mathcal{C}'.$$

Construct firstly a natural transformation

$$\Theta_n : V_n \mathcal{W}_n \rightarrow W_n.$$

We define $\Theta_n(G)$ to be the morphism of n -globular sets which coincides with

$$W_{n-1}r_{n-1} : W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}tr_{n-1}G \rightarrow W_{n-1}tr_{n-1}G$$

up to dimension $n-1$ and has

$$\Theta_n(\xi, a, \eta) = a$$

in the dimension n .

From the adjunctions $\mathcal{F}_{n-1} \dashv \mathcal{W}_{n-1}$, $F_{n-1} \dashv W_{n-1}$ we have the following mates

$$\begin{aligned} \underline{\Theta}_{n-1} : V_{n-1}\mathcal{W}_{n-1} &\rightarrow W_{n-1} \\ \underline{\Omega}_{n-1} : V_{n-1} &\rightarrow W_{n-1}\mathcal{F}_{n-1} \\ \Upsilon_{n-1} : F_{n-1}V_{n-1} &\rightarrow \mathcal{F}_{n-1}. \end{aligned}$$

We also have an A_{n-1} -algebra morphism

$$\beta : A_{n-1}tr_{n-1}C = tr_{n-1}A_nL_{n-1}tr_{n-1}C \xrightarrow{tr_{n-1}A_n\phi_{n-1}} tr_{n-1}A_nC.$$

Consider the following pair of composites

$$A_{n-1}V_{n-1}tr_{n-1} \xrightleftharpoons[A_{n-1}\epsilon_{n-1}]{\epsilon_{n-1}} A_{n-1}^2V_{n-1}tr_{n-1} \xrightarrow{\Psi} tr_{n-1}A_nV_n$$

where Ψ is the composite

$$\Psi = \beta \cdot A_{n-1}\phi \cdot A_{n-1}W_{n-1}\Upsilon_{n-1}$$

This pair is mated with a pair

$$(3.1) \quad L_{n-1}A_{n-1}V_{n-1}tr_{n-1} \xrightleftharpoons[d_1]{d_0} A_nV_n$$

which in its turn is mated to

$$F_nL_{n-1}A_{n-1}V_{n-1}tr_{n-1} \xrightleftharpoons[\partial_1]{\partial_0} F_nV_n$$

We define

$$\Upsilon_n : F_nV_n \rightarrow \mathcal{F}_n$$

to be the coequalizer of (∂_0, ∂_1) (which exists due to finitary assumption [10].)

PROPOSITION 3.1. *The functor \mathcal{F}_n is left adjoint to \mathcal{W}_n .*

PROOF. Let G be an A_n -algebra and

$$(f, \Phi) : \mathcal{C} \rightarrow \mathcal{W}_nG$$

be an A_n -computad morphism. So there exists a unique A_n -algebra morphism

$$\bar{f} : F_nV_n\mathcal{C} \rightarrow G$$

fitting commutatively into the diagram

$$\begin{array}{ccc} V_n\mathcal{C} & \xrightarrow{\epsilon_n} & A_nV_n\mathcal{C} \\ f \downarrow & & W_n\bar{f} \downarrow \\ V_n\mathcal{W}_nG & \xrightarrow{\Theta_n} & W_nG \end{array}$$

We now have to prove that

$$(3.2) \quad W_n \bar{f} \cdot d_0 = W_n \bar{f} \cdot d_1.$$

It is obvious that we need to prove (3.2) only for $(n-1)$ -truncation of these morphisms. That is we need to prove that $tr_{n-1} W_n \bar{f}$ coequalizes the diagram (3.1) after truncation.

We have the following diagram

$$\begin{array}{ccccc}
 A_{n-1}V_{n-1}tr_{n-1}\mathcal{C} & \xrightarrow{\epsilon_{n-1}} & A_{n-1}^2V_{n-1}tr_{n-1}\mathcal{C} & & \\
 \downarrow W_{n-1}\Upsilon_{n-1} & & \downarrow A_{n-1}W_{n-1}\Upsilon_{n-1} & & \\
 W_{n-1}\mathcal{F}_{n-1}tr_{n-1}\mathcal{C} & \xrightarrow{\epsilon_{n-1}} & A_{n-1}W_{n-1}\mathcal{F}_{n-1}tr_{n-1}\mathcal{C} & & \\
 \downarrow W_{n-1}\mathcal{F}_{n-1}tr_{n-1}f & \searrow \phi & \downarrow A_{n-1}\phi & & \\
 W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}tr_{n-1}G & & tr_{n-1}V_n\mathcal{C} & \xrightarrow{\epsilon_{n-1}} & A_{n-1}tr_{n-1}V_n\mathcal{C} \\
 \downarrow id & \swarrow * & \downarrow tr_{n-1}\epsilon_{n-1} & \downarrow \beta & \\
 tr_{n-1}V_n\mathcal{W}_nG & \xrightarrow{W_{n-1}r_{n-1}} & tr_{n-1}W_nG & \xrightarrow{tr_{n-1}\epsilon_{n-1}} & tr_{n-1}A_nV_n\mathcal{C} \\
 & & & \swarrow ** & \\
 & & & & tr_{n-1}W_n\bar{f}
 \end{array}$$

In this diagram the square $(*)$ commutes by definition of morphism of computads, the square $(**)$ commutes by definition of \bar{f} . The others commute by naturality. Hence, the whole diagram commutes.

