Minimal Invariant Spaces in Formal Topology

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Introduction

A standard result in topological dynamics is the existence of minimal subsystem. It is a direct consequence of Zorn's lemma: given a compact topological space X with a map $f: X \to X$, the set of compact non empty subspaces K of X such that $f(K) \subseteq K$ ordered by inclusion is inductive, and hence has minimal elements. It is natural to ask for a point-free (or formal) formulation of this statement. In a previous work [3], we gave such a formulation for a quite special instance of this statement, which is used in proving a purely combinatorial theorem (van de Waerden's theorem on arithmetical progression).

In this paper, we extend our analysis to the case where X is a boolean space, that is compact totally disconnected. In such a case, we give a point-free formulation of the existence of a minimal subspace for any continuous map $f: X \rightarrow X$. We show that such minimal subspaces can be described as *points* of a suitable formal topology, and the "existence" of such points become the problem of the consistency of the theory describing a generic point of this space. We show the consistency of this theory by building effectively and algebraically a topological model. As an application, we get a new, purely algebraic proof, of the minimal property of [3]. We show then in detail how this property can be used to give a proof of (a special case of) van der Waerden's theorem on arithmetical progression, that is "similar in structure" to the topological proof [6, 8], but which uses a simple algebraic remark (proposition 1) instead of Zorn's lemma. A last section tries to place this work in a wider context, as a reformulation of Hilbert's method of introduction/elimination of ideal elements.

1 Construction of Minimal Invariant Subspace

1.1 Algebraic formulation

The first step is to give a purely algebraic formulation of the problem of finding minimal invariant subspace. By Stone duality, the space X can be seen as a boolean algebra B, the elements of this boolean algebra being the clopen subset of the space. A continuous map $f : X \to X$ can be seen as an algebra morphism $g : B \to B$. In term of points, g is the inverse image of f.

We are now looking for a minimal non empty closed invariant subset $M \subseteq X$. We represent it as a predicate $\mu(x)$ over clopen $x \in B$, such that $\mu(x)$ expresses that the minimal closed invariant subset M is a subset of the clopen represented by x. We can characterise such a predicate without explicitly mentioning the subset M by the following properties:

1. $\mu(1)$,

2. $\neg \mu(0)$,

- 3. if $\mu(x)$ and $x \subseteq y$, then $\mu(y)$,
- 4. if $\mu(x)$ and $\mu(y)$, then $\mu(x.y)$,
- 5. if $\mu(x)$, then $\mu(g(x))$,
- 6. $\mu(1-x)$ or $\mu(\bigvee_{i < n} g^i(x))$ for some n.

It can be shown that, conversely, if μ is a predicate over B that satisfies these properties, then the closed subset that is the intersection of all clopen satisfying μ is a closed minimal invariant subset.

1.2 Space of minimal subspace

Following [15], we can see the 6 properties as describing "forcing" conditions on a point of a space. This space M can be seen as an infinitary propositional logic defined inductively by the properties

1.
$$x \vdash g(x)$$
,
2. $1 \vdash 1 - x \lor \bigvee_n (\lor_{i \le n} g^i(x))$.

A point of this space defines then exactly a closed minimal invariant subset.

We are going to show various properties of this space M, in particular that it is consistent, that is 1 is not covered by the empty set, and that it is positive, that is any cover of 1 is inhabited. The general method we follow for proving these properties is to build effectively some entailment relations over B that satisfy the two conditions above, and hence contain the entailment relation of the space M. This can be formulated as follows: we realize effectively the 6 conditions above in a topological model.

1.3 A class of intuitionistic models

This class of models is parametrised by an ideal I of the boolean algebra B which is closed under the morphism g: if $x \in I$ then $g(x) \in I$. Given such an ideal, we introduce then the predicate $Z_I(x)$ meaning that there exists n such that $\wedge_{i \leq n} g^i(x) \in I$. The next three lemmas have a direct proof.

