

# Categorical Aspects of Equivariant Homotopy

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## 1 Introduction

Recently we finished working on an article [9] that tries to lay the foundations of an infinitely lax or homotopy coherent version of category theory based on simplicially enriched categories. (That article is the overdue extended version of the preprints, [4] and [7].) The history of homotopy coherence is linked strongly to that of equivariant homotopy theory through the work of Elmendorf, May, Dwyer and Kan, and others. In this paper we will use equivariant homotopy theory as a case study for how our new methods relate to old ones, and in the process will extend results of Dwyer and Kan to an enriched setting. We will try to illustrate the uses and interpretation of the categorical results that we have proved, as we believe that these results will have significant applications in other parts of the subject.

The second author would like to thank Ronnie Brown, Marek Golasinski and Andy Tonks for conversations on the area. This work was, in part, prepared for them in order to extend the use of crossed complexes to the equivariant case and in particular to extend the theory of classifying spaces to this setting. This work on equivariant crossed complexes will be the subject of a joint publication later on.

The diagrams in this article were produced using Paul Taylor’s “Diagrams” package. The authors would like to thank him for the use of these macros.

## 2 $G$ -Sets and $OrG$ -diagrams.

The results in this section are well known, but usually are not presented in this way. We do so here to try to explain why various constructions work in later sections. For simplicity, we restrict attention to  $G$  being a discrete group.

We will denote by  $OrG$  the orbit category of  $G$ . This has as objects the various  $G$ -sets  $G/H$ ,  $H$  a subgroup of  $G$ , with a morphism from  $G/H$  to  $G/K$  being a  $G$ -equivariant map of  $G$ -sets. As  $G^{\text{op}} \cong OrG(G/1, G/1)$ , there is a functor

$$\phi : G^{\text{op}} \rightarrow OrG,$$

where on the left  $G$  is considered as a category with one object,  $*$ , say. The category of left  $G$ -sets is the category  $Sets^G$  and the category of  $OrG^{\text{op}}$ -diagrams is  $Sets^{OrG^{\text{op}}}$ . These are linked by various functors:

- $\phi^* = Sets^{\phi^{\text{op}}} : Sets^{OrG^{\text{op}}} \rightarrow Sets^G$ , defined by composition along  $\phi$ . This sends an  $OrG^{\text{op}}$ -diagram  $Y$  to  $Y(G/1)$ , which is a left  $G$ -set.
- the right Kan extension along  $\phi$  gives a right adjoint to  $\phi^*$ . This will be denoted here by  $R_\phi : Sets^G \rightarrow Sets^{OrG^{\text{op}}}$ . (Later we will tend to write simply  $R$ .) It is given by the end formula

$$R_\phi(X)(G/H) = \int_* Sets(OrG^{\text{op}}(G/H, G/1), X(*))$$

which is isomorphic to  $X^H = \{x : x \in X, h.x = x \text{ for all } h \in H\}$ . Right adjointness of  $R_\phi$  is a consequence of its construction as a right Kan extension, but is easily checked directly. The functor  $R_\phi$ , or rather its analogue in other situations, is the key to translating  $G$ -equivariant homotopy theory to a ‘diagrammatic’ form which is more amenable to analysis.

- The functor  $R_\phi$  has in its turn a right adjoint, which will be denoted by  $c : Sets^{OrG^{\text{op}}} \rightarrow Sets^G$ . This can be calculated as a left Kan extension, so that  $c(Y) = Lan_{R_\phi} Id(Y)$ . This gives a coend formula:

$$c(Y) = \int^X Sets^{OrG^{\text{op}}}(R_\phi(X), Y) \overline{\otimes} X,$$

where  $A \overline{\otimes} B$  is the  $A$ -fold copower of  $B$ .

Given the construction of coends in  $Sets^G$ , this object is given by

$$c(Y) = \coprod_X Sets^{OrG^{\text{op}}}(R_\phi(X), Y) \overline{\otimes} X / \sim$$

where  $f'R_\phi(\alpha)\overline{\otimes}x \sim f'\overline{\otimes}\alpha(x)$ , where  $\alpha : X \rightarrow X'$ ,  $f' : R_\phi(X') \rightarrow Y$ . This construction as it now stands is ‘illegal’ as its disjoint union is indexed by a proper class, but as any orbit of an  $x \in X$  is given by a map from some  $G/H$ , (sending  $H$  to  $x$ ), this difficulty can be avoided, giving

$$c(Y) = \coprod_{G/H} \text{Sets}^{OrG^{op}}(R_\phi(G/H), Y)\overline{\otimes}G/H / \sim .$$

A further reduction using the Yoneda lemma shows that

$$c(Y) \cong \left( \coprod_{G/H} Y(G/H) \times G/H \right) / \sim$$

where if  $\alpha : G/H \rightarrow G/K$  in  $OrG$ , then  $(Y(\alpha)(y), gH) \sim (y, \alpha(gH))$ .

Of particular interest is the case where  $Y = R_\phi(X)$  for a  $G$ -set  $X$ , then

$$c(R_\phi(X)) \cong \left( \coprod_{G/H} X^H \times G/H \right) / \sim$$

and the counit of the adjunction,  $\varepsilon(x) : cR_\phi(X) \rightarrow X$  is given by  $\varepsilon(x, gH) = g.x$ . Any  $(x, gH)$  is equivalent to the corresponding  $(gx, H)$  and  $\varepsilon$  is an isomorphism with inverse sending  $x$  to the class containing  $(x, H)$  where  $H = \{g \in G : gx = x\}$ .

### 3 Homotopy Coherence and Simplicial Diagrams.

(Our basic references for equivariant homotopy theory are tom Dieck, [18] and Lück, [13].)

Suppose  $G$  is now a locally compact Hausdorff group and  $f : X \rightarrow Y$  a  $G$ -equivariant map of  $G$ -equivariant CW-complexes (or  $G$ -complexes for short), then  $f$  is a  $G$ -homotopy equivalence if and only if for each closed subgroup,  $H$ , of  $G$ ,  $f^H$  is a homotopy equivalence, (cf. tom Dieck, [18]).

