POSITIVE SOLUTIONS OF NONLINEAR SCHRÖDINGER-POISSON SYSTEMS WITH RADIAL POTENTIALS VANISHING AT INFINITY

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Abstract. We deal with a weighted nonlinear Schrödinger-Poisson system, allowing the potentials to vanish at infinity.
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1. Introduction and Results

In this paper we deal with Mountain-Pass solutions for a system of Schrödinger-Poisson Equations of the form:

$$\begin{cases} -\Delta u + V(x)u + \phi u = K(x)u^p, \quad x \in \mathbb{R}^N \\ \\ -\Delta \phi = u^2. \end{cases} \tag{1}$$

Precisely, we will find solutions having the following properties:

$$u \in H^1(\mathbb{R}^N), \quad u > 0, \quad \lim_{|x|\to\infty} u = 0. \tag{2}$$

Hereafter $N \in \{3,4,5\}$ (see Section 3), $1 < p < \frac{N+2}{N-2}$ and $V, K : \mathbb{R}^N \to R_+$ are radial and smooth. On (1), existence, non-existence [12] and multiplicity results [4] have been found in the case $V = K = 1$. On the other hand we do not know any results on (1), in the presence of external potentials. V, K in (1) are assumed to satisfy the same conditions introduced in [1] in the frame of Nonlinear Schrödinger Equations (NLS). Precisely:

$$\frac{a}{1+|x|^\alpha} \le V(x) \le A \tag{3}$$

for some $\alpha \in (0,2]$, $a, A > 0$, and

$$0 < K(x) \le \frac{b}{1+|x|^\beta} \tag{4}$$

for some $\beta, b > 0$. The purpose of this paper is to extend these existence results to (1). It is convenient to introduce the following quantities:

$$\sigma(N,\alpha,\beta) := \begin{cases} \frac{N+2}{N-2} - \frac{4\beta}{\alpha(N-2)}, & \text{if } 0 < \beta < \alpha \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

and

$$\alpha^* := \frac{2(N-1)(N-2)}{3N+2}. \tag{6}$$

Definition 1. *Here and in the sequel, saying that (u, ϕ) is a non-trivial positive solution of (1) we mean that (u, ϕ) is such that both u and ϕ are non-trivial, positive and radial. Furthermore u satisfies (2).*

In order to find positive solutions of (1), we will distinguish between $2 < p < 3$ and $p \in [3, 2^* - 1)$. In the latter case we have the following

Theorem 1. *If V and K are radial and smooth satisfying (3) and (4), for any $\alpha < \alpha^*$ and any $p \in (\sigma, 2^* - 1) \cap [3, 2^* - 1)$, (1) has a non-trivial positive classical Mountain-Pass solution $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Moreover, we also have existence of positive classical solutions for p in the interval (2,3) if we assume that V and K satisfy:

$$\begin{cases} (x, \nabla V) \leq c_V^{(1)} V(x), \quad \text{and} \quad c_V^{(1)} \in (0, 2) \\ \\ (x, \nabla K) \geq c_K^{(1)} K(x), \quad \text{and} \quad c_K^{(1)} \in [2, \infty), \end{cases} \tag{7}$$

where $(\,\cdot, \cdot\,)$ denotes the scalar product in $\mathbb{R}^N$. We assume that K is such that the following condition holds:

$$\exists \varepsilon \geq 0, q \geq 1 \text{ such that } (x, \nabla K) \in L^q(\mathbb{R}^N) \text{ with } q'(p+1-\epsilon) \in [2+\frac{\alpha}{\gamma}, 2^*], \tag{8}$$

where

$$\frac{1}{q} + \frac{1}{q'} = 1 \quad \text{for} \quad q \in \mathbb{R}, \qquad q' := 1 \quad \text{for } q = \infty$$

and

$$\gamma := \frac{2(N-1) - \alpha}{4}$$

is a parameter releted to inclusions of weighted Sobolev Spaces and L^p Spaces. Furthermore, assuming V is such that the following condition holds

$$\exists \varepsilon \geq 0, r \geq 1 \text{ such that } (x, \nabla V) \in L^r(\mathbb{R}^N) \text{ and } r'(2-\epsilon) \in [2+\frac{\alpha}{\gamma}, 2^*], \tag{9}$$

where r' is defined as for q', we can state the following

Theorem 2. *If V and K are radial and smooth satisfying (3), (4), (7), (8), (9), for any $\alpha < \alpha^*$ and $p \in (\sigma, 2^* - 1) \cap (2, 3)$, then (1) has a non-trivial positive classical solution $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

If instead of (7), we assume

$$\begin{cases} (x, \nabla V) \geq c_V^{(2)} V(x), \quad \text{and} \quad c_V^{(2)} > 0 \\ \\ (x, \nabla K) \leq c_K^{(2)} K(x), \quad \text{and} \quad c_K^{(2)} \in (0, 2), \end{cases} \tag{10}$$

then, dealing with the case $p \in (2,3)$, we can state another existence result. Introducing

$$\delta := 2 + \frac{c_K^{(2)}}{2}, \tag{11}$$

we have

Theorem 3. *If V and K are radial and smooth satisfying (3), (4), (8), (9), (10), for any $\alpha < \alpha^*$ and $p \in (\sigma, 2^* - 1) \cap (\delta, 3)$, (1) has a non-trivial positive classical solution $(u, \phi) \in H^1(\mathbb{R}^N) \times \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Remark 1. *We observe that the decaying property in eq. (2) is due to the radiality of the solutions found, while the property $u \in H^1(\mathbb{R}^N)$ is proven in Lemma 6, by adapting an argument due to* [1].

