Low-dimensional linear ordering polytopes

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Abstract

In this paper we discuss some new results on the structure of linear ordering polytopes. We give a linear description of the linear ordering polytope for $n = 8$ which we believe to be complete. Furthermore, we address the question of comparing facet-defining inequalities with respect to various criteria which may have impact on the development of branch-and-cut algorithms for the linear ordering problem.

1 Introduction

Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes. A tournament T is a subset of arcs of D_n such that for every pair $i, j \in V_n$ either $ij \in T$ or $ji \in T$ but not both. Having arc weights c_a, $a \in A_n$, associated with every arc, the **linear ordering problem** consists of determining an acyclic tournament in D_n of maximum weight. It is easy to see that an acyclic tournament corresponds to a linear ordering of the nodes of D_n and vice versa.

The linear ordering problem has various applications. For a survey as well as reports on experience with branch-and-cut algorithms for solving it see [GJR84] and [Rei85]. An extension of the linear ordering problem is discussed in [CJK+97].

Here we will focus on the structure of the **linear ordering polytope** P_{LO}^n defined as the convex hull of the 0-1 characteristic vectors of all acyclic tournaments in D_n. The linear ordering polytope is not full-dimensional, because for $n \geq 2$ the system $x_{ij} + x_{ji} = 1$, for all $i, j \in V_n$, $1 \leq i < j \leq n$, is a minimal equation system for P_{LO}^n. Therefore we obtain $\dim P_{LO}^n = n(n-1) - \binom{n}{2} = \binom{n}{2}$. Due to the presence of equations, facet-defining inequalities for P_{LO}^n can be represented in various ways. But, since the equation system has this simple structure, we can easily define a standard representation. Namely, every facet can be represented uniquely by an inequality $a^T x \leq \alpha$ such that all coefficients a_{ij} are nonnegative coprime integers and, moreover, we have $a_{ij} \cdot a_{ji} = 0$ for every pair of nodes $i, j \in V_n$. We call this representation **normal form**. This makes it also very easy to decide whether two different inequalities define the same facet or not. Note that all facets are **trivially liftable**, i.e. facet-defining inequalities for $P_{LO}^{n_0}$ define facets for arbitrary $n \geq n_0$.

Many researchers have derived general results on the facet structure of P_{LO}^n. Families of facets are for example the the **3-dicycle inequalities**, the **k-fence inequalities**, the **k-wheel inequalities**, the **Z_k inequalities** the **Möbius ladder inequalities** [GJR85], the **augmented k-fence inequalities** [LL94], the **α-critical fence inequalities** [Kop95], the **r-reinforced k-fence inequalities** [LL94],[Suc91], and the **Paley inequalities** [GH96]. For a survey see [Fis92].

2 Classification of facets

We use the following concepts in order to classify facet-defining inequalities of P_{LO}^n and partition them into equivalence classes.

2.1 σ-symmetry

We call a polytope P **σ-symmetric** if it has the property that for every feasible incidence vector $\chi \in X$ and every arbitrary permutation σ of the nodes the vector $\sigma(\chi)$ defined by setting $\sigma(\chi)_{ij} = \chi_{\sigma(i)\sigma(j)}$, for all $1 \leq i, j \leq n$, is also a feasible incidence vector. In such a case, for every facet $f^T x \leq f_0$ and a permutation σ the inequality $\sigma(f)^T x \leq f_0$ also defines a facet. Obviously P_{LO}^n is σ-symmetric.

Many polytopes associated with combinatorial optimization problems are σ-symmetric. For the linear ordering polytope we can extend the notion of equivalence in two further ways.

2.2 Rotation mappings

Let $P = \text{conv}(X) \subseteq \mathbb{R}^d$ be a polytope. An affine mapping ψ of $\mathbb{R}^d$ onto itself is called a **rotation mapping** of P if $X = \psi(X)$. It is evident that a rotation mapping transforms a facet F of P to a facet $\psi(F)$ of P.

For the linear ordering polytope two rotation mappings are of interest. The **arc reversal mapping** ϕ [Rei85] defined by

$$\phi(x)_{ij} = x_{ji}, \quad \text{for all} \quad 1 \leq i, j \leq n.$$

This mapping transforms a facet $f^T x \leq f_0$ to a facet $g^T x \leq g_0$, where $g_{ij} = f_{ji}$ for all i, j and $g_0 = f_0$.