If we extend the notation  $OrG$  to apply to this situation, taking it to be the category of orbit *spaces*,  $G/H$ , with  $H$  closed in  $G$ , then the above result can be restated:  $f$  is a  $G$ -homotopy equivalence if and only if  $R(f) : R(X) \rightarrow R(Y)$  is a level homotopy equivalence in  $CW^{OrG^{op}}$ . Thus questions relating to  $G$ -homotopy are translated to ones in this diagram category modulo *levelwise* homotopy equivalence. Elmendorf, [11], Seymour, [17], and others have pointed out that using a two-sided bar construction, [14], one can reinterpret constructions in  $CW^{OrG^{op}}$  back in the category of  $G$ -complexes. The general form of such bar constructions has been thoroughly investigated by Meyer, [15], and his treatment makes it clear that they can be considered as ‘homotopy analogues’ of Kan extensions. Given the known results using Kan extensions to link  $G$ -sets and  $OrG^{op}$ -diagrams, summarised above, this suggests that the situation that pertains to the equivariant homotopy theory under study here, is another instance of a general process of translating set-based results that use category theory to space-based results using some lax or coherent generalisation of category theory. To review the evidence for this, we note:

- Vogt, [19], showed that the category  $Ho(\mathcal{T}op^{\mathcal{A}})$ , for a well pointed topological category  $\mathcal{A}$ , is equivalent to  $Coh(\mathcal{A}, \mathcal{T}op)$ , a category formed from homotopy coherent diagrams indexed by  $\mathcal{A}$  and homotopy classes of homotopy coherent maps between them. If  $\mathcal{A}$  is discrete,  $Ho(\mathcal{T}op^{\mathcal{A}})$  is obtained from  $\mathcal{T}op^{\mathcal{A}}$  by formally inverting the level homotopy equivalences.
- Following Cordier's translation, [3], of Vogt's general theory to a simplicial context, Bourn and Cordier, [1], gave a description of homotopy limits and colimits that is applicable to functors defined between simplicially enriched categories. Cordier, [5], showed the link with Vogt's definition of homotopy limits and colimits and thus, by implication, with the bar construction.
- A simplicially enriched version of Vogt's main equivalence theorem was proved by Cordier and Porter, [6], implicitly using certain homotopy Kan extensions. In more recent work, [9], as yet unpublished, we develop a theory of homotopy coherent ends and coends that allows a direct translation of categorical arguments to the simplicially enriched setting. This is briefly summarised below.

The context will thus be of a simplicially enriched category,  $\mathcal{C}$ , usually assumed to be *locally Kan*, i.e. each hom-object is a Kan complex. Typically this category  $\mathcal{C}$  will be one of:

- $\mathcal{T}op$  with its simplicial category structure

$$\underline{\mathcal{T}op}(X, Y)_n = \mathcal{T}op(X \times \Delta^n, Y),$$

(note this is locally Kan);

- $\mathcal{S}$  with its usual simplicial category structure

$$\underline{\mathcal{S}}(X, Y)_n = \mathcal{S}(X \times \Delta[n], Y),$$

and also its full subcategory  $\mathcal{K}an$  of Kan complexes;  $\mathcal{S}$  is not locally Kan but  $\mathcal{K}an$  is.

- $G\text{-}\mathcal{T}op$  for  $G$  a locally compact topological group. The simplicial structure being given by

$$\underline{G\text{-}\mathcal{T}op}(X, Y)_n = G\text{-}\mathcal{T}op(X \times \Delta^n, Y),$$

where  $\Delta^n$  is given the trivial  $G$ -action. We note that this means that  $\underline{G\text{-}\mathcal{T}op}(X, Y)$  is effectively the singular complex of the space of  $G$ -maps from  $X$  to  $Y$ .

- A category of algebraic models for homotopy types. In future applications, this will either be  $\mathcal{C}hn$ , the category of chain complexes, or  $\mathcal{C}rs$ , the category of crossed complexes, with simplicial enrichment structures that are 'clear' or will be given in future articles. Other examples might include certain reflective subcategories of the category of simplicial groupoids and, more prosaically, the category of groups.

To avoid the cumbersome terms ‘simplicially enriched category’, ‘simplicially enriched functor’, etc., we will usually abbreviate these to  $\mathcal{S}$ -category,  $\mathcal{S}$ -functor, etc. As to notation, if a category is used in both an unenriched and an  $\mathcal{S}$ -enriched form then the enriched form will usually be underlined.

Given an  $\mathcal{S}$ -category  $\mathcal{C}$  as above, a small  $\mathcal{S}$ -category  $\mathcal{A}$  and a  $\mathcal{S}$ -functor,

$$T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{C},$$

we need to define a homotopy coherent end

$$\oint_{\mathcal{A}} T(A, A),$$

and dually a homotopy coherent coend,

$$\oint^{\mathcal{A}} T(A, A).$$

These will be indexed limits and colimits respectively. (For these and other ideas from enriched category theory, the reader is referred to Kelly, [12].)

Let  $\psi(A, A')$  be the bisimplicial set defined by

$$\psi(A, A')_{n, \bullet} = \coprod_{A_0, \dots, A_n} \mathcal{A}(A, A_0)_{\bullet} \times \dots \times \mathcal{A}(A_n, A')_{\bullet}$$

where the coproduct is taken over all ordered sets of objects  $A_0, A_1, \dots, A_n$  of  $\mathcal{A}$ . Face and degeneracy operators in the left-hand counter,  $n$ , are defined as for the nerve of a category; those for the right-hand counter,  $\bullet$ , are given by those of the various simplicial sets  $\mathcal{A}(A_i, A_{i+1})$ . The indexation for the indexed limits and colimits will be given by the diagonal of this bisimplicial set, which will be denoted  $X(A, A')$ . We note that this simplicial set,  $X(A, A')$ , is defined in a natural way so that if  $\mathcal{A}$  is trivially simplicial, so that each  $\mathcal{A}(B, C)$  is a set considered as the corresponding constant simplicial set, then  $X(A, A')$  is the nerve of the category of objects under  $A$  and over  $A'$ .

