Mathematics Education Research Journal 2003, Vol. 15, No. 2, 122-137

The Role of Arithmetic Structure in the Transition from Arithmetic to Algebra

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This paper investigates students' understanding of the associative law, commutative law, and addition and division as general processes after they have completed their primary school education. All these understandings are believed to assist successful transition from arithmetic to algebra. A written test was administered to 672 students. The results identified difficulties students are experiencing with these processes. Implications for teaching algebra at both primary and secondary levels are discussed.

Studies such as the Third International Mathematics and Science Study (1998) have reported the misconceptions many students hold not only with understanding the concept of a variable, but also in solving algebraic equations and in translating word problems into algebraic symbols. This is of particular concern, given that algebra continues to play a pivotal role in the school curriculum (National Council of Teachers of Mathematics, 2000), where there is an emphasis on generalising arithmetic, applying algebraic procedures to problem solving, representing relationships between quantities, and studying algebraic structures (Thorpe, 1989). Even with the rise of technology in the algebraic domain, students still need to have a fundamental understanding of the concept of a variable and have a facility with the basic algebraic structures (Kieran, 1996).

Traditionally, beginning algebra students have been introduced to school algebra by examining the operations of arithmetic, representing the quantities and numbers with literal symbols, and operating on these literal symbols. This process approach usually involves examining equations with unknowns and representing arithmetic generalisations with variables. Activities with expressions and equations predominantly entail simplifying and solving. In spite of calls for reform (Kaput, 1999; Kieran, 1992), this approach still remains dominant in many schools in Queensland. Success for these students is heavily dependent on their past experience with arithmetic, with the transition to algebra generally occurring over a relatively short time span (usually within a six month period after seven years of operating in an arithmetic world). This research investigates the depth of understanding students have about aspects of arithmetic as they make the transition from primary to secondary school and draws implications for the learning and teaching of arithmetic and algebra.

Transition from Arithmetic to Algebra

As beginning algebra students progress from arithmetic thinking to algebraic thinking, they need to consider the numerical relations of a situation, discuss them explicitly in simple everyday language, and eventually learn to represent them with letters (Herscovics & Linchevski, 1994). This transition is believed to involve a move from knowledge required to solve arithmetic equations (operating on or with numbers) to knowledge required to solve algebraic equations (operating on or with the unknown or variable), and entails a mapping of standard mathematical symbols onto pre-existing mental models of arithmetic.

Two aspects are considered to be crucial in the transition from arithmetic to algebra. These are, first, the use of letters to represent numbers and, second, the explicit awareness of the mathematical method that is being symbolised by the use of both numbers and letters (Kieran, 1992). This involves a shift from purely numerical solutions to a consideration of method and process.

Recent mathematical reform has recommended pedagogical approaches for bridging this gap. These involve generalising the patterns found in functional situations such as number patterns, visual patterns, and tables of values; developing an understanding of the variable with concrete materials; and, using spreadsheets and computers for introducing the concept of the variable. All of these approaches tend to be used in the initial steps in more formal algebraic education with adolescents. These students seem to have varying success with these approaches, with many experiencing difficulties.

Generalising patterns calls on an array of thinking processes not generally believed to be needed for early algebra (Warren, 1997), and many students experience difficulties in expressing these patterns in natural language (Redden, 1996). MacGregor and Stacey (1995) suggested that patterns are not necessarily helpful to algebra. With regard to concrete materials, Filloy and Sutherland (1996) claimed that students' difficulties with materials and models might arise from the fact that models generally tend to hide what each means and thus problems occur in abstracting the embedded concepts. The type of model and instruction also are believed to interfere with the abstraction (Quinlan, 1994).

There are mixed findings with the use of computers in introducing algebra. It seems that computers can assist in improving students' attitudes towards mathematics but do not necessarily impact on students' achievement in mathematics (Raj & Malone, 1997). This needs to balanced against the belief that computer based languages allow students to focus on the structural aspects of algebra (Filloy & Sutherland, 1996; Kieran, 1990). Hence, in spite of the reforms in the pedagogical approaches for bridging the gap in the transitional years, students are still experiencing difficulties (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 1998). It appears that many of the difficulties students experience in algebra stem from an inadequate arithmetic knowledge base (Gallardo & Rojana, 1987; Lincheski & Hersovics, 1994). Kieran and Chalouh (1992) suggested that most students are not given the opportunity to make explicit connections between arithmetic and algebra. The need to reconceptualise the nature of algebra and algebraic reasoning is a growing concern (Carpenter & Franke, 2001; Kaput, 1998; National Council of Teachers of Mathematics, 1998), especially with regard to the artificial separation of algebra and arithmetic. We contend that knowledge of mathematical structure is essential for successful transition. This paper explores the arithmetic knowledge that children bring to algebra and draws implications for the teaching and learning of arithmetic in the early years.