In [BKG96] a second mapping ψ is presented. For arbitrary fixed $r \in \{1, \ldots, n\}$ it is defined by

$$\begin{aligned}\psi(x)_{rj} &= x_{jr}, \quad \text{for all} \quad 1 \le j \le n, j \ne r,\\ \psi(x)_{jr} &= x_{rj}, \quad \text{for all} \quad 1 \le j \le n, j \ne r,\\ \psi(x)_{ij} &= x_{ij} + x_{jr} + x_{ri} - 1, \quad \text{for all} \quad 1 \le i, j \le n, i \ne r, j \ne r.\end{aligned}$$

It is shown in [BKG96] that if $f^T x \le f_0$ defines a facet F of P_{LO}^n, then

$$\sum_{i=1,i\ne r}^{n} \sum_{j=1,j\ne r}^{n} f_{ij}\psi(x_{ij}) =$$

$$\sum_{i=1,i\ne r}^{n} \Big(\sum_{j=1,j\ne r}^{n} f_{ij}(x_{ij} + x_{jr} - x_{ir}) + f_{ir}x_{ri} + f_{ri}x_{ir}\Big) \quad \le \quad f_0$$

defines the facet $\psi(F)$ of P_{LO}^n. Note that this mapping transforms a trivial inequality into a 3-dicycle inequality and vice versa.

We now say, that two facet-defining inequalities belong to the same P_{LO}^n–**class** if one inequality can be transformed into the other inequality by permutation of the nodes or by one of the above mapping operations.

3 Linear ordering polytopes for $n = 7$ and $n = 8$

In this paper we focus on the linear description of P_{LO}^n for $n = 7$ and $n = 8$.

Complete linear descriptions of P_{LO}^n were known for $n \le 7$ ([Rei93]). The linear description of P_{LO}^7 consists of 87,472 facets and was first computed with an earlier implementation of a variant of the double-description method. The facets can be partitioned into 27 $\sigma-$classes. Table 1 shows the adjacency structure of P_{LO}^7. The notation of the facets is the same as in [Rei93], where the different facets are discussed in detail. A facet F_i^* is obtained from F_i by applying the arc reversal mapping. (In some cases arc reversal does not change the facet.) All facets of P_{LO}^7 having the same number of roots n_{rt}, i.e., the same number of incidence vectors of linear orderings which satisfy the inequality as equation, belong to the same P_{LO}^n-class. Hence the number of P_{LO}^7-classes is 7. Entries in Table 1 give the number of facets of class i that are adjacent to a facet of class j, for $1 \le i, j \le 27$. For example, a facet of type F_3 has no adjacent facets of type F_6 and 12 adjacent facets of type F_5. A facet of type F_5 has 2 adjacent facets of type F_3. The table is therefore not symmetric.

	n_{rt}	n_{adj}	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_4^*	F_5^*	F_7^*	F_8^*	F_9^*	F_{10}^*	F_{15}^*	F_{17}^*
F_1	2520	21695	30	65	360	1200	1080	1200	600	1080	1080	480	1200	1200	1320	840	660	1320	240	360	960	1200	1080	600	1080	1080	480	660	240
F_2	2520	21695	39	56	504	1440	1152	1008	648	1152	1152	576	1080	1296	1008	216	612	1224	216	504	864	1440	1152	648	1152	1152	576	612	216
F_3	126	124	18	42	6	3	12	0	6	0	0	3	0	6	0	1	0	0	0	3	0	3	12	6	0	0	3	0	0
F_4	126	124	20	40	1	6	0	4	3	6	4	2	4	4	4	2	2	6	1	0	2	4	4	0	0	2	0	3	0
F_5	67	31	9	16	2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2	0	1	0	0	0	0	0
F_6	44	28	10	14	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0
F_7	104	47	10	18	2	3	0	0	0	2	0	0	0	2	0	1	0	4	0	1	0	0	2	2	0	0	0	0	0
F_8	67	31	9	16	0	3	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
F_9	67	31	9	16	0	2	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
F_{10}	44	28	8	16	1	2	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{11}	67	31	10	15	0	2	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2	0	0	0	0	0	0	0
F_{12}	104	47	10	18	1	2	1	0	1	0	1	0	0	0	0	1	2	2	0	0	1	2	1	1	0	1	0	2	0
F_{13}	67	31	11	14	0	2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	2	0	0	0	0	0	1	0
F_{14}	126	124	42	18	1	6	0	0	3	6	6	3	0	6	0	0	3	0	0	3	0	6	0	3	6	6	3	3	0
F_{15}	104	47	11	17	0	2	0	0	0	0	2	0	0	4	2	1	0	0	0	0	0	3	0	0	0	0	0	4	1
F_{16}	104	47	11	17	0	3	0	0	2	1	0	0	2	2	0	0	0	2	0	0	1	3	0	2	1	0	0	0	0
F_{17}	28	24	8	12	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
F_{18}	28	24	6	14	1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
F_{19}	28	24	8	12	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
F_4^*	126	124	20	40	1	4	4	4	0	0	2	0	4	4	4	2	3	6	0	0	2	6	0	3	6	4	2	2	1
F_5^*	67	31	9	16	2	2	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_7^*	104	47	10	18	2	0	2	0	2	0	0	0	0	2	0	1	0	4	0	1	0	3	0	0	2	0	0	0	0
F_8^*	67	31	9	16	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	3	0	1	0	0	0	0	0
F_9^*	67	31	9	16	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	2	0	0	0	0	0	1	0
F_{10}^*	44	28	8	16	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	2	0	0	0	0	0	0	0
F_{15}^*	104	47	11	17	0	3	0	0	0	0	0	0	0	4	2	1	4	0	1	0	0	2	0	0	0	2	0	0	0
F_{17}^*	28	24	8	12	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	2	0	0	0	0	0	0	0