Lemma 1: If there exists n such that $Z_I(\wedge_{i < n} g^i(x))$, then $Z_I(x)$.

Lemma 2: If $Z_I(x)$ and $y \subseteq x$, then $Z_I(y)$.

Lemma 3: If $Z_I(x)$ and $y \in I$, then $Z_I(x \vee y)$.

Proof: Because I is closed under the morphism g.

Notice that, a priori, we cannot conclude that $Z_I(x \vee y)$ if we know only $Z_I(x)$ and $Z_I(y)$.

The main combinatorial property of this note is:

Proposition 1: If $Z_I(y.(1-x))$ and $Z_I(y. \lor_{i \le n} g^i(x))$ for all n, then $Z_I(y)$.

Proof: We have n such that $\wedge_{i \leq n} g^i(y.(1-x)) \in I$. By lemma 1, it is enough to show that $Z_I(\wedge_{i < n} g^i(y))$. This follows from lemmas 2, 3 and the inequality

$$\wedge_{i \leq n} g^{i}(y) \subseteq \wedge_{i \leq n} g^{i}(y.(1-x)) \lor y. \lor_{i \leq n} g^{i}(x).$$

This inequality is a special case of the following remark: if $w \leq u_i \lor v_i$ for all i, then $w \leq \land u_i \lor \lor v_i$. We apply this remark to the case $w = \land_{i \leq n} g^i(y)$ and $u_i = g^i(y.(1-x)), v_i = y.g^i(x)$. \Box

We introduce next a covering relation on the set of clopen of the space ${\sf X}$:

$$x \triangleleft U = (\forall z)[[(\forall y \in U)Z_I(z.y)] \Rightarrow Z_I(z.x)].$$

This defines a formal space M_I , following Sambin's definition of a formal topology [16].

Proposition 2: The relation $x \triangleleft U$ satisfies

- if $x \in U$, then $x \triangleleft U$,
- if $x \triangleleft U$, and $u \triangleleft V$ for all $u \in U$, then $x \triangleleft V$,
- if $x \triangleleft U$ and $x \triangleleft V$, then $x \triangleleft U.V$,
- if $x \triangleleft U$ then $x.y \triangleleft U$,
- $x \triangleleft g(x)$,
- $1 \triangleleft 1 x, \bigvee_n \bigvee_{i \leq n} g^i(x).$

Proof: The first three conditions do not use any special properties of the predicate Z_I . The fourth condition follows from lemma 2.

It is clear that if $Z_I(z.g(x))$, then $Z_I(z.x)$ and hence $x \triangleleft g(x)$. The last condition follows from proposition 1. \Box

1.4 An alternative presentation

For the locale-theorist reader not familiar with Sambin's definition of spaces, we add here an alternative presentation of the space M_I . If U is a subset of B, let U^{\perp} be the subset of $x \in B$ such that $Z_I(xu)$ for all $u \in U$. (It can be checked that $u \triangleleft V$ for all $u \in U$ is equivalent to $V^{\perp} \subseteq U^{\perp}$.)

The space M_I can then be described as the locales of all $U \subseteq B$ such that $U = U^{\perp \perp}$ with for meet operation the intersection, and for infinitary join $\forall U_i = (\cup U_i)^{\perp \perp}$. If we interpret $\mu(x)$ as the set $\{x\}^{\perp \perp}$, then we can use this construction to give a topological model of the predicate μ described above. It can then be checked that all 6 properties characterising the property μ are realized in this model.

1.5 The Minimal Property

As a first application, let us prove the Minimal Property as stated in [8, 3].

Proposition 3: If $1 \triangleleft I$ in M, then $1 \in I$.

Proof: If $1 \triangleleft I$ in M, we have $1 \triangleleft I$ in M_I. But it is direct that $Z_I(x)$ for all $x \in I$. Hence $1 \triangleleft I$ in M_I implies $Z_I(1)$, which implies $1 \in I$. \square

In term of points, this means that for proving that an open invariant by f is a covering of the space X, it is enough to show that all minimal point belongs to this open, where a point is minimal iff it belongs to a minimal closed invariant subset. Yet another reading is that if we can prove $\exists x \in I.\mu(x)$ for a "generic" μ satisfying the 6 properties above, then we have $1 \in I$.

