

On Computing the Minimal Labels in Time Point Algebra Networks *

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Technical Report 495

Abstract

We analyze the problem of computing the minimal labels for a network of temporal relations in the Point Algebra. van Beek proposes an algorithm for accomplishing this task which takes $O(\max(n^3, n^2 \cdot m))$ time (for n points and m relations). We show that the proof of the correctness of this algorithm given by van Beek and Cohen is faulty, and we provide a new proof showing that the algorithm is indeed correct.

Keywords: temporal reasoning, Point Algebra, constraint networks, reasoning with inequations

*The work of the first author was carried out in part during a visit at the Computer Science Department of the University of Rochester (NY) supported by the Italian National Research Council (CNR), and in part at IRST in the context of the MAIA project and CNR projects “Sistemi Informatici e Calcolo Parallelo” and “Pianificazione Automatica”. The second author was supported by Rome Lab Contract F30602-91-C-0010.

1 Introduction

The *Interval Algebra* (IA) (Allen 1983) and the *Point Algebra* (PA) (Vilain and Kautz 1986) are two fundamental approaches to representing temporal information in terms of qualitative relations between intervals and points respectively. The Interval algebra is based on thirteen basic relations that can hold between intervals; from these, 2^{13} possible disjunctive relations can be derived. The elements of the Point Algebra are the relations $\{\emptyset, <, \leq, =, >, \geq, \neq, ?\}$, where $?$ is the disjunction of the three basic relations $<, =$ and $>$. The “pointizable” interval algebra (written as SIA in (van Beek 1990)) is a restricted algebra of IA which consists of the set of the 188 relations in IA (including the empty relation) that can be translated into conjunctions of point relations between the endpoints of the intervals (Ladkin and Maddux 1988; van Beek and Cohen 1990).

A set of asserted relations in IA (PA) can be represented through a labeled graph which van Beek (1992) calls IA-network (PA-network), whose vertices represents interval variables (point variables) and whose labeled edges correspond to the asserted relations. Given a set of asserted temporal relations (a labeled graph), one of the main reasoning tasks concerns deducing new relations from those that are known and, in particular, computing the strongest entailed relations for each pair of intervals or points (called the “minimal labels” in (van Beek and Cohen 1990), the “feasible relations” in (van Beek 1992), and the “closure” of the assertions in (Vilain *et al.* 1990)).

In (Vilain and Kautz 1986) an $O(n^3)$ time algorithm for determining the closure of a set of assertions of relations in PA (for n points) is given. In (van Beek 1990; van Beek and Cohen 1990; Vilain *et al.* 1990) it is shown that this algorithm is not correct and in (van Beek 1990; van Beek 1992) another algorithm for accomplishing the same task is provided. The new algorithm takes $O(\max(n^3, n^2 \cdot m))$ time and $O(n^2)$ space (for $m \neq$ -assertions). Given a PA-network the algorithm performs two main steps: first it computes the *path-consistent network* (Montanari 1974; Mackworth 1977) of the initial network in $O(n^3)$ time; secondly it computes the minimal labels by finding and removing the *forbidden subgraphs* of the path-consistent network (see Figure 1.a) which is accomplished in $O(n^2 \cdot m)$ time. The correctness of this algorithm relies on a lemma given by van Beek and Cohen (1990) claiming that any path-consistent PA-network which does not have minimal labels contains a forbidden graph.

In this note we show that the proof of this lemma is not correct, and we provide a new proof showing that the algorithm given by van Beek (1990) is indeed correct.