As we want to form indexed limits (resp. colimits), we require the receiving category  $\mathcal{C}$  to be complete (resp. cocomplete). In this case the  $\mathcal{S}$ -category  $\mathcal{C}$  will be *cotensored* (resp. *tensored*). ‘Cotensored’ means that if  $K$  is a simplicial set and  $C$  an object of  $\mathcal{C}$ , there is an object, which we will denote by  $\overline{\mathcal{C}}(K, C)$  such that there is a natural isomorphism

$$\underline{\mathcal{C}}(K, \mathcal{C}(C', C)) \cong \mathcal{C}(C', \overline{\mathcal{C}}(K, C)).$$

In other words, one can find a mapping-space type construction in  $\mathcal{C}$ . In particular, one has a cocylinder object  $C^I$ , given by  $\overline{\mathcal{C}}(\Delta[1], C)$ . Dually if  $\mathcal{C}$  is cocomplete, then for any  $K$  in  $\mathcal{S}$  and  $C$  in  $\mathcal{C}$ , there is a *tensor*,  $K \overline{\otimes} C$  with a natural isomorphism

$$\underline{\mathcal{C}}(K, \mathcal{C}(C, C')) \cong \mathcal{C}(K \overline{\otimes} C, C').$$

When  $K \cong \Delta[1]$ , this gives a cylinder functor on  $\mathcal{C}$ . In both cases, this allows the development of a useful abstract homotopy theory internally within  $\mathcal{C}$ .

We can now define the homotopy coherent end of  $T$  by

$$\oint_A T(A, A) = \int_{A, A'} \bar{\mathcal{C}}(X(A, A'), T(A, A'))$$

(cf. Cordier and Porter, [8])

**Example**

Suppose  $F, G : \mathcal{A} \rightarrow \mathcal{C}$  are two  $\mathcal{S}$ -functors, and set  $T(A, A') = \mathcal{C}(FA, GA')$ , then  $\oint_A T(A, A)$  can be interpreted as the simplicial set of homotopy coherent transformations from  $F$  to  $G$ , which will be denoted by  $Coh(\mathcal{A}, \mathcal{C})(F, G)$  if we need to stress both domain and codomain of  $F$  and  $G$ , by  $Coh\mathcal{C}(F, G)$  if, as will often occur, the codomain is the important information to remember whilst the domain is fixed, or even shorter by  $Coh(F, G)$  if there is no danger of confusion.

As is now standard, the Bousfield-Kan homotopy limit of a diagram of simplicial sets can be given as a ‘total complex’ of a cosimplicial simplicial set constructed from the original data. If  $Y$  is a cosimplicial simplicial set,

$$TotY = \int_{[n]} \underline{\mathcal{S}}(\Delta[n], Y_\bullet^n)$$

and so is the simplicial set of natural transformations with domain the Yoneda embedding  $\Delta : \mathbf{\Delta} \rightarrow \mathcal{S}$ , considered as a cosimplicial simplicial set, and with codomain  $Y$ . Analogously if  $\mathcal{C}$  is any cotensored complete  $\mathcal{S}$ -category, one can define a ‘total object’ of a cosimplicial object  $Y$  by

$$TotY = \int_{[n]} \bar{\mathcal{C}}(\Delta[n], Y^n).$$

Given a  $\mathcal{S}$ -functor  $T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{C}$ , we can construct a cosimplicial object in  $\mathcal{C}$ , denoted  $Y(T)$ , by

$$Y(T)^n = \prod_{A_0, \dots, A_n} \bar{\mathcal{C}}(\mathcal{A}(A_0, A_1) \times \dots \times \mathcal{A}(A_{n-1}, A_n), T(A_0, A_n)).$$

The coface and codegeneracy maps are given by formulae analogous to those of the ‘cosimplicial replacement’ construction of Bousfield and Kan, [2], and are given in detail in [9].

**Lemma 3.1** [9] *Given a  $\mathcal{S}$ -functor  $T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{C}$ , as above there is a natural isomorphism*

$$\oint_A T(A, A) \cong TotY(T).$$

We will need the following results from [9]:

- If  $\mathcal{A}$  is an  $\mathcal{S}$ -category, and  $T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{S}$  is an  $\mathcal{S}$ -functor such that each  $T(A, A')$  is a Kan complex, then  $\oint_A T(A, A)$  is a Kan complex. In fact in this case  $Y(T)$  is a fibrant cosimplicial simplicial set, so by Axiom SM7 on page 277 of [2] the result follows.

- If  $S, T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{S}$  are  $\mathcal{S}$ -functors such that each  $S(A, A')$  and  $T(A, A')$  is a Kan complex, and

$$\eta(A, A') : S(A, A') \rightarrow T(A, A')$$

is a natural transformation of  $\mathcal{S}$ -bifunctors such that each  $\eta(A, A')$  is a homotopy equivalence, then  $\eta$  induces a homotopy equivalence

$$\oint_A \eta(A, A) : \oint_A S(A, A) \rightarrow \oint_A T(A, A).$$

- We will say that an object  $C$  of  $\mathcal{C}$  is *fibrant* if  $\mathcal{C}(X, C)$  is a Kan complex for all objects  $X$  in  $\mathcal{C}$ . (Thus if  $\mathcal{C}$  is locally Kan, all objects are fibrant in this sense). Analogously to the first result, if  $T : \mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{C}$  takes fibrant values then  $\oint_A T(A, A)$  is fibrant.
- The analogue of the second result above holds if  $S$  and  $T$  both take fibrant values, and  $\eta$  is an  $\mathcal{S}$ -natural homotopy equivalence between  $S$  and  $T$ .

Dually the homotopy coherent coend is defined by

$$\oint^A T(A, A) = \int^{A, A'} X(A, A') \overline{\otimes} T(A', A).$$

As one might expect, this has a description as a ‘diagonal’ of a simplicial object. Define a simplicial object,  $Z(T)$ , in  $\mathcal{C}$  by

$$Z(T)_n = \coprod_{A_0, \dots, A_n} (\mathcal{A}(A_0, A_1) \times \dots \times \mathcal{A}(A_{n-1}, A_n)) \overline{\otimes} T(A_n, A_0),$$

then we have

$$\oint^A T(A, A) \cong \int^{[n]} \Delta[n] \overline{\otimes} Z(T)_n.$$

**Remark:** Comparison of this with well known formulae for the double bar construction as studied by May, [14], Elmendorf, [11], Seymour, [17], Meyer, [15], and others, shows that the two concepts are essentially the same. The terminology we use is designed to emphasise certain universal properties of the constructions, rather than their origins in the classical bar construction.

To illustrate the use of coherent ends, we will consider the simplicial set  $Coh(F, G)$  in more detail. We will also need this later on.