1.6 The space M is consistent

If we take for I the zero ideal, we get the consistency of M. Indeed, by the minimal property, we get that $1 \triangleleft \emptyset$ implies $1 \in I$, which means 1 = 0 if I is the zero ideal. Thus we get that M is consistent if B is a consistent boolean algebra, that is $1 \neq 0$ in B.

1.7 The space M is positive

Let L be such that $1 \triangleleft L$ and let ϕ be the proposition that L is inhabited. We take for I the ideal of elements $x \in B$ such that $\phi \lor [x = 0]$. Notice that we have $\forall y \in L.\phi$ and hence $\forall y \in L.Z_I(y)$ by construction of I. Since $1 \triangleleft L$ in M, we have $1 \triangleleft L$ in M_I. In particular, we get

$$[\forall y \in L.Z_I(y)] \Rightarrow Z_I(1)$$

and hence $Z_I(1)$. This implies $\phi \vee [1 = 0]$ by definition of I, and hence ϕ if B is a consistent boolean algebra.

1.8 Generalisation to Stone spaces

All these constructions can be generalised to the case of Stone spaces $[10]^1$, that can be described as spaces of prime filters of a distributive lattice. Given such a lattice D, and a lattice morphism g, we can associate the formal propositional theory

1.
$$\mu(1)$$
,

2. $\neg \mu(0)$,

3. if $\mu(x)$ and $x \subseteq y$, then $\mu(y)$,

4. if $\mu(x)$ and $\mu(y)$, then $\mu(x.y)$,

5. if
$$\mu(x)$$
, then $\mu(g(x))$,

6. $\mu(x)$ or $\mu(\bigvee_{i < n} g^i(y))$ for some *n*, whenever $1 = x \lor y$.

We can then prove as above that the space defined by this theory is consistent and positive.

2 Applications

2.1 Application to van der Waerden's theorem

In what follows, we shall show how to give a proof of van der Waerden's theorem on arithmetical progression, that is "similar in structure" to the topological proof presented by Furstenberg and Weiss [6], but is done in an elementary meta-language, and in particular, avoids the use of Zorn's lemma. Such a remark about similarity of proofs appear already in an early appplication of point-free topology in avoiding the use of the axiom of choice [2, 13] in the development of the theory of Banach algebras. In these works the notion of "topology without points" is used to give "a theory entirely parallel to Gelfand's, such that it is possible at every stage to reach the corresponding stage in Gelfand's theory by a simple application of the axiom of choice" [13].

All we shall use of the previous sections is proposition 1.

¹Johnstone [10] call these spaces coherent spaces.

2.1.1 General Notations

We recall first some terminology extracted from [3]. A **block** is a finite sequence of 0s and 1s. We use the notation A, B, C, \ldots for blocks, and write AB for the concatenation of two blocks A and B. If $A = b_1 \ldots b_p$ then p is called the **size** of the block A. We say that A is a **subblock** of B if, and only if, B can be written B_0AB_1 , where B_0 or B_1 may be empty. If both B_0 and B_1 are empty, then A is B itself, otherwise we say that A is a **strict** subblock of B. If B_0 is empty, we say that A is an **initial** subblock of B, and if B_1 is empty, we say that A is a **final** subblock of B. If A is an initial subblock of B, we say also that B **extends** A. We say that A avoids B if, and only if, B is not a subblock of A. These relations are decidable.

A small technical improvement w.r.t. the topological proof as presented in [3, 8] is that we shall work with the space $X = \{0,1\}^N$, and not the space $\{0,1\}^Z$. A **colouring** is a point of the space X. We can define on this space the continuous map $f : X \to X$ by $f(\alpha)(n) = \alpha(n+1)$. Our analysis does not require for f to be an homeomorphism.