We have thus proved the composition $tr_{n-1} W_n \bar{f} \cdot \beta \cdot A_{n-1} \phi \cdot A_{n-1} W_{n-1} \Upsilon_{n-1}$ is an A_{n-1} -algebras morphism. So

$$tr_{n-1} W_n \bar{f} \cdot \beta \cdot A_{n-1} \phi \cdot A_{n-1} W_{n-1} \Upsilon_{n-1} = tr_{n-1} W_n \bar{f} \cdot tr_{n-1} d_0 \cdot \mu_{n-1}.$$

And we obtain

$$\begin{aligned}
 tr_{n-1} W_n \bar{f} \cdot tr_{n-1} d_1 &= tr_{n-1} W_n \bar{f} \cdot \beta \cdot A_{n-1} \phi \cdot A_{n-1} tr_{n-1} \rho_{n-1} \cdot A_{n-1} \epsilon_{n-1} = \\
 &tr_{n-1} W_n \bar{f} \cdot tr_{n-1} d_0 \cdot \mu_{n-1} \cdot A_{n-1} \epsilon_{n-1} = tr_{n-1} W_n \bar{f} \cdot tr_{n-1} d_0.
 \end{aligned}$$

Hence, equation (3.2) is proved. So the morphism \bar{f} can be extended uniquely to a morphism of A_n -algebras

$$\mathcal{F}_n \mathcal{C} \rightarrow G.$$

It is not hard to see that this construction generates a natural bijection

$$Comp_n(\mathcal{C}, \mathcal{W}_n G) \simeq Alg_n(\mathcal{F}_n \mathcal{C}, G)$$

as required. \square

DEFINITION 3.1. We say that A_n is truncable in dimension k if

$$\beta : A_k tr_k \longrightarrow tr_k A_{k+1}$$

is an invertible natural transformation. We say that A_n is truncable if it is truncable in every dimension.

For a truncable A_n we can give the following alternative construction of the left adjoint \mathcal{F}_n .

Let \mathcal{C} be an A_n -computad. Let

$$M_{-1}\mathcal{C} = L_{n-1}W_{n-1}\mathcal{F}_{n-1}tr_{n-1}\mathcal{C} , M_0\mathcal{C} = V_n\mathcal{C}$$

and

$$\phi_0 = \phi : tr_{n-1}M_{-1}\mathcal{C} \rightarrow tr_{n-1}M_0\mathcal{C}.$$

Notice that ϕ_0 transports the A_{n-1} -algebra structure from $tr_{n-1}M_{-1}\mathcal{C}$ to $tr_{n-1}M_0\mathcal{C}$.

Suppose that a globular set $M_r\mathcal{C}$, together with an A_{n-1} -algebra structure on $tr_{n-1}M_r\mathcal{C}$ and an A_{n-1} -algebras isomorphism

$$\phi_r : tr_{n-1}M_{r-1}\mathcal{C} \rightarrow tr_{n-1}M_r\mathcal{C},$$

are already constructed. Then define $M_{r+1}\mathcal{C}$ to be the pushout of the following diagram

$$\begin{array}{ccc} L_{n-1}A_{n-1}tr_{n-1}M_r\mathcal{C} & \xrightarrow{\gamma_r} & A_n M_r\mathcal{C} \\ \downarrow L_{n-1}k & & \downarrow p_r \\ L_{n-1}tr_{n-1}M_r\mathcal{C} & \xrightarrow{\alpha_{r+1}} & M_{r+1}\mathcal{C} \end{array}$$

where k is the structure morphism for the A_{n-1} -algebra $tr_{n-1}M_r\mathcal{C}$ and γ_r is the composite $\phi_{n-1} \cdot L_{n-1}\beta$. Define also

$$\phi_{r+1} = tr_{n-1}\alpha_{r+1},$$

which is invertible as β is invertible and, hence, determines an A_{n-1} -algebra structure on $tr_{n-1}M_{r+1}\mathcal{C}$.

Then we have the following sequence of morphisms

$$M_0\mathcal{C} \xrightarrow{\epsilon_n} A_n M_0\mathcal{C} \xrightarrow{p_1} M_1\mathcal{C} \xrightarrow{\epsilon_n} A_n M_2\mathcal{C} \xrightarrow{p_2} \dots$$

As A_n is finitary, the colimit of this sequence has a canonical A_n -algebra structure induced by the following commutative diagrams for every r .

$$(3.3) \quad \begin{array}{ccccc} A_n^2 M_r & \xrightarrow{A_n p_{r+1}} & A_n M_{r+1} & \xrightarrow{A_n \epsilon_n} & A_n^2 M_{r+1} \\ \downarrow \mu_n & & & & \downarrow \mu_n \\ A_n M_r & \xrightarrow{p_{r+1}} & M_{r+1} & \xrightarrow{\epsilon_n} & A_n M_{r+1} \end{array}$$

PROPOSITION 3.2. *The colim_r M_r C with A_n-algebra structure as above is canonically isomorphic to F_n C.*

PROOF. Let G be an A_n -algebra and let

$$f : \mathcal{C} \longrightarrow \mathcal{W}_n G$$

be a morphism of A_n -computads. Define H_0 to be the following composite

$$M_0 \mathcal{C} = V_n \mathcal{C} \xrightarrow{V_n f} V_n \mathcal{W}_n G \xrightarrow{\Theta_n} W_n G$$

We have the following commutative diagram after truncation

$$\begin{array}{ccccc}
 tr_{n-1} M_0 & \xrightarrow{tr_{n-1} f} & tr_{n-1} V_n \mathcal{W}_n G & \xrightarrow{tr_{n-1} \Theta_n} & tr_{n-1} W_n G \\
 \downarrow \phi & & \searrow id & & \nearrow W_{n-1} r_{n-1} \\
 W_{n-1} \mathcal{F}_{n-1} tr_{n-1} \mathcal{C} & \xrightarrow{W_{n-1} \mathcal{F}_{n-1} tr_{n-1} f} & W_{n-1} \mathcal{F}_{n-1} \mathcal{W}_{n-1} tr_{n-1} G & &
 \end{array}$$

Hence, $tr_{n-1} H_0$ is an A_{n-1} -algebras morphism.

Suppose we have constructed a morphism

$$H_r : M_r \mathcal{C} \longrightarrow W_n G$$

such that $tr_{n-1} H_r$ is an A_{n-1} -algebras morphism. Then we can lift H_r to an A_n -algebra morphism

$$\tilde{H}_r : A_n M_r \mathcal{C} \longrightarrow W_n G$$

As $tr_{n-1} H_r$ is a morphism of A_{n-1} -algebras, we have that $tr_{n-1} \tilde{H}_r \cdot \beta$ is equal to the following composite

$$A_{n-1} tr_{n-1} M_r \mathcal{C} \xrightarrow{k} tr_{n-1} M_r \xrightarrow{tr_{n-1} H_r} tr_{n-1} W_n G.$$

We get, therefore, the morphism

$$L_{n-1} tr_{n-1} M_r \mathcal{C} \longrightarrow W_n G,$$

which together with \tilde{H}_r determine the morphism

$$H_{r+1} : M_{r+1} \mathcal{C} \longrightarrow W_n G.$$

It is not hard to check that $tr_{n-1} H_{r+1}$ is an A_{n-1} -algebra morphism. Hence, the induction works and we have a morphism

$$H : colim_r M_r \mathcal{C} \longrightarrow W_n G.$$

Moreover, H is an A_n -algebra morphism as it is also the colimit of \tilde{H}_r . It is now obvious how to recover a computad morphism f for a given A_n -algebra morphism H .