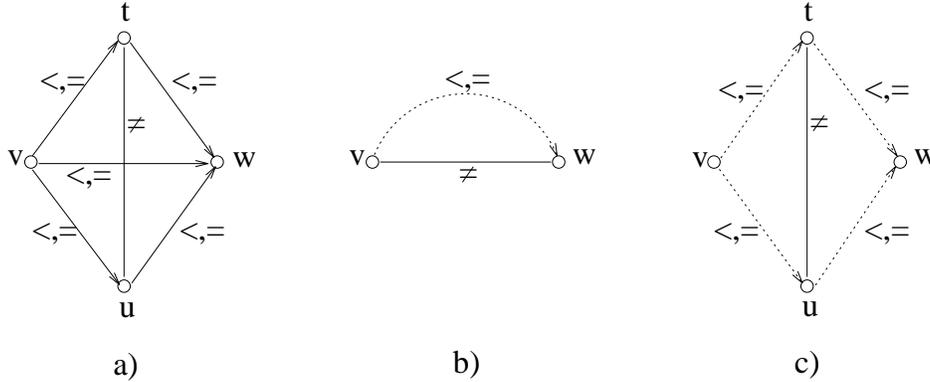


Figure 1: a) van Beek's forbidden graph; b) and c) the two kinds of implicit $<$ relation. Dotted arrows indicate paths, solid lines \neq -edges. In all the graphs there is an implicit $<$ relation between v and w .

2 Representing temporal relations through TL -graphs

In this section we first introduce the necessary terminology and theoretical background (partly taken from (Gerevini and Schubert 1993a)).

Definition 1 A *temporally labeled graph* (TL -graph) is a graph with at least one vertex and a set of labeled edges, where each edge (v, l, w) connects a pair of distinct vertices v, w . The edges are either directed and labeled \leq or $<$, or undirected and labeled \neq .

Figure 2 shows an example of TL -graph. We assume that every vertex of a TL -graph has at least one name (time point variable) attached to it. If a vertex has more than one name, then they are alternative names for the same time point. The name sets of any two vertices are required to be disjoint.

More formally, writing P for the set of time-point variables attached to a TL -graph, and V for its set of vertices, we assume that there exists a surjective function PV from P to V .

A *model* of a TL -graph is an interpretation of the vertex names as elements of a totally ordered set T (with strict ordering $<$), such that all names attached to the same vertex denote the same element and the interpretations of names attached to distinct vertices satisfy the constraints expressed by the edge(s), if any, connecting those vertices.

More formally, the notions of an interpretation and of a model for a TL -graph can be specified as follows:

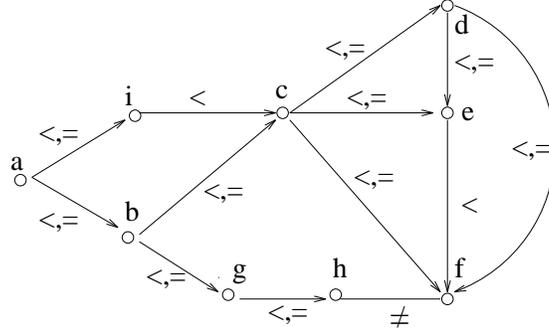


Figure 2: An example of TL -graph

Definition 2 Given a TL -graph G , an *interpretation* of G is a triple $\langle T, I, R \rangle$, where T is a totally ordered set (with ordering $<$); I is a function $I : P \rightarrow T$ such that for all $p_1, p_2 \in P$ if $PV(p_1) = PV(p_2)$ then $I(p_1) = I(p_2)$; R is a function mapping each label l on the edges of G (" $<$ ", " \leq " or " \neq ") into the corresponding binary relation $R(l)$ on T ($<$, \leq or \neq).¹

Definition 3 Given a TL -graph G , a *model* of G is an interpretation such that if (v_1, l, v_2) is an edge of G , then for all $p_i, p_j \in P$ satisfying $PV(p_i) = v_1$ and $PV(p_j) = v_2$

$$\langle I(p_i), I(p_j) \rangle \in R(l).$$

Definition 4 A TL -graph is *consistent* if and only if it has at least one model.

Definition 5 Two or more TL -graphs are *logically equivalent* if and only if they have the same models.

A path of length n from v_0 to v_n is a sequence of n triples $(v_0, l_1, v_1), \dots, (v_{n-1}, l_n, v_n)$ where v_i ($0 \leq i \leq n$) are vertices and l_j ($1 \leq j \leq n$) are labels (relations) on directed edges.

Definition 6 In a TL -graph a path is a \leq -path if each label l_j is \leq or $<$. A \leq -path is a $<$ -path if at least one of these labels is $<$.

Definition 7 A \leq -path ($<$ -path) of length n from v_0 to v_n , $n \geq 1$, is a \leq -cycle ($<$ -cycle) if $v_0 = v_n$. A TL -graph is *acyclic* if it does not contain any \leq -cycle.