Suppose as before that  $F, G : \mathcal{A} \rightarrow \mathcal{C}$  are two  $\mathcal{S}$ -functors. The simplicial set  $Coh(F, G)$  is then given by

$$\int_{A, A'} \underline{\mathcal{S}}(X(A, A'), \mathcal{C}(FA, GA'))$$

or as  $Tot(Y(F, G))$  where

$$Y(F, G)^n = \prod_{A_0, \dots, A_n} \underline{\mathcal{S}}(\mathcal{A}(A_0, A_1) \times \dots \times \mathcal{A}(A_{n-1}, A_n), \mathcal{C}(FA_0, GA_n))$$

A vertex of  $Coh(F, G)$  is then a natural transformation from the Yoneda embedding / standard cosimplicial simplicial object  $\Delta$  to  $Y(F, G)$ . It thus picks out :

- in codimension zero, a collection of morphisms from  $FA$  to  $GA$ ,  $A$  varying through  $\mathcal{A}$ ;
- in codimension 1, a map from  $\Delta[1]$  to  $Y(F, G)^1$ . Using adjointness this corresponds to families of maps

$$\mathcal{A}(A_0, A_1) \times \Delta[1] \rightarrow \mathcal{C}(FA_0, GA_1)$$

which can be thought of as the homotopies filling the squares

$$\begin{array}{ccc} F(A_0) & \longrightarrow & G(A_0) \\ \downarrow & & \downarrow \\ F(A_1) & \longrightarrow & G(A_1) \end{array}$$

indexed by the maps from  $A_0$  to  $A_1$ . (Of course as  $\mathcal{A}(A_0, A_1)$  is a simplicial set in general, this is not the whole picture.)

- in codimension 2, we get for each  $A_0, A_1, A_2$ , maps

$$\mathcal{A}(A_0, A_1) \times \mathcal{A}(A_1, A_2) \times \Delta[2] \rightarrow \mathcal{C}(FA_0, GA_2)$$

and so on.

Thus vertices of  $Coh(F, G)$  are interpretable as homotopy coherent maps from  $F$  to  $G$ . Similarly in higher dimensions.

The indexation  $X(A, A')$  is a resolution of  $\mathcal{A}(A, A')$  and composition gives a natural transformation,

$$d_0 : X(A, A') \rightarrow \mathcal{A}(A, A')$$

which is a homotopy equivalence. This induces a map

$$\int_{A, A'} \underline{\mathcal{X}}(\mathcal{A}(A, A'), \mathcal{C}(FA, GA')) \rightarrow Coh(F, G)$$

and the domain here is naturally identifiable, via the Yoneda lemma, with  $Nat(F, G)$ . Examination of the image of this mapping easily shows that it interprets natural transformations in the obvious way, so that the corresponding families of higher homotopies,

$$\mathcal{A}(A_0, A_1) \times \dots \times \mathcal{A}(A_{n-1}, A_n) \times \Delta[n] \rightarrow \mathcal{C}(FA_0, GA_n),$$

are trivial.

We note that although  $d_0$  is a natural transformation, its obvious homotopy inverses insert an identity map on the right or left of a map to get a 0-simplex in  $X(A, A')$  and hence are not natural in one of the variables.

We will need to return to this point later.

Although we will not be using this, it is worth noting that the description of  $Coh(F, G)$  as a total complex of a cosimplicial space  $Y(F, G)$  yields a spectral sequence as in Bousfield and Kan, [2], Ch X §§6 and 7.

The prime example for us here of all this theory is given by the relation between  $G$ -spaces and  $OrG^{\text{op}}$ -diagrams of simplicial sets. Following Dwyer and Kan, [10], if  $G$  is a topological group  $OrG^{\text{op}}$  will denote the orbit  $\mathcal{S}$ -category, which is the full sub- $\mathcal{S}$ -category of  $G\text{-}\mathcal{Top}$ , whose objects are the orbits  $G/H$  for  $H$  a closed subgroup of  $G$ . If  $X$  is a  $G$ -space, then

$$G\text{-}\mathcal{Top}(G/H, X) \cong \text{Sing}(X^H).$$

This defines a functor in  $X$ , which will be denoted by  $R$ . It is easy to define a left  $\mathcal{S}$ -adjoint to this functor,  $R$ . In fact examination of the left adjoint given by Dwyer and Kan, [10], shows it to be the left  $\mathcal{S}$ -enriched Kan extension of the inclusion  $\mathcal{S}$ -functor from  $OrG$  to  $G\text{-}\mathcal{Top}$  along the Yoneda embedding of  $OrG$  into  $\mathcal{S}^{OrG^{\text{op}}}$ , i.e. if  $K : OrG \rightarrow G\text{-}\mathcal{Top}$  is the inclusion, the ‘realisation’ functor of Dwyer and Kan is given by:  
for  $T : OrG^{\text{op}} \rightarrow \mathcal{S}$ ,

$$(\text{Lan}_{\text{Yon}}K)(T) \cong \int^{G/H} T(G/H) \overline{\otimes} G/H.$$

In general this does not behave well with regards to homotopy structure, but on ‘cofibrant’ objects it preserves weak homotopy equivalences and so induces an equivalence between the (weak) homotopy categories of  $G\text{-}\mathcal{Top}$  and of  $\mathcal{S}^{OrG^{\text{op}}}$ , see [10].

The functor more usually used to get back from  $\mathcal{S}^{OrG^{\text{op}}}$  to  $G\text{-}\mathcal{Top}$  is Elmendorf’s coalescence functor,  $c$ , see [11]. Its relation to the above is simple. Since the left Kan extension does not preserve homotopy equivalences, one replaces it by the *left homotopy Kan extension*, that is,

$$\oint^{G/H} T(G/H) \overline{\otimes} G/H.$$

This is essentially the same as applying the coalescence functor to  $T$  and will be denoted  $c(T)$ .