We have thus proved that $colim_r M_r$ is left adjoint to \mathcal{W}_n and, hence, that is canonically isomorphic to \mathcal{F}_n . \square

COROLLARY 3.1. *For a truncable finitary monad A_n there exists a natural isomorphism*

$$\mathcal{F}_k tr_k \longrightarrow tr_k \mathcal{F}_{k+1}$$

PROOF. The sequence of isomorphisms ϕ_r gives the desired result. \square

4. Some properties of computads

The results of this section are based on the theory developed in [4]. Recall the necessary definitions.

DEFINITION 4.1. Let $\phi : F \rightarrow G$ be a natural transformation between two functors. We call it cartesian provided every naturality square

$$\begin{array}{ccc} F(a) & \xrightarrow{\phi} & G(a) \\ F(f) \downarrow & & \downarrow G(f) \\ F(b) & \xrightarrow{\phi} & G(b) \end{array}$$

is a pullback.

It is obvious that the composition of cartesian transformations is cartesian and if the composition $\phi \cdot \psi$ is cartesian and ϕ is cartesian then ψ is cartesian as well.

The following lemma is just a slight generalization of the fact, that a pullback of a cartesian natural transformation is cartesian and may be proved by the same method.

LEMMA 4.1. *Let P be a small category, F, G two functors from a category A to the presheaf category Set^P , and $f : F \rightarrow G$ a cartesian natural transformation. Let B be a functor from A to Set and $\phi_p(a)$ be a cone from $B(a)$ to $G(a)$ which is natural in a . Then we have a functor L to the category of sets which, for a fixed object a , is the limit of the diagram*

$$\begin{array}{ccc} L(a) & \xrightarrow{f^*} & B(a) \\ \psi \downarrow & & \downarrow \phi \\ F(a) & \xrightarrow{f_a} & G(a) \end{array}$$

We have also a natural transformation

$$f^* : L \rightarrow B.$$

Then we claim that f^* is cartesian.

We shall call a *wide pullback* any limit indexed by a category which is a poset with a terminal object, in which all maximal chains have length 2.

DEFINITION 4.2. A monad $A = (A, \mu, \epsilon)$ on a category C preserves pullbacks (wide pullbacks) provided its functor part preserves them. It is called cartesian if it preserves pullbacks and ϵ and μ are cartesian natural transformations.

THEOREM 4.1. *Let A_n be a truncable finitary monad on $Glob_n$.*

(i) *If A_n preserves (finite) pullbacks then \mathcal{F}_n preserves them and $Comp_n$ is an elementary topos,*

(ii) If A_n preserves wide pullbacks and ϵ_n is a cartesian natural transformation then \mathcal{F}_n preserves them and $Comp_n$ is a presheaf topos.

PROOF. The proof generalizes the example 4.6 from [4]. We use induction on n . If $n = 0$ the statement is true by definition.

Suppose we know that the category $Comp_{n-1}$ is a topos (a presheaf topos) and, in addition, \mathcal{F}_{n-1} preserves pullbacks (wide pullbacks). Consider the functor

$$T_{n-1} : Comp_{n-1} \longrightarrow Set ,$$

which assigns to a computad \mathcal{C} the set of the parallel pairs of $(n - 1)$ -cells from $W_{n-1}\mathcal{F}_{n-1}\mathcal{C}$. Then we have the equivalence of categories

$$Comp_n \sim Set \downarrow T_{n-1}.$$

Now, prove that T_{n-1} preserves pullbacks (for (i)) or wide pullbacks (for (ii)). Notice that T_{n-1} is isomorphic to the following composite

$$Comp_{n-1} \xrightarrow{\mathcal{F}_{n-1}} Alg_{n-1} \xrightarrow{Alg_{n-1}(\mathcal{F}_{n-1}S^{n-1}, -)} Set$$

where S^{n-1} is an $n - 1$ -globular set, which has two elements $-$ and $+$ in every dimension and

$$s(-) = s(+) = - , t(-) = t(+) = +.$$

By the inductive assumption, \mathcal{F}_{n-1} preserves pullbacks (wide pullbacks), so T_{n-1} does. According to the results of [4] it is sufficient for $Set \downarrow T_{n-1}$ to be a topos (preasheaf topos).

It remains to prove that \mathcal{F}_n preserves pullbacks (wide pullbacks). Let us prove, firstly, that the functors M_r from proposition 3.2 preserve pullbacks (wide pullbacks in the case (ii)). Use induction on r . For $r = 0$ we have to prove that V_n preserves pullbacks (wide pullbacks). But that is obvious as we know that \mathcal{F}_{n-1} preserves pullbacks (wide pullbacks) by our inductive assumption.

Let $\mathcal{C} = \lim_{\alpha} \mathcal{C}_{\alpha}$ be a pullback (wide pullback) in $Comp_n$. Then

$$\begin{aligned} tr_{n-1} \lim_{\alpha} (M_{r+1} \mathcal{C}_{\alpha}) &\simeq \lim_{\alpha} (tr_{n-1} M_{r+1} \mathcal{C}_{\alpha}) \simeq \lim_{\alpha} (W_{n-1} \mathcal{F}_{n-1} tr_{n-1} \mathcal{C}_{\alpha}) \\ &\simeq W_{n-1} \mathcal{F}_{n-1} \lim_{\alpha} (tr_{n-1} \mathcal{C}_{\alpha}) \simeq tr_{n-1} M_{r+1} \lim_{\alpha} (\mathcal{C}_{\alpha}), \end{aligned}$$

as \mathcal{F}_{n-1} preserves pullbacks (wide pullbacks).