We say that a finite block A is a **subblock** of $\alpha \in X$ if, and only if, there exist n, p such that A is $\alpha(p) \dots \alpha(p + n - 1)$. A colouring β is said to be a **subcolouring** of another colouring α if, and only if, any subblock of β is a subblock of α . This defines a preorder (that is, a reflexive, transitive relation) on the set X.

Each block $A = b_0 \dots b_{n \perp 1}$ can be considered as a basic (closed) open subset of X, as the set of all sequences α such that $\alpha(0) = b_0, \dots, \alpha(n-1) = b_{n \perp 1}$. If α satisfies this condition, we say that A is an **initial** subblock of α . It is direct to check that β is a subcolouring of α iff any initial subblock of β is a subclock of α .

Let W(3, l) the set of all $\alpha \in X$ that contains three identical subblocks of size l in arithmetical progression (i.e. α has a subblock of the form BA_0BA_1B where B is of size l, and A_0, A_1 have the same size). It is clear that W(3, l) is an open of the space X. Furthermore, this open U has the property that $f(\alpha) \in U$ implies $\alpha \in U$, that is $f^{\perp 1}(U) \subset U$. This property will be used later.

For $\alpha \in X$, let $\overline{\alpha}$ be the topological closure of the set $\{f^n(\alpha) \mid n \in \mathbb{N}\}$. It is clear that β is a subcolouring of α if, and only if, β belongs to $\overline{\alpha}$ if, and only if, $\overline{\beta}$ is a subset of $\overline{\alpha}$.

Proposition 4 (Minimal Property): For any $\alpha \in X$, there exists a subcolouring β of α which is minimal.

Proof: The set of non empty closed subsets of $\bar{\alpha}$ ordered by containment is such that any chain is dominated, by compactness. By Zorn's lemma, it contains a maximal element, which is clearly of the form $\bar{\beta}$, and β is then a minimal subcolouring of α .

Using this fact allows for an elegant method for showing that a given open U of X such that $f^{\perp 1}(U) \subseteq U$ is the space X: it is enough to show that any minimal colouring is in U. Indeed, let then α be an arbitrary sequence. By proposition 4, we can find β minimal which is a subcolouring of α . We have then $\beta \in U$. Since $\beta \in \overline{\alpha}$, this implies that U meets $\{f^n(\alpha) \mid n \in \mathbb{N}\}$, and thus that there is n such that $f^n(\alpha) \in U$. Since $f^{\perp 1}(U) \subseteq U$, this in turn implies $\alpha \in U$.

2.1.2 The non constructive argument

In order to simplify the presentation, I shall limit the analysis to the non constructive proof that all sequences are in W(3, l) for an arbitrary number l. (The general case could be handled similarly, using for instance the presentation given in [3]). That is, we are going to analyse a proof of the following proposition.

Fact 1: All colourings belong to W(3, l).

Notice that W(3,l) is an open U of X such that $f^{\perp 1}(U) \subseteq U$. We can hence apply the method derived from proposition 4: in order to prove that W(3,l) = X, it is enough to show that an arbitrary minimal sequence belongs to W(3,l).

We have used proposition 4 to reduce in a non constructive way the fact 1 to the "easier" following proposition.

Fact 2: All minimal colourings belong to W(3, l).

For sake of completeness, we shall give a proof of this fact, which is directly extracted from the arguments in [8]. The result of our analysis is that it is possible to use proposition 1, which is constructive, instead of proposition 4 in order to derive the fact 1. The reader can compare our treatement with the one of Girard's [7], which uses Kreisel's no counterexample interpretation.

