

# A Linear-time Algorithm for Drawing a Planar Graph on a Grid

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## Abstract

We present a linear-time algorithm that, given an  $n$ -vertex planar graph  $G$ , finds an embedding of  $G$  into a  $(2n - 4) \times (n - 2)$  grid such that the edges of  $G$  are straight-line segments.

## 1 Introduction

We consider the problem of embedding the vertices of a planar graph into a small grid in the plane in such a way that the edges are straight, non-intersecting line segments.

The existence of such *straight-line embeddings* for planar graphs was independently discovered by Fáry [Fa48], Stein [St51], and Wagner [Wa36]; this result also follows from Steinitz's theorem on convex polytopes in three dimensions [SR34]. The first algorithms for constructing straight-line embeddings [Tu63, CYN84, CON85] required high-precision arithmetic, and the resulting drawings were not very aesthetic, since they tend to produce uneven distributions of vertices over the drawing area.

Rosenstiehl and Tarjan [RT86] noticed that it would be convenient to be able to map vertices of a planar graph into a small (polynomial size) grid, because then high-precision operations would be unnecessary. The question whether such embeddings are possible or not was left open in [RT86].

This problem was solved by de Fraysseix, Pach and Pollack [FPP88, FPP90] who proved that, for  $n \geq 3$ , each  $n$ -vertex planar graph can be drawn on the  $(2n - 4) \times (n - 2)$  grid (which has  $(2n - 3)(n - 1)$  points). They also presented an  $O(n \log n)$ -time algorithm for constructing such embeddings, but left open the problem of whether it is possible to find such an embedding in linear time.

We answer this question in the affirmative by presenting a linear-time implementation of the technique from [FPP88, FPP90]. Although we base our algorithm on the general method from

their proof, the algorithm itself differs significantly from theirs. Also, unlike their algorithm, our method does not require any sophisticated data structures. Instead, it distributes and carefully manages the information needed in the algorithm, so that we always have sufficient local information to find a tentative embedding of each new vertex — later it is moved to its correct final position using the information stored in a tree-like structure. The algorithm is very easy to implement; in fact, we give a short Pascal-like code segment for the main part of the algorithm.

The use of integer coordinates has another advantage (besides speed and accuracy): it guarantees, automatically, that the resulting picture has fairly good proportions. Jones *et al* [Jon93] show that, indeed, the algorithm presented in this paper compares favorably, in terms of aesthetics of drawings, with other graph drawing algorithms.

This algorithm was developed in 1989 [CP89], but not published. Progress on this problem has been made since that time. In particular, Schnyder [Sc90] showed a different technique that gives a linear-time embedding algorithm into a smaller  $(n - 2) \times (n - 2)$  grid. It turned out, however, that his result did not eliminate the interest in our algorithm, possibly because our approach is easier to implement, and the resulting embeddings tend to be more aesthetic.

## 2 Outline of the Algorithm

In this section we sketch the proof of de Fraysseix-Pach-Pollack theorem, on which our algorithm is based. A linear-time implementation will be presented in the next section.

Since a planar graph can be triangulated in linear time (see, for example, [Ka93, Section 6.1]), we consider only maximal (triangulated) planar graphs. From now on, let  $G$  be a fixed, but arbitrary,  $n$ -vertex triangulated planar graph. We use the following lemma, proved by de Fraysseix, Pach and Pollack in [FPP88, FPP90].

**Lemma 1** *It is possible to order the vertices of  $G$  in a sequence  $v_1, \dots, v_n$  such that for  $k = 3, 4, \dots, n - 1$ :*

- (a) *the subgraph  $G_k$  of  $G$  induced by  $v_1, \dots, v_k$  is 2-connected, internally triangulated, and the boundary of its exterior face is a cycle  $C_k$  containing the edge  $(v_1, v_2)$ ;*
- (b)  *$v_{k+1}$  is in the exterior face of  $G_{k+1}$ , it has at least two neighbors in  $G_k$ , and these neighbors are consecutive on the path  $C_k - (v_1, v_2)$ .*

*Proof:* We present only a sketch. Let  $C_n = (v_1, v_2, v_n)$  be the outer face of  $G$ . Inductively, suppose that  $v_n, v_{n-1}, \dots, v_{k+1}$  have been defined, and that graphs  $G_n, G_{n-1}, \dots, G_{k+1}$  satisfy the lemma. Then there is a vertex  $v$  in  $G_{k+1}$ ,  $v \notin \{v_1, v_2\}$ , which has exactly two neighbors in  $C_{k+1}$  (so it is not incident to any chords of  $C_{k+1}$ ). We take  $v_k = v$ .  $\square$

The ordering from Lemma 1 is called a *canonical ordering* of  $G$ . Our algorithm embeds  $G$  one vertex at a time in canonical order, at each stage adjusting the current partial embedding. With each vertex  $v_k$ , we associate a set  $L(v_k)$  of vertices that move with  $v_k$  whenever its position is adjusted.