¹For instance if $l = "<"$ then $R(<)$ is the $<$ -relation, i.e., the set of pairs $\langle t_1, t_2 \rangle$ such that $t_1 < t_2$ and $t_1, t_2 \in T$. (Analogously for " \leq " and for " \neq ".)

In (Gerevini and Schubert 1993a) we proved the following theorems about determining consistency of a TL -graph ²:

Theorem 1 *A TL -graph is consistent iff it does not contain any $<$ -cycle, or any \leq -cycle that has two vertices connected by an edge with label \neq .*

Theorem 2 *A TL -graph can be recognized as being inconsistent, or if it is consistent, collapsed into a logically equivalent acyclic TL -graph in $O(e)$ time, where e is the number of edges.*

Definition 8 A TL -graph contains an *implicit* $<$ relation $v < w$ if there is no $<$ -path from v to w and either there is an edge between v and w with label \neq and a \leq -path (but no $<$ -path) from v to w (see Figure 1.b)); or there exist two vertices u and t such that there is an edge between u and t with label \neq and \leq -paths (but no $<$ -path) from v to u , u to w , v to t , and t to w (see Figure 1.c)). The graphs isomorphic to the one given in Figure 1.c) are called *\neq -diamonds*.

Definition 9 An *explicit graph* for a given TL -graph G is an acyclic TL -graph logically equivalent to G and with no implicit $<$ relations.

Theorem 3 (Gerevini and Schubert 1993a) *An explicit TL -graph entails $v \leq w$ iff there is a \leq -path from v to w ; it entails $v < w$ iff there is a $<$ -path from v to w , and it entails $v \neq w$ iff there is a $<$ -path from v to w or from w to v , or there is an edge (v, \neq, w) .*

3 Computing the minimal labels

In this section we first introduce the notions of a path-consistent TL -graph and of a minimal TL -graph. We then provide a new proof of van Beek and Cohen's Lemma about forbidden graphs in path-consistent PA-networks. Given a PA-network \mathcal{T} , this lemma allows the computation of the minimal labels of \mathcal{T} by identifying all the forbidden graphs in the path-consistent network of \mathcal{T} and by making explicit all the implicit $<$ relations.

Definition 10 A TL -graph is *path-consistent* if there is exactly one edge joining each pair of vertices, and for every triple (u, v, w) of vertices, any of the possible relations $\{<, >, =\}$ allowed by the label of the edge joining u and v (l') is consistent with the relations allowed by the labels of the edges joining u and w (l''), and v and w (l'''). That is, there exists an interpretation I such that for all p_i, p_j, p_k satisfying $PV(p_i) = u$, $PV(p_j) = v$ and $PV(p_k) = w$

²Results similar to Theorems 1 and 2 are also given by van Beek (1990; 1992).

$$\begin{aligned}
\langle I(p_i), I(p_j) \rangle &\in R(l') \\
\langle I(p_i), I(p_k) \rangle &\in R(l'') \\
\langle I(p_j), I(p_k) \rangle &\in R(l''').
\end{aligned}$$

Definition 11 The *minimal graph* of a consistent TL -graph G is a TL -graph logically equivalent to G , containing exactly one labeled edge for each pair of distinct vertices and in which for each pair (v_1, v_2) of vertices the label l of the edge (v_1, l, v_2) corresponds to the strongest relation entailed by G ³. The minimal graph of an inconsistent TL -graph is the empty graph.

The definitions of a path-consistent TL -graph and of a minimal TL -graph are a slight departure from the notions of a *path-consistent network* and of a *minimal network* originally given in the context of constraint satisfaction problems (Montanari 1974; Mackworth 1977). Note also that not every *PA-network* as defined by van Beek and Cohen (1990) is a TL -graph; only networks without $=$ -edges translate directly into TL -graphs.