Table 1: Adjacency structure of P_{LO}^7

Using an adjacency decomposition technique ([CR97a]) we found 12,231 different $\sigma-$classes of facets for P_{LO}^8. Due to extensive memory space and CPU time requirements, this number can only be given as lower bound. A proof that it is the exact figure is missing so far, but we conjecture this bound to be tight.

Only 2.4% $\sigma-$classes of P_{LO}^8-facets are rank inequalities, while for 30.1% of the classes the maximal coefficient is 2 and for the remaining 67.5% the maximal coefficient is 3 and higher. Since no facets of P_{LO}^8 with coefficients larger than two were known before, most of the facets are first discovered by our computations.

Figure 1 shows a facet-defining inequality of P_{LO}^8 in normal form with the maximal right hand side 65. The coefficients of the inequality vary from 2 to 6.

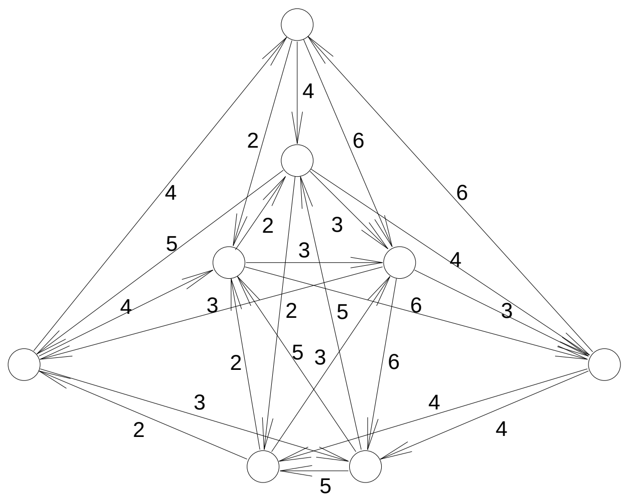

Figure 1: A facet-defining inequality for P_{LO}^8

	No. of vertices	No. of different facets	No. of facet σ-classes	No. of facet P_{LO}^n-classes
3	6	8	2	1
4	24	20	2	1
5	120	40	2	1
6	720	910	5	2
7	5,040	87,472	27	6
8	40,320	$\geq$ 488,602,996	$\geq$ 12,231	$\geq$ 1,390

Table 2: Facet structure of P_{LO}^n

Table 2 summarizes the statistics on the number of facets of small P_{LO}^n-polytopes. All facets are available in SMAPO, a "Library of linear descriptions of low-dimensional 0/1-POlytopes connected with SMAll problem instances of combinatorial optimization problems" [Chr95].

4 Quality of Facets

Besides being of theoretical interest, facet-defining inequalities are useful for branch-and-cut approaches to the linear ordering problem. From our computations of descriptions of small polytopes we now have 12,231 different classes of (necessary!) facet-defining inequalities available for all linear ordering problems on at least 8 nodes. Even on parallel hardware it does not seem to make sense to call a separation procedure for every class. In the following we therefore want to exhibit differences between the facets in order to possibly obtain insight into their usefulness for branch-and-cut algorithms.

