THE MOTION OF WHIPS AND CHAINS

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1. Abstract

We study the motion of an inextensible string fixed at one point in the absence of gravity, satisfying the equations

$$\eta_{tt} = \partial_s(\sigma\eta_s), \qquad \sigma_{ss} - |\eta_{ss}|^2\sigma = -|\eta_{st}|^2, \qquad |\eta_s|^2 \equiv 1$$

with boundary conditions $\eta(t,1) = 0$ and $\sigma(t,0) = 0$. We prove local existence and uniqueness in the space defined by the weighted Sobolev norms

$$\sum_{j=1}^{k} \int_0^1 s^j |\partial_s^j \eta_t|^2 \, ds + \int_0^1 s^{j+1} |\partial_s^{j+1}|^2 \, ds,$$

when $k \geq 3$. We do this by approximating with a discrete system sharing all the essential features, using a Galerkin method.

2. Introduction

In this paper, we explore the motion of a whip, modeled as an inextensible string. We prove that the partial differential equation describing this motion is locally well-posed in certain weighted Sobolev spaces. In addition, we are interested in the motion of a chain, modeled as a coupled system of n pendula, in the limit as n approaches infinity. We show that the motion of the chain converges to that of the whip.

Although the equations of motion are well-known and have been studied by many authors, there are few results known about the general existence and uniqueness problem. Reeken [Re2] [Re3] proved local existence and uniqueness for the *infinite* string in $\mathbb{R}^3$ with gravity and initial data sufficiently close (in H^{26}) to the vertical solution, but aside from this, we know of no other existence result. One aim of the current paper is to prove a local well-posedness theorem for arbitrary initial data for the finite string.

One reason this problem is somewhat complicated is that the equation of motion is hyperbolic, nonlinear, nonlocal, possibly degenerate on a spatial boundary, and possibly even elliptic under certain conditions.

If $\eta\colon \mathbb{R}\times[0,1] \to \mathbb{R}^m$ describes the position $\eta(t,s)$ of the whip, then one can derive that the equation of motion in the absence of gravity and under the inextensibility constraint $\langle \eta_s, \eta_s \rangle \equiv 1$ is

$$\eta_{tt}(t,s) = \partial_s\big(\sigma(t,s)\eta_s(t,s)\big). \tag{1}$$

Incorporating gravity introduces some complications; to keep things as simple as possible, we will neglect it.

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Equation (1) is a standard wave equation; however, the tension σ is determined nonlocally, as a consequence of the inextensibility constraint, by the ordinary differential equation

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) = -|\eta_{st}(t,s)|^2. \tag{2}$$

These equations are augmented with boundary conditions for η and σ, depending on the situation: periodic, two fixed ends, one fixed and one free end, or two free ends. The case of one fixed and one free end has been the most commonly studied, so we will focus on it. In this case, we have $\eta(t,1) \equiv 0$ and $\sigma(t,0) \equiv 0$, along with the compatibility condition $\partial_s\sigma(t,1) \equiv 0$.

Having done this, we compare the equations for the chain and show how they can be thought of as a spatial discretization of the equation of the whip. Unlike the typical situation, this discretization preserves the geometry of the problem, as well as conserving energy. We prove that for short time, the solutions converge as the number of links approaches infinity as long as the initial conditions converge in a sufficiently strong sense. Since the chain equations come from a vector field on a product of circles, the solution is obviously defined globally in time, and one can hope to understand the possible blowup of the whip equation in terms of the finite-dimensional evolution of the geometry.

It is surprising that not only do the position and velocity of the chain converge to those of the whip, but the acceleration also converges. In many situations, when one approximates geometrical constraints, the accelerations do not converge. (A typical example is the approximation of an incompressible fluid by a slightly compressible fluid, as discussed by Ebin [E].)

We use the energy norms

$$E_k = \sum_{j=1}^{k} \int_0^1 s^j |\partial_s^j \eta_t(t,s)|^2\,ds + \int_0^1 s^{j+1}|\partial_s^{j+1}\eta(t,s)|^2\,ds,$$

and show that for small time we have local existence and uniqueness in the E_3 norm. We prove this by showing that the corresponding discrete estimates for the chain with n links are uniformly bounded for small time; hence we get a convergent subsequence in E_2 by compactness, which must therefore be a classical solution for the whip.

Of course, one expects blowup of the whip equation, at least for some initial data, since the whole purpose of a whip is to construct the initial condition so that the velocity of the free end approaches infinity after a short time. See McMillen and Goriely [MG] for a discussion of such issues; although our model neglects some of the phenomena they consider, one expects that the situations are similar in many ways. For the heuristics of blowup in our situation, see Thess et al. [TZN]. The simplest blowup mechanism appears to be the closing off of a loop; as a loop shrinks, there appears a kink in the whip, representing blowup of both the curvature and the angular velocity. Thus from the point of view of global well-posedness, the boundary conditions do not appear to be important.

Victor Yudovich found several results on this problem, although he did not publish anything on it to my knowledge. I learned of this problem from Alexander Shnirelman, and I would like to thank him for many useful discussions about it.

3. Background

The study of the inextensible string is one of the oldest applications of calculus, going back to Galileo, and yet it is still being studied to this day. One is especially concerned about kinks in the solution and what the appropriate jump conditions should be; authors such as O'Reilly and Varadi [OV], Serre [Se], and Reeken [Re1] have discussed these issues in detail from differing points of view.

The first problem to be studied was finding the shape of a hanging chain, first solved incorrectly by Galileo and then correctly by Leibniz and Bernoulli, one of the first major applications of the calculus of variations.

The shape of small-magnitude vibrations of a chain hanging straight down (in a linear approximation) goes back to the Bernoullis and Euler [Tr], and is taught in textbooks today as an example of Bessel functions; see Johnson [J] and Schagerl-Berger [SB] for related problems. Kolodner [Ko], Dickey [D1], Luning-Perry [LP], and Allen-Schmidt [AS] studied the problem of a uniformly rotating inextensible string, one of the few other problems that can be solved more or less exactly.

Burchard and Thomas [BT] obtained a local well-posedness result for the related problem of inextensible elastica, in which there is a potential energy term reflecting a resistance to bending; however it is not clear whether the solutions are preserved in the limit as the potential term goes to zero, so this result does not help in the present situation.

Many authors have studied the problem of a vertically folded chain falling from rest; this is a classical problem that appears in several textbooks ([A], [D2], [H], and [Ros]). In recent years the problem has been debated in the physics literature, in particular the issue of whether energy is conserved and whether the tip of the chain falls at an acceleration equal to gravity or faster ([Cal] [CalMar] [CapMaz] [dSR] [HHR] [IH] [OV] [SSST] [TP] [TPG] [ST]). See Wong-Yasui [WY] or McMillen [M] for a good survey of the literature.

McMillen and Goriely ([GM] and [MG]) studied a tapered whip theoretically, numerically, and experimentally, showing that the crack comes not from the tip but rather from a loop that straightens itself out. Their model uses an elastic rod rather than a string, however, so that the resulting equation is local. Thess et al. [TZN] studied the blowup problem for the closed inextensible string, especially as a model of the blowup problem for the Euler equations for a 3D ideal fluid. They found divergence in the closing off of loops, showing numerically that $\sup_s|\eta_{st}| \simeq \frac{1}{T-t}$ and $\sup_s|\eta_{ss}| \simeq \frac{1}{(T-t)^{3/2}}$ where T is the blowup time.

The whip-chain equations are interesting partly in and of themselves, but especially as a simple model of fluids. There are many structural similarities between the equations (1) and (2) and the Euler equation for an ideal incompressible fluid, given in Lagrangian form by

$$\eta_{tt}(t,x) = -\nabla p\big(t, \eta(t,x)\big) \tag{3}$$

and

$$\Delta p = -\mathrm{Tr}\big([D\eta_t(t,x) \circ \eta^{-1}(t,x)]^2\big), \tag{4}$$

with some boundary condition to determine ∇p uniquely. (For example, for a free boundary problem there is a Dirichlet condition, while for a fixed boundary there is a Neumann condition.)

Both the whip and the ideal fluid equations are geodesic equations of a weak metric on an infinite-dimensional Riemannian manifold. See [P] for details on this approach. In addition, the forces for both equations are determined nonlocally by solving a nonhomogeneous elliptic equation whose right hand side is the square of a mixed time-space derivative. The major difference between them is that while the fluid equations represent an *ordinary* differential equation on the manifold (see Ebin and Marsden [EM] for details), the whip equation is a genuine partial differential equation; the right-hand side is not bounded in any reasonable topology. We will discuss this in [P]. However, since the whip equation is well-posed in Sobolev spaces of sufficiently high order, we can talk in the same way about Riemannian exponential maps. Thus we can hope that studying whip motion can aid in understanding fluids; for example, one could imagine a well-chosen finite-dimensional system approximating ideal fluid motion in the same way that chain motion approximates whip motion.

4. The basic equations: the whip

We will just present the equations here without derivations; the reader may refer to [P] for a detailed derivation and discussion. Schagerl et al. [SSST] and Thess et al. [TZN] also present derivations from minimum principles: the basic idea is to minimize the action $\int_0^T \int_0^1 |\frac{\partial \eta}{\partial t}|^2 \, ds \, dt$ subject to the constraint $|\frac{\partial \eta}{\partial s}|^2 \equiv 1$.

With a fixed end at $s = 1$ and a free end at $s = 0$, the evolution equation is

$$\frac{\partial^2 \eta}{\partial t^2}(t,s) = \frac{\partial}{\partial s}\left(\sigma(t,s)\,\frac{\partial \eta}{\partial s}(t,s)\right), \qquad \eta(t,1) = 0; \tag{5}$$

Differentiating $|\eta_s|^2 \equiv 1$ twice with respect to t, we find that σ is determined by the following boundary-value problem for an ordinary differential equation (for each fixed t):

$$\begin{aligned} \frac{\partial^2 \sigma}{\partial s^2}(t,s) - \left|\frac{\partial^2 \eta}{\partial s^2}(t,s)\right|^2 \sigma(t,s) &= -\left|\frac{\partial^2 \eta}{\partial s \partial t}(t,s)\right|^2, \\ \sigma(t,0) = 0, \quad \frac{\partial \sigma}{\partial s}(t,1) &= 0. \end{aligned} \tag{6}$$

The boundary conditions are compatible as long as η can be extended to an odd function through $s = 1$; in that case σ can be extended to an even function through $s = 1$. If $\sigma(t,s)$ is strictly positive for $0 < s \le 1$, then equation (5) is a hyperbolic equation with a parabolic degeneracy at $s = 0$ (since we must have $\sigma(t,0) = 0$). As such, the only condition necessary to impose at $s = 0$ is that $\eta(t,0)$ remain finite.

The same phenomenon already appears for the model equation

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial s}\left(s\,\frac{\partial \eta}{\partial s}\right), \tag{7}$$

with boundary condition $\eta(t,1) = 0$ and $\eta(t,0)$ finite: the general solution is

$$\eta(t,s) = \sum_{k=1}^{\infty} \left(a_k \cos \tfrac{\lambda_k t}{2} + b_k \sin \tfrac{\lambda_k t}{2}\right) J_0(\lambda_k \sqrt{s}), \tag{8}$$

where λ_k is the k^{th} zero of the Bessel function J_0. (This is the linearization of our equation, with gravity incorporated, about the steady state of vertical suspension, often used to demonstrate applications of Bessel functions.)

We also point out that the reason for not studying the topology of the configuration space rigorously (using Sobolev spaces, for example, as in [EM]) is that we already know the geodesic equation (5) cannot be an ordinary differential equation on an infinite-dimensional manifold: even when σ is a simple function of s as in (7), the right side is an unbounded operator in any Sobolev space. These issues are explored in [P].

As in Section 5, we can write the tension $\sigma(t,s)$ in terms of a Green function. For simplicity we suppress the dependence on t, since (6) is an ordinary differential equation in s.

Proposition 4.1. *For any fixed time t, the tension $\sigma(s)$ is given by*

$$\sigma(s) = \int_0^1 G(s,\xi)|\omega(\xi)|^2\,d\xi, \tag{9}$$

where G is the Green function given by

$$\frac{\partial^2 G}{\partial s^2}(s,\xi) - |\kappa(s)|^2 G(s,\xi) = -\delta(s-\xi), \quad G(0,\xi)=0, \quad G_s(1,\xi)=0. \tag{10}$$

For brevity we have set

$$\kappa(t,s) = \eta_{ss}(t,s) \qquad \text{and} \qquad \omega(t,s) = \eta_{ts}(t,s). \tag{11}$$

The Green function is symmetric. It satisfies $G(s,\xi) > 0$ whenever $0 < s \le 1$, $G_s(s,\xi) > 0$ for $0 < s < \xi$, and $G_s(s,\xi) \le 0$ for $\xi < s < 1$.

Proof. The existence of the Green function and the symmetry property $G(s,\xi) = G(\xi,s)$ is a well-known result of the general theory for second-order equations with homogeneous boundary conditions. See for example Courant-Hilbert [CH].

To prove positivity of the Green function, we first observe that for any $\xi \in (0,1)$, we have

$$\begin{aligned} G(\xi,\xi) &= \int_0^1 G(s,\xi)\delta(s-\xi)\,ds \\ &= \int_0^1 \kappa^2(s)G(s,\xi)G(s,\xi)\,ds - \int_0^1 G(s,\xi)G_{ss}(s,\xi)\,ds \\ &= \int_0^1 \kappa^2(s)G(s,\xi)^2\,ds + \int_0^1 G_s(s,\xi)^2\,ds, \end{aligned}$$

and since G cannot be identically zero, we must have $G(\xi,\xi) > 0$ for all $\xi \in (0,1)$. Since $G(0,\xi) = 0$, we must have $G_s(0,\xi) \neq 0$. Assume $G_s(0,\xi) < 0$; then G will be negative for small s, and hence there must be a turning point s_o where $G_s(s_o,\xi) = 0$. If s_o is the first such point, then the fact that $G(s,\xi) < 0$ for $0 < s < s_o$ implies $G_{ss}(s,\xi) < 0$, so that G_s is decreasing, a contradiction. Hence we must have $G_s(0,\xi) > 0$, and hence $G(s,\xi)$ is positive for small s. Since G_s cannot have a turning point for $s \in (0,\xi)$ (by the same reasoning), G_s must be positive in $(0,\xi)$, so that G is positive in $(0,\xi)$. A similar argument shows that G is decreasing for $s > \xi$. □

In Section 6 we will need good bounds on the quantities $\sup_{s\in[0,1]} \frac{\sigma(s)}{s}$ and $\sup_{s\in[0,1]} \frac{s}{\sigma(s)}$. The next theorem helps with this.

Theorem 4.2. *The Green function $G(s,x)$ defined by Proposition 4.1 satisfies the following bounds.*

$$\sup_{0\le s,x\le 1} |G_s(s,x)| \le 1, \text{ and } \sup_{0\le s,x\le 1} \frac{G(s,x)}{s} \le 1 \tag{12}$$

$$\inf_{0\le s,x\le 1} \frac{G(s,x)}{sx} \ge \frac{e^{-\xi}}{1+\xi} \text{ where } \xi = \int_0^1 s|\kappa(s)|^2\,ds. \tag{13}$$

Proof. The proof of (12) is easy: we just note that by Proposition 4.1, we have

$$0 \le G_s(s,x) \text{ for } s < x \text{ and } G_s(s,x) \le 0 \text{ for } s > x.$$

Furthermore G_s is increasing on $[0,x)$ and $(x,1]$, so that

$$0 \ge \lim_{r\searrow x} G_r(r,x) \ge G_s(s,x) \text{ for } s > x$$

and

$$0 \le G_s(s,x) \le \lim_{r\nearrow x} G_r(r,x) \text{ for } s < x.$$

Finally we have

$$1 + \lim_{s\searrow x} G_s(s,x) = \lim_{s\nearrow x} G_s(s,x).$$

Combining these, we obtain

$$0 \le G_s(s,x) \le 1 \text{ for } 0 \le s < x \text{ and } -1 \le G_s(s,x) \le 0 \text{ for } x < s \le 1.$$

The second part of (12) follows from $G(0,x) = 0$ for all x; we have

$$\left|\frac{G(s,x)}{s}\right| = \left|\frac{\int_0^s G_r(r,x)\,dr}{r}\right| \le \sup_{0\le r\le 1} |G_r(r,x)| \le 1.$$

The proof of (13) is a bit more involved. We first show that if $F(s,x) = \frac{G(s,x)}{sx}$ then $\inf_{s,x} F(s,x) = F(0,1)$. To do this we note that for $x < s < 1$, we have $F_s(s,x) = \frac{sG_s(s,x)-G(s,x)}{s^2x} \le 0$. In addition, if $H(s,x) = sG_s(s,x) - G(s,x)$, then $H_s(s,x) = sG_{ss}(s,x) = s|\kappa(s)|^2 G(s,x) \ge 0$ for $0 < s < x$. Since $H(0,x) = -G(0,x) = 0$ for all x, we conclude $H(s,x) \ge 0$ for $0 < s < x$, so that $F_s(s,x) \ge 0$ for $0 < s < x$.