Dealing with the case $p \in (1,2]$, the previous theorems are completed by some non-existence results in Section 4. In spite of those results, we can also have existence for $p \in (1,2)$ if we consider the Poisson term as a small perturbation. Indeed, as in [12], we can state the following

Proposition 1. *Under the assumptions (2) and (3) on V and K, for $\alpha < \alpha^*$, $p \in (\sigma, 2^* - 1) \cap (1,2)$ and $\lambda > 0$ small enough, then the problem*

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = K(x)u^p, & x \in \mathbb{R}^N \\ \\ -\Delta\phi = u^2. \end{cases} \tag{12}$$

has at least two different nontrivial positive classical solutions (u, ϕ), one of which is a Mountain-Pass.

Before proving the existence results we focus on giving the variational formulation of (1). So the next two sections deal with some functional preliminaries.

2. Notation and Functional Setting

Our aim is to use critical point theory, so let us introduce some functional spaces. We denote respectively by $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$ and $H_V(\mathbb{R}^N)$ the Hilbert Spaces defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the following norms

$$\|\phi\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\nabla\phi|^2 \, dx,$$

$$\|u\|^2_{H^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx,$$

$$\|u\|^2_{H_V(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)\,|u|^2) dx.$$

In particular we will work with the closed subspace $H \subset H_V$ defined as its restriction on radial functions:

$$\|u\|^2_H := S_N \int_0^\infty (\varphi'^2(r) + \tilde{V}(r)\varphi^2(r)) r^{N-1} dr,$$

having defined $\varphi(|x|) = u(x)$, $\tilde{V}(|x|) = V(x)$, and S_N is the Lebesgue surface measure of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. Denoting with $L^{p+1}_K(\mathbb{R}^N)$ the weighted L^{p+1} Space with norm

$$\|u\|^{p+1}_{L^{p+1}_K(\mathbb{R}^N)} := \int_{\mathbb{R}^N} K(x)\,|u|^{p+1}\,dx, \tag{13}$$

we have

Lemma 1. *The Space $H_V(\mathbb{R}^N)$ is embedded (resp. compactly embedded) in $L^{p+1}_K(\mathbb{R}^N)$ if $\sigma \le p \le \frac{N+2}{N-2}$ (resp. if $\sigma < p < \frac{N+2}{N-2}$).*

(For the proof see e.g. [11]). Due to the radiality, we can find that H is compactly embedded in $L^q(\mathbb{R}^N)$ under suitable conditions on q. More precisely we have the following extension of the Strauss Compactness Theorem (see [14]) that we give with its proof for sake of completness. See also [13] for a more general case.

Lemma 2. *Let $\gamma := \frac{2(N-1)-\alpha}{4}$. The Space H is compactly embedded in $L^q(\mathbb{R}^N)$, for any q such that $2 + \frac{\alpha}{\gamma} < q < \frac{2N}{N-2}$.*

Proof. If $N \ge 2$ and $u \in H$, then two positive constants $C, \bar{R}$ exist such that for a.e. $|x| > \bar{R}$

$$|u(x)| \le C\,|x|^{-\gamma}\,\|u\|_H\,, \quad \gamma := \frac{2(N-1)-\alpha}{4}. \tag{14}$$

By density we can test the inequality on $C^\infty_{0,rad}(\mathbb{R}^N)$. Define φ by $\varphi(|x|) = u(x)$. An integration by parts gives:

$$\varphi(r)^2 = -2\int_r^\infty \varphi'(s)\varphi(s)ds \le$$

$$\le 2\int_r^\infty s^{-(N-1)}\sqrt{\frac{1+s^\alpha}{a}}\sqrt{\frac{a}{1+s^\alpha}}\,\left|\varphi'(s)\varphi(s)ds\right|\,s^{(N-1)}ds$$

$$\le Cr^{-2\gamma}\,\|u\|^2_H$$

for some $C > 0$ and r large enough, where in the last step we have used:

$$2\sqrt{\frac{a}{1+s^\alpha}}\,\left|\varphi'(s)\varphi(s)ds\right| \le \varphi'(s)^2 + \varphi(s)^2\frac{a}{1+s^\alpha}$$

and $s^{-(N-1)}\sqrt{1+s^\alpha} \searrow 0$ as $s \to \infty$, because we are focusing on $\alpha \in (0, 2]$.

Let

$$u_n \rightharpoonup 0 \quad \text{in} \quad H.$$

Since on spheres we control the H^1 norm by the H norm and the Rellich-Kondrashov Theorem holds, it is enough to show that, passing to a subsequence, and for R large, the integral

$$\int_{|x|>R} |u_n|^q\,dx$$

can be smaller than an a priori fixed $\epsilon > 0$ uniformly for $n \geq n_0$ for some $n_0 > 0$. In the follwing $c_1, c_2, ...c_5$ are suitable positive contants. Taking into account that

$$|u_n(x)|^{q-2} \leq c_1 |x|^{-\gamma(q-2)} \|u_n\|_H^{q-2} \leq c_2 |x|^{-\gamma(q-2)}$$

and $|x|^{\alpha-\gamma(q-2)} \searrow 0$, we have

$$\int_{|x|>R} |u_n|^q \, dx \leq$$

$$\leq c_3 \int_{|x|>R} |u_n|^{q-2} |x|^\alpha \frac{a}{1+|x|^\alpha} |u_n|^2 \, dx \leq$$

$$\leq c_4 R^{\alpha-\gamma(q-2)} \|u_n\|_H^2 \leq c_5 R^{\alpha-\gamma(q-2)} \searrow 0$$

as $R \nearrow \infty$. □

Remark 2. *It is worth to point out that H is embedded in L^q for any $q \in [2 + \frac{\alpha}{\gamma}, 2^*]$ (see e.g. [13].)*

3. Variational formulation of the problem

Solutions of (1) are the critical points of the functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^N} \phi_u u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} K(x)\,|u|^{p+1}\,dx$$

(which turns out to be well defined, $C^1(H, \mathbb{R})$ and weakly lower semicontinuous, see below).