An explicit TL -graph can be transformed into a path-consistent graph by three operations: unifying multiple edges between two vertices into a single edge labeled with the intersection of the labels on the individual edges; introducing new edges in order to have exactly one edge joining each pair of vertices whose label is chosen according to path-consistency and reducing the labels on the original edges in order to satisfy path-consistency.

van Beek and Cohen (1990) give a lemma stating that any path-consistent nonminimal *PA-network* of relations taken from *PA* must include a four-vertex subgraph isomorphic to the graph in Figure 1.a). The proof they provide is not correct but the Lemma is indeed valid. The incorrectness of their proof becomes evident if one observes that the *PA-network* of figure 1.a) is not the only four-vertex *path-consistent PA-network* that, up to isomorphism, can be derived in the last step of the proof of van Beek and Cohen. In fact, the network obtained by replacing $(v, \{<, =\}, u)$ with $(v, ?, u)$ and $(u, \{<, =\}, w)$ with $(u, ?, w)$ is another possible network. This contradicts what van Beek and Cohen assert at the end of their proof.

The next theorem provides a new proof of van Beek and Cohen's Lemma. In the first part of the proof, we deal with networks without $=$ -edges, which therefore amount to TL -graphs. In the second part of the proof we relate networks with $=$ -edges to TL -graphs, by talking about the networks obtained by collapsing equal vertices. For these collapsed networks, the first part of the proof applies, and we can infer the existence of a forbidden subgraph. We then argue that the forbidden subgraph must also have been present in the network prior to collapsing it.

³Relation vRw is stronger than relation $vR'w$ if and only if vRw entails $vR'w$, but not the converse; in other words, viewed as subsets of $\{<, =, >\}$, R is a proper subset of R' .

Theorem 4 *Any path-consistent PA-network G that contains no forbidden graphs is minimal.*

Proof. Consider the case where G contains no $=$ -edges. In this case it suffices to show that such a network G is already explicit, and that an explicit, path-consistent network is minimal. To see that G is already explicit, note first that there are no implicit $<$ -relations of the first kind in G , since the presence of an edge (v, \neq, w) across a \leq -path from v to w would violate path consistency (the edge from v to w would have to be labeled $<$ rather than \neq). Moreover, there are no implicit $<$ -relations of the second kind in G , since these can exist only when there is a \neq -diamond, and in a path-consistent graph the presence of such a diamond entails the presence of a forbidden subgraph (i.e., each of the four \leq -paths making up the \neq -diamond must join two vertices whose connecting edge is also labeled \leq ; the label cannot be stronger ($<$) since otherwise the definition of a \neq -diamond is not satisfied). Also there are no \leq -cycles in a path-consistent network without $=$ -edges, and so G is indeed explicit. But an explicit, path-consistent network is minimal, since by Theorem 3 the graph entails (a) $v \leq w$ iff there is a \leq -path from v to w , (b) $v < w$ iff there is a $<$ -path from v to w , and (c) $v \neq w$ iff there is a $<$ -path from v to w or from w to v or a \neq -edge connecting v and w , and by path consistency these three cases respectively entail that the label l in edge (v, l, w) is (a) \leq only if the graph does not entail $v < w$, (b) $<$, and (c) $<$, $>$, or \neq .

This leaves us with the case where G contains $=$ -edges (but is still path-consistent and contains no forbidden subgraphs). In this case G will clearly be minimal as long as for any v, w connected by a $=$ -path (a path with label “ $=$ ” on all edges) the label of the edge connecting v and w is $=$, and the graph G' obtained by collapsing equated sets of vertices into a single vertex is minimal. The former is true by path consistency, and the latter is true by the first part of the proof (collapsing equated vertices does not destroy path consistency, does not create new forbidden graphs and preserves entailments). \square

4 Conclusions

In this note we have provided a new proof of van Beek and Cohen’s lemma about some strict orderings that cannot be derived by computing path consistency in a network of temporal relations in the Point Algebra. This lemma is fundamental in showing that the strongest entailed relations for each pair of points (the minimal labels) of a set of assertions in the Point Algebra can be computed in polynomial time ($O(\max(n^2 \cdot m, n^3))$), where n is the number of time points and m is the number \neq assertions (van Beek 1990).

The larger goal which prompted this work was the design of a temporal reasoning system called *TimeGraph-II* (TG-II) (Gerevini and Schubert 1993a; Gerevini and Schubert 1993b; Gerevini and Schubert 1993c). TG-II is descended from TG-I, an earlier system to support natural language understanding (Schubert *et al.* 1983; Schubert *et al.* 1987; Miller and Schubert 1990). TG-II is aimed at a broader class of applications, including planning.

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