Mathematical Structure

As used here, knowledge of mathematical structure is knowledge about mathematical objects and the relationship between the objects and the properties of these objects (Morris, 1999). In particular, mathematical structure is concerned with the (i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative, do inverses and identities exist); (iii) relationships between the operations (for example, does one operation distribute over the other); and (iv) relationships across the quantities (for example, transitivity of equality and inequality). In the traditional approach to algebra, it is implicitly assumed that students are familiar with these concepts from their work with arithmetic. From repeated classroom experiences in arithmetic it is assumed that students arrive at an understanding of the structure of arithmetic by inductive generalisation. Thus, knowledge of structure is considered to be at a meta-level, derived from experiences in arithmetic. Morris (1999) suggested that in many beginning algebra courses these concepts are implicitly assumed to be familiar to students from their work in arithmetic and hence given little attention. This return to a consideration of mathematical structure differs from the work of the 1970's. In this study the focus is on cognition and learning, and the abstractions children are making from their experiences in arithmetic. In the 1970's the emphasis tended to be on using the structure of mathematics to delineate and sequence content, with the belief that well-structured content resulted in high levels of mathematical cognition (Braufield, 1973; Lamon & Scott, 1970).

The difficulties students have with algebra are often mirrored in purely numerical contexts. For example, Linchevski and Livneh (1999) reported in their study with Grade 6 students that children's difficulties with algebraic structure were also found in purely numerical contexts. Students over-generalised the order of the operations, failed to perceive the cancellation of terms in equations, and displayed a static view of brackets. Linchevski and Herscovics (1996) suggested that appropriate preparation in the upper primary school would assist students to overcome many of these obstacles. This is commonly referred to in the literature as developing a 'sense of the operations'.

Operation Sense

Operation sense is the ability to use operations on at least one set of mathematical objects, for example, to add positive numbers. Slavitt's (1999) definition encompasses "conceptions that involve the operation's underlying structure, use and relationships with other mathematical operations and structures, and potential generalisations". Slavitt defined ten aspects that help delineate students' operation sense and provide insights into the beginnings of algebraic thought. The ten aspects fall into three broad groups, namely, *property* aspects, *application* aspects, and *relational* aspects. Property aspects refer to the properties that each operation possesses and involve (a) the ability to break the operation into its base components, (b) a knowledge of the operations facts (for example, $7 + 8 = 15$ since $7 + 8 = 7 + 3 + 5 = 10 + 5 = 15$), (c) an understanding of the group properties associated with the operation, and (d) an understanding of the various symbol systems that represent the operation.

Application aspects involve the ability to apply the operations in a variety of contexts, in context-free situations, and on unknowns and arbitrary units. Relational aspects entail an understanding of the relationships between the operations, and between various representations of the operation across the differing number systems (for example, whole and rational numbers). These aspects also involve an ability to move backwards and forwards between these conceptions. Thus preparation for algebra requires more than abstracting the properties of arithmetic. It also entails an understanding of the generality of the operations, for example, an ability to recognise division situations and use the operation to generate solutions. Students need to see the essence of the operations and describe and represent this essence in a symbol system. Both these understandings are considered to support the development of the variable, the meaningful solving of equations, and the notion of equivalence (Slavitt, 1999). The purpose of this study is to examine children's understanding of both of these dimensions as they conclude their arithmetic experiences, with a particular focus on drawing implications for the type of preparation that may be appropriate in the primary school.