This is due to the fact that, given $u \in H$, thanks to the Riesz Representation Theorem, a unique solution ϕ_u of the problem

$$\int_{\mathbb{R}^N} \nabla\phi\nabla v dx = \int_{\mathbb{R}^N} u^2 v dx, \ \forall v \in \mathcal{D}^{1,2}_{rad}(\mathbb{R}^N)$$

exists. Moreover, since $u^2 \in L^1_{loc}$, the following representation form holds for ϕ_u:

$$\phi_u(x) = \omega_N \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}} dy, \tag{15}$$

having denoted with ω_N the usual normalization factor of the Green function.

Now recall the Remark 2 and observe that, because of the embedding of $H_V(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$, if $u \in H$, then $u \in L^{\frac{4N}{N+2}}$, provided $\alpha \leq \alpha^* \Leftrightarrow \frac{4N}{N+2} \geq 2 + \frac{\alpha}{\gamma}$. Actually, the strict inequality has been used in order to have the compactness property stated in the following Lemma. For the same reason the restrinction on N is necessary, because it ensures that $\frac{4N}{N+2} < 2^*$.

The Hölder and the Sobolev inequality imply that, given $u \in H$, the following operator

$$L_u : v \longmapsto \int_{\mathbb{R}^N} u^2 v dx \tag{16}$$

is continuous in $\mathcal{D}^{1,2}(\mathbb{R}^N)$:

$$\left|\int_{\mathbb{R}^N} u^2 v dx\right| \leq \left\|u^2\right\|_{L^{\frac{2N}{N+2}}} \|v\|_{L^{\frac{2N}{N-2}}} = C(u)\|v\|_{\mathcal{D}^{1,2}}.$$

Introducing the notation

$$M(u) := \int_{\mathbb{R}^N} \phi_u(x) u^2 dx \tag{17}$$

we have that

Lemma 3. *If $\alpha < \alpha^*$, then M is a compact operator on H, i.e., if $u_n \rightharpoonup u$, then, up to a subsequence, $M(u_n) \to M(u)$.*

Proof. Summing and subtracting $\int_{R^N} \phi_{u_n} u^2 dx$, by the Hölder and the Sobolev inequalities we have

$$|M(u_n) - M(u)| = \left|\int_{\mathbb{R}^N} [\phi_u(x)u^2 - \phi_{u_n}(x)u_n^2]dx\right| \leq$$

$$\leq \|\phi_{u_n}\|_{L^{\frac{2N}{N-2}}} \left\|u_n^2 - u^2\right\|_{L^{\frac{2N}{N+2}}} + \|\phi_{u_n} - \phi_u\|_{L^{\frac{2N}{N-2}}} \left\|u^2\right\|_{L^{\frac{2N}{N+2}}} \leq$$

$$\leq \|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \left\|u_n^2 - u^2\right\|_{L^{\frac{2N}{N+2}}} + \|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \|u\|^2_{L^{\frac{4N}{N+2}}}.$$

Since

$$\left\|u_n^2 - u^2\right\|^{\frac{2N}{N+2}}_{L^{\frac{2N}{N+2}}} = \int_{R^N} [|u_n - u|\,|u_n + u|]^{\frac{2N}{N+2}} dx \leq$$

$$\leq \|u_n - u\|^{\frac{2N}{N+2}}_{L^{\frac{4N}{N+2}}} \|u_n + u\|^{\frac{2N}{N+2}}_{L^{\frac{4N}{N+2}}}$$

it follows

$$|M(u_n) - M(u)| \leq$$

$$\leq \|\phi_{u_n}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \|u_n - u\|_{L^{\frac{4N}{N+2}}} \|u_n + u\|_{L^{\frac{4N}{N+2}}} + \|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \|u\|^2_{L^{\frac{4N}{N+2}}}.$$

Since

$$\alpha < \alpha^* \quad \Leftrightarrow \quad \frac{4N}{N+2} > 2 + \frac{\alpha}{\gamma},$$

Lemma 2 implies $H \hookrightarrow\hookrightarrow L^{\frac{4N}{N+2}}(\mathbb{R}^N)$, hence, passing to a subsequence, $\|u_n - u\|_{L^{\frac{4N}{N+2}}} \to 0$ and therefore

$$|M(u_n) - M(u)| \leq C\|\phi_{u_n} - \phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1).$$

In order to estimate $\|\phi_{u_n} - \phi_u\|_{D^{1,2}}$ we argue as follows. One has

$$\|L_{u_n} - L_u\| \leq \sup_{\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}=1} \left\|u_n^2 - u^2\right\|_{L^{\frac{2N}{N+2}}} \|v\|_{L^{\frac{2N}{N-2}}}.$$

Since $\|u_n - u\|_{L^{\frac{4N}{N+2}}} \to 0$, passing to a subsequence, $u_n \to u$ a.e. and $|u_n|^2 \leq g$ for some $g \in L^{\frac{2N}{N+2}}$. Hence, the Dominated Convergence Theorem implies $\left\|u_n^2 - u^2\right\|_{L^{\frac{2N}{N+2}}} \to 0$, and therefore $L_{u_n} \to L_u$. The Riesz Representation Theorem implies that $L_u \in \mathcal{D}^{1,2*} \longmapsto \phi_u \in \mathcal{D}^{1,2}$ is an isometry, therefore $\phi_{u_n} \to \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. □

Lemma 3 and the compact embedding of H in L_K^{p+1} imply the weakly lower semicontinuity of I. It is standard to check also that I is a $C^1(H,\mathbb{R})$ functional. We conclude this section with a Pohozaev-like identity which shall be useful later on. For the proof see the Appendix.

