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Finite Sphere Packings and Sphere Coverings

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1 Introduction

The basic problems of finite sphere packing and sphere covering in euclidean d-space E^d are:

Determine, for a given $k \in \mathbb{N}$,

A) the minimal volume of all convex bodies, into which k unit–balls can be packed;

B) the maximal volume of all convex bodies, which can be covered by k unit–balls.

(Clearly one can replace the k balls by k translates of a given convex body, but here we only consider balls, which is the most interesting case.)

It turns out that in both cases the expected extremal configurations for $d \geq 5$ and arbitrary $k \in \mathbb{N}$ are linear arrangements, i.e. C_k a segment (cf. [4], [18]).

Although these 'sausage–conjectures' are easy to describe it seems to be hard to prove them. No complete proof is known but several partial results (e.g. cf. [1], [2], [3], [9]) support these conjectures.

For $d = 3$ and 4 the situation is more complicated. The expected behaviour is, that for small k the sausages are optimal, and that after a certain bound k_d clusters are optimal. These 'sausage–catastrophes' are supported by computer–aided experiments. No non–trivial lower bounds for k_d are known. Gandini and Wills (cf. [6], [17]) showed $k_3 \leq 56$ and investigated dense finite packings for $k > k_3$.

Further Gandini and Zucco proved $k_4 < 378.000$ and gave arguments that the expected value is not too far away. For $d = 2$ the problem is solved for packings (cf. [11], [15]), where no sausages or sausage–catastrophes occur (but open for coverings, where at least for $k < 10$ sausages seem to be optimal).

It is the purpose of this paper to understand better why the expected optimal arrangements are so differently depending on d.

For this we consider a more general packing problem which contains A) as a special case:

In the following let $B^d \subset E^d$ denote the unit–ball centered at 0, $B_i^d = B^d + c_i$, $i = 1, \ldots, k$, and $C_k = \operatorname{conv}(c_1, \ldots c_k)$. For a convex body $K \subset E^d$ let $V(K)$ denote its volume and $K + \lambda B^d$ its outer parallel body with radius $\lambda \geq 0$. We then investigate for given $k \in \mathbb{N}$ and $\lambda \geq 0$:

$$\min V(C_k + \lambda B^d),$$

where the minimum is taken over all admissible (i.e. possible) configurations C_k. The special case $\lambda = 1$ is the above mentioned problem A).

It turns out that in the general case $\lambda \geq 0$ sausages and sausage–catastrophes occur (or are at least possible) in all dimensions.

Moreover for $d = 2$ the problem is completely solved by reformulating two results by Groemer (cf. [11]) and Oler (cf. [13]), which is done in the next §.

There is no direct analogue for coverings. But generalization of B) is a similar way ('bone–problems') were already considered in [9].

2 Groemer's and Oler's circle–packing theorems

In the following E^2 denotes the euclidean plane. For a convex body $K \subset E^2$ let $V(K)$ and $P(K)$ denote its area (or 2–dim. volume) and perimeter. In 1960 H. Groemer (cf. [11]) proved

Theorem 1 (Groemer 1960). *Let $K \subset E^2$ be a convex body containing $k \geq 1$ nonoverlapping unit circles $B_i^2 = B^2 + c_i$, $i = 1, \ldots, k$. Then*

$$\sqrt{12}\, k \leq V(K) - \left(1 - \sqrt{3}/2\right) P(K) + \sqrt{12} - \pi \left(\sqrt{3} - 1\right). \tag{1}$$

Equality in (1) holds iff $K = \operatorname{conv}(B_1^2, \ldots, B_k^2)$ and one of the following conditions holds for $C_k = \operatorname{conv}(c_1, \ldots, c_k)$:

a) C_k can be dissected into (at least one) equilateral triangles of edge–length 2, and $\{c_1, \ldots, c_k\}$ is the set of its vertices (in this case we write $C_k = C_k^*$),

b) C_k is a segment S_k of length $2\,(k-1)$.

Obviously S_k exists for each $k \in \mathbb{N}$. Further it is easy to see that for each $k \in \mathbb{N}$ at least one C_k^* exists (in general more than 1).

One year after Groemer N. Oler (cf. [13]) proved the following conjecture by H. Zassenhaus (which we slightly reformulate):

Theorem 2 (Oler 1961). *Let $X \subset E^2$ be a convex body containing $k \geq 1$ points $x_1, \ldots, x_k$ with mutual distance $d(x_i, x_j) \geq 2$. Then*

$$k \leq \left(\sqrt{12}\right)^{-1} V(X) + \frac{1}{4} P(X) + 1. \tag{2}$$

Oler did not mention the equality cases, but they are implicitly in his proof.

As Wegner (cf. [15]) first pointed out, it is easy to see that both results are equivalent: First we observe that, because of the monotonicity of V and P, we can restrict ourselves in (1) to $K = C_k + B^2$ and in (2) to $X = C_k$. So we get from (1):

$$\sqrt{12}(k-1) \leq V(C_k + B^2) - \left(1 - \sqrt{3}/2\right) P(C_k + B^2) - \pi \left(\sqrt{3} - 1\right)$$

or equivalently

$$\sqrt{12}(k-1) \leq V(C_k) + P(C_k) + \pi - \left(1 - \sqrt{3}/2\right) \left(P(C_k) + 2\pi\right) - \pi \left(\sqrt{3} - 1\right)$$

or

$$\sqrt{12}(k-1) \leq V(C_k) + \sqrt{3}/2 \, P\,(C_k), \tag{3}$$

which is equivalent to (2).