4.1 Strength

Recently ([Goe95, GH96]) the notion of **strength** of a relaxation was introduced. The strength of a relaxation is meant to be a measure of how well a relaxation approximates a polyhedron in comparison to another weaker relaxation. The strength is only defined for certain types of combinatorial polyhedra, namely polyhedra of blocking type ([Goe95]) and of anti-blocking type ([GH96]). While P^n_{LO} does not meet these requirements, it is a face of the acyclic subgraph polytope P^n_{AC}, the convex hull of incidence vectors of all acyclic subdigraphs of D_n. P^n_{AC} is of anti-blocking type and for any nonnegative objective function an optimal solution of the linear ordering problem is an optimal solution of the acyclic subdigraph problem. Furthermore a facet-defining inequality for P^n_{LO} is valid, resp. facet-defining for P^n_{AC}. Hence, if we restrict ourselves to nonnegative objective functions, we can interpret a cutting-plane algorithm for solving the linear ordering problem as a cutting-plane algorithm for the acyclic subgraph problem using exclusively facet-defining inequalities for P^n_{LO}. This observation justifies the discussion of strength of relaxations given by inequalities for P^n_{LO}.

We will compute the strength of inequalities with respect to two relaxations associated with the following polytopes.

Trivial relaxation:

$$P^n_T = \{x \mid \begin{array}{ll} x_{ij} + x_{ji} = 1, & \text{for all } 1 \le i,j \le n, \\ x_{ij} \le 1, & \text{for all } 1 \le i,j \le n\} \end{array}$$

Dicycle relaxation:

$$P^n_C = P^n_T \cap \{x \mid x(C) \le 2 \quad \text{for all dicycles of length 3}\}$$

Let $f^T x \le f_0$ be a facet-defining inequality for P^n_{LO} in normal form. Following [Goe95] we define the **trivial strength** of f as

$$s_T(f) = \frac{\max\{f^T x \mid x \in P^n_T\}}{\max\{f^T x \mid x \in P^n_{LO}\}} = \frac{\max\{f^T x \mid x \in P^n_T\}}{f_0}$$

and the **dicycle strength** of f as

$$s_C(f) = \frac{\max\{f^T x \mid x \in P^n_C\}}{\max\{f^T x \mid x \in P^n_{LO}\}} = \frac{\max\{f^T x \mid x \in P^n_C\}}{f_0}.$$

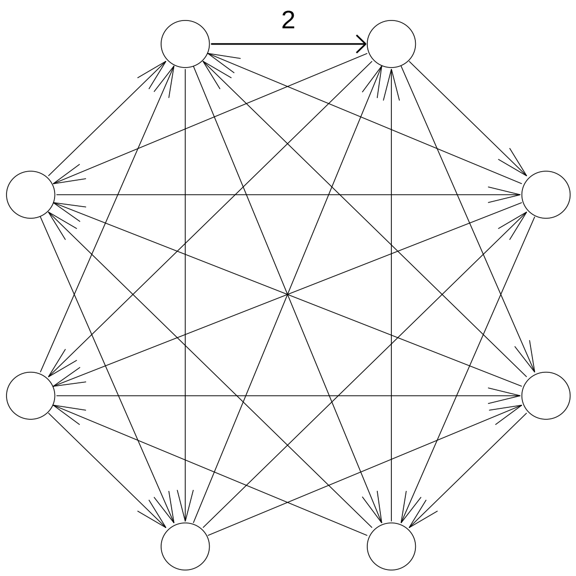

Figure 2: Facet-defining inequality for P^8_{LO} with $s_T = 1.52941$

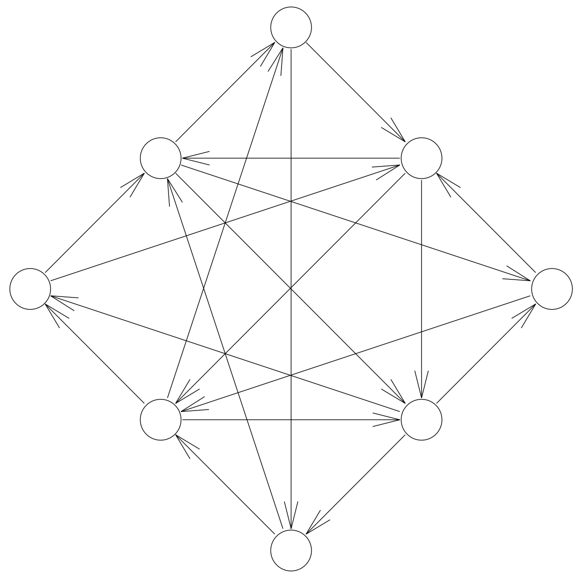

Figure 3: Facet-defining inequality for P^8_{LO} with $s_T = 1.53846$

Note that since $f^T x \leq f_0$ is in normal form we have $\max\{f^T x \mid x \in P^n_T\} = \sum_{ij \in A_n} f_{ij}$.