Proof (of fact 2): Let α be a minimal sequence. It can be checked (classically) that if A is a subblock of α , then A is a subblock of any large enough subblock of α . In this way, we can build larger and larger initial subblock of α :

- $B_0 = \alpha(0) \dots \alpha(l-1),$
- $B_1 = B_0 C_1 B_0 D_1 B'_0$, where C_1, D_1 have the same size, and B'_0 has the same size as B_0 ,
- in general $B_{k+1} = B_k C_{k+1} B_k D_{k+1} B'_k$, where C_{k+1}, D_{k+1} have the same size, and B'_k has the same size as B_k .

The construction of this sequence proceeds as follows. Since B_0 is a subblock of α , there exists n_1 such that B_0 is a subblock of any subblock of α of size $\geq n_1$. In the initial subblock of α of size $4n_1$, we can then find an initial subblock $B_1 = B_0 C_1 B_0 D_1 B'_0$, where C_1, D_1 have the same size, and B'_0 has the same size as B_0 . Similarly, we can build B_{k+1} given B_k .

Let A_k be the final subblock of B_k of size l. By the pigeon hole principle, we have $i < j \le 2^l$ such that $A_i = A_j$, and then α contains three copies of A_i in arithmetical progression.

2.1.3 A constructive proof

We now give a proof of fact 1 which is parallel to this proof of fact 2, but uses proposition 1 instead of proposition 4.

The general method that we apply here can be described as follows. In the proof of fact 2, we prove a finitary property of an arbitrary, "generic" minimal sequence. Also, an analysis of this proof reveals that all we are using of such a minimal sequence α is the predicate over blocks: A is a subblock of α . This predicate can also be defined in term of a "generic" point μ of the space M as: there exists n such that $\mu(\vee_{i \leq n} g^i(A))$. It is thus possible to interpret completely the proof of fact 2 in terms of such a "generic" point μ . We can in turn make sense of μ using the formal topological space M_I : the value of $\mu(x)$ is the basic open set of M_I defined by x.

It is in turn possible to "eliminate cuts" on this proof and obtain a direct algebraic proof of fact 1. This is such a proof that we present.

We first notice a direct corollary of proposition 1, given an ideal I of the boolean algebra B of closed open subsets of the space X which is such that $g(I) \subseteq I$, where g is the morphism of B defined by the continuous map f.

If $x \in B$ we introduce the notation S(x, n) for the element $\bigvee_{i \leq n} g^i(x)$.

Corollary (of proposition 1): if $y_1 \vee \ldots \vee y_k = 1$ then $Z_I(x)$ whenever $Z_I(x\mathsf{S}(y_j, n))$ for all n and all $1 \leq j \leq k$. We have also $Z_I(x)$ whenever $Z_I(x(1-y_1))$ and $Z_I(x\mathsf{S}(y_j, n))$ for all n and all $2 \leq j \leq k$.

Proof: We prove the first statement for k = 2, the proofs of the general case and of the other statement being similar. By proposition 1, we have $Z_I(x)$ if $Z_I(x(1-y_1))$ and $Z_I(x\mathsf{S}(y_1,n))$ for all n. By the same proposition $Z_I(x(1-y_1))$ holds whenever $Z_I(x(1-y_1)(1-y_2))$ and $Z_I(x(1-y_1)\mathsf{S}(y_2,n))$ for all n. But we have $(1-y_1)(1-y_2) = 0$, and $Z_I(0)$ holds directly, and $x(1-y_1)\mathsf{S}(y_2,n) \subseteq x\mathsf{S}(y_2,n)$. Hence the result by lemma 2. \Box

Let I be the ideal corresponding to the open set W(3,l). Since $f^{\perp 1}(W(3,l)) \subseteq W(3,l)$, we have $g(I) \subseteq I$. In algebraic term, fact 1 expresses $1 \in I$. This is directly implied by $Z_I(1)$. Now, the corollary of proposition 1 shows that for proving $Z_I(x)$, it is enough to show $Z_I(xS(A,n))$ for all n and for all block A of a given size (seeing this block as a closed open set of X).