The detailed theory of left and right coherent Kan extensions will be given in detail elsewhere, [9], here we merely note that:

if  $K : \mathcal{A} \rightarrow \mathcal{B}$  is an  $\mathcal{S}$ -functor between  $\mathcal{S}$ -categories and  $F : \mathcal{A} \rightarrow \mathcal{C}$  is an  $\mathcal{S}$ -functor with  $\mathcal{C}$  a locally Kan  $\mathcal{S}$ -category that is either complete (and thus cotensored) or cocomplete (and thus tensored), we may define

$$R_K F(B) = \oint_A \overline{\mathcal{C}}(\mathcal{B}(B, KA), FA)$$

$$L_K F(B) = \oint^A \mathcal{B}(KA, B) \overline{\otimes} FA$$

then for any  $\mathcal{S}$ -functor  $G : \mathcal{B} \rightarrow \mathcal{C}$ , there is

1. a natural isomorphism

$$Coh(\mathcal{A}, \mathcal{C})(GK, F) \cong \mathcal{C}^{\mathcal{B}}(G, R_K F)$$

(or dually

$$Coh(\mathcal{A}, \mathcal{C})(F, GK) \cong \mathcal{C}^{\mathcal{B}}(L_K F, G);$$

2. a homotopy equivalence

$$\mathcal{C}^{\mathcal{B}}(G, R_K F) \rightarrow Coh(\mathcal{B}, \mathcal{C})(G, R_K F)$$

(or dually

$$\mathcal{C}^{\mathcal{B}}(L_K F, G) \rightarrow Coh(\mathcal{B}, \mathcal{C})(L_K F, G).$$

These coherent extensions can be calculated in another way which goes some way to explaining the link with the result of Dwyer and Kan. This uses the coherent form of the mean-cotensor and mean-tensor constructions from enriched category theory.

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{A} \rightarrow \mathcal{S}$  be  $\mathcal{S}$ -functors with  $\mathcal{B}$  a complete (resp. cocomplete)  $\mathcal{S}$ -category. We define the *coherent mean cotensor* of  $F$  and  $G$  to be

$$\overline{\mathcal{B}}^{\mathcal{A}}(G, F) = \oint_A \overline{\mathcal{B}}(GA, FA),$$

(resp. the *coherent mean tensor* to be

$$G \overline{\otimes} F = \oint^A GA \overline{\otimes} FA).$$

In particular we will write  $\overline{F} : \mathcal{A} \rightarrow \mathcal{B}$  for the  $\mathcal{S}$ -functor given by  $\overline{F}(A) = \overline{\mathcal{B}}^{\mathcal{A}}(\mathcal{A}(A, -), F)$  and  $\underline{F} : \mathcal{A} \rightarrow \mathcal{B}$  for that given by  $\underline{F}(A) = \mathcal{A}(-, A) \overline{\otimes} F$ .

**Lemma 3.2** *Let  $\mathcal{C}$  be a locally Kan, complete (resp. cocomplete)  $\mathcal{S}$ -category and  $F : \mathcal{A} \rightarrow \mathcal{C}$ ,  $K : \mathcal{A} \rightarrow \mathcal{B}$  be two  $\mathcal{S}$ -functors. Then there is a natural isomorphism*

$$R_K F \cong \text{Ran}_K \overline{F}$$

(resp.

$$L_K F \cong \text{Lan}_K \underline{F}.)$$

Although this is merely a formal exercise in end calculus, we include a proof as it shows the type of argument that is used time and time again — and so will usually be left out here, although included in [9].

**Proof:** If  $B$  is an object in  $\mathcal{B}$ ,

$$\begin{aligned} \text{Ran}_K \overline{F}(B) &= \int_A \overline{\mathcal{C}}(\mathcal{B}(B, KA), \overline{F}A) \\ &\cong \int_A \overline{\mathcal{C}}\left(\mathcal{B}(B, KA), \int_{A', A''} \overline{\mathcal{C}}\left(X(A', A''), \overline{\mathcal{C}}(\mathcal{A}(A, A'), FA'')\right)\right). \end{aligned}$$

Bringing all the ends to the left and using repeatedly the adjunction of the cotensor, this is isomorphic to

$$\int_{A, A', A''} \overline{\mathcal{C}}(\mathcal{B}(B, KA) \times \mathcal{A}(A, A') \times X(A', A''), FA'').$$

Now ‘integrate’ over  $A$ ; this substitutes  $A'$  for  $A$  in  $\mathcal{B}(B, KA)$  leaving us, after another use of the adjunction (the other way), with

$$\int_{A', A''} \bar{\mathcal{C}}(X(A', A''), \bar{\mathcal{C}}(\mathcal{B}(B, KA'), FA'')).$$

i.e. with  $\oint_A \bar{\mathcal{C}}(\mathcal{B}(B, KA), FA)$ , which is the formula for  $R_K F(B)$ .

The proof for the left extensions is similar.

Another result proved in a similar way is the following :

**Proposition 3.3** *If  $F, G : \mathcal{A} \rightarrow \mathcal{B}$ , then*

(i) *if  $\mathcal{B}$  is complete, there is a natural isomorphism*

$$Coh(\mathcal{A}, \mathcal{B})(F, G) \cong \mathcal{B}^{\mathcal{A}}(F, \bar{G});$$

(ii) *if  $\mathcal{B}$  is cocomplete, there is a natural isomorphism*

$$Coh(\mathcal{A}, \mathcal{B})(F, G) \cong \mathcal{B}^{\mathcal{A}}(\underline{F}, G).$$

We refer the reader to [9] for the proof. There you will also find the detailed proof that the natural transformations  $\eta_G : G \rightarrow \bar{G}$ , resp.  $\eta^F : \underline{F} \rightarrow F$  are levelwise homotopy equivalences. As we will be needing this later, a sketch proof is included below. Analysis of the construction of  $\bar{G}$  or  $\underline{F}$  for simple examples of the  $\mathcal{S}$ -category  $\mathcal{A}$ , reveals it to be a generalisation of the construction of a fibration, resp. a cofibration, from an ordinary map. Thus although we are not seeking a Quillen model category structure in this setting, we can think of  $\bar{G}$  as  $G$  made fibrant and  $\underline{F}$  as  $F$  made cofibrant. (In a simpler case, we have worked this out in detail, see [8].) From this viewpoint, the previous proposition gives an enriched analogue of the well known result that maps to a fibrant, or from a cofibrant, object can be used to work with homotopy classes of maps. This suggests for the application in equivariant homotopy theory, that the space of coherent transformations may be a suitable way of analysing  $G$ -maps.