Figure 2 and 3 display the only two facets of P^8_{LO} with $s_T > 1.5$. The right hand sides of these facets are 17 and 13, the trivial strengths $s_T = 1.52941$ and $s_T = 1.53846$, respectively. The 3-dicycle inequality has trivial strength 1.5. Figure 4 shows the trivial strengths of the P^8_{LO}-$\sigma-$classes of facets (except for the trivial inequalities).

The 4-fence inequality is the facet with maximal dicycle strength $s_C = 1.07692$ among all facets of P^8_{LO}. Figure 5 displays the dicycle strength of the P^8_{LO}-$\sigma-$classes of facets (except for the trivial and 3-dicycle inequalities).

[GH96] show that the trivial strength of P^n_{AC} is $2 - o(1)$, where $o(1)$ is nonnegative and tends to 0 as the number of vertices n tends to infinity. They prove that the trivial strength of the known inequalities of P^n_{LO} is attained asymptotically for the augmented k-fence inequality as only 1.52777 and they present new valid inequalities, called Paley inequalities which they prove to be facet-defining for $n = 11$ and $n = 19$ with trivial strengths 1.57143 and 1.59813, respectively.

Their conclusion is that the strongest facets of the acyclic subdigraph polytope are unknown. Concerning the dicycle strength they show that the value for P^n_{AC} must be at least 4/3 since the Paley inequalities achieve this bound asymptotically.

4.2 Further Measures

As a different measure of the quality of a facet $f^T x \leq f_0$, the distance of a facet to a given relaxation P has been proposed in ([NR92]). We consider the distance with respect to the