By  $P(v)$  we will denote the current position of  $v$  in the grid; for the sake of notation  $P(v)$  is denoted  $(x(v), y(v))$ . For two grid points  $P_1 = (x_1, y_1)$ ,  $P_2 = (x_2, y_2)$  whose Manhattan distance is even, we denote by  $\mu(P_1, P_2)$  the grid point in the intersection of the line with slope  $+1$  through  $P_1$  and the line with slope  $-1$  through  $P_2$ , that is

$$\mu(P_1, P_2) = \frac{1}{2}(x_1 - y_1 + x_2 + y_2, -x_1 + y_1 + x_2 + y_2).$$

Now we describe the embedding strategy. First, we set  $L(v_i) := \{v_i\}$  for  $i = 1, 2, 3$ , and  $P(v_1) := (0, 0)$ ;  $P(v_2) := (2, 0)$ ;  $P(v_3) := (1, 1)$ .

Suppose that by stage  $k$  we have already embedded  $G_{k-1}$  (so the initial step is actually regarded as step 3) in such a way that the following conditions hold:

- (e1)  $P(v_1) = (0, 0)$  and  $P(v_2) = (2k - 4, 0)$ .
- (e2)  $x(w_1) < x(w_2) < \dots < x(w_m)$ , where  $v_1 = w_1, w_2, \dots, w_m = v_2$  is  $C_{k-1}$  in clockwise order.
- (e3) All segments  $(P(w_i), P(w_{i+1}))$ ,  $i = 1, \dots, m - 1$ , have slopes either  $+1$  or  $-1$ .

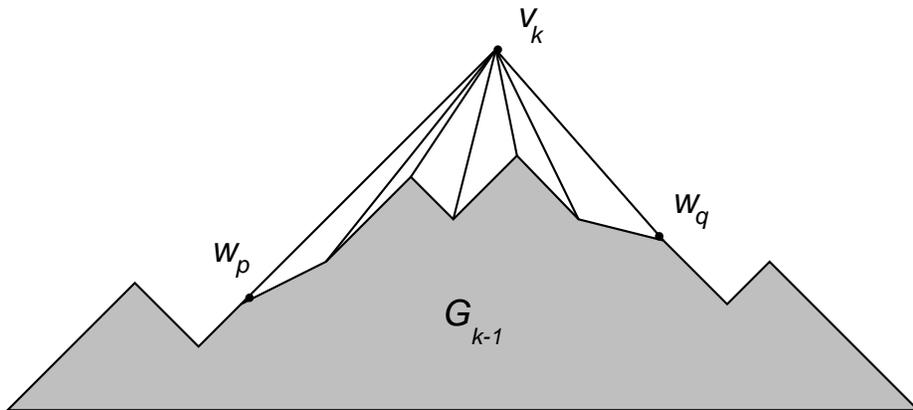


Figure 1: Embedding  $v_k$ .

From now on, we will refer to the sequence  $w_1, \dots, w_m$  (or, sometimes rather informally, to their embedding) as the *contour* of  $G_{k-1}$ . Let  $w_p, \dots, w_q$  be the neighbors of  $v_k$  in  $C_{k-1}$ . We will say that the vertex  $v_k$  *covers* the vertices  $w_{p+1}, \dots, w_{q-1}$  (See Figure 1). Note that by property (e3), the Manhattan distance between any two  $w_i$ 's is even, so  $\mu(w_p, w_q)$  is always a grid point. We now shift some portions of the embedding to the right, by one or two, and install  $v_k$  as follows:

*Step 1: for each*  $v \in \bigcup_{i=q}^m L(w_i)$  **do**  $x(v) := x(v) + 2$ ;

*Step 2: for each*  $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$  **do**  $x(v) := x(v) + 1$ ;

*Step 3:*  $P(v_k) := \mu(P(w_p), P(w_q))$ ;

*Step 4:*  $L(v_k) := \{v_k\} \cup \bigcup_{i=p+1}^{q-1} L(w_i)$ .

