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Complementary Sets and Beatty Functions

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Abstract : A classical theorem of S. Beatty states that the set of positive integers can be covered by a union of two disjoint sets of the form $\{\lfloor \alpha n \rfloor\}$ and $\{\lfloor \beta n \rfloor\}$ for appropriate irrationals α, β. This result has been subsequently generalized by various authors, in particular, by A. McD. Mercer using a relatively simple analytic approach. Here we extend Mercer's method to deal with the non-homogeneous case and obtain extensions of corresponding A. Fraenkel's results.

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1 Introduction

Let U be a subset of $\mathbb{Z}$. We call two non-empty sets A, B *complementary* with respect to U if $A \cap B = \emptyset$ and $A \cup B = U$. A classical result of 1926 due to S. Beatty [1] states that if α and β are positive irrational numbers with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ $(n \in \mathbb{N})$ are complementary with respect to $\mathbb{N}$, where $\lfloor x \rfloor$ denotes the greatest integer function of real x. This result has since then been generalized by a number of authors, see e.g., [2], [4], [5], [6] and [7].

In particular, Mercer [7] replaces the two complementary sets of Beatty by sets of the form $\{\lfloor f(n) \rfloor\}, \{\lfloor g(n) \rfloor\}$ for suitably defined functions f and g, while Fraenkel [2] replaces them by sets of the form $\{\lfloor n\alpha + \gamma \rfloor\}$ and $\{\lfloor n\beta + \delta \rfloor\}$, which is referred to as non-homogeneous case, and carries out an extensive investigation on their complementary properties related to the rationality of α and β. Note that the class of functions f and g in Mercer's work does not contain the non-homogeneous case of Fraenkel.

In this work, we define a class of functions, named *Beatty pairs*, which includes both of Mercer and Fraenkel classes and establish their complementary properties. In contrast to Fraenkel's results which provide necessary and sufficient conditions for being complementary, our results extend the class of possible functions but only yield sufficient conditions. We note in passing that complementary sets are closely related to the so-called Wythoff's game [8]. Indeed the winning positions of Wythoff's game are those elements in the Beatty complementary sequences with α being the golden number, while a special case of the non-homogeneous complementary sequences gives winning positions of a generalized Wythoff's game

[3].

2 Upper complementary sets

For a fixed integer N, by an $N-$*upper Beatty pair*, we mean a pair of functions $f : [N, \infty) \to \mathbb{R}$ and $g : [N-1, \infty) \to \mathbb{R}$ having the following properties :

1. $\lfloor f(N) \rfloor \geq \lfloor g(N-1) \rfloor$,
2. for all integers $m, n \geq N$ such that $\lfloor f(m) \rfloor = \lfloor g(n) \rfloor$, if $f(m) \leq g(n)$, then $f(m) \notin \mathbb{Z}$, but if $g(n) \leq f(m)$, then $g(n) \notin \mathbb{Z}$,
3. g is a strictly increasing function,
4. f' exists and $1 \leq f'(x) \leq 2$.

Note that from this description, inverse functions of both elements of an N-upper Beatty pair always exist. Our defining conditions of Beatty pairs are originated from the work of Mercer ([7]), but differ markedly in the conditions 1, 2, and the parameter N.

Following Fraenkel ([2]), let

$$S_N := \{\lfloor f(n) \rfloor;\ n \in \mathbb{Z}, n \geq N\},\ T_N := \{\lfloor g(n) \rfloor;\ n \in \mathbb{Z}, n \geq N\}.$$

We say that S_N and T_N are $N-$*upper complementary* if

(i) $S_N \cap T_N = \emptyset$,

(ii) $S_N \cup T_N = \{k \in \mathbb{Z}; k \geq \lfloor f(N) \rfloor\}$, and

(iii) no integer appears more than once in the sequence $\{\lfloor f(n) \rfloor, \lfloor g(n) \rfloor; n \geq N\}$.

Our first main theorem is :

Theorem 2.1 *Let N be a fixed integer, (f, g) an $N-$upper Beatty pair, and F, G the inverse functions of f, g, respectively. If there exists an integer c such that*

$$F(x) + G(x) = x + c,\ \ x \in (f(N), \infty),$$

then S_N and T_N are N-upper complementary.