**Proposition 3.4** *Given an  $\mathcal{S}$ -functor  $T : OrG^{\text{op}} \rightarrow \mathcal{S}$  and a  $G$ -space  $Y$ , there is a natural isomorphism*

$$Coh(OrG^{\text{op}}, \mathcal{S})(T, R(Y)) \cong \underline{G\text{-Top}}(c(T), Y).$$

**Proof.**

$$\begin{aligned} Coh(T, R(Y)) &\cong \int_{G/H} \underline{\mathcal{S}}(T(G/H), \underline{G\text{-Top}}(G/H, Y)) \\ &\cong \int_{G/H} \underline{G\text{-Top}}(T(G/H) \bar{\otimes} G/H, Y) \\ &\cong \underline{G\text{-Top}}\left(\int^{G/H} T(G/H) \bar{\otimes} G/H, Y\right) \end{aligned}$$

as required.

This is an enriched analogue of Elmendorf's proof, [11], which shows that the coalescence is a left adjoint at the homotopy level. Note however that Elmendorf also proves that at the level of homotopy classes there is a right adjunction. His proof does not generalise as such to this enriched setting, as it uses the fact that in the topological setting, as in general in the discrete case,  $R$  has an inverse, which is not true in our context. An analogue of his result is however given below.

First suppose given  $T : OrG^{\text{op}} \rightarrow \mathcal{S}$ , a  $G$ -space  $Y$ , and  $G/H$  in  $OrG$ . The simplicial adjunction proved above gives two elements:

$$\eta(T) = \eta \in Coh(OrG^{\text{op}}, \mathcal{S})(T, Rc(T))_0$$

and

$$\varepsilon(Y) = \varepsilon \in G\text{-Top}(cR(Y), Y)_0.$$

It is easily checked that

$$c(T)^H \cong \int^{G/L} X(G/H, G/L) \overline{\otimes} T(G/L)$$

whilst

$$T(G/H) \cong \int^{G/L} OrG(G/H, G/L) \overline{\otimes} T(G/L).$$

The map  $\eta$  can be realised as follows:

As we noted earlier, the indexing functor  $X(G/H, G/L)$  is constructed like the nerve of the category  $G/H \downarrow OrG \downarrow G/L$  and is exactly that nerve if  $G$  is discrete. It therefore comes together with a natural 'augmentation' map,  $d_0$ , to  $OrG(G/H, G/L)$ . This map has a homotopy inverse given by adding the identity on  $G/H$  on the start of any simplex. This is an 'extra' degeneracy, but is not natural in  $G/H$ . It is this extra degeneracy,  $s_{-1}$ , that induces  $\eta$ . The natural map,  $d_0$ , induces a natural transformation

$$\eta' : Rc(T) \rightarrow T,$$

which composes with  $\eta$  to give the identity on  $T$ . This is a 'strong deformation retraction' of  $Rc(T)$  onto  $T$ , but its homotopy inverse and the homotopy between  $\eta\eta'$  and the identity on  $Rc(T)$  are only homotopy coherent. The difficulty of proving this is due to the fact that, in general, homotopy coherent maps compose only 'up to homotopy' (cf. Vogt, [19] or Cordier and Porter, [6]), so we must make a slightly different attack on the problem.

**Proposition 3.5** (i) *There is a natural isomorphism*

$$Rc(T) \cong \underline{T}.$$

(ii) *The natural transformation*

$$\eta' : Rc(T) \rightarrow T$$

*is such that  $\eta'(G/H)$  is a homotopy equivalence for any  $G/H$  in  $OrG$ .*

**Proof:** By definition,

$$\begin{aligned}\underline{T}(G/H) &= OrG^{\text{op}}(-, G/H) \overline{\otimes} T \\ &= \int^{G/K} OrG(G/H, G/K) \overline{\otimes} T(G/K) \\ &\cong Rc(T)(G/H)\end{aligned}$$

This proves (i). The proof of (ii) is by first noting that  $d_0 s_{-1} = id$ , so  $\eta' \eta = id$ , and then using the homotopy  $s_{-1} d_0 \simeq id$ , which is natural in  $G/L$  (but not in  $G/H$ ), to induce a homotopy from  $\eta(G/H) \eta'(G/H)$  to the identity on  $c(T)^H$ .

**Remark:** This homotopy is constant on the image of  $T(G/H)$ . This corresponds to well known results from the theory of double bar constructions mentioned by Elmendorf, [11].

Using this proposition we can easily prove

**Proposition 3.6** *For any  $\mathcal{S}$ -functors  $T, T' : OrG^{\text{op}} \rightarrow Kan$ , there are homotopy equivalences*

$$Coh(T', \underline{T}) \rightarrow Coh(T', T)$$

and

$$Coh(T, T') \rightarrow Coh(\underline{T}, T')$$

(natural in  $T'$ ).

**Proof:** The natural maps  $\eta'$  induce natural transformations

$$\underline{\mathcal{S}}(T'(A), \underline{T}(A')) \rightarrow \underline{\mathcal{S}}(T'(A), T(A'))$$

and

$$\underline{\mathcal{S}}(T(A), T'(A')) \rightarrow \underline{\mathcal{S}}(\underline{T}(A), T'(A'))$$

which are homotopy equivalences. The result follows from earlier comments, since it is assumed that  $T(A)$  and  $T'(A)$  are Kan complexes and, for instance,

$$Coh \underline{\mathcal{S}}(T, T') = \int_A \underline{\mathcal{S}}(TA, T'A).$$

Although the details of the next result are not strictly necessary for the application to equivariant homotopy theory, it seems worth noting :

**Proposition 3.7** *If  $T : OrG^{\text{op}} \rightarrow \mathcal{S}$  takes Kan values, then the natural map  $\eta' : \underline{T} \rightarrow T$  is a coherent homotopy equivalence in the following sense : there is an element  $\eta \in Coh(T, \underline{T})_0$  (the same  $\eta$  as before in fact) and 1-simplices,  $h \in Coh(T, T)_1$ ,  $k \in Coh(\underline{T}, \underline{T})_1$ , such that, writing*

$$\eta'_* : Coh(T, \underline{T}) \rightarrow Coh(T, T)$$

$$\eta'^* : Coh(T, \underline{T}) \rightarrow Coh(\underline{T}, \underline{T})$$

for the induced maps, we have

$$\begin{aligned} d_0 h &= id_T & d_1 h &= \eta'_*(\eta) \\ d_0 k &= id_{\underline{T}} & d_1 k &= \eta'^*(\eta) \end{aligned}$$

Moreover  $k$  can be chosen to be  $s_0(id_{\underline{T}})$  as, in fact,  $\eta'^*(\eta) = id_{\underline{T}}$ .