Assume we know that M_r preserves pullbacks (wide pullbacks). Then, in dimension n

$$\lim_{\alpha} (M_{r+1} \mathcal{C}_{\alpha})_n \simeq \lim_{\alpha} (A_n M_r \mathcal{C}_{\alpha})_n \simeq (A_n M_r \mathcal{C})_n \simeq (M_{r+1} \mathcal{C})_n.$$

Now, in case (i),

$$\begin{aligned} W_n \mathcal{F}_n \mathcal{C} &\simeq \text{colim}_r (M_r \mathcal{C}) \simeq \text{colim}_r (\lim_{\alpha} (M_r \mathcal{C}_{\alpha})) \\ &\simeq \lim_{\alpha} (\text{colim}_r (M_r \mathcal{C}_{\alpha})) \simeq \lim_{\alpha} W_n \mathcal{F}_n (\mathcal{C}_{\alpha}) \end{aligned}$$

as finite limits in $Glob_n$ commutes with filtered colimits.

In case (ii) we can not use the above calculation but we have the additional property that the unit of A_n is a cartesian natural transformation. Using the usual induction on n we can assume that $tr_{n-1} W_n \mathcal{F}_n \mathcal{C} \simeq W_{n-1} \mathcal{F}_{n-1} tr_{n-1} \mathcal{C}$ is isomorphic to $tr_{n-1} \lim_{\alpha} (W_n \mathcal{F}_n \mathcal{C}_{\alpha})$. It remains to establish this isomorphism in dimension n .

But in dimension n every morphism

$$(p_r)_n : (A_n M_r)_n \longrightarrow (M_{r+1})_n$$

is an isomorphism. And, for every projection $pr_{\alpha} : C_{\alpha} \rightarrow C_0$, we have a pullback

$$\begin{array}{ccc}
(M_r \mathcal{C}_\alpha)_n & \xrightarrow{\epsilon_n} & (A_n M_r \mathcal{C}_\alpha)_n \\
pr_\alpha \downarrow \swarrow & & pr_\alpha \downarrow \swarrow \\
(M_r \mathcal{C}_0)_n & \xrightarrow{\epsilon_n} & (A_n M_r \mathcal{C}_0)_n
\end{array}$$

Hence, for every α, r and k we have a pullback

$$(4.1) \quad \begin{array}{ccc}
(M_r \mathcal{C}_\alpha)_n & \xrightarrow{c_{r,r+k}} & (M_{r+k} \mathcal{C}_\alpha)_n \\
pr_\alpha \downarrow \swarrow & & pr_\alpha \downarrow \swarrow \\
(M_r \mathcal{C}_0)_n & \xrightarrow{c_{r,r+k}} & (M_{r+k} \mathcal{C}_0)_n
\end{array}$$

We now construct a map

$$lim_\alpha colim_r (M_r \mathcal{C}_\alpha)_n \rightarrow colim_r lim_\alpha (M_r \mathcal{C}_\alpha)_n$$

inverse to the canonical map. Indeed, let

$$\{m_\alpha\} \in lim_\alpha colim_r (M_r \mathcal{C}_\alpha)_n.$$

Here $m_\alpha \in colim_r (M_r \mathcal{C}_\alpha)_n$. Moreover, as the unit of A_n is a monomorphism (because ϵ_n is cartesian) we know there exists a minimal finite $r(\alpha)$ such that $m_\alpha \in (M_{r(\alpha)} \mathcal{C}_\alpha)_n$. And, in addition, for $r_{\alpha_1} \leq r_{\alpha_2}$

$$c_{r(\alpha_1), r(\alpha_2)} \cdot pr_{\alpha_1} m_{\alpha_1} = pr_{\alpha_2} m_{\alpha_2}.$$

Using the pullback square (4.1) we see that r_{α_1} must be equal to r_{α_2} . This, therefore, determines an element from $lim_\alpha (M_{r(\alpha)} \mathcal{C}_\alpha)_n$ and, finally, we have an element from $colim_r lim_\alpha (M_r \mathcal{C}_\alpha)_n$. The map thus constructed is obviously inverse to the canonical map. Hence, part (ii) is proved. \square

THEOREM 4.2. *If A_n is a truncable cartesian finitary monad then the monad a_n on $Comp_n$, induced by the adjunction $\mathcal{W}_n \dashv \mathcal{F}_n$, is cartesian.*

Let us prove, firstly, the following lemma:

LEMMA 4.2. *Let*

$$\begin{aligned}
r'_n &= r_n \cdot \mathcal{F}_n : \mathcal{F}_n \mathcal{W}_n \mathcal{F}_n \longrightarrow \mathcal{F}_n, \\
\Theta'_n &= \Theta_n \cdot \mathcal{F}_n : V_n \mathcal{W}_n \mathcal{F}_n \longrightarrow W_n \mathcal{F}_n.
\end{aligned}$$

Under the conditions of the theorem, the natural transformations r'_n, Θ'_n are cartesian.

PROOF. It is easy to check for a truncable monad that A_n is cartesian implies A_k is cartesian. Hence, we can use induction.

For $k = 0$ the lemma is obviously true. Let us suppose it true for $k = n - 1$. We prove firstly, that Θ'_n is cartesian.

It is sufficient to check it for $(n-1)$ -truncation and in dimension n separately. But $tr_{n-1}\Theta'_n$ by definition is

$$W_{n-1}r_{n-1} \cdot tr_{n-1}\mathcal{F}_n : W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}tr_{n-1}\mathcal{F}_n \longrightarrow W_{n-1}tr_{n-1}\mathcal{F}_n,$$

which is, by corollary 3.1, isomorphic to

$$W_{n-1}r'_{n-1}tr_{n-1} : W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}\mathcal{F}_{n-1}tr_{n-1} \longrightarrow W_{n-1}\mathcal{F}_{n-1}tr_{n-1},$$

and, hence, is cartesian by the inductive assumption.