By moving some of the points  $P(w_i)$  in Steps 1 and 2, we ensure that all neighbors of  $v_k$  will be visible from  $P(v_k)$ . Clearly, the conditions (e1), (e2), and (e3) are satisfied. We have to show that  $G_k$  is straight-line embedded. This follows from the following lemma, proved by de Fraysseix, Pach and Pollack in [FPP88, FPP90].

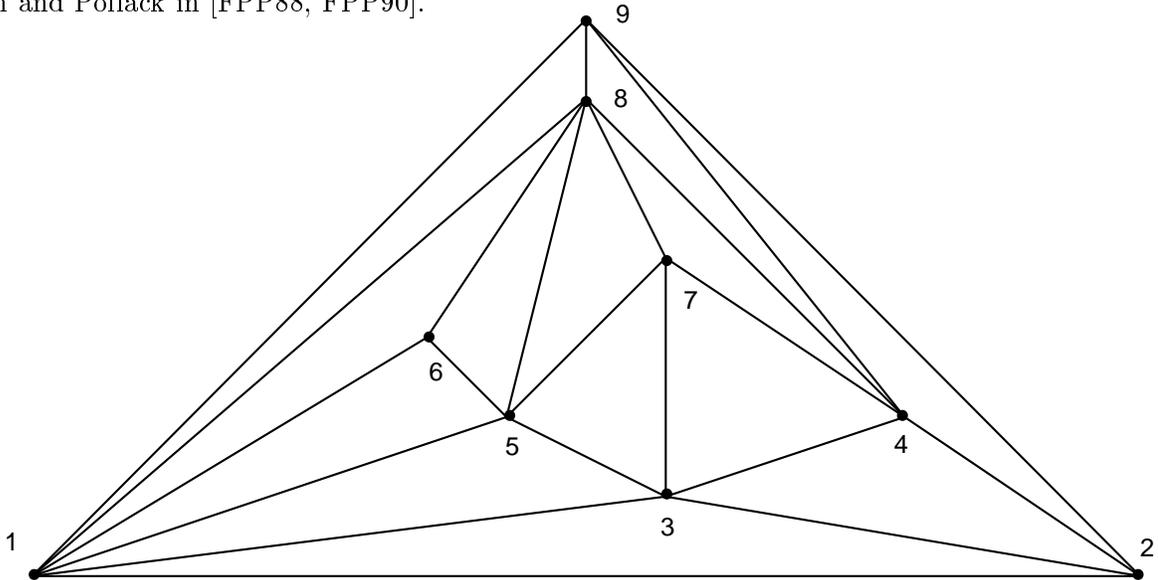


Figure 2: An example of a canonical ordering, sets  $L(w_i)$ , and a partial embedding.

**Lemma 2** *Let  $G_k$  be straight-line embedded, as described above. Suppose we are given a non-decreasing sequence of non-negative integers  $\rho_1 \leq \rho_2 \leq \dots \leq \rho_m$ . If, for each  $i$ , we translate the points in  $L(w_i)$  by  $\rho_i$  to the right, we again obtain a straight-line embedding.*

*Proof:* The proof is by induction on  $k$ . For  $G_3$  the lemma is obvious. So suppose that it holds for  $G_{k-1}$ . As in the algorithm, let the contour of  $G_{k-1}$  be  $w_1, \dots, w_m$ . We are about to add  $v_k$ . Let

$$\rho_1 \leq \dots \leq \rho_p \leq \rho \leq \rho_q \leq \dots \leq \rho_m$$

be a fixed sequence of non-negative integers, and translate each  $L(w_i)$  by  $\rho_i$  and  $L(v_k)$  by  $\rho$ . We show that  $G_k$  remains straight-line embedded.

Take the sequence  $\rho'_1 \leq \rho'_2 \leq \dots \leq \rho'_m$  defined by

$$\rho'_i = \begin{cases} \rho_i & \text{for } i = 1, \dots, p \\ \rho + 1 & \text{for } i = p + 1, \dots, q - 1 \\ \rho_i + 2 & \text{for } i = q, \dots, m. \end{cases}$$

By induction, when we translate the sets  $L(w_i)$  of  $G_{k-1}$  by the  $\rho'_i$ ,  $G_{k-1}$  remains straight-line embedded. Thus  $G_k$  is straight-line embedded, because  $v_k$  moves rigidly with  $w_p, \dots, w_q$ .  $\square$

So, in the end we have a straight-line embedding of  $G$  such that  $P(v_1) = (0, 0)$  and  $P(v_2) = (2n - 4, 0)$ . By (e3),  $P(v_n) = (n - 2, n - 2)$ . Therefore, the whole graph is embedded in the  $(2n - 4) \times (n - 2)$  grid.