So for any fixed x, we see $F(s,x)$ is increasing on $(0,x)$ and decreasing on $(x,1)$; therefore we know

$$F(s,x) \ge F(0,x) \text{ for } 0 < s < x, \tag{14}$$

$$F(s,x) \ge F(1,x) \text{ for } x < s < 1. \tag{15}$$

Letting x approach 1 in (14), we get $F(s,1) \ge F(0,1)$; letting x approach 0 in (15), we get $F(s,0) \ge F(1,0)$. Thus finally, using $F(s,x) = F(x,s)$, we have $F(s,x) \ge F(0,x) = F(x,0) \ge F(1,0)$ for $s < x$, and by symmetry this is also true for $s > x$.

So the minimum of F is $F(1,0)$, and clearly

$$F(1,0) = \lim_{s\to 0} \frac{G(s,1)}{s} = G_s(0,1).$$

Now, as $x \to 1$, we have $G(s,x) \to \varphi(s)$ where $\varphi''(s) - |\kappa(s)|^2\varphi(s) = 0$, $\varphi(0) = 0$, and $\varphi'(1) = 1$. The above shows that $\inf_{0\le s,x\le 1} G(s,x) = \varphi'(0)$, so our goal is now to find a lower bound for $\varphi'(0)$.

Clearly $\varphi(s) > 0$ for $0 < s \le 1$, so we can define $\gamma(s) = \ln[\varphi(s)/s]$. We have $\gamma(0) = \ln \varphi'(0)$ and $\gamma(1) = \ln \varphi(1)$. Furthermore $\gamma'(s) = \frac{\varphi'(s)}{\varphi(s)} - \frac{1}{s}$, so that $\lim_{s\to 0} s\gamma'(s) = 0$ and $\gamma'(1) = \frac{1}{\varphi(1)} - 1$. Finally we see that

$$\gamma''(s) + \frac{2\gamma'(s)}{s} + \gamma'(s)^2 = |\kappa(s)|^2,$$

from which we conclude

$$\frac{d}{ds}\big[s^2\gamma'(s)\big] = -s^2\gamma'(s)^2 + s^2|\kappa(s)|^2 \tag{16}$$

$$\frac{d}{ds}\big[s(1-s)\gamma'(s)\big] + \gamma'(s) = -s(1-s)\gamma'(s)^2 + s(1-s)|\kappa(s)|^2 \tag{17}$$

Integrating (16) from $s = 0$ to $s = 1$ yields

$$\gamma'(1) = -\int_0^1 s^2\gamma'(s)^2\,ds + \int_0^1 s^2|\kappa(s)|^2\,ds \le \int_0^1 s|\kappa(s)|^2\,ds = \xi,$$

so that

$$\varphi(1) \ge \frac{1}{1+\xi}. \tag{18}$$

Integrating (17) yields

$$\begin{aligned}\gamma(1) - \gamma(0) &= -\int_0^1 s(1-s)\gamma'(s)^2\,ds + \int_0^1 s(1-s)|\kappa(s)|^2\,ds \\ &\le \int_0^1 s|\kappa(s)|^2\,ds = \xi,\end{aligned}$$

so that

$$\ln\frac{\varphi'(0)}{\varphi(1)} \ge -\xi. \tag{19}$$

Combining (18) and (19), we get

$$\varphi'(0) \ge \frac{e^{-\xi}}{1+\xi}.$$

□

Remark 4.3. We could get an estimate of the form (13) much more easily if we simply assumed an upper bound on $|\kappa(s)|$, using standard Sturm-Liouville comparison arguments. However, we prefer the weaker assumption that $\int_0^1 s|\kappa(s)|^2\,ds < \infty$, since it allows for the possibility of the curvature at the free end of the whip approaching infinity (a possibility not precluded by the equations due to the degeneracy there).

It is also worth noting that we do not get a lower bound if we assume only that $\int_0^1 s^{1+\alpha}|\kappa(s)|^2\,ds < \infty$ for some $\alpha > 0$. A counterexample is furnished by $|\kappa(s)| = \frac{\kappa_0}{s}$, where the equation becomes

$$\sigma''(s) - \frac{\kappa_0^2}{s^2}\sigma(s) = -|\omega(s)|^2;$$

we can easily solve this equation explicitly, and a straightforward computation confirms that its Green function has $\inf_{0\le s,x\le 1} \frac{G(s,x)}{sx} = 0$.

5. The basic equations: the chain

We now derive the equations for the finite model, consisting of n particles in $\mathbb{R}^m$, each of mass $\frac{1}{n}$. The particles are assumed to be joined by rigid links of length $\frac{1}{n}$, whose mass is negligible. The position of the i^{th} particle is $\eta_i(t)$ for $1 \le i \le n$. The configuration space is thus homeomorphic to $(S^{m-1})^n$, and is naturally embedded in $\mathbb{R}^{mn}$. We assume one end is fixed and one end is free; it is a bit more convenient to assume the fixed end is the $(n+1)^{\text{st}}$ particle, so that $\eta_{n+1}(t) \equiv \mathbf{0}$ for all t.

The kinetic energy in $\mathbb{R}^{mn}$ is

$$K = \frac{1}{2n}\sum_{i=1}^n |\dot{\eta}_i|^2. \tag{20}$$

In addition the constraints are given by

$$h_i(\eta_1, \ldots, \eta_n) = \frac{1}{2}|\eta_{i+1} - \eta_i|^2 = \frac{1}{2n^2}, \qquad 1 \le i \le n.$$

We can use Lagrange multipliers to obtain the equations of motion, obtaining

$$\ddot{\eta}_i = n^2\sigma_i(\eta_{i+1} - \eta_i) - n^2\sigma_{i-1}(\eta_i - \eta_{i-1}) \tag{21}$$

for $1 \le i \le n$. The scaling is chosen so that $\sigma_i(t)$ converges (under some assumptions) to a function $\sigma(t,s)$ as $n \to \infty$. The numbers σ physically represent the tensions in each link. We set $\sigma_0 = 0$ so the same equation is valid when $i = 1$.

The constraint equations determine the σ. We get

$$\begin{aligned} -|\dot{\eta}_{i+1} - \dot{\eta}_i|^2 = n^2\sigma_{i+1}\langle \eta_{i+2} - \eta_{i+1}, \eta_{i+1} - \eta_i\rangle - 2\sigma_i \\ + n^2\sigma_{i-1}\langle \eta_i - \eta_{i-1}, \eta_{i+1} - \eta_i\rangle \end{aligned}$$

for $1 \le i < n$ (again using $\sigma_0 = 0$), while for $i = n$ we get

$$-|\dot{\eta}_n|^2 = -\sigma_n - n^2\sigma_{n-1}\langle \eta_n, \eta_n - \eta_{n-1}\rangle.$$

A slightly less cumbersome notation is to set $\delta_i = n(\eta_{i+1} - \eta_i)$; then each δ_i is a unit vector and the tension equation is

$$\begin{aligned} -\frac{|\dot{\delta}_i|^2}{n^2} &= \langle \delta_i, \delta_{i+1}\rangle\sigma_{i+1} - 2\sigma_i + \langle \delta_{i-1}, \delta_i\rangle\sigma_{i-1}, \qquad 1 \le i \le n-1, \\ -\frac{|\dot{\delta}_n|^2}{n^2} &= -\sigma_n + \langle \delta_{n-1}, \delta_n\rangle\sigma_{n-1}. \end{aligned} \tag{22}$$

Proposition 5.1. *The solution of the constraint equations* (22) *is*

$$\sigma_k = \frac{1}{n}\sum_{j=1}^n G_{kj}\dot{\delta}_j^2, \tag{23}$$

where $\delta_j = n(\eta_{j+1} - \eta_j)$ and the discrete Green function G_{kj} is constructed by

$$G_{kj} = \frac{1}{n}\sum_{i=1}^{\min(j,k)} \frac{p_{ij}p_{ik}}{b_i}, \text{ where } p_{ij} = \begin{cases} \prod_{m=i}^{j-1} \frac{a_m}{b_{m+1}} & j > i, \\ 1 & j = i, \end{cases}$$

$$a_i = \langle \delta_{i+1}, \delta_i\rangle, \quad \text{and} \quad b_n = 1, \quad b_i = 2 - \frac{a_i^2}{b_{i+1}} \text{ for } 1 \le i \le n-1. \tag{24}$$

The tensions σ_k are nonnegative for every choice of $\dot\delta$ if and only if $a_i \geq 0$ for every i.

Proof. In terms of $a_i = \langle \delta_i, \delta_{i+1}\rangle$, we can set up (22) as a matrix equation:

$$\begin{pmatrix} 2 & -a_1 & 0 & \cdots & 0 & 0 \\ -a_1 & 2 & -a_2 & \cdots & 0 & 0 \\ 0 & -a_2 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 2 & -a_{n-1} \\ 0 & 0 & 0 & \cdots & -a_{n-1} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} = \frac{1}{n^2} \begin{pmatrix} |\dot\delta_1|^2 \\ |\dot\delta_2|^2 \\ |\dot\delta_3|^2 \\ \vdots \\ |\dot\delta_{n-1}|^2 \\ |\dot\delta_n|^2 \end{pmatrix}. \tag{25}$$

Gaussian elimination yields

$$\begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 & 0 \\ -a_1 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} = \begin{pmatrix} c_1/n \\ c_2/n \\ c_3/n \\ \vdots \\ c_{n-1}/n \\ c_n/n \end{pmatrix},$$

where b is defined as in (24), recursively, by

$$b_n = 1 \qquad \text{and} \qquad b_i = 2 - \frac{a_i^2}{b_{i+1}}, \tag{26}$$

and c is defined recursively by

$$c_n = \frac{|\dot\delta_n|^2}{n}, \qquad c_i = \frac{|\dot\delta_i|^2}{n} + \frac{a_i}{b_{i+1}} c_{i+1}. \tag{27}$$

The σ are then given recursively by

$$\sigma_1 = \frac{c_1}{nb_1}, \qquad \sigma_k = \frac{c_k}{nb_k} + \frac{a_{k-1}}{b_k}\sigma_{k-1}. \tag{28}$$

Obviously we have $1 \leq b_i \leq 2$ for every i, although (26) will generally be too difficult to solve exactly. However (27) and (28) are linear difference equations, and their solution is easily written in the form (24).

Now we analyze the effect of the sign of a. If any a_i are zero, then clearly the matrix (25) splits into disjoint blocks: if for example a_q and a_r are zero and $q < r$, then the Green function (24) can be written as a sum over each block:

$$\sigma_k = \frac{1}{n}\sum_{j=q+1}^{r} G_{kj}|\dot\delta_j|^2; \qquad G_{kj} = \frac{1}{n}\sum_{i=q+1}^{\min(j,k)} \frac{p_{ij}p_{ik}}{b_i},$$

the only difference being that the starting condition for b is $b_r = 2$ if $r \neq n$, while $b_n = 1$.

Hence by working within each block, we can assume that all a are nonzero. Thus if all a are positive, the discrete Green function matrix (24) will be positive-definite, and any choice of $\dot\delta$ which is not identically zero will lead to all tensions being strictly positive. On the other hand, if $a_i < 0$ for some i (representing a sharp kink, with two portions of the chain meeting at an acute angle), then there will be some choice of $\dot\delta$ so that some σ is negative. For example, suppose $a_i > 0$

for $i < q$ and $a_q < 0$. Then setting $\dot{\delta}_1$ to be any unit vector and $\dot{\delta}_i = 0$ for $i > 1$ gives

$$\sigma_k = \frac{1}{n^2}\frac{p_{1k}}{b_1} = \frac{1}{n^2 b_1}\prod_{m=1}^{k-1}\frac{a_m}{b_{m+1}},$$

and therefore $\sigma_{q+1} < 0$. □

Now we want to prove the basic estimates for the discrete Green function defined in Proposition 5.1, analogously to those in Theorem 4.2.

Theorem 5.2. *Suppose G_{kj}, η_k, δ_k, a_k, and b_k are defined as in Proposition 5.1. Assume the η_k are such that $a_k = \langle \delta_{i+1}, \delta_i\rangle \geq 0$ for all k.*

Then the discrete Green function G_{kj} satisfies the following bounds.

First, if $1 \leq j, k \leq n$, and if $\xi_{kj} \equiv n(G_{kj} - G_{k-1,j})$ with $G_{0j} = 0$, then

$$|\xi_{kj}| \leq 1 \text{ for every } k \text{ and } j. \tag{29}$$

Second, if $|\delta_{k+1} - \delta_k| \leq 1$ for all k, we have

$$G_{kj} \geq \frac{e^{-\xi}}{1+\xi} \qquad \text{where } \xi = \frac{1}{n^2}\sum_{k=1}^{n-1} k|\kappa_k|^2 \tag{30}$$

and $\kappa_k \equiv n(\delta_{k+1} - \delta_k)$ is the discrete curvature.

Proof. From Proposition 5.1 we know that $G_{kj} \geq 0$ for all k and j.

Assume first that $j \neq n$. Then the discrete Green function satisfies the equation

$$a_{k+1}G_{k+1,j} - 2G_{kj} + a_{k-1}G_{k-1,j} = -\frac{1}{n}\delta_{kj}$$

for $1 \leq k < n-1$, while

$$-G_{nj} + a_{n-1}G_{n-1,j} = 0$$

Since $a_k = \langle \delta_k, \delta_{k+1}\rangle$ with $|\delta_k| = 1$, we have $a_k < 1$. Thus for $k \neq j$ we have

$$G_{k+1,j} - 2G_{kj} + G_{k-1,j} = (1 - a_{k+1})G_{k+1,j} + (1 - a_{k-1})G_{k-1,j} \geq 0. \tag{31}$$

Thus $\xi_{k+1,j} - \xi_{kj} \geq 0$ for all $k \neq j$. Since $\xi_{1j} = n(G_{1j} - G_{0j}) = nG_{1j} \geq 0$, we conclude that $\xi_{kj} \geq \xi_{1j} \geq 0$ as long as $k < j$.

On the other hand, if $j \neq n$ then we have $-G_{nj} + G_{n-1,j} = (1 - a_{n-1})G_{n-1,j} \geq 0$, so that $\xi_{nj} \leq 0$, and thus $\xi_{kj} \leq \xi_{nj} \leq 0$ for all $k > j$.

Finally when $k = j$ we have

$$G_{j+1,j} - 2G_{jj} + G_{j-1,j} = -\frac{1}{n} + (1 - a_{j+1})G_{j+1,j} + (1 - a_{j-1})G_{j-1,j} \geq 0.$$

Thus $\xi_{j+1,j} - \xi_{jj} \geq -1$. We conclude that

$$0 \leq \xi_{1j} \leq \cdots \leq \xi_{jj} \leq 1 + \xi_{j+1,j} \leq \cdots \leq 1 + \xi_{nj} \leq 1.$$

Hence we must have $|\xi_{kj}| \leq 1$ for all k, as long as $j \neq n$.

If $j = n$, the situation is slightly different; in that case we have

$$-G_{nn} + a_{n-1}G_{n-1,n} = -\frac{1}{n},$$

so that $\xi_{nn} = 1 - n(1 - a_{n-1})G_{n-1,n} \leq 1$, and we still get

$$0 \leq \xi_{1n} \leq \xi_{2n} \leq \cdots \leq \xi_{n-1,n} \leq \xi_{nn} \leq 1.$$

To obtain (30), we use the same tricks as in Theorem 4.2. That is, we first show that $G_{kj} \ge nG_{1n}$, then get an estimate for G_{1n}. To do this, define $F_{kj} = \frac{n^2}{kj} G_{kj}$, and observe that for $1 < k \le n$ we have

$$\begin{aligned} F_{kj} - F_{k-1,j} &= \frac{n^2}{j}\left(\frac{G_{kj}}{k} - \frac{G_{k-1,j}}{k-1}\right) \\ &= \frac{n^2}{k(k-1)}\left[(k-1)(G_{kj} - G_{k-1,j}) - G_{k-1,j}\right]. \end{aligned}$$

Now if $k > j$ we know from above that $\xi_{kj} = n(G_{kj} - G_{k-1,j}) \le 0$, so that $G_{kj} \le G_{k-1,j}$, and thus $F_{kj} - F_{k-1,j} \le 0$. Thus $F_{kj} \ge F_{nj}$ if $k \ge j$.