Proof. (i) We first show that $S_N \cap T_N = \phi$.
Let M be an integer $\geq \lfloor f(N) \rfloor$ such that $M \in S_N \cap T_N$. Then $M = \lfloor f(m) \rfloor = \lfloor g(n) \rfloor$ for some integers $m, n \geq N$. Writing

$$f(m) = M + e,\ g(n) = M + d,\ 0 \leq e, d < 1,$$

we have $m + n = F(M + e) + G(M + d)$.

If $f(m) \leq g(n)$, then using also the condition 2 of Beatty pair, we get $0 < e \leq d$. Since F and G are increasing, we have

$$F(M + e) + G(M + e) \leq F(M + e) + G(M + d) \leq F(M + d) + G(M + d),$$

which by hypothesis implies $M + e + c \le m + n \le M + d + c$. Thus $M + c < m+n < M+c+1$, which is impossible. The case $g(n) \le f(m)$ is treated similarly.

(ii) Next we show that $S_N \cup T_N = \{k \in \mathbb{Z}; k \ge \lfloor f(N) \rfloor\}$.

Assume to the contrary that there is an integer $h \ge \lfloor f(N) \rfloor$ for which $h \notin S_N$ and $h \notin T_N$. Thus $h > \lfloor f(N) \rfloor$. We claim that for each integer $k \ge N$, either $\lfloor f(k+1) \rfloor = \lfloor f(k) \rfloor + 1$ or $\lfloor f(k) \rfloor + 2$.

To verify this claim, write $f(k) = \lfloor f(k) \rfloor + e$, $0 \le e < 1$. By the mean-value theorem, we have $f(k+1) = \lfloor f(k) \rfloor + e + f'(k+r)$ for some $0 < r < 1$. Using the condition 4 of Beatty pair, we see that $1 \le e + f'(k+r) < 3$, which immediately yields the claim. Returning to the proof, since $h > \lfloor f(N) \rfloor$, there is an integer $n \ge N$ such that $\lfloor f(n) \rfloor < h < \lfloor f(n+1) \rfloor$. Using the claim, we deduce $\lfloor f(n) \rfloor = h - 1$ and $\lfloor f(n+1) \rfloor = h + 1$. From $\lfloor f(N) \rfloor \ge \lfloor g(N-1) \rfloor$, let m be the greatest integer for which $\lfloor g(m) \rfloor \le \lfloor f(n) \rfloor$. Thus $\lfloor g(m) \rfloor < h$ and $\lfloor g(m+1) \rfloor \ge h + 2$, which yields

$$f(n) < h, \quad g(m) < h, \tag{2.1}$$

and

$$f(n+1) > h+1, \quad g(m+1) \ge h+2. \tag{2.2}$$

¿From (2.1), we get $n + m < F(h) + G(h) = h + c$, and so

$$n + m + 1 - c \le h. \tag{2.3}$$

¿From (2.2), we get

$$n + m + 2 > F(h+1) + G(h+2) > F(h+1) + G(h+1) = h + 1 + c,$$

and so $n + m + 1 - c > h$, which contradicts (2.3).

(iii) Finally we show that no integer appears more than once in the sequence $\{\lfloor f(n) \rfloor, \lfloor g(n) \rfloor; n \ge N\}$.

Observe that from the claim in (ii), we have $\lfloor f(k) \rfloor < \lfloor f(k+1) \rfloor$ for each integer $k \ge N$. From the condition 4 of Beatty pair, we deduce that $\frac{1}{2} \le F'(x) \le 1$. Thus the main hypothesis yields $0 \le G'(x) \le \frac{1}{2}$, which infers that $g'(x) \ge 2$. Using this inequality and the same proof as for f, we similarly have $\lfloor g(k) \rfloor < \lfloor g(k+1) \rfloor$ for each integer $k \ge N$. Both statements together imply what we want to show, and this completes the proof of our first main theorem. □

We now show that the necessary part of Fraenkel's Theorem II ([2]) is a special case of our Theorem 2.1. To do so, we find it convenient to use the following technical result, which is part of Fraenkel's Lemma 3 ([2]).