Lemma 4. *Assume that V and K satisfy (3),(4),(8) and (9). If $u \in H_V(\mathbb{R}^N) \cap H^2_{loc}(\mathbb{R}^N)$ is a radial solution of the problem (1), then u satisfies the following identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} (x, \nabla V(x))u^2 dx +$$

$$+\frac{N+2}{4}\int_{\mathbb{R}^N} \phi_u u^2 dx = \frac{N}{p+1}\int_{\mathbb{R}^N} K(x)\,|u|^{p+1}\, dx + \frac{1}{p+1}\int_{\mathbb{R}^N} (x, \nabla K(x))\,|u|^{p+1}\, dx.$$

4. Proofs

Because we will use the Mountain-Pass Theorem (see [3], [2]), we will need the following

Lemma 5. *I has the Mountain-Pass Geometry for $p > 2$.*

Proof. The continuous embedding of H in L_K^{p+1} gives:

$$I(u) = \frac{1}{2}\|u\|_H^2 + o(\|u\|_H^2), \quad u \to 0 \tag{18}$$

which shows that I possesses a strict local minimum at the origin. Furthermore, let us show that I attains negative values on curves $u_t(x) := t^\lambda u(t^\mu x)$ for a suitable choice of $u \in H$, positive λ, μ and large values of t. The case $3 < p < 2^* - 1$ is standard and it can be checked taking any $u \in H \setminus \{0\}$ and putting $\mu = 0, \lambda = 1$. The case $p \in (2, 3]$ can be treated as follows. Fix $u \in H \cap L^2 \cap L^{p+1}$. Because of the integrability of u and the boundedness of V and K, the Dominated Convergence Theorem yields the following asymptotics for $t \to \infty$:

$$\|u_t\|_H^2 = t^{2(\lambda+\mu)-\mu N}\,\|\nabla u\|_{L^2}^2 + t^{2\lambda-\mu N}\int_{\mathbb{R}^N} V(t^{-\mu}x)\,|u|^2\, dx$$

$$\approx t^{2(\lambda+\mu)-\mu N}, \tag{19}$$

$$\int_{\mathbb{R}^N} K(x)\,|u_t|^{p+1}\, dx = t^{\lambda(p+1)-\mu N}\int_{\mathbb{R}^N} K(t^{-\mu}x)\,|u|^{p+1}\, dx \approx t^{\lambda(p+1)-\mu N}. \tag{20}$$

Moreover, since

$$\phi_{u_t}(x) = \omega_N \int_{\mathbb{R}^N} t^{2\lambda}u^2(t^\mu y)\frac{t^{\mu(N-2)}}{|t^\mu x - t^\mu y|^{N-2}} dy = t^{2\lambda+\mu(N-2)-\mu N}\phi_u(t^\mu x),$$

then

$$\int_{\mathbb{R}^N} \phi_{u_t}(x)u_t^2(x)dx = t^{4\lambda+\mu(N-2)-2\mu N}\int_{\mathbb{R}^N} \phi_u(x)u^2(x)dx \approx t^{4\lambda-\mu(N+2)}. \quad (21)$$

Summing up (19),(20) and (21) we get

$$I(u_t) \approx t^{2(\lambda+\mu)-\mu N} + t^{4\lambda-\mu(N+2)} - t^{\lambda(p+1)-\mu N}.$$

With the choice $\lambda = 2\mu \Rightarrow$ (19)$\approx$(21) and for $p > 2$, we have (20) $\gg$ (21), so $I(u_t) \to -\infty$ if $t \to \infty$, hence the functional has the Mountain-Pass geometry.

□

PROOF OF THEOREM 1.

Step 1 : For $p \geq 3$, *I satisfies the Palais-Smale Condition.* Take a sequence such that

$$I(u_n) < C, \qquad I'(u_n) \to 0.$$

We write

$$(p+1)I(u_n) - (I'(u_n), u_n) =$$
$$= \frac{p-1}{2}\|u_n\|_H^2 + \frac{p-3}{4}\int_{\mathbb{R}^N} \phi_{u_n}(x)u_n^2 \geq$$
$$\geq \frac{p-1}{2}\|u_n\|_H^2$$

iff $p \geq 3$. This shows that u_n is bounded in H. Hence, passing to a subsequence we have

$$u_n \rightharpoonup u \quad \in H$$

and

$$u_n \longrightarrow u, \quad \text{in } L_K^{p+1}, \quad p \in (\sigma, 2^* - 1).$$

So we write

$$o(1) = (I'(u_n), (u_n - u)) = \|u_n\|_H^2 - \|u\|_H^2 + o(1) +$$
$$+ \int_{\mathbb{R}^N} \phi_{u_n}(x)u_n(u_n - u)dx + \int_{\mathbb{R}^N} K(x)|u_n|^p(u_n - u)dx. \quad (22)$$

For the Poisson term we have

$$\left|\int_{\mathbb{R}^N} \phi_{u_n}(x)u_n(u_n - u)dx\right| \leq \|\phi_{u_n}\|_{D^{1,2}} \left\|u_n u - u_n^2\right\|_{L^{\frac{2N}{N+2}}}.$$

Now notice that, because of Lemma 3, ϕ_{u_n} is bounded in $\mathcal{D}^{1,2}$. Moreover, because of the compact embedding in $L^{\frac{4N}{N+2}}$, passing to a subsequence we have $u_n \to u$ a.e. and $\left|u_n u - u_n^2\right| \leq u\sqrt{g} + g \in L^{\frac{2N}{N+2}}$ for some $g \in L^{\frac{2N}{N+2}}$. Now, using the Dominated Convergence Theorem we infer that $\left\|u_n u - u_n^2\right\|_{L^{\frac{2N}{N+2}}} \to 0$, and thus

$$\left|\int_{\mathbb{R}^N} \phi_{u_n}(x)u_n(u_n - u)dx\right| \to 0.$$

In the same fashion, using Lemma 1, we see that the p-term tends to zero. From this and (22), it follows that $\|u_n\|_H - \|u\|_H \to 0$ and hence $u_n \to u$ strongly in H.

Step 2 : Conclusion.