On the other hand, if we define for $1 \le k \le n$ the quantity $H_{kj} = (k-1)(G_{kj} - G_{k-1,j}) - G_{k-1,j}$, then it is easy to compute that

$$H_{k+1,j} - H_{kj} = k\big[G_{k+1,j} - 2G_{kj} + G_{k-1,j}\big],$$

and we conclude using (31) that if $k < j$ then $H_{k+1,j} - H_{kj} \ge 0$. Since $H_{1j} = 0$, this shows that $H_{kj} \ge 0$ as long as $k \le j$. Then since $F_{kj} - F_{k-1,j} = \frac{n^2}{k(k-1)} H_{kj}$ for $k > 1$, we have $F_{kj} \ge F_{k-1,j}$ as long as $1 < k \le j$, and hence $F_{kj} \ge F_{1j}$ for $1 \le k \le j$.

Using $F_{kj} = F_{jk}$, we conclude that if $k \ge j$ then $F_{kj} \ge F_{nj} = F_{jn}$, and since $j \le n$ we have $F_{jn} \ge F_{1n}$. Hence this is also true if $k \le j$, and we get $\min_{1\le j,k\le n} F_{kj} = F_{1n} = nG_{1n}$.

Now using the formula (24) we have that

$$nG_{1n} = \frac{p_{11}p_{1n}}{b_1} = \frac{1}{b_1}\prod_{m=1}^{n-1}\frac{a_m}{b_{m+1}} = \frac{\prod_{k=1}^{n-1} a_k}{\prod_{l=1}^{n} b_l}. \tag{32}$$

First we get an upper estimate for $\prod_{l=1}^{n} b_l$. Writing $b_k = 1 + g_k$, we observe that $g_n = 0$, that $0 \le g_k \le 1$ for every k, and that

$$(1 + g_k)(1 + g_{k+1}) = 2(1 + g_{k+1}) - a_k^2.$$

Now if $\kappa_k = n(\delta_{k+1} - \delta_k)$ for $1 \le k \le n-1$, then

$$a_k = \langle \delta_{k+1}, \delta_k\rangle = 1 - \frac{1}{2}|\delta_{k+1} - \delta_k|^2 = 1 - \frac{1}{2n^2}|\kappa_k|^2,$$

and we quickly find that

$$g_k - g_{k+1} = -g_k g_{k+1} + \frac{|\kappa_k|^2}{n^2} - \frac{|\kappa_k|^4}{4n^4},$$

from which we conclude that

$$g_j = \sum_{k=j}^{n-1}(g_k - g_{k+1}) \le \frac{1}{n^2}\sum_{k=j}^{n-1}|\kappa_k|^2$$

for $1 \le j \le n-1$.

Now using the inequality between geometric and arithmetic means, $\left(\prod_{j=1}^{n-1} c_j\right)^{1/(n-1)} \le \sum_{j=1}^{n-1} \frac{c_j}{n-1}$ for $c_j = 1 + g_j$, we get

$$\begin{aligned}\prod_{j=1}^{n} b_j = \prod_{j=1}^{n-1}(1+g_j) &\le \prod_{j=1}^{n-1}\left(1 + \tfrac{1}{n^2}\sum_{k=j}^{n-1}\lvert\kappa_k\rvert^2\right)\\ &\le \left(1 + \frac{1}{n^2(n-1)}\sum_{j=1}^{n-1}\sum_{k=j}^{n-1}\lvert\kappa_k\rvert^2\right)^{n-1}\\ &= \left(1 + \frac{1}{n^2(n-1)}\sum_{k=1}^{n-1}k\lvert\kappa_k\rvert^2\right)^{n-1}.\end{aligned}$$

Then the formula $(1+\frac{c}{n-1})^{n-1} \le e^c$ yields that

$$\prod_{j=1}^{n} b_j \le \exp\left(\frac{1}{n^2}\sum_{k=1}^{n-1}k\lvert\kappa_k\rvert^2\right).$$

Next we get a lower estimate for $\prod_{k=1}^{n-1} a_k$. Using $(1-x)^{n-1} \ge 1-(n-1)x$ for every $n \ge 1$ whenever $0 \le x$, we have

$$1 - \frac{\lvert\kappa_k\rvert^2}{2n^2} \ge \left(1 - \frac{(n-1)\lvert\kappa_k\rvert^2}{2n^2}\right)^{1/n-1},$$

and thus

$$\prod_{k=1}^{n-1} a_k = \prod_{k=1}^{n-1}\left(1 - \frac{\lvert\kappa_k\rvert^2}{2n^2}\right) \ge \left[\prod_{k=1}^{n-1}\left(1 - \frac{(n-1)\lvert\kappa_k\rvert^2}{2n^2}\right)\right]^{1/n-1}.$$

Next, using the fact that $1 - \frac{x}{2} \ge \frac{1}{1+x}$ whenever $0 \le x \le 1$, along with the assumption that $\lvert\kappa_k\rvert \le n$ for all k, we get

$$\prod_{k=1}^{n-1} a_k \ge \frac{1}{\left[\prod_{k=1}^{n-1}\left(1 + \frac{(n-1)\lvert\kappa_k\rvert^2}{n^2}\right)\right]^{1/n-1}}.$$

Again using the inequality between geometric and arithmetic means, we get

$$\begin{aligned}\prod_{k=1}^{n-1} a_k &\ge \frac{1}{\frac{1}{n-1}\sum_{k=1}^{n-1}\left(1 + \frac{(n-1)\lvert\kappa_k\rvert^2}{n^2}\right)}\\ &= \frac{1}{1 + \frac{1}{n^2}\sum_{k=1}^{n-1}\lvert\kappa_k\rvert^2} \ge \frac{1}{1 + \frac{1}{n^2}\sum_{k=1}^{n-1}k\lvert\kappa_k\rvert^2}.\end{aligned}$$

□

All of the above computations have assumed that there is no gravity; when there is a uniform gravitational field $-ge_m$, it is not difficult to see that the equations are modified only at the boundary points. The new Lagrangian is

$$\mathcal{L} = \frac{1}{2n}\sum_{i=1}^{n}\lvert\dot{\eta}_i\rvert^2 - \frac{1}{n}\sum_{i=1}^{n} g\langle\eta_i, e_m\rangle.$$

The corrected versions of (21) and (22) in this case are

$$\ddot{\eta}_i = n^2\sigma_i(\eta_{i+1} - \eta_i) - n^2\sigma_{i-1}(\eta_i - \eta_{i-1}) - ge_m$$

along with

$$\begin{aligned} -\frac{|\dot{\delta}_i|^2}{n^2} &= \langle \delta_i, \delta_{i+1}\rangle\sigma_{i+1} - 2\sigma_i + \langle \delta_{i-1}, \delta_i\rangle\sigma_{i-1}, \qquad 1 \le i \le n-1, \\ -\frac{|\dot{\delta}_n|^2}{n^2} &= -\sigma_n + \langle \delta_{n-1}, \delta_n\rangle\sigma_{n-1} - \frac{g}{n}\langle e_m, \delta_n\rangle. \end{aligned}$$

Proposition 5.1 is then easily modified to

$$\sigma_k = \frac{1}{n}\sum_{j=1}^{n} G_{kj}|\dot{\delta}_j|^2 + gG_{kn}\langle e_m, \delta_n\rangle, \tag{33}$$

so that the conditions for all tensions to be nonnegative is that all $\langle \delta_i, \delta_{i+1}\rangle \ge 0$ and in addition $\langle e_m, \delta_n\rangle \ge 0$. In other words, the link at the fixed point should be at or below the horizontal. Physically, this condition is understandable; the equilibrium with the chain hanging straight down, $\delta_i = e_m$, corresponds to strictly positive tension, while the unstable equilibrium with the chain suspended above the fixed point, $\delta_i = -e_m$, results in all links being compressed.

The analogous condition for the whip holds as well, as can easily be derived.

6. A priori estimates for the whip

In this section we derive estimates for equations (5) and (6), assuming the existence of a smooth solution. The actual construction of a solution will come from a limiting process in the chain equation in Section 7; however the estimates here prove uniqueness. They can also be used to understand persistence of smooth solutions, although we will leave a more detailed analysis of this to future research. The estimates in this section also suggest the correct form of the corresponding estimates for the chain, which we will explore in Section 7.

Because of the parabolic degeneracy of (5) at the free end $s = 0$, we need weighted estimates which are not completely standard; hence we will derive our own.

We define the weighted inner products U_k and V_k by the formulas

$$U_k = \int_0^1 s^k|\partial_s^k \eta_t|^2\,ds \qquad \text{and} \qquad V_k = \int_0^1 s^{k+1}|\partial_s^{k+1}\eta|^2\,ds. \tag{34}$$

We will see that the estimates close up at $k = 3$; in other words, the quantities $U_0, U_1, U_2, U_3, V_0, V_1, V_2, V_3$ can all be bounded in terms of each other, and all higher norms can be bounded in terms of these. The reason so many derivatives are needed is partly due to the degeneracy, and partly due to the nonlinearity of the tension constraint (6).

The basic estimates we need are the following weighted analogues of the one-dimensional Poincaré inequality and Sobolev inequality.

Theorem 6.1. *Let $f\colon [0,1] \to \mathbb{R}^m$ be C^∞. Then for any $r > 0$ we have*

$$\int_0^1 s^{r-1}|f(s)|^2\,ds \lesssim \int_0^1 s^r|f(s)|^2\,ds + \int_0^1 s^{r+1}|f'(s)|^2\,ds, \tag{35}$$

as well as

$$\sup_{x\in[0,1]} x^r|f(x)|^2 \lesssim \int_0^1 s^r|f(s)|^2\,ds + \int_0^1 s^{r+1}|f'(s)|^2\,ds, \tag{36}$$

where the constants are independent of f.

Proof. The starting point is the following basic equation, valid for any $p, q \in \mathbb{R}$ and any $a < b$ in $(0,1)$, an easy integration by parts:

$$\int_a^b s^{q+1}|f'(s)|^2\,ds = -ps^q|f(s)|^2\Big|_{s=a}^{s=b} + p(q-p)\int_a^b s^{q-1}|f(s)|^2\,ds + \int_a^b s^{q-2p+1}\left|\frac{d}{ds}\big(s^p f(s)\big)\right|^2\,ds. \tag{37}$$

Now set $p = q = r$, $a = x$, and $b = 1$ in (37) to get

$$x^r|f(x)|^2 \le |f(1)|^2 + \frac{1}{r}\int_0^1 s^{r+1}|f'(s)|^2\,ds. \tag{38}$$

Next, set $p = \frac{r}{2}$, $q = r$, $a = 0$, and $b = 1$ to get

$$\int_0^1 s^{r-1}|f(s)|^2\,ds \le \frac{2}{r}|f(1)|^2 + \frac{4}{r^2}\int_0^1 s^{r+1}|f'(s)^2|\,ds. \tag{39}$$

Choose $q = r+1$ and $p = -(r+1)$ with $a = x$ and $b = 1$, so we have

$$|f(1)|^2 \le x^{r+1}|f(x)|^2 + 2(r+1)\int_0^1 s^r|f|^2\,ds + \frac{1}{r+1}\int_0^1 s^{r+2}|f'|^2\,ds. \tag{40}$$

Thus combining (38) and (40), we get

$$x^r(1-x)|f(x)|^2 \le 2(r+1)\int_0^1 s^r|f|^2\,ds + \frac{1}{r+1}\int_0^1 s^{r+2}|f'|^2\,ds + \frac{1}{r}\int_0^1 s^{r+1}|f'|^2\,ds,$$

for all $x < 1$. Hence as long as the right side is bounded, we have

$$\lim_{x\to 0} x^r|f(x)|^2 \le 2(r+1)\int_0^1 s^r|f|^2\,ds + \frac{2}{r}\int_0^1 s^{r+1}|f'|^2\,ds.$$

Hence if both $\int_0^1 s^r|f|^2\,ds < \infty$ and $\int_0^1 s^{r+1}|f'|^2\,ds < \infty$, then we must have $\lim_{x\to 0} x^r|f(x)|^2 < \infty$ and so $\lim_{x\to 0} x^{r+1}|f(x)|^2 = 0$. Then since

$$\int_0^1 s^{r+2}|f'|^2\,ds \le \int_0^1 s^{r+1}|f'|^2\,ds,$$

equation (40) implies

$$|f(1)|^2 \le 2(r+1)\int_0^1 s^r|f(s)|^2\,ds + \frac{1}{r+1}\int_0^1 s^{r+1}|f'(s)|^2\,ds. \tag{41}$$

Combining (39) with (41), we get (35). Combining (38) with (41), we get (36). $\square$

Remark 6.2. The estimate (39), in the case $f(1) = 0$, is a very special case of the weighted Hardy inequality. See [KP] and the many references therein for discussion of it and various generalizations.

The example $f(s) = \operatorname{arcsinh}(\ln s)$ demonstrates that the inequalities (35) and (36) cannot be extended to $r = 0$: in that case we have $f(1) = 0$, $\int_0^1 |f(s)|^2\,ds < \int_0^1 (\ln s)^2\,ds < \infty$, and $\int_0^1 s|f'(s)|^2\,ds = \frac{\pi}{2} < \infty$, while $\int_0^1 \frac{1}{s}|f(s)|^2\,ds$ and $\sup_{x\in[0,1]}|f(x)|^2$ are both infinite.

Now to get good estimates on the quantities U_k and V_k defined in (34), it is more natural to derive equations for the time-dependent inner products

$$\widetilde{U}_k = \int_0^1 \sigma^k |\partial_s^k \eta_t|^2\,ds \qquad \text{and} \qquad \widetilde{V}_k = \int_0^1 \sigma^{k+1}|\partial_s^{k+1}\eta|^2\,ds, \tag{42}$$

since $\sigma(s)$ rather than s is what appears in equation (5).

We now derive estimates for the higher derivatives of σ, in terms of spatial derivatives of η and η_t.

Theorem 6.3. *Let $\eta(t, s)$ be a function on $J \times [0, 1]$ for some time interval J satisfying $|\eta_s(t, s)| \equiv 1$, and set*

$$U_k = \int_0^1 s^k |\partial_s^k \eta_t|^2\,ds, \qquad V_k = \int_0^1 s^{k+1}|\partial_s^{k+1}\eta|^2\,ds$$

for every $k \in \mathbb{N}$. Set

$$P_k = \sum_{j=1}^k U_j \qquad \textit{and} \qquad Q_k = \sum_{j=1}^k V_j.$$

If $\sigma\colon [0, 1] \to \mathbb{R}$ satisfies

$$\sigma_{ss} - |\eta_{ss}|^2 = -|\eta_{st}|^2, \qquad \sigma_s(1) = 0, \quad \sigma(0) = 0, \tag{43}$$

then setting $A = \sup_{s\in[0,1]}|\sigma_s|$, we have

$$\sup_{s\in[0,1]} \frac{\sigma(s)}{s} \le A \lesssim P_2, \tag{44}$$

In addition, for any $\alpha > 0$ and any integer $k \ge 1$, let

$$D_k = \int_0^1 s^{k+\alpha}\left|\tfrac{d^{k+1}}{ds^{k+1}}\sigma\right|^2\,ds.$$

Then for $k = 1$ or $k = 2$ we have

$$D_k \lesssim A^2 Q_3 V_k + P_3 U_k, \tag{45}$$

while for $k \ge 3$ we have

$$D_k \lesssim A^2 Q_3 V_k + P_3 U_k + A^2 Q_{k-1}^2 + P_{k-1}^2 + Q_{k-1}^2 \sum_{j=0}^{k-3} (D_{k-2-j} + D_{k-1-j}), \tag{46}$$

where all the inequalities involve only numerical constants.

Thus for $k \ge 4$, we have

$$D_k \lesssim F_1 U_k + F_2 V_k + F_3, \tag{47}$$

where F_1, F_2, and F_3 depend only on P_{k-1} and Q_{k-1}.

Proof. We first observe that

$$\sup_s \frac{\sigma(s)}{s} = \sup_s \frac{\int_0^s \sigma_x(x)\,dx}{s} \le \sup_s |\sigma_s| = A.$$

Next we have by Theorem 4.2 that $\sup_s |G_s(s,x)| \le 1$, so that

$$\begin{aligned}\sup_s|\sigma_s(s)| = \sup_s \Big| \int_0^1 |G_s(s,x)||\eta_{xt}(x)|^2\,ds &\le \int_0^1 |\eta_{st}(s)|^2\,ds \\ &\lesssim \int_0^1 s|\eta_{st}|^2 + \int s^2|\eta_{sst}|^2 = P_2,\end{aligned}$$

by (35). Thus we obtain (44).