**Proof:**

The first thing we examine is the last statement. First a lemma:

**Lemma 3.8** *For any  $\mathcal{S}$ -functors  $T, U : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}$ , the map  $\eta'^* : Coh(T, U) \rightarrow Coh(\underline{T}, U)$  can be written as the composite*

$$Coh(T, U) \cong Nat(\underline{T}, U) \rightarrow Coh(\underline{T}, U).$$

**Proof:** Both maps are induced by the augmentation  $d_0$  of the resolution  $X(A, A')$  of  $\mathcal{A}(A, A')$ .

Returning to the proof of the proposition:

The last statement is now easy, as  $\eta \in Coh(T, \underline{T})$  corresponds to  $id_{\underline{T}} \in Nat(\underline{T}, \underline{T})$  under the adjunction isomorphism.

The next stage is to consider the commutative diagram:

$$\begin{array}{ccccc} & & Coh(T, \underline{T}) & \xrightarrow{\eta'_*} & Coh(T, T) \\ & \swarrow \cong & \downarrow \eta'^* & & \downarrow \eta'^* \\ Nat(\underline{T}, \underline{T}) & & & & Nat(\underline{T}, T) \\ & \searrow & \downarrow \eta'^* & & \downarrow \eta'^* \\ & & Coh(\underline{T}, \underline{T}) & \xrightarrow{\eta'_*} & Coh(\underline{T}, T) \end{array}$$

Taking  $\eta \in Coh(T, \underline{T})$ , and using our previous calculation, we have:

$$\eta'^* \eta'_*(\eta) = \eta'_* \eta'^*(\eta) = \eta'_*(id_{\underline{T}}).$$

From proposition 3.6,  $\eta'_*$  is a homotopy equivalence, and as  $Coh(T, T)$  is a Kan complex, we obtain that there is an  $h \in Coh(T, T)_1$  as required, i.e.  $\eta'_*(\eta)$  is in the same connected component of  $Coh(T, T)$  as  $id_T$ .

As a corollary of this sequence of results, we can note:

**Corollary 3.9** *For any  $OrG^{\text{op}}$ -diagrams  $S, T$  in  $\mathcal{S}$ , with  $T$  taking Kan values, there is a natural homotopy equivalence*

$$\underline{G\text{-Top}}(c(S), c(T)) \xrightarrow{\cong} Coh(S, T).$$

**Remarks:**

1. Of course, the homotopy inverse may not be natural in  $T$ .
2. If  $T$  does not take Kan values, then the natural transformation is a weak homotopy equivalence. In applications  $T$  will often be  $R(Y)$  for some  $G$ -space  $Y$ , and  $R(Y)$  is always Kan-valued.

The final result we need, before stating our version of the Elmendorf and Dwyer-Kan results, concerns the coherent counit

$$\varepsilon(Y) : cR(Y) \rightarrow Y.$$

**Proposition 3.10** *If  $Y$  is a  $G$ -complex, then*

$$\varepsilon(Y) : cR(Y) \rightarrow Y$$

*is a  $G$ -homotopy equivalence.*

**Proof:** The natural augmentation  $d_0$  of  $X(A, A')$  for  $A, A' \in OrG^{\text{op}}$  induces, as was seen before, the inclusion of the natural transformations between two functors as the coherent maps with trivial homotopy coherence data. In the simplicial adjunction,

$$Coh(R(Y), R(Y)) \xrightarrow{\cong} \underline{G\text{-Top}}(cR(Y), Y),$$

the  $G$ -map  $\varepsilon(Y)$  corresponds to the identity map on  $R(Y)$ , which is, of course, a natural transformation. It is easily checked to be partially induced by  $d_0$  in the following way:

$$\begin{aligned} cR(Y) &= \int^{G/H} Sing(Y^H) \overline{\otimes} G/H \\ &\cong \int^{G/H_0, G/H_1} |X(G/H_1, G/H_0)| \times |Sing(Y^{H_0})| \times G/H_1, . \end{aligned}$$

The augmentation gives a map to the coend

$$\int^{G/H_0, G/H_1} |OrG(G/H_1, G/H_0)| \times |Sing(Y^{H_0})| \times G/H_1 \cong \int^{G/H} |Sing(Y^H)| \times G/H,$$

by the Yoneda lemma, whilst  $Y$  has a decomposition as

$$\int^{G/H} Y^H \times G/H$$

and  $\varepsilon(Y)$  on each ‘cofactor’ is induced by the usual counit  $|Sing(X)| \rightarrow X$ . A gluing lemma for  $G$ -homotopy equivalences would then give the desired result. There is however

a neater way to conclude this argument, namely by using the description of  $G$ -homotopy equivalences of  $G$ -equivariant CW-complexes mentioned at the start of this section. Using this, it is sufficient to check  $R\varepsilon(Y)(G/H)$  is a homotopy equivalence for each  $G/H$ . (Note the proof of this fact basically checks that  $R\varepsilon \cdot \eta R$  is homotopic to the identity map on  $R(Y)$ , and so shows that the pair  $(c, R)$  gives a coherent adjunction in some sense.)

$$\begin{aligned} RcR(Y)(G/K) &\cong \int^{G/H_0, G/H_1} |X(G/H_1, G/H_0)| \times |Sing(Y^{H_0})| \times |OrG(G/K, G/H_1)| \\ &\cong \int^{G/H} |X(G/K, G/H)| \times |Sing(Y^H)| \end{aligned}$$

by the Yoneda lemma. The augmentation  $d_0$  gives a homotopy equivalence between this and

$$\int^{G/H} |OrG(G/K, G/H)| \times |Sing(Y^H)|$$

and another use of the Yoneda lemma yields  $|Sing(Y^K)|$ , i.e.  $R(Y)(G/K)$ , which completes the proof.

**Remark:** If we follow Moerdijk and Svensson, [16], and other authors, in defining a  $G$ -map  $f : X \rightarrow Y$  to be a weak  $G$ -equivalence if, for each closed subgroup  $H$ , the restricted map  $f^H : X^H \rightarrow Y^H$  is a weak homotopy equivalence, then the above proof shows that

- for any  $G$ -space  $Y$ ,  $\varepsilon(Y)$  is a weak  $G$ -homotopy equivalence
- any  $G$ -space is weakly  $G$ -homotopic to a  $G$ -complex since the formula for  $cR(Y)$  clearly expresses that space as being a  $G$ -complex.