Now define $\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0, I(\gamma(1)) < 0\}$. The previous steps and Lemma 5 show that the hypothesis of the Mountain-Pass Theorem are satisfied, hence

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

is a critical level of I corrisponding to a nontrivial weak solution in H. The Bootstrap process can be made (see the Lemma below) and by a Maximum Principle argument it can be shown that we can actually get a positive classical solution. □

Because we will use Lemma 4, we need to show that $H-$solutions actually belong to H^2_{loc}. More precisely we can state the following

Lemma 6. *Let u be a weak solution in H of the problem (1). Then $u \in H^2_{loc}(\mathbb{R}^N)$. Moreover, $u \in L^2(\mathbb{R}^N)$, i.e. $u \in H^1(\mathbb{R}^N)$.*

Proof. For convenience sake we write the first equation in (1) as $-\Delta u = a(x)u$, having defined $a(x) := K(x)u^{p-1} - V(x) - \phi(x)$. By standard elliptic regularity theory it's enough to show that $a(x)u \in L^2_{loc}$. We now claim that $u \in L^q_{loc}$, for any $2 \leq q < \infty$. In order to prove that we use the Brezis-Kato result (see e.g. [9], pag 48), since $a_-u \in L^1_{loc}$ and $a_+ \in L^{\frac{N}{2}}$. Observe that the former claim is trivial, while, dealing with the latter simply observe that $(p-1)\frac{N}{2} < 2^* \Leftrightarrow p < 2^* - 1$. As a consequence, $\phi \in W^{2,q}_{loc}$ and by the Morrey embedding theorem, $\phi \in C^{0,\alpha}_{loc}$. Thanks to the local boundedness of V, K and ϕ, the L^2_{loc} regularity of $a(x)u$ follows, hence the conclusion.

Now we prove that, actually, $u \in H^1$. In order to do that, first observe that ϕ is a positive continuous radial function vanishing at infinity. This is a consequence of the fact that $\phi \in C^{0,\alpha}_{loc}$ plus the following decaying estimate (see [5], pag 340):

$$|\phi(x)| \leq C_N|x|^{(2-N)/2}\|\phi\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad |x| \geq 1. \tag{23}$$

This observation allows us to define the auxiliary potential $V_u(x) = V(x) + \phi_u(x)$, satisfying the condition:

$$\frac{a}{1+|x|^\alpha} \leq V_u(x) \leq A' \tag{24}$$

which is identical to (3). Observe now that u is a solution of the equation

$$-\Delta u + V_u(x)u = K(x)u^p$$

which is formally the same as the one studied in [1]. More precisely it can be shown that

$$\int_{\mathbb{R}^N\setminus B_R(0)} V_u(x)u^2dx \approx \exp(-cR^{1-\alpha/2}) \quad R \gg 1,\, c > 0 \tag{25}$$

where $B_r(y) := \{x \in \mathbb{R}^N : |x-y| < r\}$. Now observe that, as a consequence of (24) we have

$$\int_{B_1(y)} u^2 dx \leq c_1 |y|^\alpha \int_{B_1(y)} V_u(x) u^2 dx. \tag{26}$$

By repeating the same argument in [1] pag 14, the equations (25) and (26) yield the existence of a partition $\{B_{r_k}(y_k)\}_{k\geq 1}$ of $\mathbb{R}^N \setminus B_2(0)$ such that

$$\int_{\mathbb{R}^N \setminus B_2(0)} u^2 dx \leq \sum_k \int_{B_{r_k}(y_k)} u^2 dx \leq c_2 \sum_k |y_k|^\alpha \exp(-C|y_k|^{1-\alpha/2}) < \infty$$

and with this we are done. □

PROOF OF THEOREM 2.
We point out that, for $p \in (2,3)$, the PS condition is not known for I, even in the case $V = K = 1$, although the Mountain-Pass geometry holds. This is due to the difficulty in proving the boundedness for Palais-Smale sequences. In order to overcome this obstacle, we use a method introduced by Struwe (see e.g.[15] and also [4], [8]).
Let us consider a perturbation of I :

$$I_\mu(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi_u u^2 dx - \frac{\mu}{p+1} \int_{\mathbb{R}^N} K(x) \, |u|^{p+1} \, dx \tag{27}$$

for $\mu \in [1/2, 1]$.

Following [4] (Proposition 2.3), it is possibile to define min-max levels for I_μ, which we denote by c_μ, such that the following properties are satisfied:

i) $\mu \mapsto c_\mu$ *is non-increasing (hence differentiable a.e. in* $[1/2, 1]$*) and left-continuous.*
ii) Denoting with $\mathcal{I}$ *the set of* μ *for whom* c_μ *is differentiable, then* $\forall \mu \in \mathcal{I}$ *there exist a Palais-Smale seqence for* I_μ *at the level* c_μ*.*
iii) Almost every $\mu \in [1/2, 1]$ c_μ *is a critical level for* I_μ*.*

We remark that thanks to Lemma 5, I has the Mountain Pass Geometry and we are allowed to use this arguments.
We denote by $\mathcal{C}$ the set of the values of μ for whom c_μ is a critical level for I_μ. Now take a sequence $\mu_n \nearrow 1$ in $\mathcal{C}$ and a sequence $u_n \in H$ of critical points for I_{μ_n}. It is easy to see that, if this sequence is bounded, then Theorem 2 follows. Actually, we can now repeat the same argument carried out in Step 1 above: up to a subsquence, we have $u_n \rightharpoonup u$ in H and

$$u_n \longrightarrow u, \quad \text{in } L_K^{p+1}, \qquad p \in (\sigma, 2^* - 1)$$

hence, from $I'(u_n)(u_n - u) = \|u_n\|_H^2 - \|u\|_H^2 + o(1)$ and $\mu_n \nearrow 1$, we find again that $u_n \to u$ in H and thus $I'(u) = 0$.
To prove that the sequence u_n is bounded we use Lemma 4. First we define

the following quantities:

$$\begin{array}{ll} \chi_{1,n} := \int_{\mathbb{R}^N} |\nabla u_n|^2 & \chi_{2,n} := \int_{\mathbb{R}^N} V(x) u_n^2 \\ \chi_{3,n} := \int_{\mathbb{R}^N} \phi_{u_n} u_n^2 & \chi_{4,n} := \mu_n \int_{\mathbb{R}^N} K(x) |u_n|^{p+1} . \end{array}$$