Now, consider a category P which has as objects the pairs (k, σ) , where k is a natural number between 0 and $n-1$ and σ is one of the symbols $+$ or $-$. We have also the generating morphisms

$$s_- : (k, -) \longrightarrow (k-1, -), \quad t_- : (k, -) \longrightarrow (k-1, +)$$

$$s_+ : (k, +) \longrightarrow (k-1, -), \quad t_+ : (k, +) \longrightarrow (k-1, +)$$

and relations

$$t_- \cdot s_- = t_+ \cdot t_-, \quad s_- \cdot s_+ = s_+ \cdot t_+.$$

Then we can construct two functors $F, G : Comp_{n-1} \longrightarrow Set^P :$

$$F(\mathcal{C})(k, -) = F(\mathcal{C})(k, +) = (W_{n-1}\mathcal{F}_{n-1}\mathcal{W}_{n-1}\mathcal{F}_{n-1}tr_{n-1}\mathcal{C})_k$$

$$G(\mathcal{C})(k, -) = F(\mathcal{C})(k, +) = (W_{n-1}\mathcal{F}_{n-1}tr_{n-1}\mathcal{C})_k$$

and the values on s_-, s_+, t_-, t_+ are the corresponding source and target morphisms. We have also a natural transformation $f : F \rightarrow G$ which is equal to $W_k r'_k$ on the objects $(k, +)$ and $(k, -)$. By the previous argument f is cartesian.

We have, in addition, the functor $B(\mathcal{C}) = (W_n \mathcal{F}_n \mathcal{C})_n$ and a cone $\phi : B(\mathcal{C}) \longrightarrow G(\mathcal{C})$ given by the source and target functions.

We are, therefore, in the conditions of lemma 4.1. It means that we have the cartesian natural transformation

$$f^* : L \longrightarrow B.$$

It is not difficult to check that f^* coincides with the n -dimensional component of Θ'_n .

It remains to prove lemma 4.2 for r'_n . Recall that r'_n is the mate of $id : \mathcal{W}_n \mathcal{F}_n \rightarrow \mathcal{W}_n \mathcal{F}_n$. We can, therefore, apply the inductive construction of proposition 3.2 to the identity transformation and check that it gives a cartesian natural transformation

$$H_r : M_r \mathcal{W}_n \mathcal{F}_n \longrightarrow W_n \mathcal{F}_n$$

at every step. Then the colimiting natural transformation will be cartesian as well.

For $r=0$ we see that H_0 coincides with Θ'_n and, hence, is cartesian. Suppose we have already proved that H_r is cartesian. By the construction of H_{r+1} we have to lift, firstly, H_r to a A_n -algebra morphism \tilde{H}_r . But

$$\tilde{H}_r = k \cdot A_n H_r$$

where k is the A_n -algebra structure morphism for $W_n \mathcal{F}_n$. It is cartesian because the multiplication in A_n is cartesian and k is determined by the colimit of the squares (3.3). Hence, \tilde{H}_r is cartesian.

Now, $tr_{n-1}H_{r+1}$ is isomorphic to r'_{n-1} and $(H_{r+1})_n = (\tilde{H}_r)_n$ (see the construction in 3.2)). Hence, H_{r+1} is cartesian and the lemma is proved. \square

PROOF OF THE THEOREM 4.2. (a). The monad \mathcal{A}_n preserves finite pullbacks by theorem 4.1(i).

(b). The multiplication in \mathcal{A}_n is cartesian, because it is obtained as $\mathcal{W}_n r'_n$.

(c). There is a natural transformation

$$q : V_n \longrightarrow W_n \mathcal{F}_n$$

induced by the coprojection $M_0 \longrightarrow \text{colim}_r M_r$. It is cartesian, because all the transformations $M_r \longrightarrow M_{r+1}$ are cartesian. The unit ν of the monad \mathcal{A}_n mates with the identity transformation for \mathcal{F}_n and, hence, q fits commutatively into the diagram

$$\begin{array}{ccc} V_n & & \\ \downarrow V_n \nu & \searrow q & \\ V_n \mathcal{W}_n \mathcal{F}_n & \xrightarrow{\Theta'_n} & W_n \mathcal{F}_n \end{array}$$

(see the construction of proposition 3.2). So $V_n \nu$ is cartesian. Using the inductive assumption we get that $tr_{n-1} \nu$ is cartesian. Hence, ν is cartesian and we have finished the proof of the theorem. \square

COROLLARY 4.1. *For an n -operad \mathcal{A}_n in Span_n the category of \mathcal{A}_n -computads is a preasheaf topos and the corresponding monad \mathcal{A}_n is cartesian.*

PROOF. By a result of R.Street [17] every such monad is cartesian and preserves wide pullbacks. \square

5. Presentation of algebras via computads.

THEOREM 5.1. *Let \mathcal{A}_n be a truncable, cartesian monad on Glob_n then the functor $\mathcal{W}_n : \text{Alg}_n \rightarrow \text{Comp}_n$ is monadic.*

We use for this the Beck's monadicity theorem in the form given in [2]. So we need to check that

1. \mathcal{W}_n has a left adjoint;
2. \mathcal{W}_n reflects isomorphisms;
3. Alg_n has coequalizers of \mathcal{W}_n -contractible pairs and \mathcal{W}_n preserves them.

We have shown already that \mathcal{W}_n has a left adjoint. The second condition can be easily verified using induction. So it remains to check the third.

LEMMA 5.1. *The natural transformation Θ_n has a right inverse*

$$\iota_n : W_n \rightarrow V_n \mathcal{W}_n.$$

PROOF. We will construct ι_n by induction. Define $\iota_0 = id$. Assume that ι_{n-1} is already defined and $\Theta_{n-1} \cdot \iota_{n-1} = id$. Let B be an \mathcal{A}_n -algebra. Then

$$tr_{n-1} W_n B = W_{n-1} tr_{n-1} B$$

and

$$tr_{n-1} V_n \mathcal{W}_n B = W_{n-1} \mathcal{F}_{n-1} \mathcal{W}_{n-1} tr_{n-1} B$$

by definition. Then define $tr_{n-1}\iota_n$ to be the following composite

$$W_{n-1}tr_{n-1}B \xrightarrow{\iota_{n-1}} V_{n-1}W_{n-1}tr_{n-1}B \xrightarrow{V_{n-1}\nu_{n-1}} V_{n-1}W_{n-1}\mathcal{F}_{n-1}W_{n-1}tr_{n-1}B \\ \xrightarrow{\Theta_{n-1}} W_{n-1}\mathcal{F}_{n-1}W_{n-1}tr_{n-1}B$$

where ν_{n-1} is the unit of the adjunction $\mathcal{F}_{n-1} \dashv W_{n-1}$.