If $k \ge 1$ we have

$$\begin{aligned}
\textstyle\int s^{k+\alpha}|\partial_s^{k+1}\sigma|^2 = \int s^{k+\alpha}\Big| &-\textstyle\sum_{i=0}^{k-1}\binom{k-1}{i}\langle \partial_s^{i+1}\eta_t, \partial_s^{k-i}\eta_t\rangle \\
&+ \textstyle\sum_{j=0}^{k-1}\sum_{i=0}^{j}\binom{k-1}{j}\binom{j}{i}(\partial_s^{k-1-j}\sigma)^2\langle\partial_s^{i+2}\eta, \partial_s^{j-i+2}\eta\rangle\Big|^2 \\
&\lesssim \textstyle\sum_{j=0}^{k-1}\sum_{i=0}^{j}\int s^{k+\alpha}(\partial_s^{k-1-j}\sigma)^2|\partial_s^{i+2}\eta|^2|\partial_s^{j-i+2}\eta|^2 \\
&\quad + \textstyle\sum_{i=0}^{k-1}\int s^{k+\alpha}|\partial_s^{i+1}\eta_t|^2|\partial_s^{k-i}\eta_t|^2 \\
&\lesssim \textstyle\sum_{i=0}^{k-1}\int s^{k+\alpha}\sigma^2|\partial_s^{i+2}\eta|^2|\partial_s^{k+1-i}\eta|^2 && \text{(I)} \\
&\quad + \textstyle\sum_{i=0}^{k-2}\int s^{k+\alpha}\sigma_s^2|\partial_s^{i+2}\eta|^2|\partial_s^{k-i}\eta|^2 && \text{(II)} \\
&\quad + \textstyle\sum_{j=0}^{k-3}\sum_{i=0}^{j}\int s^{k+\alpha}(\partial_s^{k-1-j}\sigma)^2|\partial_s^{i+2}\eta|^2|\partial_s^{j-i+2}\eta|^2 && \text{(III)} \\
&\quad + \textstyle\sum_{i=0}^{k-1}\int s^{k+\alpha}|\partial_s^{i+1}\eta_t|^2|\partial_s^{k-i}\eta_t|^2. && \text{(IV)}
\end{aligned}$$

Let us analyze these terms separately. We use (35) and (36) freely.

First, in part (I) we have by (44) that

$$\begin{aligned}
&\textstyle\sum_{i=0}^{k-1}\int s^{k+\alpha}\sigma^2|\partial_s^{i+2}\eta|^2|\partial_s^{k+1-i}\eta|^2 \\
&\quad\lesssim A^2\textstyle\sum_{i=0}^{k-1}\int s^{k+2+\alpha}|\partial_s^{i+2}\eta|^2|\partial_s^{k+1-i}\eta|^2 \\
&\quad\lesssim A^2\Big[s^{k+2+\alpha}|\partial_s^{k+1}\eta|^2|\eta_{ss}|^2 + \textstyle\sum_{i=1}^{k-2}s^{k+2+\alpha}|\partial_s^{i+2}\eta|^2|\partial_s^{k+1-i}\eta|^2\Big] \\
&\quad\lesssim A^2\Big[\big(\sup_s s|\eta_{ss}|^2\big)\big(\textstyle\int s^{k+1}|\partial_s^{k+1}\eta|^2\big) \\
&\qquad\qquad + \textstyle\sum_{i=1}^{k-2}\big(\sup_s s^{k+1-i}|\partial_s^{k+1-i}\eta|^2\big)\big(\int s^{i+1}|\partial_s^{i+2}\eta|^2\big)\Big] \\
&\quad\lesssim A^2\Big[(V_1+V_2+V_3)V_k + \textstyle\sum_{i=1}^{k-2}(V_{i+1}+V_{i+2})(V_{k-i}+V_{k+1-i})\Big] \\
&\quad\lesssim A^2\big[Q_3V_k + Q_{k-1}^2\big]
\end{aligned}$$

if $k > 2$. (If $k = 1$ or $k = 2$ the last sum disappears.)

Term (II) is zero if $k=1$; otherwise we have

$$\begin{aligned}
&\textstyle\sum_{i=0}^{k-2} \int s^{k+\alpha}\sigma_s^2|\partial_s^{i+2}\eta|^2|\partial_s^{k-i}\eta|^2 \\
&\quad\lesssim A^2 \textstyle\sum_{i=0}^{k-2} \int s^{k+\alpha}|\partial_s^{i+2}\eta|^2|\partial_s^{k-i}\eta|^2 \\
&\quad\lesssim A^2\Big[\big(\sup s|\eta_{ss}|^2\big)\big(\int s^{k-1}|\partial_s^k\eta|^2\big) \\
&\qquad\qquad + \textstyle\sum_{i=1}^{k-3}\big(\sup s^{k-i-1}|\partial_s^{k-i}\eta|^2\big)\big(\int s^{i+1}|\partial_s^{i+2}\eta|^2\big)\Big] \\
&\quad\lesssim A^2\Big[Q_3(V_{k-1}+V_k) \\
&\qquad\qquad + \sum_{i=1}^{k-3}(V_{i+1}+V_{i+2})(V_{k-i-1}+V_{k-i}+V_{k-i+1})\Big] \\
&\quad\lesssim A^2\left[Q_3V_k + Q_{k-1}^2\right].
\end{aligned}$$

(Again if $k=2$ the last sum is zero.)

Term (III) is zero if $k=1$ or $k=2$. Otherwise,

$$\begin{aligned}
&\textstyle\sum_{j=0}^{k-3}\sum_{i=0}^{j} \int s^{k+\alpha}(\partial_s^{k-1-j}\sigma)^2|\partial_s^{i+2}\eta|^2|\partial_s^{j-i+2}\eta|^2 \\
&\quad\lesssim \textstyle\sum_{j=0}^{k-3}\Big[\sup_s s^{k-3-j+\alpha}(\partial_s^{k-1-j}\sigma)^2\Big]\cdot \\
&\qquad\quad \textstyle\sum_{i=0}^{j}\Big(\sup_s s^{j-i+2}|\partial_s^{k-1-i}\eta|^2\Big)\Big(\int s^{i+1}|\partial_s^{i+2}\eta|^2\Big) \\
&\quad\lesssim \textstyle\sum_{j=0}^{k-3}\Big[\int s^{k-2-j+\alpha}(\partial_s^{k-1-j}\sigma)^2 + \int s^{k-2-j+\alpha}(\partial_s^{k-j}\sigma)^2\Big]\cdot \\
&\qquad\quad \big[\int s^{i+2}|\partial_s^{i+2}\eta|^2 + s^{i+3}|\partial_s^{i+3}\eta|^2\big]\cdot \\
&\qquad\quad \big[\int s^{j-i+2}|\partial_s^{j-i+2}\eta|^2 + s^{j-i+3}|\partial_s^{j-i+3}\eta|^2\big] \\
&\quad\lesssim \textstyle\sum_{j=0}^{k-3}[D_{k-2-j}+D_{k-1-j}]\sum_{i=0}^{j}[V_{i+1}+V_{i+2}][V_{j-i+1}+V_{j-i+2}] \\
&\quad\lesssim Q_{k-1}^2\textstyle\sum_{j=0}^{k-3}[D_{k-2-j}+D_{k-1-j}].
\end{aligned}$$

Notice this is the first time we use the fact that $\alpha>0$, to estimate $\sup_s s^\alpha(\sigma_{ss})^2$ when $j=k-3$ in terms of the integrals using equation (36).

Finally for term (IV) we have

$$\begin{aligned}
&\textstyle\sum_{i=0}^{k-1} \int s^{k+\alpha}|\partial_s^{i+1}\eta_t|^2|\partial_s^{k-i}\eta_t|^2 \\
&\quad\lesssim \textstyle\int s^{k+\alpha}|\partial_s^k\eta_t|^2|\eta_{st}|^2 + \sum_{i=1}^{k-2}\int s^{k+\alpha}|\partial_s^{i+1}\eta_t|^2|\partial_s^{k-i}\eta_t|^2 \\
&\quad\lesssim \big(\sup_s s^\alpha|\eta_{st}|^2\big)U_k + \textstyle\sum_{i=1}^{k-2}\big(\sup_s s^{k-i+\alpha}|\partial_s^{k-i}\eta_t|^2\big)\big(\int s^i|\partial_s^{i+1}\eta_t|^2\big) \\
&\quad\lesssim \Big(\int s^\alpha|\eta_{st}|^2 + s^{1+\alpha}|\eta_{sst}|^2\Big)U_k \\
&\qquad + \textstyle\sum_{i=1}^{k-2}\Big(\int s^{i+1}|\partial_s^{i+1}\eta_t|^2 + \int s^{i+2}|\partial_s^{i+2}\eta_t|^2\Big)\cdot \\
&\qquad\qquad\qquad \Big(\int s^{k-i}|\partial_s^{k-i}\eta_t|^2 + \int s^{k-i+1}|\partial_s^{k-i+1}\eta_t|^2\Big) \\
&\quad\lesssim P_3U_k + \textstyle\sum_{i=1}^{k-2}(U_{i+1}+U_{i+2})(U_{k-i}+U_{k-i+1}) \\
&\quad\lesssim P_3U_k + P_{k-1}^2.
\end{aligned}$$

Here again the last sum disappears if $k=1$ or $k=2$. Furthermore it is essential to have $\alpha>0$ to get a bound on $\sup_s s^\alpha|\eta_{st}|^2$ in terms of P_3.

Putting (I)–(IV) together, we obtain (46) when $k \geq 3$, with the special cases (45) when the sums disappear for $k = 1$ or $k = 2$.

Equation (47) is easy inductively from (46). □

We can now do the energy estimates: we just need an upper bound on $\sup_{0 \leq s \leq 1} \frac{\sigma_t(t,s)}{s}$ because the appropriate energy norms are time-dependent.

Proposition 6.4. *Suppose η and σ are as in Theorem 6.3. Then $C(t) = \sup_{0 \leq s \leq 1} \frac{\sigma_t(t,s)}{s}$ satisfies*

$$C^2 \lesssim P_2^3 Q_3, \tag{48}$$

where P_k and Q_k are the energy norms as in Theorem 6.3.

Proof. We first observe the following consequences of Theorem 6.1.

$$\int |\eta_{ss}|^2 \lesssim \int s^2 |\eta_{sss}|^2 \lesssim \int s^3 |\eta_{sss}|^2 + s^4 |\eta_{ssss}|^2 \leq Q_3 \tag{49}$$

$$\int |\eta_{st}|^2 \lesssim \int s|\eta_{st}|^2 + s^2 |\eta_{sst}|^2 \leq P_2 \tag{50}$$

$$\int s^2 |\eta_{sss}|^2 \lesssim \int s^3 |\eta_{sss}|^2 + s^4 |\eta_{ssss}|^2 \leq Q_3 \tag{51}$$

Next we differentiate the equation for σ in time to obtain

$$\begin{aligned} \sigma_{tss} - |\eta_{ss}|^2 \sigma_t &= 2\langle \eta_{ss}, \eta_{sst} \rangle \sigma - 2\langle \eta_{st}, \eta_{stt} \rangle \\ &= 2\langle \eta_{ss}, \eta_{sst} \rangle \sigma - 2\langle \eta_{st}, \eta_{sss} \rangle \sigma - 4\langle \eta_{st}, \eta_{ss} \rangle \sigma_s. \end{aligned}$$

Since $\sigma_{ts}(1) = 0$ and $\sigma_t(0) = 0$, we see that σ_t satisfies the same equation as σ with the same boundary conditions, and hence we can write, using Proposition 4.1,

$$\sigma_t(s) = 2 \int_0^1 G(s,x) \big[\langle \eta_{xx}, \eta_{xxt} \rangle \sigma(x) - \langle \eta_{xt}, \eta_{xxx} \rangle \sigma(x) - 2\langle \eta_{xt}, \eta_{xx} \rangle \sigma_x \big] \, dx.$$

Now using (12) we get

$$\sup_s \frac{\sigma_t(s)}{s} \leq 2A \int_0^1 s|\eta_{ss}||\eta_{sst}| + s|\eta_{st}||\eta_{sss}| + 2|\eta_{st}||\eta_{ss}| \, ds,$$

and hence

$$\begin{aligned} \frac{C^2}{A^2} &\lesssim \left(\textstyle\int |\eta_{ss}|^2 \right) \left(\textstyle\int s^2 |\eta_{sst}|^2 \right) + \left(\textstyle\int |\eta_{st}|^2 \right) \left(\textstyle\int s^2 |\eta_{sss}|^2 \right) \\ &\qquad + \left(\textstyle\int |\eta_{st}|^2 \right) \left(\textstyle\int |\eta_{ss}|^2 \right) \\ &\lesssim Q_3 P_2, \end{aligned}$$

using (49), (50), and (51). □

Finally we compute the energy estimates.

Theorem 6.5. *Let $\eta(t,s)$ be a smooth function on $J \times [0,1]$ for some time interval J satisfying $|\eta_s(t,s)| \equiv 1$, along with $\eta(t,1) = 0$, and suppose that η is odd through $s = 1$ (i.e., that $\partial_s^k \eta(t,1) = 0$ for every even integer k).*

Suppose $\sigma \colon J \times [0,1] \to \mathbb{R}$ is a smooth function with $\sigma(t,0) = 0$ and σ even through $s = 1$ (i.e., $\partial_s^k \sigma(t,1) = 0$ for every odd integer k).

Let $\alpha > 0$, set

$$U_k = \int_0^1 s^k |\partial_s^k \eta_t|^2 \, ds, \qquad V_k = \int_0^1 s^{k+1} |\partial_s^{k+1} \eta|^2 \, ds,$$

$$\text{and } D_k = \int_0^1 s^{k+\alpha} |\partial_s^{k+1} \sigma|^2 \, ds$$

for every $k \in \mathbb{N}$, and set

$$P_k = \sum_{j=1}^k U_j, \qquad Q_k = \sum_{j=1}^k V_j, \quad \text{and} \quad R_k = \sum_{j=1}^k D_j.$$

Finally set

$$\tilde{U}_k = \int_0^1 \sigma^k |\partial_s^k \eta_t|^2 \, ds, \qquad \tilde{V}_k = \int_0^1 \sigma^{k+1} |\partial_s^{k+1} \eta|^2 \, ds.$$

Then with $A = \sup_s |\sigma_s|$ and $C = \sup_s \frac{|\sigma_t|}{s}$, we have

$$\frac{d}{dt}(\tilde{U}_1 + \tilde{V}_1) \lesssim CU_1 + CAV_1, \tag{52}$$

$$\frac{d}{dt}(\tilde{U}_2 + \tilde{V}_2) \lesssim CAU_2 + CA^2V_2 + A^2\sqrt{R_2P_2Q_3}, \tag{53}$$

$$\frac{d}{dt}(\tilde{U}_3 + \tilde{V}_3) \lesssim CA^2U_3 + CA^3V_3 + A^3\sqrt{R_3P_3Q_3}, \tag{54}$$

where all the inequalities involve only numerical constants.

Furthermore, for $k \geq 4$, we have

$$\frac{d}{dt}(\tilde{U}_k + \tilde{V}_k) \lesssim G_1U_k + G_2V_k + G_3D_k + G_4, \tag{55}$$

where G_1, G_2, G_3, and G_4 depend only on P_{k-1}, Q_{k-1}, and R_{k-1}.