And also $\xi_{V,n} := \int_{\mathbb{R}^N} (x, \nabla V(x)) u_n^2$ and $\xi_{K,n} := \mu_n \int_{\mathbb{R}^N} (x, \nabla K(x)) |u_n|^{p+1}$. Notice that u_n are solutions of the problem $(1)_{\mu_n}$, obtained substituing K with $\mu_n K$ in (1). Now we can use Lemma 4, having already checked the H^2_{loc} regularity in Lemma 6, yielding:

$$\frac{N-2}{2}\chi_{1,n} + \frac{N}{2}\chi_{2,n} + \frac{N+2}{4}\chi_{3,n} - \frac{N}{p+1}\chi_{4,n} = \frac{1}{p+1}\xi_{K,n} - \frac{1}{2}\xi_{V,n}. \quad (28)$$

By definition, we have

$$\frac{1}{2}\chi_{1,n} + \frac{1}{2}\chi_{2,n} + \frac{1}{4}\chi_{3,n} - \frac{1}{p+1}\chi_{4,n} = c_{\mu_n}. \quad (29)$$

Eliminating $\chi_{3,n}$ in the system (28)-(29) we obtain

$$2\chi_{1,n} + \chi_{2,n} - \frac{1}{2}\xi_{V,n} = (N+2)c_{\mu_n} + \frac{1}{p+1}(2\chi_{4,n} - \xi_{K,n}). \quad (30)$$

Using (7), (30) implies

$$2\chi_{1,n} + \frac{2 - c_V^{(1)}}{2}\chi_{2,n} \leq (N+2)c_{\mu_n} + \frac{1}{p+1}(2 - c_K^{(1)})\chi_{4,n}. \quad (31)$$

Since $2 - c_V^{(1)} > 0$, $2 - c_K^{(1)} \leq 0$, and c_{μ_n} is bounded, (31) implies now that $\chi_{1,n}$ and $\chi_{2,n}$ are bounded, namely that $\|u_n\|_H \leq C$, hence the conclusion. □

PROOF OF THEOREM 3.
The proof is the same as the previous one, being reduced to checking the boundedness of u_n. Multiplying the first equation of the problem $(1)_{\mu_n}$ by u and integrating by parts, we find:

$$\chi_{1,n} + \chi_{2,n} + \chi_{3,n} - \chi_{4,n} = 0. \quad (32)$$

Let us solve the system (29)-(32) with respect to $\chi_{3,n}$ and $\chi_{4,n}$. If $D = (3-p)/[4(p+1)]$ denotes the determinant of the system (since we are considering $p \in (2,3)$, then D is positive), we obtain:

$$\begin{cases} \chi_{3,n} = \frac{1}{D}\left[\frac{p-1}{2(p+1)}(\chi_{1,n} + \chi_{2,n}) - c_{\mu_n}\right] \\ \\ \chi_{4,n} = \frac{1}{D}\left[\frac{1}{4}(\chi_{1,n} + \chi_{2,n}) - c_{\mu_n}\right]. \end{cases} \quad (33)$$

Using (10) in (28), we have:

$$\frac{N-2}{2}\chi_{1,n} + \left(\frac{N}{2} + \frac{c_V^{(2)}}{2}\right)\chi_{2,n} + \frac{N+2}{4}\chi_{3,n} - \left(\frac{N}{p+1} + \frac{c_K^{(2)}}{p+1}\right)\chi_{4,n} \leq 0. \quad (34)$$

Substituting (33) into (34) we get

$$\left[\frac{N-2}{2}+\frac{N+2}{4D}\cdot\frac{p-1}{2(p+1)}-\frac{1}{4D}\left(\frac{N}{p+1}+\frac{c_K^{(2)}}{p+1}\right)\right]\chi_{1,n}+$$

$$+\left[\frac{N}{2}+\frac{c_V^{(2)}}{2}+\frac{N+2}{4D}\cdot\frac{p-1}{2(p+1)}-\frac{1}{4D}\left(\frac{N}{p+1}+\frac{c_K^{(2)}}{p+1}\right)\right]\chi_{2,n}\leq$$

$$\leq\left[\frac{N+2}{4D}-\frac{1}{D}\left(\frac{N}{p+1}+\frac{c_K^{(2)}}{p+1}\right)\right]c_{\mu_n}.$$

It is easy to check that, since $p>\delta:=2+\frac{c_K^{(2)}}{2}$, the coefficient of $\chi_{1,n}$ is positive. For the same reason the coefficient of $\chi_{2,n}$ is also positive. Furthermore it can be verified that the coefficient of c_{μ_n} is positive for $p>\frac{4c_K^{(2)}+3N-2}{N+2}$, which is less then δ. Hence we get the boundedness of u_n as above. □

Proof of Proposition 1.
The proof is based on the Mountain-Pass Theorem and the Ekeland Variational Principle and it is almost the same as for Theorem 4.3 and Corollary 4.4 in [12]. Precisely, it can be shown that:

i) $I>-\infty$

ii) I satisfies the Palais-Smale condition.