In particular, for proving $Z_I(1)$ it is enough to prove $Z_I(S(E_0, n_1))$ for all n_1 and all block E_0 of size l. By using the second statement of the corollary, and writing $x = S(E_0, n_1)$, for showing $Z_I(x)$ it is enough to show $Z_I(x(1-E_0))$ and $Z_I(xS(E_1, n_2))$ for all block E_1 of size $4n_1$ that extends E_0 . Indeed, if y_2, \ldots, y_k is the list of all block of size $4n_1$ that extends E_0 , then $(1-E_0) \vee y_2 \ldots \vee y_k = 1$ and we can hence apply the second part of corollary 2.

But $Z_I(x(1-E_0))$ is directly proved, because we have $\wedge_{i \leq n_1} g^i(x(1-E_0)) = 0$.

Hence, to prove $Z_I(x)$, it is enough to prove $Z_I(xS(E_1, n_2))$ for all n_2 and all block E_1 of size $4n_1$ that extends E_0 . In the same way, for proving

$$Z_I(\mathsf{S}(E_0, n_1)\mathsf{S}(E_1, n_2))$$

it is enough to prove

$$Z_I(S(E_0, n_1)S(E_1, n_2)S(E_2, n_3))$$

for all n_3 and all block E_2 of size $4n_2$ that extends E_1 .

Proceeding similarly, we get that it is enough to prove

$$Z_I(\mathsf{S}(E_0, n_1)\mathsf{S}(E_1, n_2)\ldots\mathsf{S}(E_p, n_{p+1}))$$

for p large enough, where E_j extends $E_{j\perp 1}$ and is of size $4n_j$.

We have then reduced the problem to find p, n large enough such that the following element of B

$$\wedge_{i < n} g^{i}(\mathsf{S}(E_{0}, n_{1})\mathsf{S}(E_{1}, n_{2}) \dots \mathsf{S}(E_{p}, n_{p+1}))$$

belongs to the ideal I, where E_j extends $E_{j\perp 1}$ and is of size $4n_j$.

We can think of such an element as a finite information about an infinite sequence α , and this information, in some sense, is a finitary version of the fact that α is minimal. For $2^l \leq p$ and n large enough, we can find a sequence B_0, \ldots, B_{2^l} as in the proof of the fact 2 such that

$$\wedge_{i\leq n}g^{i}(\mathsf{S}(E_0,n_1)\mathsf{S}(E_1,n_2)\ldots\mathsf{S}(E_p,n_{p+1}))\subseteq B_{2^{l}}$$

The proof of the fact 2 shows in a constructive way $B_{2^l} \in I$. Hence we get that

$$\wedge_{i < n} g^{i}(\mathsf{S}(E_{0}, n_{1})\mathsf{S}(E_{1}, n_{2}) \dots \mathsf{S}(E_{2^{l}}, n_{2^{l}+1}))$$

is in I for n large enough. Hence $Z_I(1)$ and $1 \in I$.

2.2 A special filter of functions

Another application is the intuitionistic construction of a special kind of non principal ultrafilter. We start with the boolean algebra $B = 2^N$, together with the function g(x)(n) = x(n+1). A point μ defines then a special kind of filter of functions, that has interesting combinatorial properties. In term of points, g correspond to a continuous map f on the space X of ultrafilters, and μ corresponds to an invariant non empty invariant subset. Any point of this subset defines a non principal ultrafilter. We refer to the paper [4] for a discussion on the analysis of the notion of ultrafilters in formal topology.

3 A Reformulation of Hilbert's Program

One important component of Hilbert's program [9] is the following justification of non effective reasoning. One sees the non effective components of a proof of a theorem as purely "ideal" objects, having no "real existence", and the problem is to show how to eliminate the uses of these fictive objects in a given concrete instance of this theorem. For instance, talking about the axiom of choice, Hilbert says that the theory he is proposing does not intend to show that it is actually possible to make a choice, but that we can always proceed "as if" such a choice was possible [9]. If this can be done in general, this will ensure that no contradiction can follow from the uses of these "ideal" objects. To take an example given by Hilbert, if we prove the statement

$$\forall n > 2, x, y, z[x^n + y^n \neq z^n],$$

using some ideal elements, we will be sure that, for any concrete instance x_0, y_0, z_0, n_0 we do have $x_0^{n_0} + y_0^{n_0} \neq z_0^{n_0}$.