Proof. We just compute:

$$\begin{aligned}
\frac{d}{dt}(\tilde{U}_k + \tilde{V}_k) &= \frac{d}{dt} \int \sigma^k |\partial_s^k \eta_t|^2 + \sigma^{k+1} |\partial_s^{k+1} \eta|^2 \\
&= \int k\sigma^{k-1}\sigma_t |\partial_s^k \eta_t|^2 + (k+1)\sigma^k \sigma_t |\partial_s^{k+1}\eta|^2 \\
&\qquad + 2\int \sigma^k \langle \partial_s^k \eta_t, \partial_s^{k+1}(\sigma\eta_s)\rangle + \sigma^{k+1}\langle \partial_s^{k+1}\eta_t, \partial_s^{k+1}\eta\rangle \\
&\leq \int k\sigma^{k-1}\sigma_t |\partial_s^k \eta_t|^2 + (k+1)\sigma^k \sigma_t |\partial_s^{k+1}\eta|^2 \\
&\qquad + \int \partial_s\big(\sigma^k \langle \partial_s^k \eta_t, \partial_s^{k+1}\eta\rangle\big) \\
&\qquad + \textstyle\sum_{j=1}^{k-1} \binom{k+1}{j} \int \sigma^k (\partial_s^{k+1-j}\sigma) \langle \partial_s^k \eta_t, \partial_s^{j+1}\eta\rangle \\
&\qquad + \int \sigma^k (\partial_s^{k+1}\sigma)\langle \partial_s^k \eta_t, \eta_s\rangle \\
&= \phi_1 + \phi_2 + \phi_3 + \phi_4.
\end{aligned}$$

First, for ϕ_1, we use Proposition 6.4 and Theorem 6.3 to get $|\sigma_t| \leq Cs$ and $|\sigma| \leq As$, so that

$$\begin{aligned}
\phi_1 &= \big| \int k\sigma^{k-1}\sigma_t |\partial_s^k \eta_t|^2 + (k+1)\sigma^k \sigma_t |\partial_s^{k+1}\eta|^2 \big| \\
&\lesssim A^{k-1}C \int s^k |\partial_s^k \eta_t|^2 + A^k C \int s^{k+1} |\partial_s^{k+1}\eta|^2 \\
&= A^{k-1}CU_k + A^kCV_k.
\end{aligned}$$

We then observe that

$$\phi_2 = \sigma^k \langle \partial_s^k \eta_t, \partial_s^{k+1}\eta\rangle\Big|_{s=0}^{s=1} = 0$$

since $\sigma(t,0)=0$ and η and η_t are both odd through $s=1$, so that one of $\partial_s^k\eta_t$ or $\partial_s^{k+1}\eta$ is zero at $s=1$ for every integer k.

Next, we observe that $\phi_3 = 0$ if $k=1$, and if $k\ge 2$ we have

$$\begin{aligned}
\phi_3^2 &\lesssim \textstyle\sum_{j=1}^{k-1} \big(\int \sigma^k (\partial_s^{k+1-j}\sigma)\langle \partial_s^k\eta_t, \partial_s^{j+1}\eta\rangle\big)^2 \\
&\lesssim A^{2k} \textstyle\sum_{j=1}^{k-1} \big(\int s^k |\partial_s^k \eta_t|^2\big)\big(\int s^k |\partial_s^{k+1-j}\sigma|^2 |\partial_s^{j+1}\eta|^2\big) \\
&\lesssim A^{2k} U_k \Big[\big(\sup_s s\sigma_{ss}^2\big) \textstyle\int s^{k-1}|\partial_s^k\eta|^2 \\
&\qquad + \textstyle\sum_{j=1}^{k-2} \big(\sup_s s^j |\partial_s^{j+1}\eta|^2\big) \int s^{k-j}|\partial_s^{k+1-j}\sigma|^2 \Big] \\
&\lesssim A^{2k} U_k \Big[\big(\textstyle\int s\sigma_{ss}^2 + s^2\sigma_{sss}^2\big)\big(\int s^k|\partial_s^k\eta|^2 + s^{k+1}|\partial_s^{k+1}\eta|^2\big) \\
&\qquad + \textstyle\sum_{j=1}^{k-2} \big(\int s^j |\partial_s^{j+1}\eta|^2 + s^{j+1}|\partial_s^{j+2}\eta|^2\big)\cdot \\
&\qquad\quad \big(\textstyle\int s^{k-j+1}|\partial_s^{k-j+1}\sigma|^2 + s^{k-j+2}|\partial_s^{k-j+2}\sigma|^2\big)\Big] \\
&\lesssim A^{2k} U_k \big[(D_1 + D_2 + D_3)(V_{k-1} + V_k) \\
&\qquad + \textstyle\sum_{j=1}^{k-2} (V_j + V_{j+1} + V_{j+2})(D_{k-j} + D_{k-j+1})\big].
\end{aligned}$$

Finally for ϕ_4, we first observe that $|\eta_s|^2 \equiv 1$ implies $\langle \eta_s, \eta_{st}\rangle \equiv 0$. Hence

$$0 = \partial_s^{k-1}\langle \eta_s, \eta_{st}\rangle = \textstyle\sum_{j=0}^{k-1} \binom{k-1}{j} \langle \partial_s^{j+1}\eta, \partial_s^{k-j}\eta_t\rangle,$$

so that

$$\langle \partial_s^k \eta_t, \eta_s\rangle = -\textstyle\sum_{j=1}^{k-1}\binom{k-1}{j}\langle \partial_s^{j+1}\eta, \partial_s^{k-j}\eta_t\rangle.$$

Again we notice this is zero if $k=1$. If $k\ge 2$ we have (choosing $\alpha = \frac12$ in Theorem 6.3 for the σ estimates D_k):

$$\begin{aligned}
\phi_4^2 &= \big(\textstyle\int \sigma^k (\partial_s^{k+1}\sigma)\langle \partial_s^k \eta_t, \eta_s\rangle\big)^2 \\
&\lesssim A^{2k}\textstyle\sum_{j=1}^{k-1}\big(\int s^k (\partial_s^{k+1}\sigma)\langle \partial_s^{j+1}\eta, \partial_s^{k-j}\eta_t\rangle\big)^2 \\
&\lesssim A^{2k} \textstyle\int s^{k+1/2}(\partial_s^{k+1}\sigma)^2 \sum_{j=1}^{k-1}\int s^{k-1/2}|\partial_s^{j+1}\eta|^2|\partial_s^{k-j}\eta_t|^2 \\
&\lesssim A^{2k} D_k \Big[\big(\sup_s s^{1/2}|\eta_{st}|^2\big)\textstyle\int s^{k-1}|\partial_s^k\eta|^2 \\
&\qquad + \textstyle\sum_{j=1}^{k-2}\big(\sup_s s^{j+1/2}|\partial_s^{j+1}\eta|^2\big)\int s^{k-j-1}|\partial_s^{k-j}\eta_t|^2\Big] \\
&\lesssim A^{2k} D_k \big[(U_1+U_2+U_3)(V_{k-1}+V_k) \\
&\qquad + \textstyle\sum_{j=1}^{k-2}(V_j + V_{j+1} + V_{j+2})(U_{k-j} + U_{k-j+1})\big].
\end{aligned}$$

Putting all of these inequalities together, we obtain (52)–(54). Using $\sqrt{ab} \lesssim a+b$, we easily conclude (52)–(55). $\square$

The main complication in this proof and in the proof of Theorem 6.5 is that, by Remark 6.2, we cannot get an estimate for quantities like $\sup_s |\eta_{st}|^2$ directly using the estimates above; instead we need some weighting like $\sup_s s^\alpha |\eta_{st}|^2$ for some $\alpha > 0$ to get an estimate in terms of P_3. Of course, the standard version of the one-dimensional Sobolev inequality would apply here to get a supremum in terms of integrals of derivatives, but to get the proper weighted norms we would need to go up to P_4.

Theorems 6.3 and 6.5 imply that the estimates close up at $k = 3$; we just need to be able to estimate U_k and V_k in terms of $\tilde{U}_k$ and $\tilde{V}_k$ respectively. To do this, we just need an upper bound on $\frac{s}{\sigma}$, or equivalently a lower bound on $\frac{\sigma}{s}$.

Lemma 6.6. *Let σ and η solve* (57). *Then*

$$U_0 = \int_0^1 |\eta_t(t,s)|^2\,ds$$

is constant in time, and if $U_0 = 0$ then $\eta(t,s)$ is constant in time. In addition

$$\frac{1}{B} \equiv \inf_{0\le s\le 1} \frac{\sigma(s)}{s} \ge \frac{e^{-Q_2}}{1+Q_2}\frac{U_0}{4}, \tag{56}$$

where

$$Q_2 = \int_0^1 s^2|\eta_{ss}|^2\,ds + \int_0^1 s^3|\eta_{sss}|^3\,ds.$$

Proof. To prove that U_0 is constant in time, we use

$$\begin{aligned}\frac{d}{dt}\int_0^1 |\eta_t(t,s)|^2\,ds &= 2\int_0^1 \langle \eta_t(t,s), \eta_{tt}(t,s)\rangle\,ds \\ &= 2\int_0^1 \langle \eta_t(t,s), \partial_s(\sigma\eta_s)\rangle\,ds \\ &= 2\int_0^1 \frac{\partial}{\partial s}\big(\sigma\langle \eta_t, \eta_s\rangle\big)\,ds - 2\int_0^1 \sigma\langle \eta_{st}, \eta_s\rangle \\ &= 0,\end{aligned}$$

using the fact that $\sigma(0) = 0$, $\eta_t(1) = 0$, and $\langle \eta_{ts}, \eta_s\rangle = 0$ since $|\eta_s|^2 \equiv 1$. Hence U_0 is constant in time. If U_0 is zero, then $\eta_t(t,s) = 0$ for all t and s, so that $\eta(t,s) = \eta(0,s) = \gamma(s)$, and the chain never moves.

Now to prove (56), we use Proposition 4.1 and Theorem 4.2 to get

$$\begin{aligned}\frac{1}{B} = \inf_s \frac{\sigma(s)}{s} &= \inf_s \int_0^1 \frac{G(s,x)}{s}|\eta_{xt}(x)|^2\,dx \\ &\ge \inf_{s,x} \frac{G(s,x)}{sx}\int_0^1 s|\eta_{st}(s)|^2\,ds \ge \frac{e^{-\xi}}{1+\xi}U_1,\end{aligned}$$

where $\xi = \int s|\eta_{ss}|^2$. We have $\xi \le Q_2$ by (35) and $U_1 \ge \frac{U_0}{4}$ by (39) (since $\eta_t(1) = 0$), and (56) follows. $\square$

Combining Theorems 6.3 and 6.5 with Lemma 6.6, we see that all the estimates close up at $k = 3$, and that all higher norms are controlled by P_3 and Q_3. Hence we obtain the following criterion for blowup of smooth solutions.

Corollary 6.7. *Suppose η, σ is a solution of the system*

$$\begin{gathered}\eta_{tt} = \partial_s(\sigma\eta_s), \qquad \sigma_{ss} - |\eta_{ss}|^2 = -|\eta_{st}|^2, \qquad |\eta_s|^2 \equiv 1, \\ \eta(t,1) = 0,\ \sigma_s(1) = 0,\ \sigma(0) = 0, \quad \eta(0,s) = \gamma(s),\ \eta_t(0,s) = w(s),\end{gathered} \tag{57}$$

where we assume that $|\gamma'(s)|^2 = 1$ and $\langle \gamma'(s), w'(s)\rangle = 0$ for all s.

Assume that $U_0 > 0$ and that in some time interval $[0,T]$, the norms $P_3(t)$ and $Q_3(t)$ are bounded uniformly. Assume further that $U_k(0)$ and $V_k(0)$ are bounded for all k. Then $U_k(t)$ and $V_k(t)$ are also bounded in $[0,T]$ for all k.

Proof. Combining (46) with (55), we see that

$$\frac{d}{dt}(\tilde{U}_k + \tilde{V}_k) \le H_1 U_k + H_2 V_k + H_3$$

where H_1, H_2, and H_3 are bounded. Furthermore since Q_2 is bounded, so is $B(t) = \sup_s \frac{s}{\sigma(t,s)}$ by Lemma 6.6, and thus

$$\frac{d}{dt}(\tilde{U}_k + \tilde{V}_k) \le B^k H_1 \tilde{U}_k + B^{k+1} H_2 \tilde{V}_k + H_3.$$

So by Gronwall's inequality, $\tilde{U}_k(t)$ and $\tilde{V}_k(t)$ are bounded on $[0,T]$ in terms of $\tilde{U}_k(0)$ and $\tilde{V}_k(0)$. Thus finally $U_k(t)$ and $V_k(t)$ are also bounded on $[0,T]$. □

Remark 6.8. Certainly the estimates in Theorems 6.3 and 6.5 seem to demand a lot of the functions; they only close up at $k = 3$, which essentially requires that away from the free end, the whip's position must be locally H^4, so that at minimum it has three continuous derivatives. This is partly due to the basic structure of the equations, but mostly due to the lack of any constraint at the free end. The main issue is that we need to keep increasing the weighting of the norms, because if we were to take any other norm

$$\widetilde{U}_k + \widetilde{V}_k = \int_0^1 \sigma^p |\partial_s^k \eta_t|^2 + \sigma^{p+1} |\partial_s^{k+1} \eta|^2$$

and differentiate in time (even if σ were constant in time) we would end up with

$$\begin{aligned}
\frac{d}{dt}(\widetilde{U}_k + \widetilde{V}_k) &= \int_0^1 \sigma^p \langle \partial_s^k \eta_t, \partial_s^{k+1}(\sigma \eta_s)\rangle + \sigma^{p+1} \langle \partial_s^{k+1}\eta, \partial_s^{k+1}\eta_t\rangle \\
&= \int_0^1 \sigma^{p+1} \langle \partial_s^k \eta_t, \partial_s^{k+2}\eta\rangle + (k+1)\sigma_s \sigma^p \langle \partial_s^k \eta_t, \partial_s^{k+1}\eta\rangle \\
&\qquad + \sigma^{p+1}\langle \partial_s^{k+1}\eta, \partial_s^{k+1}\eta_t\rangle + \text{ lower order terms} \\
&= \int_0^1 (k-p)\sigma_s \sigma^p \langle \partial_s^k \eta_t, \partial_s^{k+1}\eta\rangle + \text{ lower order terms},
\end{aligned}$$

and if $k \ne p$, then it's easy to see there is no way to bound this term by $(\widetilde{U}_k + \widetilde{V}_k)$.

If we knew σ were always positive, as for example for a whip with fixed endpoints or a periodic loop, a bound in terms of P_2 and Q_2 would be sufficient. With more sophisticated techniques we could perhaps reduce the dependence to P_1 and Q_1; but of course, we could not expect to go beyond that. The tension equation clearly could not be expected to make sense unless η_{ss} were in L^2, so we need at least Q_1 to be bounded.

The phenomenon in Remark 6.8 is perhaps explained by the following observation, which is much simpler in dimension two.

Proposition 6.9. *(η, σ) is a smooth solution of (57) in $\mathbb{R}^2$ if and only if the functions $\theta\colon D^4 \to S^1$ and $\alpha\colon D^4 \to \mathbb{R}^+$ defined by*

$$\begin{aligned}\big(\cos\theta(x), \sin\theta(x)\big) &= \eta_s(|x|^2)\\ \alpha(x) &= \frac{\sigma(|x|^2)}{4|x|^2}\end{aligned}$$

are spherically symmetric solutions of the equations

$$\begin{aligned}\theta_{tt} &= \alpha\Delta\theta + \langle\nabla\alpha, \nabla\theta\rangle\\ \Delta\alpha - |\nabla\theta|^2\alpha &= -|\theta_t|^2.\end{aligned} \tag{58}$$

with Neumann boundary condition $\frac{\partial\theta}{\partial\nu} = 0$ for θ and Robin boundary condition $\frac{\partial\alpha}{\partial\nu} + 2\alpha = 0$ for α on $\partial D^4 = S^3$. Furthermore any smooth solution has $\alpha > 0$ everywhere, so that the hyperbolic equation for θ is nondegenerate.

Proof. Since η_s is a unit vector, we can write $\eta_s = (\cos\theta, \sin\theta)$ for some $\theta \in S^1$. Differentiating the equation for η with respect to s and using this formula, we obtain the following equation for θ:

$$\theta_{tt} = \sigma\theta_{ss} + 2\sigma_s\theta_s.$$

Now if we set $\sigma(s) = 4s\alpha(s)$, the equations change to

$$\begin{aligned}\theta_{tt} &= 4\alpha(s\theta_{ss} + \theta_s) + 4s\alpha_s\theta_s\\ -|\theta_t|^2 &= 4s\alpha_{ss} + 8\alpha_s - 4s|\theta_s|^2\alpha.\end{aligned}$$

Now changing variables by $s = r^2$, we easily see that these equations are

$$\begin{aligned}\theta_{tt} &= \alpha\left(\theta_{rr} + \tfrac{3}{r}\theta_r\right) + 2\alpha_r\theta_r\\ -|\theta_t|^2 &= \alpha_{rr} + \tfrac{3}{r}\alpha_r - |\theta_r|^2\alpha.\end{aligned}$$

Now the operator $\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$ is familiar as the spherically symmetric Laplacian on $\mathbb{R}^n$, and hence we recognize both terms above as coming from the Laplacian on $\mathbb{R}^4$ under the assumption that α and θ are both spherically symmetric.