In order to do that, since we work on H, (14) and Lemma 1 must be used instead of the Strauss Inequality and the Strauss Embedding Theorem. The restriction on α is also needed in order to use the continuity property stated in Lemma 3. □

For λ large enough, Proposition 1 does not hold anymore. Indeed we have the following

Proposition 2. *Assume $\sigma\in(1,2]$, $p\in[\sigma,2]$, $\alpha\leq\alpha^*$ and suppose V and K are radial and smooth satisfying (3) and (4), then:*

i) For $p=2$: if $K(x)\leq 1$, then (1) has no nontrivial positive solution $(u,\phi)\in H\times\mathcal{D}^{1,2}(\mathbb{R}^N)$.

ii) For $p\in[\sigma,2)$: if V and K are such that $V(x)\geq(C_pK(x))^{\frac{1}{2-p}}$, where $C_p=(p-1)^{p-1}(2-p)^{2-p}$, then (1) has no non-trivial positive solution $(u,\phi)\in H\times\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof . Here we follow [10] and [12]. Because of the assumptions on p and α, H is continuously embedded in L_K^{p+1} and $L^{\frac{4N}{N+2}}$, hence all the following integrals are well defined. Now observe that, as a consequence of the trivial inequalty $ab\leq a^2+\frac{b^2}{4}$, it follows

$$\int_{\mathbb{R}^N}u^3dx=\int_{\mathbb{R}^N}\nabla\phi\nabla u dx\leq\int_{\mathbb{R}^N}\left(|\nabla u|^2+\frac{1}{4}|\nabla\phi|^2\right)dx. \qquad (35)$$

Now we argue by contradiction, assuming that $(u,\phi) \in H \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ is a nontrivial positive solution. Then we have

$$0 = (I'(u),u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + \int_{\mathbb{R}^N} \left(V(x)u^2 + \phi_u u^2 dx - K(x)\,|u|^{p+1} \right) dx$$

$$\geq \int_{\mathbb{R}^N} \left(u^3 - \frac{1}{4}\,|\nabla\phi|^2 \right) dx + \int_{\mathbb{R}^N} \left(V(x)u^2 + \phi_u u^2 - K(x)\,|u|^{p+1} \right) dx.$$

Since $\int_{R^N} \phi u^2 dx = \int_{R^N} |\nabla\phi|^2\, dx$ we infer

$$0 \geq \int_{\mathbb{R}^N} u^3 dx + \int_{\mathbb{R}^N} \left(\frac{3}{4}\,|\nabla\phi|^2 + V(x)u^2 - K(x)\,|u|^{p+1} \right) dx$$

$$\geq \int_{\mathbb{R}^N} (u^3 + V(x)u^2 - K(x)u^{p+1}) dx = \int_{\mathbb{R}^N} u^2(u + V(x) - K(x)u^{p-1}) dx. \quad (36)$$

Now define $f(u) := u + V(x) - K(x)u^{p-1}$. If $p = 2$, since $K(x) \leq 1$, then the function f is strictly increasing, hence f is strictly positive for $u > 0$. Therefore, from (36), it follows $u \equiv 0$ and this is a contradiction. Now consider the case $p \in (1,2)$. Observe that the function f has an absolute minimum point $u_m = (K(x)(p-1))^{\frac{1}{2-p}}$. Now defining $C_p = (p-1)^{p-1}(2-p)^{2-p}$ and observing that

$$f(u) \geq f(u_m) = V(x) - (C_p K(x))^{\frac{1}{2-p}} \geq 0$$

we get a contradiction as above.

Remark 3. *We remark that the condition $V(x) \geq (C_p K(x))^{\frac{1}{2-p}}$ is compatible with the case $\sigma \in (1,2]$. Therefore, under this condition, we have non-existence although we also have compactness.*
As a final remark we also consider

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = K(x)u^p, & x \in \mathbb{R}^N \\ \\ -\Delta\phi = u^2. \end{cases} \quad (37)$$

For $\lambda \geq 1/4$, by repeating the same proof, it is easy to see that Proposition 2 holds true, extending the result of the Theorem 4.1 in [12] *to the case of NLSP with radial potentials vanishing at infinity.*

5. APPENDIX

Proof of Lemma 4. The proof of this identity follows a standard method in the literature, therefore we only sketch the main steps. Consider $\{\eta^s(x)\}_{s>0} \subset C^\infty_{rad}(\mathbb{R}^N)$ with the following properties:

1) $0 \leq \eta^s(x) \leq 1$ 3) $\eta^s(x) = 1, \quad x \in B(0,s/2)$

2) $|\nabla\eta^s(x)| \leq \frac{C}{s}$ 4) $\eta^s(x) = 0, \quad x \in \mathbb{R}^N \setminus B(0,s)$

where $B(0,s) := \{x \in \mathbb{R}^N)\ |x| < s\}$ and for some positive constant C. Multipliply the first equation in (1) by $x_i\partial_i u(x)\eta^s(x)$, integrate on $B(0,s)$ and sum up on i. Observe that, since $\operatorname{supp}\eta^s$ is contained in $\{x : |x| \leq s\}$, then $|\nabla\eta^s(x)| \leq \frac{C}{s} \leq \frac{C}{|x|}$. Thanks to the Dominated Convergence Theorem there exist a sequence $s_n \to \infty$ (we simply write $s \to \infty$) (see e.g. [5], [6], [7], [9] Section 3) such that:

$$-\sum_i \int_{B(0,s)} \Delta u x_i(\partial_i u)\eta^s dx = \frac{2-N}{2}\int_{B(0,s)} |\nabla u|^2 dx + o(1). \tag{38}$$

In order to perform the calculation for the K-term, integrating by parts, observe that

$$\int_{B(0,s)} K(x)u^p x_i(\partial_i u)\eta^s dx = \frac{1}{p+1}\int_{B(0,s)} K(x)(\partial_i u^{p+1})x_i\eta^s dx =$$

$$= -\frac{1}{p+1}\int_{B(0,s)} \eta^s u^{p+1}K(x)dx - \frac{1}{p+1}\int_{B(0,s)} x_i(\partial_i\eta^s)u^{p+1}K(x)dx -$$

$$-\frac{1}{p+1}\int_{B(0,s)} \eta^s u^{p+1}x_i\partial_i K(x)dx.$$

In the last step the boundary term has been neglected since $\eta^s(\partial B(0,s)) = 0$. Since $0 \leq \eta^s \leq 1$ and $\eta^s \to 1$, $|x_i\partial_i\eta^s| \leq C$ and $\partial_i\eta^s \to 0$, by the Dominated Convergence Theorem the second integral in the last step tends to zero. Hence

$$\int_{B(0,s)} K(x)u^p x_i\partial_i u\eta^s dx = -\frac{1}{p+1}\int_{B(0,s)} \eta^s u^{p+1}K(x)dx -$$

$$-\frac{1}{p+1}\int_{B(0,s)} \eta^s u^{p+1}x_i\partial_i K(x)dx + o(1). \tag{39}$$