In dimension n we define

$$\iota_n(x) = (\iota_{n-1}s_{n-1}x, x, \iota_{n-1}t_{n-1}x).$$

Now we only need to prove that

$$W_{n-1}r_{n-1} \cdot tr_{n-1}\iota_n = id,$$

where r_{n-1} is the counit of the adjunction $\mathcal{F}_{n-1} \dashv W_{n-1}$ (see the definition of Θ_n).

Consider the following diagram

$$\begin{array}{ccc} V_{n-1}W_{n-1}tr_{n-1}B & \xrightarrow{V_{n-1}\nu_{n-1}} & V_{n-1}W_{n-1}\mathcal{F}_{n-1}W_{n-1}tr_{n-1}B \\ & \searrow id & \swarrow V_{n-1}W_{n-1}r_{n-1} \\ & & V_{n-1}U_{n-1}tr_{n-1}B \\ \Theta_{n-1} \downarrow & \swarrow \Theta_{n-1} & \downarrow \Theta_{n-1} \\ W_{n-1}tr_{n-1}B & \xleftarrow{W_{n-1}r_{n-1}} & W_{n-1}\mathcal{F}_{n-1}W_{n-1}tr_{n-1}B \end{array}$$

In this diagram the top triangle commutes by a triangle equation for an adjunction. The inner quadrangle commutes by naturality. Hence, the whole diagram commutes as well. So we have

$$W_{n-1}r_{n-1} \cdot tr_{n-1}\iota_n = W_{n-1}r_{n-1} \cdot \Theta_{n-1} \cdot V_{n-1}\nu_{n-1} \cdot \iota_{n-1} = \\ = \Theta_{n-1} \cdot \iota_{n-1} = id.$$

□

LEMMA 5.2. *Let*

$$f, g : X \rightarrow Y$$

be a W_n -contractible coequalizer pair of morphisms of A_n -algebras. Then the pair is W_n -contractible.

PROOF. Let

$$s : W_n Y \rightarrow W_n X$$

be a contraction of $(W_n f, W_n g)$. Then a trivial verification shows that the following composite

$$W_n Y \xrightarrow{\iota_n} V_n W_n Y \xrightarrow{V_n s} V_n W_n X \xrightarrow{\Theta_n} W_n A.$$

determines a contraction of the pair $(W_n f, W_n g)$. □

Now let $d : Y \rightarrow C$ be a coequalizer of (f, g) which exists due to our finitary assumption. Hence, by Beck's theorem ,

$$W_n d : W_n Y \rightarrow W_n C$$

is a coequalizer of $(W_n f, W_n g)$.

Let $\delta : W_n Y \rightarrow Z$ be a coequalizer of $(W_n f, W_n g)$. Then we have the morphism

$$m : Z \rightarrow W_n C$$

of computads which makes commutative the following diagram.

$$(5.1) \quad \begin{array}{ccc} w_n Y & \xrightarrow{\delta} & Z \\ w_n d \searrow & & \swarrow m \\ & w_n C & \end{array}$$

We can reformulate now our theorem in the following form:

m is an invertible morphism of computads.

The following four lemmas provide a proof of this statement.

LEMMA 5.3. *Let $F : A \rightarrow B$ be a functor and let $G : B \rightarrow A$ be its right adjoint. Suppose the unit*

$$\epsilon : I \rightarrow GF$$

is a monomorphism. Then F reflects mono and epimorphisms.

PROOF. Let $f : X \rightarrow Y$ be such that Ff is an epimorphism. Then, for an arbitrary object Z of A , we have a commutative diagram of sets

$$\begin{array}{ccc} A(Y, Z) & \xrightarrow{\epsilon^*} & A(Y, GFZ) \\ f^* \downarrow & & \downarrow f^* \\ A(X, Z) & \xrightarrow{\epsilon^*} & A(X, GFZ) \end{array}$$

In this diagram the morphisms ϵ^* are monomorphisms, and the right-side f^* is isomorphic by adjunction to

$$B(FY, FZ) \xrightarrow{(Ff)^*} B(FX, FZ)$$

which is a monomorphism. Hence, f^* is monomorphism and f is an epimorphism.

Suppose that Ff is a monomorphism. Then, by adjunction, the induced morphism of sets

$$A(Z, GFZ) \rightarrow A(Z, GFY)$$

is a monomorphism for an arbitrary Z and, hence, GFf is a monomorphism. By naturality of ϵ we have

$$\epsilon \cdot f = GFf \cdot \epsilon.$$

Thus f is a monomorphism. □

LEMMA 5.4. *The functor V_n reflects isomorphisms.*

PROOF. Let

$$f : \mathcal{C} \rightarrow \mathcal{D}$$

be a morphism of A_n -computads such that $V_n f$ is an isomorphism. We thus have

$$tr_{n-1} V_n f \simeq W_{n-1} \mathcal{F}_{n-1} tr_{n-1} f$$

is an isomorphism. But W_{n-1} reflects isomorphisms. Let us prove that \mathcal{F}_{n-1} reflects isomorphisms too.

By theorem 4.2 the unit of the adjunction $\mathcal{F}_{n-1} \dashv \mathcal{W}_{n-1}$ is a cartesian natural transformation and, hence, is a monomorphism as $Comp_{n-1}$ has a terminal object.

By lemma 5.3 this implies that \mathcal{F}_{n-1} reflects mono and epimorphisms. But by theorem 4.1 the category $Comp_{n-1}$ is an elementary topos and, hence, mono+epi implies iso in $Comp_{n-1}$, which finishes the proof. \square

Using the induction we can assume now that $tr_{n-1}V_n m$ is an isomorphism. So it remains to prove that $V_n m$ is an isomorphism in dimension n .

LEMMA 5.5. $(V_n m)_n$ is a monomorphism.

PROOF. Let $a, b \in (V_n \mathcal{Z})_n$ be such that $V_n m(a) = V_n m(b)$. Then

$$(5.2) \quad s_{n-1}a = s_{n-1}b, \quad t_{n-1}a = t_{n-1}b$$

by the inductive assumption.