We are now able to state and prove our enriched version of the result of Elmendorf, [11]. Other authors have obtained variants of this, notably Seymour, [17], and Dwyer and Kan, [10].

**Theorem 3.11** *There is a pair of  $\mathcal{S}$ -functors*

$$R : G\text{-}\mathcal{T}op \rightarrow \mathcal{S}^{OrG^{op}}$$

and

$$c : \mathcal{S}^{OrG^{op}} \rightarrow G\text{-}\mathcal{T}op$$

such that

(i) for any  $G$ -space  $Y$  and  $OrG^{op}$ -diagram  $T$  there is a natural isomorphism of simplicial sets,

$$\underline{G\text{-}\mathcal{T}op}(c(T), Y) \cong Coh\underline{\mathcal{S}}(T, R(Y))$$

and if  $Y$  is a  $G$ -complex, and  $T$  takes Kan values, a homotopy equivalence,

$$\underline{G\text{-}\mathcal{T}op}(Y, c(T)) \simeq Coh\underline{\mathcal{S}}(R(Y), T);$$

(ii) for any  $G$ -spaces  $X, Y$  with  $X$  a  $G$ -complex, there is a homotopy equivalence

$$\underline{G\text{-Top}}(X, Y) \simeq \text{Coh}\underline{\mathcal{S}}(R(X), R(Y));$$

(iii) for any  $\text{Or}G^{\text{op}}$ -diagrams  $T, T'$ , with  $T$  and  $T'$  taking Kan values, there is a homotopy equivalence

$$\text{Coh}\underline{\mathcal{S}}(T, T') \simeq \underline{G\text{-Top}}(c(T), c(T')).$$

**Proof:** The key point that is yet to be proved is that there is a homotopy equivalence

$$\underline{G\text{-Top}}(Y, c(T)) \simeq \text{Coh}\underline{\mathcal{S}}(R(Y), T).$$

Replacing  $Y$  by  $cR(Y)$  in  $\underline{G\text{-Top}}(Y, c(T))$  gives a homotopy equivalence

$$\underline{G\text{-Top}}(Y, c(T)) \rightarrow \underline{G\text{-Top}}(cR(Y), c(T))$$

induced by  $\varepsilon(Y)$ . The result then follows from the natural homotopy equivalence of corollary 3.9

$$\underline{G\text{-Top}}(cR(Y), c(T)) \rightarrow \text{Coh}\underline{\mathcal{S}}(R(Y), T).$$

The other parts are similar.

**Remark:** It does not seem worthwhile at present to try to formulate the above as a ‘homotopy equivalence of  $\mathcal{S}$ -categories’ along the lines of Dwyer and Kan, [10]. Coherent transformations only compose up to homotopy and the resulting structure is still mysterious. The advantage of this result over that of Dwyer and Kan is that it gives quite an explicit combinatorial or geometric description of the homotopy type of  $\underline{G\text{-Top}}(X, Y)$  by means of a total complex or a coherent end. This allows one to construct classifying spaces etc. in this context with little difficulty and will possibly allow calculations to be made via spectral sequences.

## References

- [1] Bourn, D. and Cordier, J.-M., A general formulation of homotopy limits, J. Pure Applied Algebra, 29, (1983), 129-141.
- [2] Bousfield, A.K. and Kan, D.M., Homotopy limits, completions and localizations, Lecture Notes in Math. 304 (Springer-Verlag, Berlin, 1972).
- [3] Cordier, J.-M., Sur la notion de diagramme homotopiquement cohérent, Cahiers Topologie et Géom. Différentielle Catégoriques, 23, (1982), 93-112.
- [4] Cordier, J.-M., Extensions de Kan simplicialement cohérentes, Prépublications, Amiens, (1985).
- [5] Cordier, J.-M., Sur les limites homotopiques de diagrammes homotopiquement cohérents, Comp. Math. 62, (1987), 367-388.

- [6] Cordier, J.-M. and Porter, T., Vogt's theorem on categories of homotopy coherent diagrams, *Math. Proc. Cambridge Philos. Soc.*, 100, (1986), 65-90.
- [7] Cordier, J.-M., and Porter, T., Coherent Kan Extensions, (i). Simplicially Enriched Ends and Coends, U.C.N.W. Pure Maths. Preprint 86.19, (1986).
- [8] Cordier, J.-M. and Porter, T., Fibrant diagrams, rectifications and a construction of Loday, *J. Pure Applied Algebra*, 67, (1990), 111-124.
- [9] Cordier, J.-M. and Porter, T., Homotopy coherent category theory, (preprint 1995).
- [10] Dwyer, W.G. and Kan, D.M., Singular functors and realization functors, *Proc. Kon. Akad. van Wetensch. A87=Ind. math.* 46, (1984), 147 - 153.
- [11] Elmendorf, A., Systems of Fixed Point sets, *Trans. Amer. Math. Soc.*, 277, (1983), 275-284.
- [12] Kelly, G.M., *The Basic Concepts of Enriched Category Theory*, LMS Lecture Notes 64, Cambridge University Press, 1983.
- [13] Lück, W., *Transformation Groups and Algebraic K-Theory*, Lecture Notes in Math. 1408 (Springer-Verlag, Berlin, 1989).
- [14] May, J.P., *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. No. 155 (1975).
- [15] Meyer, J.-P., Bar and Cobar Constructions, I, *J. Pure Applied Algebra*, 33 (1984) 163-207.
- [16] Moerdijk, I, and Svensson, J.-A., Algebraic classification of equivariant homotopy 2-types, *J. Pure Applied Algebra*, 89 (1993) 187-216.
- [17] Seymour, R.M., Some functorial constructions on  $G$ -spaces, *Bull. London Math. Soc.* 15 (1983), 353-359.
- [18] tom Dieck, T., *Transformation Groups*, de Gruyter Studies in Mathematics, 8, 1987, de Gruyter, Berlin-New York.
- [19] Vogt, R. M., Homotopy limits and colimits, *Math. Z.* 134 (1973), 11-52.