Previous research has documented ways in which students' arithmetic experiences constitute obstacles for the learning of algebra. Most of this research has focussed on the differences between the two systems, for example, differing syntaxes (Lodholz, 1993), closure (Collis, 1974; Kieran, 1992), use of letters as shorthand (Booth, 1989), manipulations (Booth, 1989), unknowns and pronumerals (Collis, 1974; Filloy & Rojono, 1989), and equality (Wagner & Parker, 1993). Research in this area has tended to involve small samples of students (Herscovics & Linchevski, 1994; Linchevski & Herscovics 1996; Pillay, Wilss, & Boulton-Lewis, 1998) or students who were already participating in algebra courses (Booth, 1984; Lee & Wheeler, 1989). While such studies are important in giving insights into students' mathematical knowledge, for planning purposes, they need to be balanced with studies that provide insights into the general student body before they begin to participate in formal algebra courses. Also many of the studies have determined students' structural knowledge through the correctness or otherwise of their responses. The study reported here probes students' understanding of these structural aspects by their ability to provide further examples and discuss the underlying structure in everyday language. More specifically, the present study investigated Grade 7 and Grade 8 students' abilities to:

(1) exhibit an understanding of addition and division as general operations, given that the former is the first of the four operational concepts to be introduced in the primary school (Grade 1) and the latter, the last (Grade 3);
(2) recognise the commutative and associative laws;
(3) discuss these laws in everyday language; and
(4) represent these laws symbolically.

Methodology

Instrument

A written test consisting of six tasks was developed. The four tasks examined here were designed to ascertain students' understanding of (1) the property aspects of arithmetic, that is, their ability to break addition and division into their base components; and (2) the associative and commutative properties of the two operations. The four tasks are presented in Appendix A.

The tasks required the students to:

(a) indicate, where appropriate, whether the initial use of the operation and property was correct;
(b) generate more instances of the operation or property;
(c) discuss the operation or property explicitly in simple everyday language; and
(d) where appropriate, represent the generality in symbols.

Tasks 1 and 2 reflect property and application aspects of addition and division, two of the three groups of operation sense (Slavitt, 1999). Tasks 3 and 4 focus purely on the group property aspects of the four operations, in particular, associativity and commutativity. It was considered important to include a language component (Tasks 2, 3, & 4) as previous research (e.g., Herscovics & Linchevski, 1994; Redden, 1996) had indicated that expressing patterns in natural language is necessary for representing patterns in algebraic notation. All tasks were trialled with a group of 94 children (Warren, 2000) and adjustments were made according to the results. The structure of task 2 displayed the characteristics of "good questions" (Sullivan & Clarke, 1991; Warren, 2000). Such questions are believed to cater for a range of student abilities and allow insights into student understanding of concepts, and hence their inclusion in this study.

Participants

The written test was administered to 672 students aged from 11 years to 14 years, with 82% of the sample aged 12 or 13 years. They attended six different co-educational schools in Brisbane. Each school was located in a low to medium socio-economic area. The sample was spread across two different grade levels, Grade 7 (N = 169) and Grade 8 (N = 503). In Queensland students complete 7 years of primary school before commencing secondary school. Formal algebraic experiences do not occur until the beginning secondary school (in Grade 8). As the data were collected in the first six weeks of the school year, none of the groups had had formal experience with algebraic concepts.

Coding of Responses

The first two components of each task were scored according to the correctness of the response and the number of instances of the operation or property generated by the student. For example, Task 1 required students to identify that there are an infinite number of sums that add to a particular number. Hence the responses were classified according to how many more sums students could generate. Statements such as "*twenty*" or "*nine*" were coded as finite, and a statement such as "*they go on for ever*" represented infinite. Some students simply stated that there were "*lots*" or "*many*" and these statements were coded accordingly.

Language responses. Categories for classifying the responses to the language component of each task ("Explain in writing … ") were developed from an examination of the students' scripts. Scripts were sorted according to correctness and completeness. The resultant categories differed from task to task.

Task 2 involved seeing that the operation required was division and that the application of this operation will generate many more answers. Four categories were identified, namely: no response, trivial, incomplete, and valid. Responses such as *"Divide $15.40 by how many friends you want and then split the money"*, *"You divide the number of friends to $15 and then afterwards you can divide the 40 between them"*, and *"If you have 4 friends and you divide $15.40 between them so each of your four friends get money"* were considered to be valid explanations. Examples of incomplete explanations were: *"Try dividing more numbers into $15.40"*, *"I just divided the number by 4. I guessed I got it right (write)"*, and *"Half it … . Quarter it … . Third it … ."*

Responses to Task 3 and 4 were categorised according to completeness. Six categories were delineated from the scripts. These were: no explanation; statements such as "*goes in patterns*"; numbers only used (for example, 7 + 8 = 8 + 7 or 2 + (3 + 9) = (2 + 3) + 9); attempted to generalise but incorrectly (for example, "*the answer before the equal sign is the same as the answer after the equal sign*"); valid explanation without including the operation (for example, "*they are the same just turned around*"); and valid explanation including the operations (for example, "*As long as it is + or* x *the numbers can be first or second and still give the same answer*").