Let now

$$a' = \Theta_n \cdot \tau(a), \quad b' = \Theta_n \cdot \tau(b)$$

where $\tau : V_n \mathcal{Z} \rightarrow V_n \mathcal{W}_n Y$ is right inverse to $V_n \delta$ (recall that δ is the coequalizer of a contractible pair and hence splits). Then we have

$$V_n m \cdot V_n \delta \cdot \tau(a) = V_n m \cdot V_n \delta \cdot \tau(b).$$

Using the definition of m (see the triangle (5.1)), we get

$$V_n \mathcal{W}_n d \cdot \tau(a) = V_n \mathcal{W}_n d \cdot \tau(b).$$

Applying Θ_n to both sides of the last equation and using naturality of Θ_n we get

$$W_n d \cdot \Theta_n \cdot \tau(a) = \Theta_n \cdot V_n \mathcal{W}_n d \cdot \tau(a) = \Theta_n \cdot V_n \mathcal{W}_n d \cdot \tau(b) = W_n d \cdot \Theta_n \cdot \tau(b).$$

So we have

$$(5.3) \quad W_n d(a') = W_n d(b').$$

Suppose that $a' \neq b'$. Consider an n -globular set $\{0, 1\}_n$ which has exactly one element in every dimension less than n and two different elements 0 and 1 in dimension n . Then there exists a unique map

$$\lambda : V_n \mathcal{C} \rightarrow \{0, 1\}_n,$$

such that $\lambda(x) = 0$ for every $x \in (V_n \mathcal{C})_n, x \neq b$ and $\lambda(b) = 1$. Construct also the unique map

$$l : W_n Y \rightarrow \{0, 1\}_n,$$

with $l(x) = 0, x \neq b'$ and $l(b') = 1$. It is clear now that

$$\lambda \cdot V_n \delta = l \cdot \Theta_n.$$

From surjectivity of Θ_n (lemma 5.1) we deduce that

$$l \cdot W_n f = l \cdot W_n g.$$

This means that there should exist a map $\eta : W_n \mathcal{C} \rightarrow \{0, 1\}_n$ such that $l = \eta \cdot W_n d$. But this is impossible because of (5.3). Hence, $a' = b'$. This means, that $\tau(a) = \tau(b)$ because of (5.2). But τ is monomorphism so $a = b$ and we have proved the lemma. \square

LEMMA 5.6. $V_n m$ is a split epimorphism.

PROOF. From the triangle 5.1 we see that it is sufficient to prove that $V_n \mathcal{W}_n d$ is a split epimorphism. The following commutative square

$$\begin{array}{ccc}
F_n W_n Y & \xrightarrow{F_n W_n d} & F_n W_n C \\
k \downarrow & & k \downarrow \\
Y & \xrightarrow{d} & C
\end{array}$$

(where k is the counit of the adjunction $W_n \dashv F_n$) shows that we need to exhibit a splitting of

$$(5.4) \quad V_n \mathcal{W}_n F_n W_n d$$

and

$$(5.5) \quad V_n \mathcal{W}_n k.$$

By lemma 5.2 $W_n d$ is a coequalizer of a contractible pair and therefore splits. Hence, (5.4) splits as well.

To exhibit a splitting of (5.5) we construct a natural transformation

$$e_n : \mathcal{W}_n \longrightarrow \mathcal{W}_n F_n W_n$$

which is a right inverse for $\mathcal{W}_n k$. It is not hard to do using induction.

Indeed, put e_0 to be the unit ϵ_0 of the monad A_0 . As $\mathcal{W}_0 = W_0$ we see that it indeed is a splitting of $\mathcal{W}_0 k$.

Assume that we have already defined e_k up to dimension $n-1$ with an additional property that the diagram

$$\begin{array}{ccc}
W_k \mathcal{F}_k \mathcal{W}_k C & \xrightarrow{W_k r_k} & W_k C \\
W_k \mathcal{F}_k e_k \downarrow & & \epsilon_k \downarrow \\
W_k \mathcal{F}_k \mathcal{W}_k A_k C & \xrightarrow{W_k r_k} & W_k A_k C
\end{array}$$

commutes. Then define

$$e_n = (e, E) : \mathcal{W}_n C \longrightarrow \mathcal{W}_n A_n C$$

as follows. The morphism E of A_{n-1} -computads is the composite

$$\mathcal{W}_{n-1} tr_{n-1} C \xrightarrow{e_{n-1}} \mathcal{W}_{n-1} A_{n-1} tr_{n-1} C \xrightarrow{\mathcal{W}_{n-1} \beta} \mathcal{W}_{n-1} tr_{n-1} A_n C$$

and the morphism e in dimension n is defined by

$$e(\xi, a, \eta) = (e'(\xi), \epsilon_n a, e'(\eta))$$

where $e' = (W_{n-1} \mathcal{F}_{n-1} E)_{n-1}$. It is not hard to verify now using the inductive assumption that we thus get the required computad morphism. So $\mathcal{W}_n k$ splits and 5.5 is proved. \square

6. Generalized pasting and some examples

EXAMPLE 6.1. Let Cat_ω be the category of (strict) ω -categories. Then the forgetful functor

$$Cat_\omega \rightarrow Glob$$

has a left adjoint and is monadic [3]. The corresponding monad on $Glob$ was denoted by D_s . It is not hard to see that our notion of $(D_s)_n$ -computad coincides with the notion of n -computad of R.Street [12, 14].

Let \mathcal{D}_n be a functor of free $(D_s)_n$ -computad. For an n -category K the counit of the adjunction $\mathcal{D}_n \dashv \mathcal{W}_n$

$$r_n : \mathcal{D}_n \mathcal{W}_n K \rightarrow K$$

was called in [14] the *pasting operation* for K . This pasting operation allows the calculation of the composite for any pasting diagram in K .

Let us return to the general case. By analogy with the above example we give the following definitions

DEFINITION 6.1. The pasting operation for an A_n -algebra G is the counit

$$r_n : \mathcal{F}_n \mathcal{W}_n G \rightarrow G$$

DEFINITION 6.2. The n -dimensional pasting scheme for a finitary monad A_n is an n -cell in $\mathcal{F}_n 1_n$, where 1_n is the terminal A_n -computad.