Remarks.

1. C.A. Rogers (cf. [14]) proved already in 1951 a more general result, which in the case of circles leads to the weaker inequality

$$\sqrt{12}(k-1) \leq V(C_k) + P(C_k).$$

2. Generalizations of (2) (and hence of (1)) are due to Folkman and Graham (cf. [5]) and Graham, Witsenhausen and Zassenhaus (cf. [8]).

3. Groemer's proof requires 10 pages and Oler's proof even 30 pages, which makes clear that the proof of (1) or (2) is far from being simple.

4. In the next section we give an equivalent formulation of (1) and (2), which leads to new insight into the 'sausage–problems'.

3 A general sausage result in $\mathbf{E^2}$.

Let again S_k denote the segment of length $2(k-1)$. Then (3) and hence (1) and (2) are equivalent to

$$V\left(S_k + \frac{\sqrt{3}}{2} B^2\right) \leq V\left(C_k + \frac{\sqrt{3}}{2} B^2\right) \tag{4}$$

with equality for $C_k = S_k$ or $C_k = C_k^*$.

With Steiner's formula (cf. [1]) for the outer parallel body one has for $\lambda \geq 0$:

$$V(C_k + \lambda B^2) = V(C_k) + \lambda P(C_k) + \lambda^2 \pi$$

and

$$V(S_k + \lambda B^2) = \lambda\, 2\,(k-1) + \lambda^2 \pi,$$

hence (4) implies

$$V(S_k + \lambda B^2) < V(C_k + \lambda B^2) \text{ for } 0 \leq \lambda < \sqrt{3}/2. \tag{5}$$

In other words: For a packing of $k \geq 1$ unit circles the area of the outer parallel body of C_k with radius $\lambda \in \left[0, \sqrt{3}/2\right)$ is minimal iff $C_k = S_k$. For $\lambda > \sqrt{3}/2$ follows with the same argument and the fact that for each k a $C_k^* \neq S_k$ exists:

$$\min V(C_k + \lambda B^2) \leq V(C_k^* + \lambda B^2) < V(S_k + \lambda B^2),$$

i.e. for no $k \in \mathbb{N}$ the sausage is minimal.

Now (5) (and (4)) gives a minimality property of a linear arrangement of circles. So it is a 'sausage–theorem', which in particular is best possible. The value $\lambda = \sqrt{3}/2$, where the shape of the optimal arrangement changes drastically, is called the critical radius of the outer parallel body of C_k.

4 Critical radii in higher dimensions.

In contrast to the previous sections we mention the results without proofs. Again we ask for given $d \geq 2$, $k \in \mathbb{N}$ and $\lambda \geq 0$ for $\min V(C_k + \lambda B^d)$, where the minimum is taken over all admissible $C_k \subset E^d$.

In particular we want to know when

$$V(S_k + \lambda B^d) \leq V(C_k + \lambda B^d) \tag{6}$$

for all admissible C_k.

The following basic result on the critical radii is easy to prove:

Theorem 3 . *For each $d \geq 2$ there are numbers λ_d and λ_d' with $0 < \lambda_d \leq \lambda_d' < 2$, such that (6) is*
a) true for all $k \in \mathbb{N}$, if $0 \leq \lambda \leq \lambda_d$
b) not true for at least one $k \in \mathbb{N}$, if $\lambda > \lambda_d$
c) not true for all but finitely many $k \in \mathbb{N}$, if $\lambda > \lambda_d'$.

Hence for each $d \geq 2$ the sausage property holds, if $\lambda \in [0, \lambda_d]$.

The upper bound 2 for λ_d' stems from the Minkowski–Hlawka–Theorem in Geometry of Numbers and shows the close relation to classical packing problems.

We do not have good lower bounds for λ_d, but we conjecture that λ_d grows with d, hence $\lambda_d \geq \sqrt{3}/2$. Moreover we conjecture $\lambda_d = \lambda'_d$, i.e. that the critical radii coincide.

Fejes Tóth's sausage conjecture says that $\lambda_d \geq 1$ for $d \geq 5$.

For small d the upper bound can easily be dropped; in particular:

$$\lambda'_3 \leq 8^{1/4} / \pi^{1/2} = 0.95\ldots \quad \text{and} \quad \lambda'_4 \leq 0.98\ldots.$$

Proofs and more details will be published in a forthcoming paper.

As mentioned in the introduction, all investigations can also be made for centrally symmetric convex bodies K via $\min V(C_k + \lambda K)$.

For some special cases the problem is simple, but in general it is even harder than for B^d. The case $\lambda = 1$ was investigated by Rogers in 1950 (cf. [9] or [10]).

5 Sausage–skin.

If one replaces the volume in the basic problems A) and B) by the surface, one obtains interesting analogous problems in the covering case (cf. [3] and [9]). In particular sausages occur and for $d = 2$ the problem is completely solved.

In the packing case the problem is less attractive although some investigations were made (cf. [2], [12]). The problem becomes more attractive if one considers only $\dim C_k \leq i$ when $V_i(C_k + B^d)$ (or more general $V_i\,(C_k + \lambda B^d)$ is investigated.

A general survey on all these and related problems is [10].

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