Lemma 2.2 *Let α, β be positive numbers with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, and γ, δ be real numbers*

(i) Suppose that

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = q \in \mathbb{Z}.$$

Then there exists an integer m such that $m\alpha + \gamma = K \in \mathbb{Z}$ if and only if there exists an integer n such that $n\beta + \delta = K$.

(ii) If

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \lfloor N\alpha + \gamma \rfloor - 2N + 1, \tag{2.4}$$

then $(N-1)\beta + \delta \leq \lfloor N\alpha + \gamma \rfloor$.

Example Let α, β be positive irrational numbers, γ, δ be real numbers and N a fixed integer. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = \alpha x + \gamma$, $g(x) = \beta x + \delta$. Assume that

(a) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

(b) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \lfloor f(N) \rfloor - 2N + 1$ and

(c) if $n \in \mathbb{Z}$ is such that $n\beta + \delta \in Z$, then $n < N$.

Then S_N, T_N are N-upper complementary.

By switching the role of f with g, if necessary, we may assume without loss of generality that $\alpha < \beta$. We first show that (f,g) is an N-upper Beatty pair. By Lemma 2.2(ii), we have $\lfloor f(N) \rfloor \geq (N-1)\beta + \delta = g(N-1)$, which fulfils condition 1.

To check condition 2, let m, n be integers such that $m, n \geq N$, and

$$\lfloor \alpha m + \gamma \rfloor = \lfloor f(m) \rfloor = \lfloor g(n) \rfloor = \lfloor \beta n + \delta \rfloor. \tag{2.5}$$

Consider the case $f(m) \leq g(n)$. From (2.5), we have

$$\alpha m + \gamma \leq \beta n + \delta < \alpha m + \gamma + 1. \tag{2.6}$$

If $f(m) = \alpha m + \gamma \in \mathbb{Z}$, then by Lemma 2.2(i), there is an integer h such that

$$h\beta + \delta = \alpha m + \gamma. \tag{2.7}$$

Substituting this back into (2.6), we get $h \leq n < h+1$, and so $n = h$. By (2.7), we then deduce that $\beta n + \delta = \alpha m + \gamma$, which by (c) yields $n < N$, contradicting the fact that $n \geq N$. The case $g(n) \leq f(m)$ is similarly dealt with, and so condition 2 of Beatty pair holds.

Since $g'(x) = \beta > 1$, g is then strictly increasing on $[N-1, \infty)$, which is condition 3. Condition 4 follows from $\alpha < \beta$ and $f'(x) = \alpha$.

It remains to show the existence of an integer c fulfilling the main hypothesis of Theorem 2.1. Since

$$F(x) = f^{-1}(x) = \frac{x-\gamma}{\alpha}, \quad G(x) = g^{-1}(x) = \frac{x-\delta}{\beta},$$

using the given conditions (a) and (b), we have $F(x) + G(x) = x + c$ with $c = -\lfloor f(N) \rfloor + 2N - 1$. The assertion now follows from Theorem 2.1.

Note that in the case where $\gamma = \delta = 0$ and $\lfloor \alpha N \rfloor = 2N-1, N > 0$, the conditions (b) and (c) can be omitted, which yields an extension one half of Fraenkel's Theorem IX ([2]).

3 Lower complementary sets

Parallel to upper complementary notion is the concept of lower complementary sets, which we now describe. For a fixed integer N, by an $N-$*lower Beatty pair*, we mean a pair of functions $f : (-\infty, N] \to \mathbb{R}$ and $g : (-\infty, N-1] \to \mathbb{R}$ having the following properties:

1. $\lfloor f(N)\rfloor \geq \lfloor g(N-1)\rfloor$,
2. for all integers $m, n < N$ such that $\lfloor f(m)\rfloor = \lfloor g(n)\rfloor$, if $f(m) \leq g(n)$, then $f(m) \notin \mathbb{Z}$, but if $g(n) \leq f(m)$, then $g(n) \notin \mathbb{Z}$,
3. g is a strictly increasing function,
4. f' exists and $1 \leq f'(x) \leq 2$.