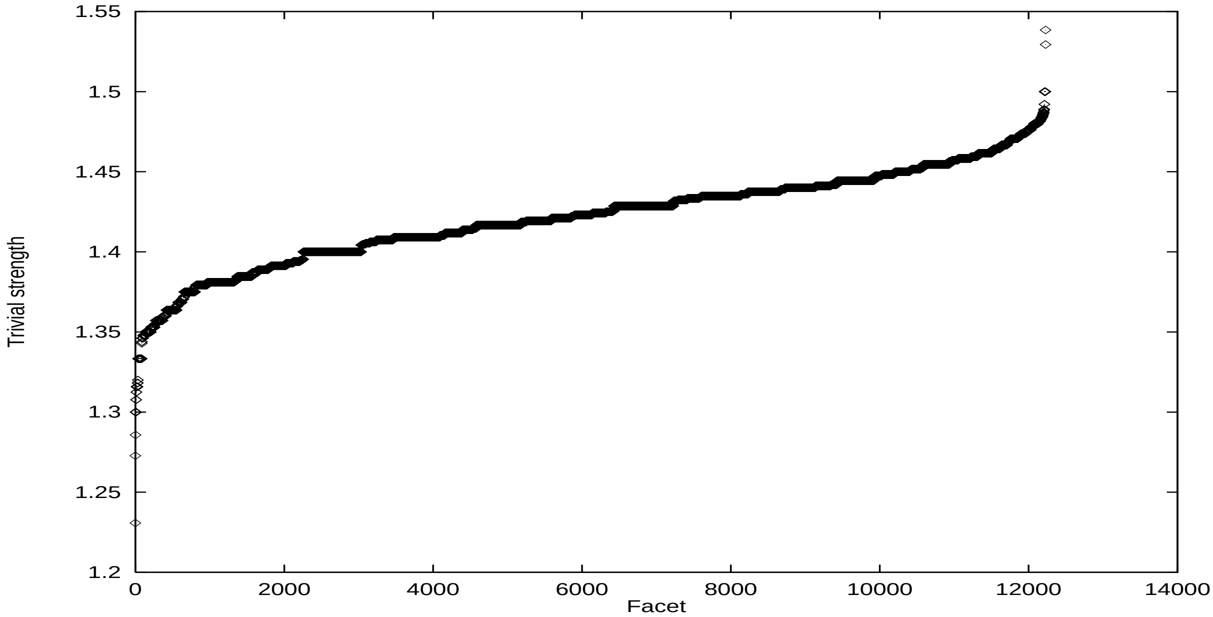

Figure 4: Trivial strength of P_{LO}^{8}-facets

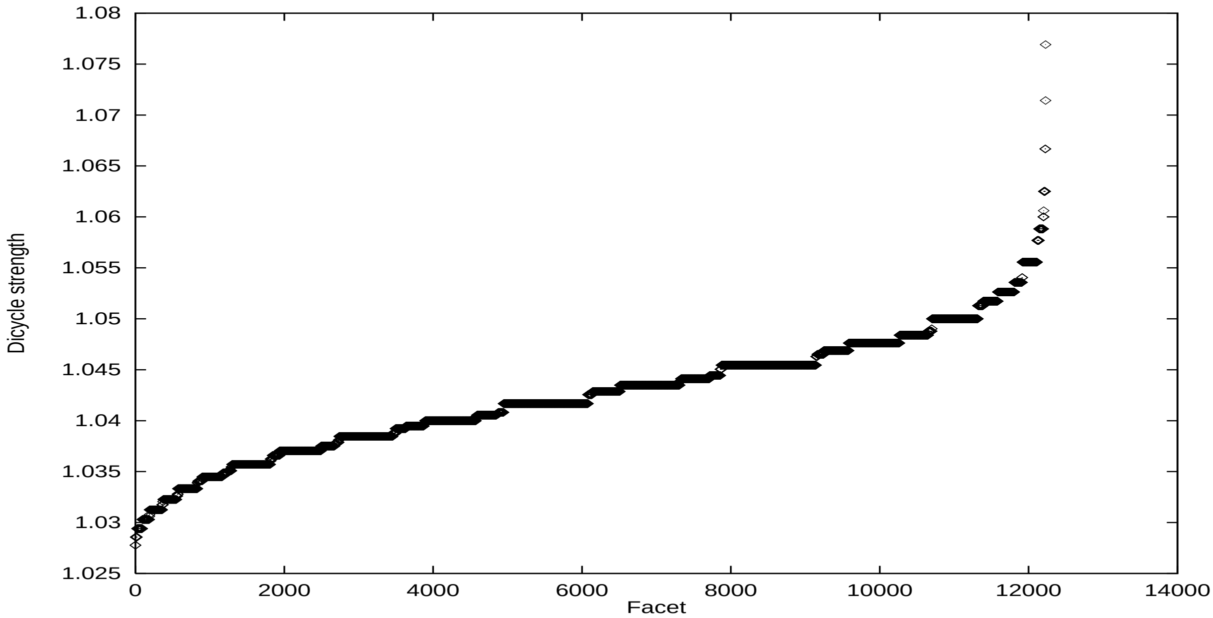

Figure 5: Dicycle strength of P_{LO}^{8}-facets

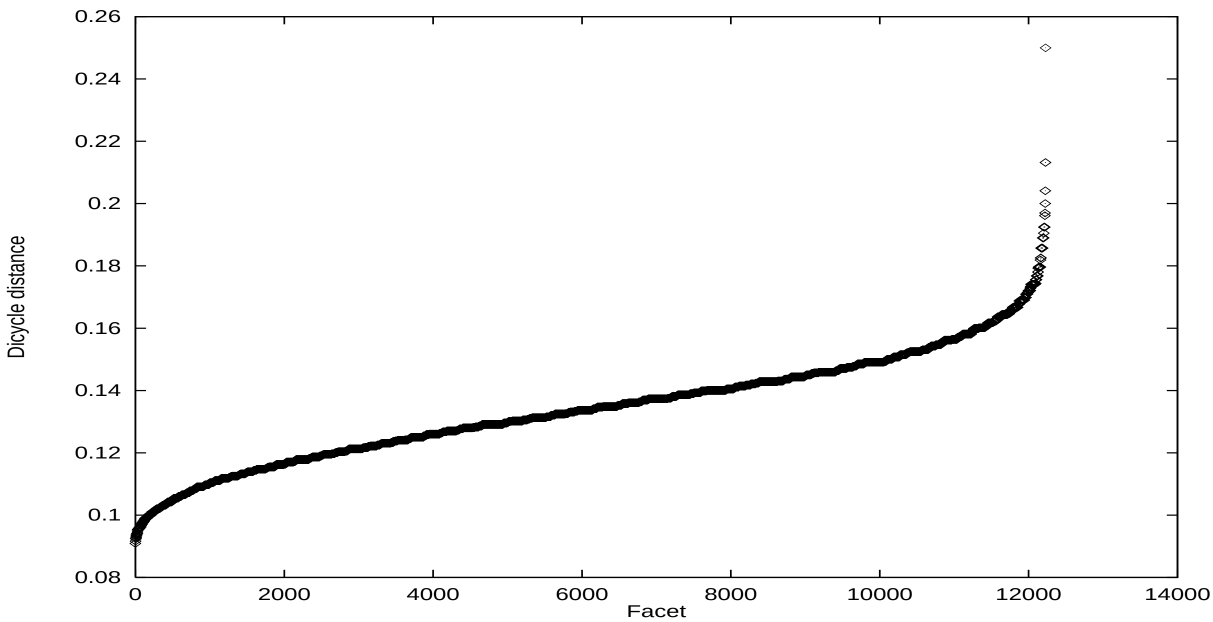

Figure 6: Distance of facets relative to P_C^n

dicycle relaxation, i.e.

$$d_c(f) = \frac{f^T x^* - f_0}{||f||}, \quad \text{where} \quad f^T x^* = \max\{f^T x \mid x \in P_C^n\}$$

Figure 6 shows this distances for the facets of P_{LO}^8 (except for the trivial and 3-dicycle inequalities). The strongest facet in this sense is the 4-fence inequality with dicycle distance $d_c = 0.25$.