EXAMPLE 6.2. The most interesting case for us is that of monads generated by higher order operads from [3, 16]. Recall that an n -operad is an n -collection $\{A_T\}_{T \in \mathcal{T}_k^n}$ equipped with a monoid structure with respect to a ‘substitutional’ tensor product of n -collections. Every n -operad A generates a monad on $Glob_n$

$$A(X)_k = \coprod_{T \in \mathcal{T}_k^n} A_T \times X^T$$

where the coproduct is taken in an internal globular sense (see [3, 16]). The category of algebras for this monad is isomorphic to the category of algebras of the operad.

Example 1 is also from this class of monads as $(D_s)_n$ is generated by the terminal n -operad M . The terminality of M implies the *pasting theorem* in this case: the result of pasting depends only on the shape of the diagram and not on the method of calculation (see also [9]).

EXAMPLE 6.3. Let A be the following 2-collection. It has just one point for the unique 0-tree. For the 1-stage tree T with n -leaves $A_T(1)$ has the set \mathcal{S}_n , where \mathcal{S} is the free nonsymmetric operad generated by the pointed collection of sets which has exactly one element in each of the dimensions 0, 1 and 2 and has no element in all higher dimensions. Finally, for any 2-stage tree T ,

$$A_T(2) = A_{\partial T}(1) \times A_{\partial T}(1).$$

It was shown in [3] that this 2-collection has the structure of a 2-operad. The category of algebras in $Glob_2$ of this operad is isomorphic to the category of bicategories and strict lax-functors (i.e. the structure cells for such a lax-functor are identities).

So an A -computad consists of a 2-globular set C such that the 1-skeleton of C is isomorphic to the free A_1 -algebra generated by a 1-globular set (i.e. directed graph) D . It is not hard to see that the typical 1-cell in $A_1 D$ is a chain of cells of

D and identities together with a chosen binary bracketing. For example, we could have

$$a \xrightarrow{\alpha} b \xrightarrow{id} b \xrightarrow{\alpha} c \xrightarrow{\gamma} d \xrightarrow{id} d \xrightarrow{id} d \xrightarrow{\xi} e$$

with the following bracketing

$$((\xi \cdot id) \cdot id) \cdot (\gamma \cdot (\beta \cdot (id \cdot \alpha))).$$

Hence, the pasting diagram for a bicategory consists of a usual 2-dimensional pasting diagram P , an indication of the 1-dimensional identities in P , and a binary bracketing of the 1-dimensional source and target for every 2-cell in P . In this case the pasting composite of the diagram is well defined. So we have rediscovered a result of D.Verity [19].

EXAMPLE 6.4. Another useful monad considered in [3] is the monad on $Glob_3$ generated by the 3-operad G which has as algebras the Gray-categories [6]. The 2-dimensional skeleton of G is the monad $S\bar{s}q$ whose algebras are sesquicategories (i.e. 2-categories without interchange law [14]). The 1-skeleton of G coincides with $(D_s)_1$. Hence, a G -computad consists of a 3-globular set C whose 2-skeleton is isomorphic to a free sesquicategory generated by a 2-computad (in the sense of [14]). This notion of 3-dimensional computad was used by M.McIntyre and T.Trimble in their calculus of progressive 3D-diagrams [11]. They called it a Gray-computad.

It is clear that the corresponding pasting diagram is the usual three-dimensional pasting diagram together with information about the order of horizontal composition of 2-cells. The existence of such pasting diagrams appropriate for pasting in a Gray category was recently conjectured by S.Crans [5].

References

- [1] Baez J., Dolan J., Higher Dimensional Algebra III: n -Categories and the Algebra of Opetopes, Adv. Math. 135 (1998), pp.145-206.
- [2] Barr M., Wells C., Toposes, Triples and Theories, Springer-Verlag, Berlin, 1985.
- [3] Batanin M.A., Monoidal globular categories as a natural environment for the theory of weak n -categories, Adv. Math. 136 (1998), pp. 39-103.
- [4] Carboni A., Johnstone P., Connected limits, familial representability and Artin glueing, Mathematical Structures in Computer Science 5 (1995) 441-459.
- [5] Crans S., Pasting potpourri, talk in the Australian Category Seminar, 25 June 1997, <http://www-math.mpce.mq.edu.au/~coact/abstracts/acs970625a.html>
- [6] Gordon R., Power A.J., Street R., Coherence for Tricategories. Memoirs of the AMS, v.117, n.558, 1995.
- [7] Hermida C., Makkai M., Power J., On weak higher dimensional categories, preprint, 1997.
- [8] Joyal A., Street R., The geometry of tensor calculus, I, Adv. Math. 88 (1991), pp.55-112.
- [9] Johnson M., The combinatorics of n -categorical pasting, Journal of Pure and Appl. Algebra, 1989, 62(3), pp. 211-225.
- [10] Kelly G.M., A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves and so on, Bulletin of the Australian Math. Soc., 1980, v.22(1), pp.1-84.
- [11] McIntyre M., Trimble T., Surface diagrams for Gray-categories,(submitted), 1997.
- [12] Street R., Limits indexed by category-valued 2-functors, J.Pure and Appl. Algebra, 8, pp.149-181, 1976.
- [13] Street R., The algebra of oriented simplexes, J.Pure and Appl. Algebra, 43, pp.235-242, 1986.
- [14] Street R., Categorical Structures, in Handbook of Algebra, ed. M.Hazewinkel, Elsevier, pp.529-577, 1996.
- [15] Street R., Parity complexes, Cahiers Topologie Geom. Differentielle Categoriqes 35 , pp.283-235, 1991; Corrigenda 35, pp.359-363, 1994.
- [16] Street R., The role of Michael Batanin's monoidal globular categories, to appear in Proc. Northwestern Conference, available at www-math.mpce.mq.edu.au/~coact/street_nw97.ps

- [17] Street R., The petit topos of globular sets, Macquarie Math. Report 98/232, (submitted) 1998.
- [18] Tamsamani Z., Sur des notion de n -categorie et n -groupoide non-stricte via des ensemble multi-simpliciaux, PhD. thesis, Universite Paul Sabatier, Toulouse, France, 1995.
- [19] Verity D., Enriched categories, internal categories and change of base, PhD thesis, Fitzwilliam College, Cambridge, 1992, also Macquarie Math. Reports 93/123, 1993.

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