The boundary conditions are easy to check: the condition for α comes from $\frac{d\sigma}{ds}\big|_{s=1} = 0$, which translates into $\alpha(1) + \frac{1}{2}\alpha_r(1) = 0$, and the condition on θ comes from the fact that a smooth solution of (57) will be odd through $s = 1$, which forces θ to be even through $s = 1$. Positivity of α comes from the fact that $\frac{\sigma(s)}{s}$ is bounded below, using Lemma 6.6. □

The fact that the degeneracy can be removed if we work in a higher-dimensional space, and thus in some sense the equations naturally "live" there, is essentially the reason why we need higher than usual Sobolev order for the estimates to close.

7. Convergence as the chain approaches the whip

We now show how to construct the solution of the partial differential equations (57), by showing that solutions of the ordinary differential equations (22) and (21) converge in the limit as the number of links approaches infinity. It is easy to see that this is true formally, in the sense that if we knew $\sigma_i = \sigma(\frac{i}{n})$ and $\eta_i = \eta(\frac{i-1}{n})$ for smooth functions σ and η, then the limit of (21) and (22) would be (57). The main requirement is to show that there are finite-dimensional analogues of each of the energy norms in Theorems 6.3 and 6.5, and that they are bounded by numbers independent of n, so that we can find a convergent subsequence.

As a consequence, we have that the motion of a chain converges to the motion of a whip as n approaches infinity, in the sense that position, velocity, and acceleration all converge.

The analogues of the energy U_k and V_k from (34) are the discrete energy norms

$$\begin{aligned} u_k &= \frac{1}{n}\sum_{j=1}^{n-k+1} \frac{\Gamma(j+k)}{n^k\Gamma(j)} \lvert n^k \Delta^k \dot{\eta}_j\rvert^2, \\ v_k &= \frac{1}{n}\sum_{j=1}^{n-k} \frac{\Gamma(j+k+1)}{n^{k+1}\Gamma(j)} \lvert n^{k+1}\Delta^{k+1}\eta_j\rvert^2, \end{aligned} \tag{59}$$

and the analogues of (42) are the time-dependent energy norms

$$\begin{aligned} \widetilde{u}_k &= \frac{1}{n}\sum_{j=1}^{n-k+1} \sigma_j\sigma_{j+1}\cdots\sigma_{j+k-1}\lvert n^k\Delta^k\dot{\eta}_j\rvert^2, \\ \widetilde{v}_k &= \frac{1}{n}\sum_{j=1}^{n-k}\sigma_j\sigma_{j+1}\cdots\sigma_{j+k}\lvert n^{k+1}\Delta^{k+1}\eta_j\rvert^2. \end{aligned} \tag{60}$$

Here Δ^k is the difference operator, defined recursively by

$$\Delta^0\eta_j = \eta_j, \qquad \Delta^{k+1}\eta_j = \Delta^k\eta_{j+1} - \Delta^k\eta_j.$$

We replace the powers s^q with gamma function terms $\frac{\Gamma(k+q)}{n^q\Gamma(k)}$, which reduce when q is an integer to $\frac{k(k+1)\cdots(k+q-1)}{n^q}$. These behave much more nicely in difference equations than the more obvious analogue $(\frac{k}{n})^q$, though they are the same in the limit as n approaches infinity.

Clearly if $\eta_j = \eta(\frac{j-1}{n})$ for every j, and j_n is a sequence such that $\frac{j_n}{n} \to s$, then we have

$$\lim_{n\to\infty} n^k\Delta^k\eta_{j_n} = \frac{\partial^k\eta}{\partial s^k}.$$

Hence the weighting by powers of n is to ensure

$$\lim_{n\to\infty} u_k = U_k, \quad \lim_{n\to\infty} v_k = V_k, \quad \lim_{n\to\infty}\widetilde{u}_k = \widetilde{U}_k, \quad \lim_{n\to\infty}\widetilde{v}_k = \widetilde{V}_k$$

as long as $\eta_j = \eta(\frac{j-1}{n})$ for some smooth function η.

Now we need to construct discrete analogues of all the estimates in Section 6. The starting point is the following analogue of Theorem 6.1.

Theorem 7.1. *Let f_k be a finite sequence in $\mathbb{R}^m$, for $1 \le k \le n$. Denote $\Delta f_k = f_{k+1} - f_k$ for $1 \le k \le n-1$. Then for any $r > 0$, we have the discrete weighted Poincaré inequality*

$$\sum_{k=1}^n \frac{\Gamma(k+r-1)}{\Gamma(k)}\lvert f_k\rvert^2 \lesssim \sum_{k=1}^n \frac{\Gamma(k+r)}{\Gamma(k)}\lvert f_k\rvert^2 + \sum_{k=1}^{n-1}\frac{\Gamma(k+r+1)}{\Gamma(k)}\lvert\Delta f_k\rvert^2 \tag{61}$$

and the discrete weighted Sobolev inequality

$$\max_{1\le m\le n}\frac{\Gamma(m+r)}{\Gamma(m)}\lvert f_m\rvert^2 \lesssim \sum_{k=1}^n \frac{\Gamma(k+r)}{\Gamma(k)}\lvert f_k\rvert^2 + \sum_{k=1}^{n-1}\frac{\Gamma(k+r+1)}{\Gamma(k)}\lvert\Delta f_k\rvert^2. \tag{62}$$

Proof. The main thing here is a discrete analogue of the basic formula (37). Let p and q be any real numbers. We have

$$|(p\,+\,k)f_{k+1}\,-\,kf_k|^2\;=\;p^2|f_{k+1}|^2\,+\,k(k\,+\,p)|\Delta f_k|^2\,+\,pk\big(|f_{k+1}|^2\,-\,|f_k|^2\big)$$

We multiply through by $\frac{\Gamma(k+q)}{\Gamma(k+1)}$, then observe that the terms on the right side can be simplified a bit, using the formula $\Gamma(x+1)=x\Gamma(x)$, to

$$p^2\frac{\Gamma(k+q)}{\Gamma(k+1)}|f_{k+1}|^2+pk\frac{\Gamma(k+q)}{\Gamma(k+1)}\big(|f_{k+1}|^2-|f_k|^2\big)$$
$$=p(p-q)\frac{\Gamma(k+q)}{\Gamma(k+1)}|f_{k+1}|^2+p\Delta\left(\frac{\Gamma(k+q)}{\Gamma(k)}|f_k|^2\right)$$

Now let i and j be any integers with $1\le i<j\le n$. Summing all the terms from $k=i$ to $k=j-1$ and using the telescope formula $\sum_{k=i}^{j-1}\Delta b_k=b_j-b_i$, we obtain

$$\sum_{k=i}^{j-1}\frac{\Gamma(k+q)}{\Gamma(k+1)}|(k+p)f_{k+1}-kf_k|^2=p(p-q)\sum_{k=i}^{j-1}\frac{\Gamma(k+q)}{\Gamma(k+1)}|f_{k+1}|^2$$
$$+\sum_{k=i}^{j-1}(k+p)\frac{\Gamma(k+q)}{\Gamma(k)}|\Delta f_k|^2+p\frac{\Gamma(j+q)}{\Gamma(j)}|f_j|^2-p\frac{\Gamma(i+q)}{\Gamma(i)}|f_i|^2. \quad (63)$$

The rest of the proof proceeds exactly as the proof of Theorem 6.1, and we omit it. □

Now we prove the discrete analogues of Theorems 6.3 and 6.5.

Theorem 7.2. *Let $\eta_k(t)\in\mathbb{R}^m$, $\sigma_k(t)\in\mathbb{R}$ be functions of time defined for $1\le k\le n$, with $\eta_{n+1}\equiv 0$ and $\sigma_0\equiv 0$ for convenience. Set $\delta_k=n(\eta_{k+1}-\eta_k)$ for $1\le k\le n$. Suppose that η_k and σ_k solve the system*

$$\begin{aligned}\ddot\eta_k&=n\sigma_k\delta_k-n\sigma_{k-1}\delta_{k-1},\qquad 1\le k\le n,\\ -\frac{|\dot\delta_k|^2}{n^2}&=\langle\delta_{k+1},\delta_k\rangle\sigma_{k+1}-2\sigma_k+\langle\delta_k,\delta_{k-1}\rangle\sigma_{k-1}\\ \eta_k(0)=\gamma_k,\quad&\dot\eta_k(0)=w_k,\quad \eta_{n+1}=0,\quad \sigma_0=0,\end{aligned}\tag{64}$$

where we assume that $|\gamma_{k+1}-\gamma_k|^2=\frac{1}{n^2}$ and $\langle\gamma_{k+1}-\gamma_k,w_{k+1}-w_k\rangle=0$ for $1\le k\le n$.

For $\alpha\in(0,1)$, set

$$d_k=\frac{1}{n}\sum_{j=0}^{n-k-1}\frac{\Gamma(j+k+\alpha)}{n^{k+\alpha}\Gamma(j)}|n^{k+1}\Delta^{k+1}\sigma_j|^2.$$

With u_k and v_k defined as in (59), set

$$p_k=\sum_{j=1}^k u_j,\quad q_k=\sum_{j=1}^k v_k,\quad \textit{and } r_k=\sum_{j=1}^k d_j. \tag{65}$$

Set

$$a = \max_{1\le k\le n} n|\sigma_k - \sigma_{k-1}|,$$
$$b = \max_{1\le k\le n} \frac{k}{n\sigma_k},$$
$$c = \max_{1\le k\le n} \frac{n|\dot{\sigma}_k|}{k}.$$

Then $|\delta_k|^2 \equiv 1$*, so that* $v_0(t) \equiv \frac{n+1}{2n}$ *for all* t*. In addition we have the following a priori estimates, involving only numerical constants (independent of* σ_k *and* η_k*).*

First we have

$$a \lesssim p_2, \quad b \lesssim 4e^{q_2}(1+q_2)/u_0, \quad c \lesssim p_2^{3/2} q_3^{1/2}. \tag{66}$$

We also have

$$d_1 \lesssim a^2 q_3 v_1 + p_2 u_1, \tag{67}$$
$$d_2 \lesssim a^2 q_3 v_2 + p_3 u_2, \tag{68}$$
$$d_3 \lesssim a^2 q_3 v_3 + p_3 u_3 + q_2^2(a^2 q_3 q_2 + p_3 p_2). \tag{69}$$

Finally we have

$$\frac{d}{dt}(\tilde{u}_1 + \tilde{v}_1) \lesssim c u_1 + c a v_1, \tag{70}$$
$$\frac{d}{dt}(\tilde{u}_2 + \tilde{v}_2) \lesssim c a u_2 + c a^2 v_2 + a^2\sqrt{r_2 p_2 q_3}, \tag{71}$$
$$\frac{d}{dt}(\tilde{u}_3 + \tilde{v}_3) \lesssim c a^2 u_3 + c a^3 v_3 + a^3\sqrt{r_3 p_3 q_3}. \tag{72}$$

Proof. These are generally proved in the same way as the estimates in Section 6.

The analogue of (44) is the following:

$$\max_{1\le k\le n} \frac{n\sigma_k}{k} = \max_{1\le k\le n} \frac{1}{k}\sum_{j=1}^k n(\sigma_j - \sigma_{j-1}) \le \frac{1}{k}\sum_{j=1}^k a = a. \tag{73}$$

That $|\delta_k|^2 \equiv 1$ for all k and t is automatic, since we are dealing with ordinary differential equations for which existence and uniqueness are easy. Hence we get $v_0 = \frac{1}{n^2}\sum_{k=1}^n k|\delta_k|^2 = \frac{n+1}{2n}$.

For u_0 we have

$$\begin{aligned}
\tfrac{d}{dt}\textstyle\sum_{k=1}^n |\dot{\eta}_k|^2 &= 2\textstyle\sum_{k=1}^n \langle \dot{\eta}_k, \ddot{\eta}_k\rangle \\
&= 2n\textstyle\sum_{k=1}^n \langle \dot{\eta}_k, \sigma_k\delta_k - \sigma_{k-1}\delta_{k-1}\rangle \\
&= 2n\textstyle\sum_{k=1}^n \langle \dot{\eta}_k, \sigma_k\delta_k\rangle - 2n\textstyle\sum_{k=2}^n \langle \dot{\eta}_{k-1}, \sigma_{k-1}\delta_{k-1}\rangle \\
&\qquad - 2n\textstyle\sum_{k=2}^n \langle \dot{\eta}_k - \dot{\eta}_{k-1}, \sigma_{k-1}\delta_{k-1}\rangle \\
&= 2n\sigma_n\langle \dot{\eta}_n, \delta_n\rangle - \textstyle\sum_{k=2}^n \sigma_{k-1}\tfrac{d}{dt}|\delta_k|^2 \\
&= 0
\end{aligned}$$

since $n\dot{\eta}_n = -\dot{\delta}_n$ and $|\delta_k|^2$ is constant in time for all k. If u_0 is zero, then $\dot{\eta}_k = 0$ for every k, so the chain doesn't move.

The proofs of the bound for b in (66) is clearly exactly the same as the proof of Lemma 6.6.

To prove the bound for c we differentiate the equation for σ in time, using $\ddot{\delta}_k = n^2\sigma_{k+1}\delta_{k+1} - 2n^2\sigma_k\delta_k + n^2\sigma_{k-1}\delta_{k-1}$, to obtain

$$\begin{aligned}
&-2\sigma_{k+1}\langle\dot{\delta}_k, \delta_{k+1}-\delta_k\rangle - 2\sigma_{k-1}\langle\dot{\delta}_k, \delta_{k-1}-\delta_k\rangle \\
&\qquad = (\langle\dot{\delta}_{k+1}, \delta_k\rangle + \langle\delta_{k+1}, \dot{\delta}_k\rangle)\sigma_{k+1} + \langle\delta_{k+1}, \delta_k\rangle\dot{\sigma}_{k+1} \\
&\qquad\qquad - 2\dot{\sigma}_k + (\langle\dot{\delta}_k, \delta_{k-1}\rangle + \langle\delta_k, \dot{\delta}_{k-1}\rangle)\sigma_{k-1} + \langle\delta_k, \delta_{k-1}\rangle\dot{\sigma}_{k-1},
\end{aligned}$$

for $1 \le k \le n-1$, again using $\sigma_0 \equiv 0$ for convenience. For $k = n$ we have

$$\dot{\sigma}_n - \langle\delta_{n-1}, \delta_n\rangle\dot{\sigma}_{n-1} = \sigma_{n-1}\big(\langle\delta_n, \dot{\delta}_{n-1}\rangle + 3\langle\delta_{n-1}, \dot{\delta}_n\rangle\big).$$

Thus we have

$$\begin{aligned}
\dot{\sigma}_k = n\sum_{j=1}^{n-1} G_{kj}\Big(&\big[\langle\dot{\delta}_{j+1}, \delta_j\rangle + 3\langle\delta_{j+1}, \dot{\delta}_j\rangle\big]\sigma_{j+1} \\
&+ \big[3\langle\dot{\delta}_j, \delta_{j-1}\rangle + \langle\delta_j, \dot{\delta}_{j-1}\rangle\big]\sigma_{j-1}\Big) \\
&+ nG_{kn}\sigma_{n-1}\big[\langle\delta_n, \dot{\delta}_{n-1}\rangle + 3\langle\delta_{n-1}, \dot{\delta}_n\rangle\big] \\
= n\sum_{j=2}^{n-1} G_{kj}\Big(&\Delta\sigma_j\big[-\langle\dot{\delta}_{j+1}, \Delta\delta_j\rangle + 3\langle\Delta\delta_j, \dot{\delta}_j\rangle\big] \\
&- \Delta\sigma_{j-1}\big[\langle\Delta\delta_{j-1}, \dot{\delta}_{j-1}\rangle - 3\langle\dot{\delta}_j, \Delta\delta_{j-1}\rangle\big] \\
&+ \sigma_j\big[2\langle\dot{\delta}_j, \Delta^2\delta_{j-1}\rangle - \langle\Delta\delta_j, \Delta\dot{\delta}_j\rangle - \langle\Delta\delta_{j-1}, \Delta\dot{\delta}_{j-1}\rangle\big]\Big) \\
&+ nG_{k1}\sigma_2\big[-\langle\dot{\delta}_2, \Delta\delta_1\rangle + 3\langle\delta_1, \dot{\delta}_1\rangle\big] \\
&\quad + nG_{kn}\sigma_{n-1}\big[\langle\Delta\delta_{n-1}, \dot{\delta}_{n-1}\rangle - 3\langle\Delta\delta_{n-1}, \dot{\delta}_n\rangle\big]
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{a}\frac{n|\dot{\sigma}_k|}{k} \lesssim n^2\sum_{j=2}^{n-1}\Big(&|\Delta\dot{\eta}_{j+1}||\Delta^2\eta_j| + |\Delta^2\eta_j||\Delta\dot{\eta}_j| + |\Delta\dot{\eta}_{j-1}||\Delta^2\eta_{j-1}| \\
&+ |\Delta^2\eta_{j-1}||\Delta\dot{\eta}_j| + j|\Delta^3\eta_{j-1}||\Delta\dot{\eta}_j| + j|\Delta^2\eta_j||\Delta^2\dot{\eta}_j| \\
&+ j|\Delta^2\eta_{j-1}||\Delta^2\dot{\eta}_{j-1}|\Big) + n^2|\Delta\dot{\eta}_2||\Delta^2\eta_1| + n^2|\Delta^2\eta_1||\Delta\dot{\eta}_1| \\
&+ n^3|\Delta^2\eta_{n-1}||\Delta\dot{\eta}_{n-1}| + n^3|\Delta^2\eta_{n-1}||\Delta\dot{\eta}_n| \\
\lesssim n^2\Bigg(&\sum_{j=1}^{n-1}|\Delta\dot{\eta}_j||\Delta^2\eta_j| + \sum_{j=1}^{n-2} j|\Delta^3\eta_j||\Delta\dot{\eta}_j| + \sum_{j=1}^{n-1} j|\Delta^2\eta_j||\Delta^2\dot{\eta}_j|\Bigg) \\
&+ n^3|\Delta^2\eta_{n-1}||\Delta\dot{\eta}_n|.
\end{aligned}$$

From this the bounds follow as before.