[LM94] introduce a distance function based on the volume for pairs of polytopes and derive the asymptotic behaviour of that function for some special settings of combinatorial optimization problems. Another approach to classify the importance of a facet could be to determine the volume of the facet. Intuitively, a facet should be the more helpful for practical computations the larger it is. Since we had no means available for computing the volume of a facet we take the number of its roots as a rough measure for its size. Figure 7 displays the number of roots of inequalities other than trivial or 3-dicycle inequalities.

The maximal number of roots of a facet of P_{LO}^8 is 1008. It is attained for the facets which are obtained by trivial lifting of facets of P_{LO}^7 having the maximal number of roots for $n = 7$ (namely facets F_3, F_4, F_4^*, F_{14} of Table 1.) Note that trivial and dicycle facets have 20,160 roots.

4.3 Comparison of measures

Figures 8–13 give a pairwise comparison of the different measures for the facets of P_{LO}^8. Most obvious is a correlation between the dicycle strength and the dicycle distance (Figure 8).

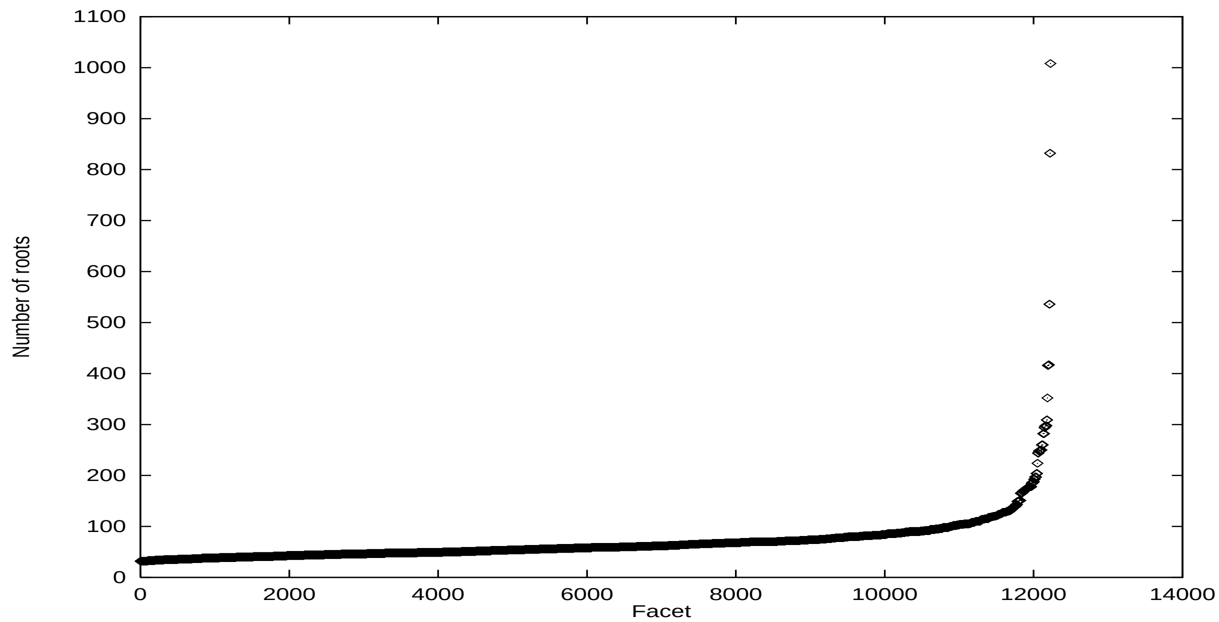

Figure 7: Number of roots of facets

A very interesting observation is that there seems to be a negative correlation between the trivial strength and the dicycle strength (Figure 9), i.e., a facet with large dicycle strength has small trivial strength and vice versa. The two observations imply the negative correlation between the trivial strength and the dicycle distance (Figure 10). Moreover, there seems to be a weak correlation between the number of roots and the trivial strength (Figure 11), while no correlation can be found between the number of roots and the dicycle strength (Figure 12), and between the number of roots and dicycle distance (Figure 13), respectively.