The instrument was administered under test conditions. Students were allowed ample time to complete each of the tasks and the test was calculator supported.

Results

Initially, Grade 7 and Grade 8 students' results were analysed separately. Chi-square tests were used to ascertain the differences between the two groups. On the whole, the groups were not significantly different. Thus, for reporting purposes, the two groups were combined (N = 672). The number of responses for the various tasks ranged from 100% to 50%, with only 50% of the students responding to Task 2. This could reflect students' adversity to answering word problems (the only word problem on the test). The style of the problem was also atypical of the type of question students commonly deal with on a day-to-day basis, as it was open-ended allowing for multiple answers (Sullivan, Warren, & White, 2000). Tables 1 and 2 summarise the results for Task 1.

Table 1
Percentage Responses to: "Find the missing numbers."

Activity	No. of Students (percentage)		
Find the missing number	Correct	Incorrect	No Response
23 = 2 + 5 + + 4 + 6	455 (67%)	65 (10%)	162 (24%)
23 = 1 + 5 + 3 + + 2 + $\triangle$	433 (64%)	76 (11%)	163 (25%)

Note. N = 672.

Table 2
Percentage Responses to: "Write some other sums that add to 23."

No. of sums provided	No. of students (percentage)
0	154 (24%)
1 – 4	289 (43%)
5 – 9	78 (14%)
≥ 10	131 (19%)

There was little difference between students' ability to find one unknown or two unknowns in an equation. An examination of the scripts indicated that most of the incorrect responses were due to simple computational errors. For example, for the second equation common responses for the unknowns were 6 and 5.

An examination of the scripts indicated that nearly all of the students who answered the question used two whole numbers to create other sums that add to 23. Only 15 students included decimal numbers in their responses and all of these students created at least 20 more examples. When asked "*How many can you write?*" only 5% stated that "*they could go on forever*". Seventeen percent indicated that they could write "*many*" or "*lots*". The rest of the sample provided a finite number of explicit solutions (44%) or simply failed to respond (34%).

Overall, only one hundred and eight students (16%) correctly found the missing numbers, wrote some other sums that added to 23, and stated that they could write lots / many or an infinite number of examples.

Task 2 focussed on the division operations and consisted of two parts. It required children to recognise an open-ended word problem as a division problem, apply the division concept to generate a number of answers, and to describe the process used to generate answers in general. Tables 3 and 4 summarise the responses to this task.

Table 3
Percentage of Responses to: "How many people are there and how much might each get?"

No. of correct responses	No. of students (percentage)
0	341 (50%)
1 - 3	339 (48%)
4 - 5	11 (2%)

Table 4
Percentage Responses to: "Explain in writing how to work out more answers."

Explanation	No. of students (percentage)
No explanation	426 (63%)
Trivial explanation	28 (4%)
Incomplete explanation	137 (20%)
Valid explanation	90 (13%)

Only 50% of the students responded to this part of the task, the lowest response rate for all four tasks. Students were asked to explain in writing how to work out more answers, this also could have influenced their willingness to respond. The results for this are presented in Table 4.

Two hundred and ninety two students (43%) failed to answer both parts of the question, indicating that some students answered only the first part and some answered only the second part. Only 90 students (13%) offered a valid explanation *"Divide $15.40 by how many friends you want and then split the money"* with 11 of these giving 4 or 5 examples of possible solutions. A follow up interview with a sample of respondents (32 students) indicated that the context of the problem played a role in identifying division as a general process. Some believed that sharing entailed each friend receiving *"exactly"* the same amount, thus you could only share amongst 2, 4, 5, 7, or 10 people, the factors of $15.40. Others discarded solutions such as $3.08 because: *"We no longer have 1- and 2-cent pieces"*.

The next two tasks related to gauging students' understanding of the commutative and associative law. Students were initially asked to indicate for which operations the laws were true. These results are summarised in Table 5.