Now to approximate the norms d_j, recall that the defining equation for σ_j is

$$a_j\sigma_{j+1} - 2\sigma_j + a_{j-1}\sigma_{j-1} = -\tfrac{1}{n^2}|\dot{\delta}_j|^2.$$

Using $\kappa_j = n(\delta_{j+1} - \delta_j)$ as in the proof of Theorem 5.2, we see that $a_j = 1 - \frac{1}{2n^2}|\kappa_j|^2$, so that

$$\sigma_{j+1} - 2\sigma_j + \sigma_{j-1} = -\tfrac{1}{2n^2}\left(2|\dot{\delta}_j|^2 + |\kappa_j|^2\sigma_{j+1} + |\kappa_{j-1}|^2\sigma_{j-1}\right)$$

for $1 \le j \le n-2$, while

$$a_{n-1}\sigma_{n-1} - \sigma_n = -\tfrac{1}{n^2}|\dot{\delta}_n|^2.$$

We then get

$$\begin{aligned}
&\frac{1}{n}\sum_{j=0}^{n-3} \tfrac{\Gamma(j+1+\alpha)}{n^{1+\alpha}\Gamma(j)}|n^2\Delta^2\sigma_j|^2 \\
&\quad\lesssim n^{-2-\alpha}\textstyle\sum_{j=0}^{n-3}\frac{\Gamma(j+1+\alpha)}{\Gamma(j)}\Big(|\dot{\delta}_{j+1}|^4 + |\kappa_{j+1}|^4\sigma_{j+2}^2 + |\kappa_j|^4\sigma_j^2\Big) \\
&\quad\lesssim \tfrac{1}{n^{2+\alpha}}\Big(\textstyle\sum_{j=1}^{n-2}\frac{\Gamma(j+\alpha)}{\Gamma(j-1)}|n\Delta\dot{\eta}_j|^4 + a^2\sum_{j=1}^{n-2}\frac{(j+1)^2}{n^2}\frac{\Gamma(j+\alpha)}{\Gamma(j-1)}|n^2\Delta^2\eta_j|^4\Big) \\
&\quad\lesssim \big(\max_{1\le j\le n}\tfrac{\Gamma(j+\alpha-1)}{\Gamma(j-1)}|n\Delta\dot{\eta}_j|^2\big)\tfrac{1}{n}\textstyle\sum_{j=1}^{n-1}(j+\alpha-1)|n\Delta\dot{\eta}_j|^2 \\
&\qquad\quad + a^2\big(\max_{1\le j\le n-1}\tfrac{j}{n}|n^2\Delta^2\eta_j|^2\big)\tfrac{1}{n}\textstyle\sum_{j=1}^{n-2}\frac{j(j+1)}{n^2}|n^2\Delta^2\eta_j|^2 \\
&\quad\lesssim p_2^2 + a^2 q_2^2,
\end{aligned}$$

using (35) and (36).

We'll skip the details for the higher norms of σ_j, but the basic estimates are proved exactly as in Theorem 6.3. The only thing to worry about is the analogue of the formulas for σ_{sss} and σ_{ssss}, which we do now. For the third difference, we clearly have

$$\begin{aligned}
\Delta^3\sigma_{j-1} &= \sigma_{j+2} - 3\sigma_{j+1} + 3\sigma_j - \sigma_{j-1} \\
&= -\tfrac{1}{2n^2}\big(2|\dot{\delta}_{j+1}|^2 - 2|\dot{\delta}_j|^2 + |\kappa_{j+1}|^2\sigma_{j+2} \\
&\qquad\quad + |\kappa_j|^2\sigma_j - |\kappa_j|^2\sigma_{j+1} - |\kappa_{j-1}|^2\sigma_{j-1}\big) \\
&= -\tfrac{1}{2n^2}\big(2\langle\dot{\delta}_{j+1}+\dot{\delta}_j, \Delta\dot{\delta}_j\rangle + \Delta\sigma_{j+1}|\kappa_{j+1}|^2 + \Delta\sigma_{j-1}|\kappa_{j-1}|^2 \\
&\qquad\quad \langle\kappa_{j+1}+\kappa_j, \Delta\kappa_j\rangle\sigma_{j+1} + \langle\kappa_j+\kappa_{j-1}, \Delta\kappa_{j-1}\rangle\sigma_j\big).
\end{aligned}$$

For the fourth difference, we have

$$\begin{aligned}
\Delta^4\sigma_{j-1} &= \sigma_{j+3} - 4\sigma_{j+2} + 6\sigma_{j+1} - 4\sigma_j + \sigma_{j-1} \\
&= -\tfrac{1}{n^2}\Big(|\Delta\dot{\delta}_{j+1}|^2 + |\Delta\dot{\delta}_j|^2 + 2\langle\dot{\delta}_{j+1}, \Delta^2\dot{\delta}_j\rangle\Big) \\
&\qquad - \tfrac{1}{2n^2}\Big(|\kappa_{j+2}|^2\Delta^2\sigma_{j+1} + 2\Delta\sigma_{j+1}\langle\kappa_{j+2}+\kappa_{j+1}, \Delta\kappa_{j+1}\rangle \\
&\qquad\qquad + |\kappa_{j-1}|^2\Delta^2\sigma_{j-1} + 2\Delta\sigma_j\langle\kappa_j+\kappa_{j-1}, \Delta\kappa_{j-1}\rangle\Big) \\
&\qquad - \tfrac{\sigma_{j+1}}{2n^2}\Big(|\Delta\kappa_{j+1}|^2 + |\Delta\kappa_{j-1}|^2 + 2\langle\Delta\kappa_j, \Delta\kappa_{j+1}\rangle \\
&\qquad\qquad + 2\langle\kappa_j, \Delta^2\kappa_j\rangle + 2\langle\kappa_j, \Delta^2\kappa_{j-1}\rangle\Big).
\end{aligned}$$

For (70) we have

$$\begin{aligned}
\frac{d}{dt}(\widetilde{u}_1+\widetilde{v}_1) &= \frac{d}{dt}\Big(n\textstyle\sum_{j=1}^{n}\sigma_j|n\Delta\dot{\eta}_j|^2 + n^3\sum_{j=1}^{n-1}\sigma_j\sigma_{j+1}|n^2\Delta^2\eta_j|^2\Big)\\
&= n\textstyle\sum_{j=1}^{n}\dot{\sigma}_j|\Delta\dot{\eta}_j|^2 + n^3\sum_{j=1}^{n-1}(\dot{\sigma}_j\sigma_{j+1}+\sigma_j\dot{\sigma}_{j+1})|\Delta^2\eta_j|^2\\
&\quad + 2n\textstyle\sum_{j=1}^{n-1}\sigma_j\langle\sigma_{j+1}\delta_{j+1}-2\sigma_j\delta_j+\sigma_{j-1}\delta_{j-1},\dot{\delta}_j\rangle\\
&\quad + \sigma_j\sigma_{j+1}\langle\delta_{j+1}-\delta_j,\dot{\delta}_{j+1}-\dot{\delta}_j\rangle - 2n^2\sigma_n\sigma_{n-1}\langle\dot{\eta}_n,\delta_{n-1}\rangle\\
&\le cu_1+2cav_1+2n\textstyle\sum_{j=2}^{n-1}\sigma_j\sigma_{j-1}\langle\delta_{j-1},\dot{\delta}_j\rangle\\
&\quad - 2n\textstyle\sum_{j=1}^{n-1}\sigma_j\sigma_{j+1}\langle\delta_j,\dot{\delta}_{j+1}\rangle + 2n\sigma_{n-1}\sigma_n\langle\dot{\delta}_n,\delta_{n-1}\rangle\\
&= cu_1+2cav_1
\end{aligned}$$

For the proofs of (71) and (72), we will just write the basic difference formulas; again the proofs work the same as for Theorem 6.5.

We easily verify, after some manipulations, that

$$\begin{aligned}
&\frac{d}{dt}(\widetilde{u}_2+\widetilde{v}_2)\\
&= \tfrac{1}{n}\textstyle\sum_{j=1}^{n-1}\frac{d}{dt}(\sigma_j\sigma_{j+1})|n^2\Delta^2\dot{\eta}_j|^2 + \frac{1}{n}\sum_{j=1}^{n-2}\frac{d}{dt}(\sigma_j\sigma_{j+1}\sigma_{j+2})|n^3\Delta^3\eta_j|^2)\\
&\quad + n^3\textstyle\sum_{j=1}^{n-2}\Big[\sigma_j\sigma_{j+1}\sigma_{j+2}\langle\delta_{j+2}-2\delta_{j+1}+\delta_j,\dot{\delta}_{j+2}-2\dot{\delta}_{j+1}+\dot{\delta}_j\rangle\\
&\qquad + \sigma_j\sigma_{j+1}\langle\Delta\dot{\delta}_j,\sigma_{j+2}\delta_{j+2}-3\sigma_{j+1}\delta_{j+1}+3\sigma_j\delta_j-\sigma_{j-1}\delta_{j-1}\rangle\Big]\\
&\quad + n^3\sigma_{n-1}\sigma_n\langle\dot{\delta}_{n-1},-2\sigma_n\delta_n+3\sigma_{n-1}\delta_{n-1}-\sigma_{n-2}\delta_{n-2}\rangle\\
&= \tfrac{1}{n}\textstyle\sum_{j=1}^{n-1}\frac{d}{dt}(\sigma_j\sigma_{j+1})|n^2\Delta^2\dot{\eta}_j|^2 + \frac{1}{n}\sum_{j=1}^{n-2}\frac{d}{dt}(\sigma_j\sigma_{j+1}\sigma_{j+2})|n^3\Delta^3\eta_j|^2)\\
&\qquad + n^3\textstyle\sum_{j=1}^{n-1}\sigma_j\sigma_{j+1}\Big[\langle\Delta\delta_j,\Delta\dot{\delta}_j\rangle(2\sigma_{j+2}-3\sigma_{j+1}+\sigma_{j-1})\\
&\qquad\quad + \langle\delta_j,\dot{\delta}_{j+1}-\dot{\delta}_j\rangle(\sigma_{j+2}-3\sigma_{j+1}+3\sigma_j-\sigma_{j-1})\Big].
\end{aligned}$$

Analyzing the differences of the σ_j, we observe that

$$\begin{aligned}
2\sigma_{j+2}-3\sigma_{j+1}+\sigma_{j-1} &= 2\Delta^2\sigma_j+\Delta^2\sigma_{j-1},\\
\sigma_{j+2}-3\sigma_{j+1}+3\sigma_j-\sigma_{j-1} &= \Delta^3\sigma_{j-1}.
\end{aligned}$$

Now these terms can be estimated in terms of d_1 and d_2.

For the third discrete norm we have, after a lot of algebra,

$$\begin{aligned}
\frac{d}{dt}(\widetilde{u}_3+\widetilde{v}_3) &= n^5\tfrac{d}{dt}\textstyle\sum_{j=1}^{n-2}\sigma_j\sigma_{j+1}\sigma_{j+2}|\Delta^3\dot{\eta}_j|^2\\
&\quad + n^7\tfrac{d}{dt}\textstyle\sum_{j=1}^{n-3}\sigma_j\sigma_{j+1}\sigma_{j+2}\sigma_{j+3}|\Delta^4\eta_j|^2\\
&= n^5\textstyle\sum_{j=1}^{n-2}\frac{d}{dt}(\sigma_j\sigma_{j+1}\sigma_{j+2})|\Delta^3\dot{\eta}_j|^2\\
&\quad + n^7\textstyle\sum_{j=1}^{n-3}\frac{d}{dt}(\sigma_j\sigma_{j+1}\sigma_{j+2}\sigma_{j+3})|\Delta^4\eta_j|^2\\
&\quad + n^7\textstyle\sum_{j=1}^{n-2}\sigma_j\sigma_{j+1}\sigma_{j+2}\Big((3\sigma_{j+3}-4\sigma_{j+2}+\sigma_{j-1})\langle\Delta^2\delta_j,\Delta^2\dot{\delta}_j\rangle\\
&\qquad + (3\sigma_{j+3}-8\sigma_{j+2}+6\sigma_{j+1}-\sigma_{j-1})\langle\Delta\delta_j,\Delta^2\dot{\delta}_j\rangle\\
&\qquad + (\sigma_{j+3}-4\sigma_{j+2}+6\sigma_{j+1}-4\sigma_j+\sigma_{j-1})\langle\delta_j,\Delta^2\dot{\delta}_j\rangle\Big)\\
&\quad + n^7\sigma_n\langle-2\delta_n+3\delta_{n-1}-\delta_{n-2},\dot{\delta}_n-2\dot{\delta}_{n-1}+\dot{\delta}_{n-2}\rangle.
\end{aligned}$$

We clearly have

$$
\begin{aligned}
3\sigma_{j+3} - 4\sigma_{j+2} + \sigma_{j-1} &= 3\Delta^2\sigma_{j+1} + 2\Delta^2\sigma_j + \Delta^2\sigma_{j-1},\\
3\sigma_{j+3} - 8\sigma_{j+2} + 6\sigma_{j+1} - \sigma_{j-1} &= 3\Delta^3\sigma_j + \Delta^3\sigma_{j-1},\\
\sigma_{j+3} - 4\sigma_{j+2} + 6\sigma_{j+1} - 4\sigma_j + \sigma_{j-1} &= \Delta^4\sigma_{j-1},
\end{aligned}
$$

and so using the bounds on d_1, d_2, and d_3, we can estimate all these terms as in the proof of Theorem 6.5. □

Now we can finally prove the local existence theorem for the system (57) of partial differential equations. The fact that all the smooth a priori estimates from Theorems 6.3 and 6.5 have discrete analogues in Theorem 7.2 allows us to construct the solution as a limit of a subsequence of discrete solutions as $n \to \infty$, following the technique of Ladyzhenskaya [L] and references therein. First, some lemmas.

Lemma 7.3. *Suppose $\gamma\colon [0,1] \to \mathbb{R}^m$ and $w\colon [0,1] \to \mathbb{R}^m$ are functions with bounded weighted Sobolev norms:*

$$\int_0^1 s^2|\gamma''(s)|^2 + s^3|\gamma'''(s)|^2 + s^4|\gamma^{iv}(s)|^2\,ds < \infty$$

and

$$\int_0^1 s|w'(s)|^2 + s^2|w''(s)|^2 + s^3|w'''(s)|^2\,ds < \infty,$$

and that in addition we have

$$|\gamma'(s)|^2 \equiv 1 \qquad \text{and} \quad \langle \gamma'(s), w'(s)\rangle \equiv 0$$

for all $s \in [0,1]$, along with the compatibility conditions $\gamma(1) = 0$ and $w(1) = 0$.

For $n \in \mathbb{N}$ define the discrete approximations γ_n and w_n to these initial conditions as follows:

$$
\begin{aligned}
(\gamma_n)_i &= -\tfrac{1}{n}\sum_{j=i}^{n} \gamma'(\tfrac{j}{n}),\\
(w_n)_i &= -\tfrac{1}{n}\sum_{j=i}^{n} w'(\tfrac{j}{n}).
\end{aligned}
\tag{74}
$$

Let $(\eta_n)_i(t)$ and $(\sigma_n)_i(t)$, for $1 \le i \le n$, be the solution of equations (64) with $\eta_n(0) = \gamma_n$ and $\dot{\eta}_n(0) = w_n$.