We now consider the last integral in (39). Since u is a radial function in $H_V(\mathbb{R}^N)$ the Strauss type inequality (14) holds:

$$|u(x)| \leq c\,|x|^{-\gamma}\,\|u\|_{H_V}. \tag{40}$$

a.e. in $\mathbb{R}^N \setminus B^c(0,s)$ for large s. Since $1-\eta^s = 0$ on $B(0,s)$ and using (40) we get

$$\left|\int_{\mathbb{R}^N} u^{p+1}x_i\partial_i K(x)dx - \int_{\mathbb{R}^N} \eta^s u^{p+1}x_i\partial_i K(x)dx\right| \leq$$

$$\leq c's^{-(N-1)\varepsilon/2}\int_{\mathbb{R}^N\setminus B(0,s)} (1-\eta^s)u^{p+1-\varepsilon}|x_i\partial_i K(x)|dx. \tag{41}$$

Notice that, because of (8), since $q'(p+1-\epsilon) \in [2+\dfrac{\alpha}{\gamma}, 2^*]$, there exist a constant $C_{p,q',\varepsilon}$ such that $\|u\|_{L^{q'(p+1-\epsilon)}(\mathbb{R}^N)} \leq C_{p,q',\varepsilon}\|u\|_{H_V(\mathbb{R}^N)}$. Therefore, since $0 \leq 1-\eta^s \leq 1$ and using the Hölder inequality, we have:

$$\int_{\mathbb{R}^N\setminus B(0,s)} (1-\eta^s)u^{p+1-\varepsilon}|x_i\partial_i K(x)|dx \leq \int_{\mathbb{R}^N\setminus B(0,s)} u^{p+1-\epsilon}|x_i\partial_i K(x)|dx \leq$$

$$\leq \|u\|_{L^{q'(p+1-\epsilon)}(\mathbb{R}^N)}\|(x,\nabla K)\|_{L^q(\mathbb{R}^N)} \leq C_{p,q',\varepsilon}\|u\|_{H_V(\mathbb{R}^N)}\|(x,\nabla K)\|_{L^q(\mathbb{R}^N)} < \infty. \tag{42}$$

Observe that (41) and (42) imply that $\int_{\mathbb{R}^N} u^{p+1}(x, \nabla K)dx < \infty$ and

$$\int_{B(0,s)} \eta^s u^{p+1} x_i \partial_i K(x)dx = \int_{B(0,s)} u^{p+1} x_i \partial_i K(x)dx + o(1). \tag{43}$$

Finally, from (39), (43) and summing up on i we have

$$\sum_i \int_{B(0,s)} K(x)u^p x_i(\partial_i u)\eta^s dx = -\frac{N}{p+1}\int_{B(0,s)} K(x)u^{p+1}dx -$$

$$-\frac{1}{p+1}\int_{B(0,s)} (x, \nabla K)u^{p+1}dx + o(1) \tag{44}$$

In the same fashion as in (44), because of the assumptions on V, we are allowed to use the Dominated Convergence Theorem and we get

$$\sum_i \int_{B(0,s)} V(x)u^2 x_i(\partial_i u)\eta^s dx = -\frac{N}{2}\int_{B(0,s)} V(x)u^2 dx - \frac{1}{2}\int_{B(0,s)} (x, \nabla V)u^2 dx + o(1). \tag{45}$$

Moreover, as in (39):

$$\sum_i \int_{B(0,s)} \phi_u u x_i(\partial_i u)\eta^s dx = -\frac{N}{2}\int_{B(0,s)} \phi_u u^2 dx - \frac{1}{2}\int_{B(0,s)} (x, \nabla \phi_u)u^2\eta^s dx + o(1). \tag{46}$$

From the first equation in (1) and (38), (44), (45), (46), we finally have , as $s \to \infty$:

$$\frac{2-N}{2}\int_{B(0,s)} |\nabla u|^2 dx - \frac{N}{2}\int_{B(0,s)} V(x)u^2 dx - \frac{1}{2}\int_{B(0,s)} (x, \nabla V)u^2 dx -$$

$$-\frac{N}{2}\int_{B(0,s)} \phi_u u^2 dx - \frac{1}{2}\int_{B(0,s)} (x, \nabla \phi_u)u^2\eta^s dx + o(1) =$$

$$= -\frac{N}{p+1}\int_{B(0,s)} K(x)u^{p+1}dx - \frac{1}{p+1}\int_{B(0,s)} (x, \nabla K)u^{p+1}dx \tag{47}$$

In the same way as above, we now multiply the second equation in (1) by $(x, \nabla \phi_u)\eta^s$ and integrate on $B(0, s)$, obtaining:

$$\frac{2-N}{2}\int_{B(0,s)} |\nabla \phi_u|^2 dx = \int_{B(0,s)} (x, \nabla \phi_u)u^2\eta^s dx + o(1). \tag{48}$$

Eliminating $\int_{B(0,s)}(x, \nabla \phi_u)u^2\eta^s dx$ from (47) and (48), letting $s \to \infty$ and using $\int_{\mathbb{R}^N} |\nabla \phi_u|^2 dx = \int_{\mathbb{R}^N} \phi_u u^2 dx$, we get the conclusion. □

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