Table 5
Percentage Distribution of Responses for Commutative and Associative Laws Question

	Commutative Law		Associative Law	
Operation	True	False	True	False
Addition	637 (94%)	35 (6%)	546 (80%)	115 (17%)
Subtraction	107 (16%)	565 (84%)	145 (21%)	516 (76%)
Division	125 (18%)	547 (82%)	139 (20%)	517 (76%)
Multiplication	634 (93%)	38 (7%)	539 (78%)	128 (19%)

For the commutative law, most students recognised that the number sentences were valid for addition and multiplication. Of concern was the number of students who believed that the statements for subtraction and division were also true, especially as these students were in the final stages of their primary school experience. A further analysis of the data indicated that there were 57 students (8%) who believed that all four number sentences were true, and most of these offered answers for other components of the task, suggesting that their answers were not simply guesses.

The responses indicate that students found the associative law more difficult than the commutative law. Fewer correctly identified the addition and multiplication number sentences as correct with significantly more students stating that they were incorrect. This could reflect the increased complexity of the number sentences under consideration (that is, the inclusion of brackets). This trend continued throughout (see Tables 6, 7 and 8).

Students were asked to create two more examples for the operations they believed were true. The results are summarised in Table 6.

Table 6
Percentage Distribution of Responses to: "Create two more examples".

	No. of examples for Commutative Law			No. of examples for Associative Law		
Operations	0	1	2	0	1	2
Addition	173 (27%)	389 (57%)	110 (16%)	307 (46%)	278 (41%)	86 (13%)
Subtraction	672 (100%)			672 (100%)		
Division	667 (99%)	5 (1%)		672 (100%)		
Multiplication	228 (34%)	359 (53%)	85 (13%)	400 (60%)	228 (34%)	44 (7%)

An analysis of Table 5 with Table 6 gives some indication of students' understanding of the commutative law question. While up to 18% of students stated that the equations for subtraction and division were true, only 5 students produced one more example for division. Only 73% of the 94% of students who stated that the equation was true for addition created one or two examples of a similar kind. The responses for multiplication exhibited a similar decline (93% to 66%).

The next segment required students to explain in their own words the patterns they had discovered in the commutative and associative laws. Students' responses fell into six broad categories delineated from the data. The categories appeared to represent increasingly more sophisticated levels of response. Table 7 describes the categories and summarises the percentage of responses in each category.

Table 7
Percentage Responses to: "Explain in writing what patterns you have discovered."

Explanation	Commutative Law	Associative Law
1. No explanation	174 (27%)	280 (42%)
2. Statement of the form "it goes in patterns"	106 (15%)	157 (23%)
3. Used only numbers in the explanation	32 (5%)	14 (2%)
4. Attempted to generalise but incorrect	31 (5%)	50 (7%)
5. Valid explanation without the inclusion of the operation	233 (34%)	115 (17%)
6. Valid explanation with the inclusion of the operation	92 (14%)	56 (8%)

A typical category 6 response to the commutative law component was: "The reason that add and multiply are true is because that when you multiply or add something then turn it around it will always equal the same thing, and when you subtract and divide the numbers and then turn them around it won't". One student responded by focussing on the number sentences that were not true: "Because 1 - 3 will give you an answer of –2. While 3 - 1 will equal 2. And 1 ÷3 will give you an answer of 0.33 while 3 ÷ 1 will give you 3. While the other two operations are true." Forty-eight percent of the sample supplied a valid explanation. Fourteen percent included the specific operations in their response.

Compared with the commutative law, even more students experienced difficulties in expressing the patterns of the associative law in language with 65% of the sample proffering no explanation or "goes up in patterns".

Table 8 summarises the percentage of students who were able to express their patterns in symbols. The tasks suggested students consider using symbols such as ♥ and \ to represent the numbers.

Table 8
Percentage Distribution of Responses to using Symbols to Represent the Commutative and Associative Laws

Operation	Commutative Law		Associative Law	
	Correct	Incorrect	Correct	Incorrect
Addition	183 (27%)		148 (22%)	
Subtraction				
Division		4 (1%)		
Multiplication	122 (18%)		63 (9%)	

Of the 183 students who correctly represented the commutative law for addition in symbols, 46 gave level 6 explanations, 84 gave level 5 responses and the rest (63) gave either level 0, 1, 2 or 3 explanations. A similar pattern existed for successfully representing the commutative law for multiplication. It seems that correctly expressing a generalisation in language may not be a prerequisite for successfully representing the generalisation in symbols.