Let

$$S'_N := \Big\{\lfloor f(n)\rfloor; n \in \mathbb{Z}, n < N\Big\},\ T'_N := \Big\{\lfloor g(n)\rfloor; n \in \mathbb{Z}, n < N\Big\}.$$

We say that S'_N and T'_N are $N-$*lower complementary* if

(i) $S'_N \cap T'_N = \emptyset$,

(ii) $S'_N \cup T'_N = \{k \in \mathbb{Z}; k < \lfloor f(N)\rfloor\}$ and

(iii) no integer appears more than once in the sequence $\{\lfloor f(n)\rfloor, \lfloor g(n)\rfloor; n < N\}$.

Using similar consideration as in the last section, we have:

Theorem 3.1 *Let N be a fixed integer, (f, g) an $N-$lower Beatty pair, and F, G the inverse functions of f and g, respectively. If there exists an integer c such that*

$$F(x) + G(x) = x + c,\quad x \in (-\infty, f(N)),$$

then S'_N and T'_N are N-lower complementary.

¿From this theorem, an extension of Fraenkel's Theorem I ([2]) follows.

Example Let α, β be positive irrational numbers, γ, δ be real numbers and N a fixed integer. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = \alpha x + \gamma,\ g(x) = \beta x + \delta$. Assume that

(a) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

(b) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = \lfloor f(N)\rfloor - 2N + 1$ and

(c) if $n \in \mathbb{Z}$ is such that $n\beta + \delta \in \mathbb{Z}$, then $n \geq N$.

Then S'_N, T'_N are N-lower complementary.

As before in the case where $\gamma = \delta = 0$ and $\lfloor \alpha N\rfloor = 2N - 1, N \leq 0$, the conditions (b) and (c) can be omitted, which yields an extension of one half of Fraenkel's Theorem VIII ([2]).

4 Complementary sets

Note that if there is a pair of functions (f, g) and a fixed integer N for which both S_N, T_N are N-upper complementary and S'_N, T'_N are N-lower complementary, then their corresponding sets $S := \{\lfloor f(n) \rfloor; n \in \mathbb{Z}\}$ and $T := \{\lfloor g(n) \rfloor; n \in \mathbb{Z}\}$ are complementary with respect to $\mathbb{Z}$. Combining Theorems 2.1 and 3.1, we have the following generalization embracing both homogeneous and non-homogeneous cases.

Theorem 4.1 *Let f and $g : \mathbb{R} \to \mathbb{R}$. Assume that:*

1. *for all $m, n \in \mathbb{Z}$ such that $\lfloor f(m) \rfloor = \lfloor g(n) \rfloor$, if $f(m) \le g(n)$, then $f(m) \notin \mathbb{Z}$, but if $g(n) \le f(m)$, then $g(n) \notin \mathbb{Z}$,*
2. *g is a strictly increasing function,*
3. *f' exists and $1 \le f'(x) \le 2$,*
4. *there exists an integer c such that*

$$F(x) + G(x) = x + c, \quad x \in \mathbb{R},$$

where F and G the inverse functions of f and g, respectively.

Then $S := \{\lfloor f(n) \rfloor; n \in \mathbb{Z}\}$ and $T := \{\lfloor g(n) \rfloor; n \in \mathbb{Z}\}$ are complementary with respect to $\mathbb{Z}$.

The following particular case is an extension of one half of Fraenkel's Theorem XI ([2]), which also contains the classical Beatty's theorem.

Example Let α, β be positive irrational numbers, and γ, δ be real numbers. Define $f(x) = \alpha x + \gamma$ and $g(x) = \beta x + \delta$. Assume that

(a) $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

(b) $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} \equiv 0 \ (mod\ 1)$ and

(c) there is no $n \in \mathbb{Z}$ such that $n\beta + \delta \in \mathbb{Z}$.

Then the sets $\{\lfloor \alpha n + \gamma \rfloor; n \in \mathbb{Z}\}$ and $\{\lfloor \beta n + \delta \rfloor; n \in \mathbb{Z}\}$ are complementary with respect to $\mathbb{Z}$.

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