We think that some techniques of point-free, or formal, topology provide an illustration of this method reformulated in a constructive framework. Here the "ideal" objects are special kind of objects: namely points of a formal space X. In usual applications, the formal basic open of X are concrete object, and a point of X is a predicate over X. Thus, the ideal object that we try to use is a certain predicate over a set of concrete objects, and in most cases, it can be shown that this predicate cannot be defined effectively.

A formal space can be described as a set of (forcing) conditions on a point (see for instance [15]). As we have just said, this space may fail to have any effective point. However, even if such points may fail to exist "absolutely", they exist always in a "relative" sense, namely in the sense of the logic defined by the space X. By "changing logic", we can then proceed as if a given formal space had a point. This technique can be rightly described as one of the main tool of topos theory.²

Thus, it is always possible to "explain" the meaning of these special "ideal" objects, and to introduce a point of the space X. The connection with Hilbert's program is now clear, and in order to illustrate further this connection, we have to show how to eliminate the use of the assumption that the space X has a point. This can be expressed as follows: if a concrete statement (like the statement above) is valid in a "relativised" sense, namely interpreted in the

²See for instance [14] for one example of this technique; as shown in [5], this method can be used even in cases where, even classically, the formal space fail to have any point. An example is the formal space of surjective functions from natural numbers to a set X. This formal space is always consistent; but it has no point if X is for instance the function set $\{0, 1\}^{\mathbb{N}}$.

logic defined by the space X, is this statement valid "absolutely"? This is reduced to a question that concerns only the formal space X, namely that this space is not covered by the empty set.

In such a case, we say that the space X is *consistent*. If a space is consistent, we can transfer the truth of a concrete statement relative to the space X to an "absolute" truth. A stronger form of consistency that we shall meet is that any covering of the space X is inhabited. We say then that the space X is *positive*. It gives a stronger form of transfer for purely existential statements.

The method we have used here to show the consistency and positivity of a space is the following. We build effectively a positive topological model of the geometric theory describing a point of this space. In other terms, we build a positive space Y with a continuous map $Y \rightarrow X$. The consistency (resp. positivity) of Y implies then the consistency (resp. positivity) of X.

This method of "eliminating the use of points" seems extremely general. In this paper, we illustrate its use by giving an intuitionistic explanation of the existence of minimal invariant subspace. We describe a formal space X such that a minimal invariant subspace corresponds exactly to a point of this space, and we show then that this space is consistent and positive. These are purely syntactical properties that can be shown in a relatively weak constructive metalanguage. This can be seen as an illustration of some remarks contained in [1].

Conclusion

We have given a completely elementary and algebraic proposition that replaces in a given concrete application the existence of minimal invariant closed subset of a boolean space with a continuous map over itself. We have furthermore explicited such a reduction in a concrete instance of the use of minimal invariant closed subset. We can see that as an illustration of the power of algebraic reasoning. The proof of this algebraic property is not so different from the Minimal Property proved in a combinatorial way in [3], but much simpler and general.

We have used only the simplest kind of topological models provided by formal topology. There exists a subtler notion, based on the notion of sites [5]. It seems likely that this notion will also be relevant to the constructive analysis of mathematical proofs. In any case, we could not apply the method presented in this paper to the analysis of proofs that use *non separable* spaces, such as the infinitary proofs of Hindman's theorem in the reference [8].

We hope to have illustrated in this note a statement that we believe is quite general (see [4] for another illustration): via the use of formal topology, it is often possible to transform a non constructive proof that uses the axiom of choice in a simple, direct and purely algebraic proof.

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