Profile of the Group

From the results of this study it appears that many students fail to see addition and division as general solution processes. For addition, a significant percentage of students were unable to find the unknown(s) in simple arithmetic equations. Only a relatively small percentage of students exhibited some understanding of addition as a general process with the inclusion of decimals in their solutions to the open-ended question "*Write some numbers that add to 23: How many can you write*". Responses to the division question exhibited similar trends. Only 13% recognised division as a general process used to generate many valid solutions to the problem.

The four components of tasks 3 and 4 provide insights into the depth of knowledge that these students have with regard to the associative and commutative laws. Most students indicated that the appropriate arithmetic representations of the laws were correct, but up to 20% of the students also stated that the subtraction and division arithmetic statements were also correct. As the students proceeded through the tasks there was a rapid decline in correct responses. Up to three quarters of the sample gave further arithmetic examples of each law. Less than half could explain the laws in their own language with many of these students failing to indicate which operations they were referring to in their response. Up to 28% of the sample represented the laws in a symbolic format.

With regard to correct responses to both tasks, 30 (4%) students correctly responded to the commutative task, that is, they stated that the examples given for addition and multiplication were true, offered some more examples, correctly described the relationship and represented the law in symbols. For the associative law the number of complete responses was even less (22 students). Only 13 students responded correctly to both tasks. Four of these were in Grade 7, and 9 were in Grade 8.

Discussion and Conclusion

The results of this study highlight a number of concerns and implications for teaching and research.

First, most students did not seem to exhibit an understanding of addition and division as generalised processes. For addition, while they could successfully identify missing components of an addition sentence, they failed to consistently decompose a composite unit into its unit parts (Steffe & Olive, 1996), that is, they generated limited examples of sums that add to 23. They also did not seem to understand that there were an infinite number of such examples. Most of the examples they gave involved whole numbers, thus it seems that even though they have had experiences with decimals, whole number examples seem to trigger whole number responses. These limitations could be explained in terms of relational aspects, that is, the students appeared unsuccessful in connecting the relationship between the differing number systems (Whole and Rational) and the operation (Slavitt, 1999). This has implications for developing understanding of the variable as generalised number, including decimals and negatives. In early algebra courses, often the variable is introduced through situations involving addition (for example, exploring the meaning of $x + 3$ or $y = x + 3$). Yet, many students appear to be completing primary school without understanding addition as a generalised process.

The responses from the follow up interviews with regard to the division task seem to indicate that if we want students to abstract operations as general processes we need to explore problem types that lead to general thinking and that lead to a countable number of answers. The context of the problem seems to play an important role in seeing the generality of the operation and maybe it is only in strictly context free situations that full generality is attainable for students. This conjecture needs further research.

Second, the results for the commutative and associate law tasks add to Pillay, Wilss, and Boulton-Lewis' (1998) findings that many students not only fail to understand the commutative law in general terms at the end of their primary school experiences but also fail to understand the associative law. There were a significant number of students who believed that the number sentences for subtraction and division were correct. Some possible explanations for these misconceptions are:

(a) a focus in early mathematics on discovering and inducing relationships (for example, area of the rectangle = length x breadth, 5 + 7 is the same as 7 + 5. This needs to be balanced with the exploration of, for example, $7 - 5 \neq 5 - 7$ and area of a rectangle $\neq$ length + breadth);

(b) limited opportunities may be being provided for students to explore their own conjectures and inductions and make explicit the connections (Kieran & Chalouh, 1992); and

(c) mathematics being taught in non-calculator supported environments (especially in the lower grades) making it difficult to explore the non-relationships that exist in subtraction and division.

The increased difficulty students experienced with the associative law seems to confirm the finding of Linchevski and Herscovics (1994) that students tend to hold a static view of brackets. The main difference between the two laws is the inclusion of brackets and the position of the brackets changing instead of the position of the numbers changing.

Third, many students experienced difficulties in finding more examples. Most could only find one more example, even though they were asked to find two. One reason for this could be that they made decisions in the first segment of the tasks purely on computational grounds without recognising the mathematical structure being represented by the number sentences. This could also reflect limited experience with the use of "=" as equality. Falkner, Levi, and Carpenter (1999) reported that children as young as Grade 1 have enduring misconceptions about the equals sign. The reason for this, they suggest, is that the equals sign is typically used in equations where it is followed by just one number.