Then there is a $T > 0$ such that the discrete energy norm $e_3(t) = 1 + p_3(t) + q_3(t)$ (where p_3 and q_3 are defined as in (65)) is bounded uniformly on $[0,T]$ and uniformly in n.

Proof. Note that the definition (74) ensures that $n|(\gamma_n)_{i+1} - (\gamma_n)_i| = 1$ for each $1 \le i \le n$, and also $\langle (\gamma_n)_{i+1} - (\gamma_n)_i, (w_n)_{i+1} - (w_n)_i\rangle = 0$, so that γ_n and w_n will serve as proper initial conditions for the ODEs (64).

Let $q_2(0) = v_1(0) + v_2(0)$ be the norm of the initial data γ_n, and choose any $R > q_2(0)$. Choose any $\beta > \max\{1, 4e^R(1+R)/u_0(0)\}$. Then as long as $q_2(t) < R$, we will have the corresponding $b(t) = \sup_{1\le k\le n} \frac{k}{n\sigma_k(t)} < \beta$ by (66), and this allows us to uniformly relate the norms $\tilde{e}_3(t)$ and $e_3(t)$ for short time.

Now with $\tilde{e}_3(t) = 1 + \sum_{i=1}^3 \tilde{u}_i(t) + \tilde{v}_i(t)$, we combine all the estimates of Theorem 7.2, finding a numerical constant M independent of all the data such that the following crude estimate holds:

$$\frac{d\tilde{e}_3}{dt} \le M e_3(t)^7.$$

Next, we have $e_3(t) \le \max\{b(t), 1\}^4 \tilde{e}_3(t) \le \beta^4 \tilde{e}_3(t)$, so that $\tilde{e}_3$ satisfies the differential inequality

$$\frac{d\tilde{e}_3}{dt} \le M\beta^{28}\tilde{e}_3(t)$$

as long as $q_2(t) < R$. Gronwall's inequality then implies

$$\tilde{e}_3(t) \le \tilde{e}_3(0)[1 - 6M\beta^{28}\tilde{e}_3(0)^6 t]^{-1/6}.$$

Let $T_1 = (12M\beta^{28}\tilde{e}_3(0)^6)^{-1}$; then we have a uniform bound on $\tilde{e}_3(t)$ for $t \in [0, T_1]$ as long as $q_2(t) < R$.

To finish, we need a bound on $q_2(t)$ to confirm our assumption. So we compute

$$\begin{aligned}
\frac{dq_2}{dt} &= 2\int_0^1 s^2 \langle \eta_{ss}, \eta_{sst}\rangle \, ds + 2\int_0^1 s^3 \langle \eta_{sss}, \eta_{ssst}\rangle \, ds \\
&\le 2\sqrt{\left(\int s^2|\eta_{ss}|^2\right)\left(\int s^2|\eta_{sst}|^2\right)} + 2\sqrt{\left(\int s^3|\eta_{sss}|^2\right)\left(\int s^3|\eta_{ssst}|^2\right)} \\
&\le 2\sqrt{q_2 p_3} \\
&\le 2\beta^{3/2}\sqrt{q_2\tilde{p}_3}.
\end{aligned}$$

We thus have

$$\sqrt{q_2(t)} \le \sqrt{q_2(0)} + \beta^{3/2}\int_0^t \sqrt{\tilde{p}_3(\tau)} \, d\tau.$$

Now choose T_2 small enough so that the right hand side is less than $\sqrt{R}$.

Then for $T = \min\{T_1, T_2\}$, we have $q_2(t) < R$, so that $b(t) < \beta$, and $\tilde{e}_3(t)$ is uniformly bounded. Hence on $[0, T]$, we also know $e_3(t) \le \beta^4 \tilde{e}_3(t)$ is uniformly bounded, and all these bounds are independent of n. □

Next we need to find an interpolating function for the discrete data, for any fixed time: the important thing is to bound the derivatives of the interpolating function in terms of the differences, with constants independent of n. The proof is a simple consequence of a theorem of Kunkle.

Lemma 7.4. *Let $f_1, \cdots, f_n$ be a finite sequence in $\mathbb{R}^m$, and adjoin $f_{n+1} = 0$. Then we can find a function $f\colon [0,1] \to \mathbb{R}^m$ such that $f(\frac{i-1}{n}) = f_i$ for $1 \le i \le n+1$, and constants independent of n such that*

$$\int_0^1 s^k |f^{(k)}(s)|^2 \, ds \le M_k \frac{1}{n}\sum_{i=1}^{n-k} \frac{\Gamma(i+k)}{n^k\Gamma(i)} |n^k \Delta^k f_i|^2 \tag{75}$$

for $1 \le k \le 4$.

Proof. The result of Kunkle [Ku] is that for any integer N one can find a constant C such that for any such data, there is an interpolating functions such that for $0 \le k \le N$ we have $|f^{(k)}(s)| \le C|n^k \Delta^k f_j$ if $\frac{j-1}{n} \le s \le \frac{j+N-1}{n}$. So we just integrate both sides over intervals $[\frac{i}{n}, \frac{i+1}{n}]$ and sum. □

Our final lemma is a weighted version of the Rellich theorem.

Lemma 7.5. *Let $N_k[0,1]$ be the completion of the space of smooth functions from $[0,1]$ to $\mathbb{R}^m$, in the norm given by*

$$\|f\|_{N_k}^2 = \sum_{j=0}^{k} \int_0^1 s^j \left|\tfrac{d^j f}{ds^j}\right|^2 ds. \tag{76}$$

Then N_{k+1} is compactly embedded in N_k for each nonnegative integer k.

Proof. By (35), we know that there is a constant λ_k such that

$$\|f\|_{N_k}^2 \le \lambda_k \sum_{j=0}^{k+1} \int_0^1 s^{j+1} \left|\tfrac{d^j f}{ds^j}\right|^2 ds.$$

Now we use a trick from [OK]. For any $n \in \mathbb{N}$, the operator

$$I_n(f)(x) = \begin{cases} f(x) & x \ge \frac{1}{n} \\ 0 & x < \frac{1}{n} \end{cases}$$

is certainly compact as an operator from W to V, by Rellich's lemma on $[\frac{1}{n}, 1]$ (in ordinary Sobolev spaces). So to prove $I\colon N_{k+1} \to N_k$ is compact, we just need to show I_n converges in norm to I, since a limit of compact operators is compact. Now

$$\begin{aligned}\|I_n(f) - If\|_{N_k}^2 &= \sum_{j=0}^{k} \int_0^{1/n} s^j \left|\tfrac{d^j f}{ds^j}\right|^2 ds \\ &\le \lambda_k \sum_{j=0}^{k+1} \int_0^{1/n} s^{j+1} \left|\tfrac{d^j f}{ds^j}\right|^2 ds \le \frac{\lambda_k}{n} \|f\|_{N_{k+1}}^2,\end{aligned}$$

so that $I_n \to I$ in $L(W, V)$. □

Theorem 7.6. *Given initial conditions γ and w as in Lemma 7.3, there is a $T > 0$ such that there is a unique solution η of the system* (57) *in $L^\infty([0,T], N_3[0,1]) \cap W^{1,\infty}([0,T], N_2[0,1])$.*

Proof. For each fixed t and each $n \in \mathbb{N}$, construct the piecewise interpolating polynomial as in Lemma 7.4, and call it $\tilde{\eta}_n(t)$. Then by Lemma 7.3 we get a uniform bound on on $E_3(t)$ in some short time interval $[0,T]$; in other words, the family $\tilde{\eta}_n$ is bounded in $L^\infty([0,T], N_4[0,1]) \cap W^{1,\infty}([0,T], N_3[0,1])$.

By the compactness Lemma 7.5, there is a subsequence $\tilde{\eta}_{n_k}$ which converges strongly in $L^\infty([0,T], N_3[0,1]) \cap W^{1,\infty}([0,T], N_2[0,1])$. For any $\varepsilon > 0$ the convergence is in $H^3[\varepsilon, 1]$, and thus by the usual Sobolev embedding theorem also in $C^2[\varepsilon, 1]$. So we can take the limit of the system (64) pointwise to see that we have a solution of (57).

Uniqueness follows from the energy conservation result $\int_0^1 |\eta_t|^2\, ds = \text{constant}$, the derivation of which is valid since $\eta_t \in N_2$. □

Combining Theorem 7.6 and Corollary 6.7, we have the following Corollary.

Corollary 7.7. *Suppose γ and w are C^∞ initial conditions for* (57) *which are odd through $s = 1$, i.e., all even derivatives are zero at $s = 1$. Then* (57) *has a C^∞ solution on some time interval $[0,T]$, and $T < \infty$ if and only if $\int_0^T E_3(t)\, dt = \infty$.*

8. Future research

In this paper we considered the whip with one fixed and one free end as boundary conditions. The other possibilities are to have two free ends, to have two fixed ends, and to have a periodic loop. All of the estimates in this paper have analogues in those cases. When there are two free ends, the tension must satisfy $\sigma(0) = 0$ and $\sigma(1) = 0$, so the appropriate weighted norms look like $\int_0^1 s^k(1-s)^k |f^{(k)}(s)|^2\,ds$; we would expect to need at least as many derivatives as in the one-fixed, one-free case and possibly more to get the estimates to close up. When there are two fixed ends, or when the whip is periodic, the problem becomes simpler since we can use ordinary Sobolev spaces for the estimates, and one could probably get estimates in lower-order norms.

The addition of gravity brings some complications. One is that the boundary conditions change, and oddness through the fixed point is no longer enough to satisfy the conditions automatically. The other is that, as mentioned, if the whip is above the fixed point, the tension may become negative. In that case the evolution equation becomes elliptic.

The blowup criterion $\int_0^T E_3(t)\,dt$ can certainly be improved; once we know a solution exists, we can use alternative methods to get better a priori bounds on it. Thess et al. have speculated that blowup for the periodic loop might be controlled by the L^∞ norms of $|\eta_{ss}|$ and $|\eta_{st}|$, analogous to the way blowup for the ideal Euler equations is controlled by the L^∞ norm of vorticity. This is an interesting problem to study, since we have a much greater handle on all aspects of this one-dimensional problem. We will explore this in a future paper.

In addition, the geometry of the space of inextensible curves is interesting in its own right. Although the geometric objects are not continuous in the Sobolev topology (unlike on the group of volumorphisms), the curvature formulas still make sense, and one can compute that all sectional curvatures are nonnegative. We can thus study stability of the motion from the geometric point of view (as in [AK]), as well as the geometry of blowup. See [P] for details on this.

Finally, the technique of approximating a continuous system with a discrete system preserving the geometry may be interesting to apply to fluids directly. For example, in two dimensions we could consider a rectangular grid on a torus, the vertices of which are free to move as long as all quadrilateral areas are preserved. Although such a model may not have global existence (as edges of a quadrilateral may collapse to give a triangle without changing the area), we might still get some useful insight out of it.

References

AS. T.J. Allen and J.R. Schmidt, *Vibrational modes of a rotating string,* Can. J. Phys. **76** no. 12, 965–975 (1998).

A. S.S. Antman, *Nonlinear problems of elasticity,* Springer, 1995.

AK. V. Arnold and B. Khesin, *Topological methods in Hydrodynamics,* Springer, 1998.

BT. A. Burchard and L.E. Thomas, *On the Cauchy problem for a dynamical Euler's elastica,* Comm. Partial Differential Equations **28** no. 1 and 2, 271–300 (2003).

CalMar. M.G. Calkin and R.H. March, *The dynamics of a falling chain: I,* Am. J. Phys. **57** 154–157 (1989).

Cal. M.G. Calkin, *The dynamics of a falling chain: II,* Am. J. Phys. **57** 157–159 (1989).

CapMaz. G. Capriz and G. Mazzini, *An apparent paradox in the mechanics of strings,* Meccanica **28** no. 2, 91–95 (1993).

CH. R. Courant and D. Hilbert, *Methods of mathematical physics, volume 1,* Wiley-Interscience, New York, 1953.

dSR. C.A. de Sousa and V.H. Rodrigues, *Mass redistribution in variable mass systems,* Eur. J. Phys. **25** 41–49 (2004).

D1. R.W. Dickey, *Dynamic behavior of the inextensible string,* Quart. Appl. Math. **62** part 1, 135–161 (2004).

D2. R.W. Dickey, *Bifurcation problems in nonlinear elasticity,* Pitman, New York (1977).

E. D.G. Ebin, *The motion of slightly compressible fluids viewed as a motion with strong constraining force,* Ann. Math. **105** no. 1, 141–200 (1977).

EM. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. **92** 102–163 (1970).

GM. A. Goriely and T. McMillen, *Shape of a cracking whip*, Phys. Rev. Lett. **88** no. 24 (2002).

H. G. Hamel, *Theoretische Mechanik, 2nd ed.,* Springer-Verlag, Berlin, 1949.

HHR. W.A. Heywood, H. Hurwitz, Jr., and D.Z. Ryan, *Whip effect in a falling chain*, Am. J. Phys., **23**, no. 5, (1955).

IH. H. Irschik and H.J. Holl, *The equations of Lagrange written for a non-material volume,* Acta Mech. **153** 231–248 (2002).

J. H.L. Johnson, *The existence of a periodic solution of a vibrating hanging string,* SIAM J. Appl. Math. **16** no. 5, 1048–1058 (1968).

Ko. I.I. Kolodner, *Heavy rotating string — a nonlinear eigenvalue problem,* Comm. Pure Appl. Math. **8** no. 3, 395–408 (1955).

Ku. T. Kunkle, *Lagrange interpolation on a lattice: bounding derivatives by divided differences,* J. Approx. Theory, **71** no. 1, 94–103 (1992).

KP. A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type,* World Scientific, Hackensack, NJ, 2003.

L. O.A. Ladyzhenskaya, *The boundary value problems of mathematical physics,* Springer-Verlag, New York, 1985.

LP. C.D. Luning and W.L. Perry, *Iterative solutions of a non-linear boundary value problem for a rotating string,* Internat. J. Non-Linear Mech. **19** no. 1, 83–92 (1984).

M. T. McMillen, *On the falling (or not) of the folded inextensible string,* unpublished, accessed via http://math.fullerton.edu/tmcmillen/

MG. T. McMillen and A. Goriely, *Whip waves,* Phys. D **184** 192–225 (2003).

OK. B. Opic and A. Kufner, *Remark on compactness of imbeddings in weighted spaces,* Math. Nachr. **133** 63–69 (1987).

OV. O.M. O'Reilly and P. Varadi, *A treatment of shocks in one-dimensional thermomechanical media,* Continuum Mech. Thermodyn. **11** 339–352 (1999).

P. S.C. Preston, *The geometry of whips and chains,* to appear.

Re1. M. Reeken, *The equation of motion of a chain,* Math. Z. **155** no. 3, 219–237 (1977).

Re2. M. Reeken, *Classical solutions of the chain equation I,* Math. Z. **165** 143–169 (1979).

Re3. M. Reeken, *Classical solutions of the chain equation II,* Math. Z. **166** 67–82 (1979).

Ros. R.M. Rosenberg, *Analytic mechanics of discrete systems,* Plenum Press, New York, 1977.

SB. M. Schagerl and A. Berger, *Propagation of small waves in inextensible strings,* Wave Motion **35** 339–353 (2002).

SSST. M. Schagerl, A. Steindl, W. Steiner, and H. Troger, *On the paradox of the free falling folded chain,* Acta Mechanica **125** 155–168 (1997).

Se. D. Serre, *Un modèle relaxé pour les câbles inextensibles,* RAIRO, Modélisation Math. Anal. Numér. **25**, no. 4, 465–481 (1991).

ST. W. Steiner and H. Troger, *On the equations of motion of the folded inextensible string,* Z. Angew. Math. Phys. **46** no. 6, 960–970 (1995).

TZN. A. Thess, O. Zikanov, and A. Nepomnyashchy, *Finite-time singularity in the vortex dynamics of a string,* Phys. Rev. E **59** no. 3 (1999).

TP. W. Tomaszewski and P. Pieranski, *Dynamics of ropes and chains: I. the fall of the folded chain,* New J. Phys. **7**, no. 45 (2005).

TPG. W. Tomaszewski, P. Pieranski, and J.-C. Geminard, *The motion of a freely falling chain tip,* Am. J. Phys. **74** no. 9, 776–783 (2006).

Tr. C. Truesdell, *The rational mechanics of flexible or elastic bodies, 1638–1788*, Leonhardi Euleri Opera Omnia Ser. II, vol. XI, Orell Füssli, Zürich, 1960.

WY. C.W. Wong and K. Yasui, *Falling chains,* Am. J. Phys. **74** no. 6, 490–496 (2006).

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