### 3 Linear-time Algorithm

It is easy to implement the method of the preceding section in time  $O(n^2)$ . In [FPP88, FPP90] this time bound was reduced to  $O(n \log n)$  using data structures for rectangle-range queries. Unlike that algorithm, ours follows the method of the proof from Section 2, and the linear-time complexity is achieved by appropriately distributing information in the vertices of the graph.

We assume that  $G$  is already triangulated and embedded in the plane, and that a canonical ordering  $v_1, \dots, v_n$  of  $G$  is given. A planar embedding of  $G$  can be found in linear time using standard algorithms, for example [HT74] or [BL76]. It is quite easy to triangulate a given planar graph in linear-time, given its planar embedding (see, for example, [Ka93, Section 6.1]). Also, the existence proof for canonical orderings presented in the previous section can be implemented in linear time by maintaining a queue of vertices that have degree 2 in  $C_k$  (see also [FPP88, FPP90]).

We view  $G_k$  as a forest consisting of trees  $L(w_1), \dots, L(w_m)$  rooted at the members of  $C_k = \{w_1, \dots, w_m\}$ . The children of a node are the vertices that it covers, its neighbors that leave the contour when it is installed. We follow the common practice of representing this forest as a binary tree  $T$ , where the left  $T$ -child of a node is its leftmost child (if any), and the right  $T$ -child of a node is its next sibling to the right (if any).

The root of  $T$  is  $v_1$ , and  $C_k = \{w_1, \dots, w_m\}$  consists of:  $v_1$ , its right  $T$ -child, its right  $T$ -child's right  $T$ -child, etc.  $L(w_i)$  consists of  $w_i$  and its left  $T$ -subtree. Thus, the  $T$ -subtree rooted at  $w_i$  consists of  $\bigcup_{j \geq i} L(w_j)$ .

The crucial observation is that, when we embed  $v_k$ , it is not really necessary to know the exact positions of  $w_p$  and  $w_q$ . If we know only their  $y$ -coordinates and their relative  $x$ -coordinates (that is,  $x(w_q) - x(w_p)$ ), then we can compute  $y(v_k)$ , and the  $x$ -coordinate of  $v_k$  relative to  $w_p$ , that is  $x(v_k) - x(w_p)$ .

For each vertex  $v \neq v_1$ , the  $x$ -offset of  $v$  is defined as  $\Delta x(v) = x(v) - x(w)$ , where  $w$  is

the  $T$ -parent of  $v$ . More generally, if  $w$  is an ancestor of  $v$ , the  $x$ -offset between  $w$  and  $v$  is  $\Delta x(w, v) = x(v) - x(w)$ .

With each vertex  $v$  we store the following information:

$$\begin{aligned} \text{Left}(v) &= \text{the left } T\text{-child of } v, \\ \text{Right}(v) &= \text{the right } T\text{-child of } v, \\ \Delta x(v) &= \text{the } x\text{-offset of } v \text{ from its } T\text{-parent,} \\ y(v) &= \text{the } y\text{-coordinate of } v. \end{aligned}$$

The algorithm consists of two phases. In the first phase, we add new vertices one by one, and each time we add a vertex we compute its  $x$ -offset and  $y$ -coordinate, and update the  $x$ -offsets of one or two other vertices. In the second phase, we traverse the tree and compute final  $x$ -coordinates by accumulating offsets.

The first phase is implemented as follows: First we initialize the values stored at  $v_1, v_2, v_3$ :

$$\begin{aligned} \textit{Initialize: } \Delta x(v_1), y(v_1), \text{Right}(v_1), \text{Left}(v_1) &:= 0, 0, v_3, \text{nil}; \\ \Delta x(v_2), y(v_2), \text{Right}(v_2), \text{Left}(v_2) &:= 1, 0, \text{nil}, \text{nil}; \\ \Delta x(v_3), y(v_3), \text{Right}(v_3), \text{Left}(v_3) &:= 1, 1, v_2, \text{nil}; \end{aligned}$$