Fourth, the segment of the tasks relating to expressing patterns in everyday language was where almost all students experienced difficulties, with most proffering no answer or a trivial answer, such as, "*it goes in patterns*". The importance of natural language to the algebraic domain has been acknowledged in previous research (Herscovics & Linchevski, 1994; Redden, 1996), with the conjecture that natural language is a necessary prerequisite to symbolic notation. Of interest was the disparity between the percentage of students who gave a valid explanation with the inclusion of the operation and correctly represented the law in symbols. While these students were not strictly using algebraic notation to represent the generalisations, it seems that, for the commutative and associative laws, successfully expressing generalisations in natural language was not a necessary prerequisite to representing the generalisation in symbols. This raises a number of questions needing further research.

- How important is expressing patterns and generalisations in everyday language to the successful transition from arithmetic to algebra?
- What is the relationship between language development and being able to describe mathematical relationships? MacGregor and Price (2000) indicate that this relationship is not straightforward.
- If expressing patterns in everyday language were important for the successful transition into algebra, what activities would assist students in this area of language development?

Fifth, there was no significant difference between the responses for the two groups of students. The extra year with arithmetic did not seem to make students more aware of the structure of arithmetic or the arithmetic operations as general processes. This has implications for the present curriculum. Linchevski and Herscovics (1996) suggested that appropriate preparation in the upper primary school would assist students to overcome many obstacles experienced by beginning algebra students. We believe that the preparation should occur from the early years. For example, students need to continually revisit the operations as they are introduced to new number systems. In these instances, the focus would include informally examining the group properties and the operations as general processes.

From this study, it seems that the majority of students are leaving primary school with limited awareness of the notion of mathematical structure and of arithmetic operations as general processes: from the instances they are experiencing in arithmetic, they have failed to abstract the relationships and principles needed for algebra.

This has implications for both the primary and secondary school curriculums. There needs to be more of a balance between calculations and searching for the implicit patterns in the operations. Students not only need many instances of relationships, they also need to explicitly discuss these relationships in everyday language. Present schooling does not appear to be reaching this balance. Students also need to explore false relationships, such as, 2 ÷ 3 = 3 ÷ 2. They also need broader experiences in arithmetic encompassing activities where an equals sign is used in equivalent situations (for example, 5 + 7 = ? - 2).

While it is acknowledged that a limitation of this study is conjecturing from pen and paper responses about what children know and do not know, the results provide many insights into the types of primary school learning activities that could assist students to move successfully from arithmetic thinking to algebraic thinking. Further research is needed to decide on the appropriate learning experiences throughout the primary years that would assist students to gain access to the world of algebra.

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Appendix

Task 1

a. Find the missing number

$23 = 2 + 5 + \quad + 4 + 6$

b. Find the missing numbers

$23 = 1 + 5 + 3 + \quad + 2 + \Delta$

c. Write some other sums that add to 23.

d. How many can you write?

Task 2

Sarah shares $15.40 among some of her friends. She gives the same amount to each person.

(a) How many people might there be and how much would each receive? (Give at least 3 answers)

(b) Explain in writing how to work out more answers.

Task 3

2 ♣ 3 = 3 ♣ 2

Note: ♣ could be + (addition), - (subtraction), ÷ (division) or x (multiplication)

For which operations (+, -, ÷, x) is the above true ?

2 + 3 = 3 + 2	True / False
2 - 3 = 3 - 2	True / False
2 ÷ 3 = 3 ÷ 2	True / False
2 x 3 = 3 x 2	True / False

b. For the operations that are TRUE

(i) Create two more examples

(ii) Explain in writing what patterns you have discovered.

(iii) Show me the pattern using the symbols and \ for the numbers.

Task 4

(2 ♣ 5) ♣ 8= 2 ♣ (5 ♣8)

Note: ♣ could be + (addition), - (subtraction), ÷ (division) or x (multiplication)

a. For which operations (+, -, ÷, x) is the above true ?

(i)	(2 + 5) + 8 = 2 + (5 + 8)	True / False
(ii)	(2 - 5) - 8 = 2 - (5 – 8)	True / False
(iii)	(2 x 5) x 8 = 2 x (5 x 8)	True / False
(iv)	(2 ÷ 5) ÷ 8 = 2 ÷ (5 ÷ 8)	True / False

b. For the operations that are TRUE

(i) Create two more examples

(ii) Explain in writing what patterns you have discovered.

(iii) Show me the pattern using the symbols ♥, \ and ♠ for the numbers.