The comparisons do not give a clear picture so far and extensive computational experiments have to be conducted to find out if and how these measures have implications for the selection of these inequalities for solving linear ordering problems in practice. This issue will be discussed in [CR97b].

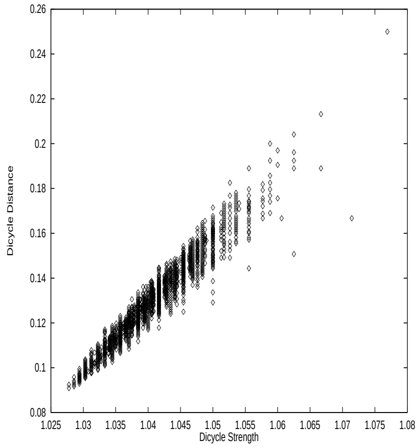

Figure 8: P_{LO}^{8} - Dicycle strength vs. dicycle distance

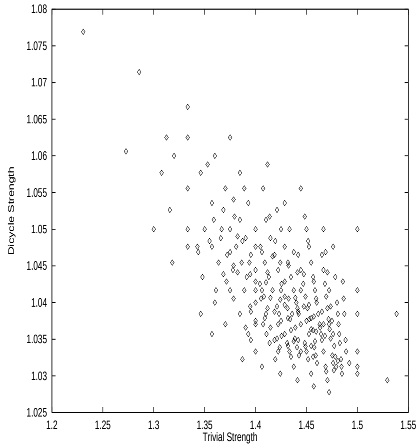

Figure 9: P_{LO}^{8} - Trivial strength vs. dicycle strength

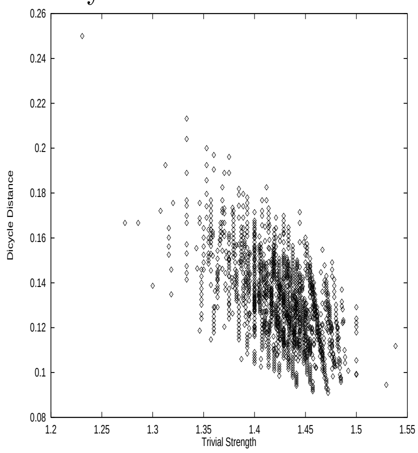

Figure 10: P_{LO}^{8} - Trivial strength vs. dicycle distance

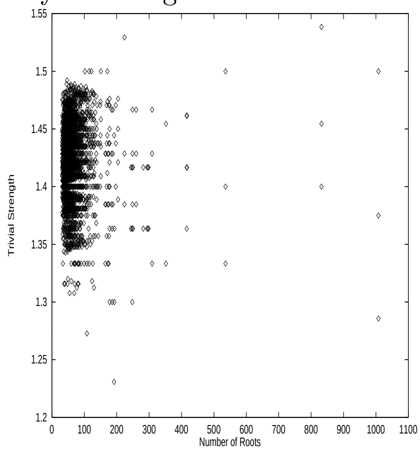

Figure 11: P_{LO}^{8} - Number of roots vs. trivial strength

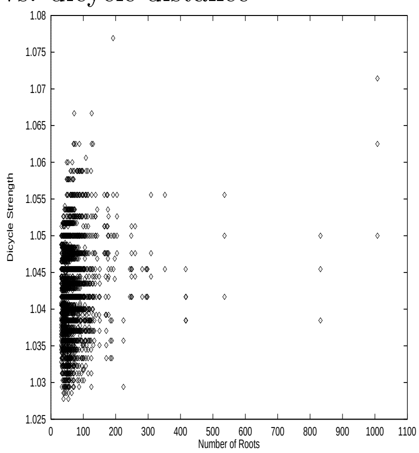

Figure 12: P_{LO}^{8} - Number of roots vs. dicycle strength

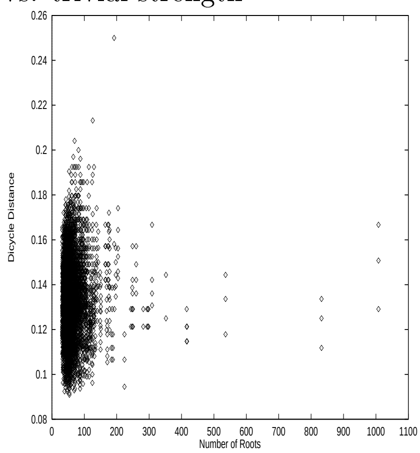

Figure 13: P_{LO}^{8} - Number of roots vs. dicycle distance

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