Then, we embed other vertices, one by one:

```

for  $k := 4$  to  $n$  do begin
  Notation:      let  $w_1, \dots, w_m$  be the contour of  $G_{k-1}$ ;
                   let  $w_p, \dots, w_q$  be the neighbours of  $v_k$  in  $G_{k-1}$ ;
  Stretch gaps:  $\Delta x(w_{p+1}) := \Delta x(w_{p+1}) + 1$ ;
                    $\Delta x(w_q) := \Delta x(w_q) + 1$ ;
  Adjust offsets:  $\Delta x(w_p, w_q) := \Delta x(w_{p+1}) + \dots + \Delta x(w_q)$ ;
                    $\Delta x(v_k) := \frac{1}{2} [-y(w_p) + \Delta x(w_p, w_q) + y(w_q)]$ ;
                    $y(v_k) := \frac{1}{2} [y(w_p) + \Delta x(w_p, w_q) + y(w_q)]$ ;
                    $\Delta x(w_q) := \Delta x(w_p, w_q) - \Delta x(v_k)$ ;
                   if  $p + 1 \neq q$  then  $\Delta x(w_{p+1}) := \Delta x(w_{p+1}) - \Delta x(v_k)$ ;
  Install  $v_k$ :   $\text{Right}(w_p) := v_k$ ;
                    $\text{Right}(v_k) := w_q$ ;
                   if  $p + 1 \neq q$  then begin
                      $\text{Left}(v_k) := w_{p+1}$ ;
                      $\text{Right}(w_{q-1}) := \text{nil}$ 
                   end
                   else  $\text{Left}(v_k) := \text{nil}$ ;
end;

```

In the second phase, we invoke `AccumulateOffsets( $v_1, 0$ )`; where the procedure `AccumulateOffsets` is defined as follows:

```

procedure AccumulateOffsets( $v$ : vertex,  $\delta$ : integer);
begin
  if  $v \neq \text{nil}$  then begin
     $\Delta x(v) := \Delta x(v) + \delta$ ;
    AccumulateOffset(Left( $v$ ),  $\Delta x(v)$ );
    AccumulateOffset(Right( $v$ ),  $\Delta x(v)$ )
  end
end;

```

In order to prove correctness, it is sufficient to show that the offsets are computed correctly.

To see that the *stretch* step works correctly, recall that the  $T$ -subtree rooted at  $w_i$  consists of  $\bigcup_{j \geq i} L(w_j)$ . So, incrementing the offset of  $w_i$  increments the cumulative offset from  $v_1$  of each member of that  $T$ -subtree, i.e., shifts them all to the right.

During the *adjustment* step of  $v_k$ , only the offsets of  $w_q$  and possibly  $w_{p+1}$  get changed. But  $w_p$  remains an ancestor of each, and their offsets from  $w_p$  remain unchanged, by simple algebra. It follows that the cumulative offsets of all vertices already in the graph remain unchanged by the adjustment step.

As for complexity, we have already mentioned that the triangulation and the canonical ordering can be found in time  $O(n)$ . In first phase, when we add  $v_k$ , the cost is at most  $O(\deg(v_k))$ . So the time complexity of the first phase is  $O(n)$ . Obviously, the second phase runs in linear time.

## 4 Final Comments

As mentioned in the introduction, more progress has been made recently. Schnyder [Sc90] has developed another technique that yields a linear-time embedding algorithm into an  $(n-2) \times (n-2)$  grid. He also pointed out (private communication) that our method can be improved to use a grid of this size: when we install  $v_k$ , we move all vertices  $w_{p+1}, \dots, w_m$  by one to the right, and then place  $v_k$  at  $P(v_k) = (x, y)$ , where  $x = x(w_p) + 1$  and  $y = y(w_q) + x(w_q) - x$ , that is, the edge  $(v_k, w_q)$  will have slope  $-1$ . The invariant will be different: all slopes must belong to the set  $[0, \infty) \cup \{-1\}$ . The proof of correctness and the linear-time implementation are similar. The grid size is smaller, but the embedding occupies the triangle  $(0, 0) - (n-2, 0) - (1, n-2)$  (similarly to the one from [Sc90]), and the overall drawing is asymmetric and may not be very aesthetic. Depending on the application, either of these two methods may be preferable.

The problem of convex drawing of planar graphs has recently attracted attention, see [CYN84, CON85, Ka92]. This is due to the fact that the method from [FPP88, FPP90], as well as Schnyder's algorithm [Sc90], triangulate the given graph before embedding it and remove the added edges

afterwards — this may produce unaesthetic drawings when the initial graph is sparse. Goos Kant [Ka92] gave a linear algorithm, based on our technique, that produces convex drawings of planar graphs in a  $(2n - 4) \times (n - 2)$  grid. Recently, Chrobak and Kant [CK93] observed that, using Schnyder’s improvement of our algorithm, it can be modified to produce convex embeddings in a  $(n - 2) \times (n - 2)$  grid. A similar result has been obtained independently by Schnyder and Trotter [ST92], using the technique from [Sc90].

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