DEFINITE INTEGRALS, RIEMANN SUMS, AND AREA UNDER A CURVE: WHAT IS NECESSARY AND SUFFICIENT?

Vicki Sealey
Arizona State University
vicki@mathpost.asu.edu

A teaching experiment was conducted in a calculus class to determine what it means to understand definite integrals. One interesting result was based on students' use of area under a curve as a tool for computing definite integrals. Results show that in the problems presented in this study, students' use of area under a curve was helpful in problem solving only when a deeper understanding of the structure behind the definite integral was present.

The purpose of this research was to examine student understanding of Riemann sums and definite integrals. These concepts are imperative for students to understand for three main reasons. First, many real world applications involve functions that do not have an antiderivative that can be expressed in terms of elementary functions. For example, the antiderivative of the function $f(x) = e^{x^2}$ cannot be expressed in terms of elementary functions. Thus, the Fundamental Theorem of Calculus could not be applied, and other methods for evaluating the definite integral, such as Riemann sums would be needed.

This leads to the second reason that students need to have an understanding of the structure of Riemann sums. While Riemann sums may not be the most efficient method for approximating a definite integral, other methods, such as the trapezoid rule, midpoint rule, or Simpson's method are based on the structure of the Riemann sum. Thus, an understanding of the structure of Riemann sums will help students to understand these other methods as well.

Finally, I hypothesize that an understanding of Riemann sums is needed even when a function has an antiderivative that *can* be expressed in terms of elementary functions. Setting up the appropriate definite integral requires the student to know what to integrate, and an understanding of the structure of the Riemann sum will give the student the tools he/she needs. In all cases, it is possible to imagine the definite integral being represented by the area under a curve. This research begins to examine what is necessary for students to be able to use area under a curve as a powerful tool for solving problems that involve definite integrals.

Background

There are several pieces of literature that focus on mathematical topics that build the definite integral: $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. Multiplication, rate of change, sequences and series, limits, and functions are all incorporated into the definite integral, and several research studies have been done to understand these topics. In addition, there are two pieces of literature that focus on the concept of integration. Orton (1983) mainly focuses on methods of evaluating definite integrals. As is common in many calculus classes, many of the definite integral problems in Orton's study involve finding the area under the curve. He discusses the structural and calculational/executive errors that students made when finding the area under the curve in several situations. Artigue (1991) discussed Orton's studies of calculus students' understanding of differentiation and integration. The study found that many students could perform routine procedures for finding

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

the area under a curve, but the students rarely could explain their procedures, and some even admitted that they "didn't really know why they were doing it" (Artigue, 1991). Orton's study does not attend to understanding *why* area under a curve is equal to the definite integral of a function. My research will provide data that shows that when solving real world problems, students need to understand why this relationship between area and the definite integral holds. "For students to see "area under a curve" as representing a quantity other than area [i.e. velocity], it is imperative that they understand how the quantities being accumulated are created" (P. Thompson & Silverman, 2006).

Thompson's (1994) study focuses on student understanding of the Fundamental Theorem of Calculus. As part of his teaching experiment, he developed and implemented a module to help students understand Riemann sums in a way that develops the Fundamental Theorem. Thompson noted a distinction between accumula*tion* and accumulat*ing*, and stressed the idea of quantities accumulat*ing* for his work. Thompson also describes a younger student, Sue, who is able to construct a Riemann sum to approximate distance traveled. Sue was also able to explain that she could get better approximations by using smaller time intervals.

For Orton's study, students were given a definite integral and asked to evaluate it or were given a graph and asked to determine the function's definite integral by finding the area under the curve. In Thompson's study, students were asked to examine aspects of the definite integral that related to the Fundamental Theorem of Calculus. My study focuses on a different aspect of the definite integral. Specifically, my students were asked to look at problems where they either needed to set up a definite integral or use Riemann sums to approximate a total accumulation

Theoretical Perspective

The theoretical perspective that I have used to analyze the data is taken from the work of Piaget (1970, 1975). The basic idea is a type of constructivism with the premise that we construct not at free will, but within certain constraints. The system in which we construct is subject to certain laws, specifically reversibility, wholeness, transformation, and self-regulation (Piaget, 1970).

Possibly the most important aspect of Piaget's constructivism, structuralism, is the concept of reflective abstraction. Abstraction "in the ordinary sense of the word" refers to something being "′drawn out' from things which have that property" (Piaget, 1970). For example, a child learns what "red" is by seeing lots of objects that are red. The child may be shown a red ball, a red crayon, a red shirt, and a red block and the child eventually learns the meaning of "red". Reflective abstraction is a type of abstraction that comes from "acting on things" and ways in which we coordinate actions. Reflective abstraction deals with the elements *and* the operations we perform on them. Specifically with the definite integral, students cannot understand definite integrals simply by looking at a lot of them. Instead, the students need to *do* something with the components of definite integrals to be able to reflectively abstract and understand the structure of the definite integral.

Piaget's structuralism is a response to both Platonic and atomistic views. Within the Platonic view, knowledge is something that already exists (Piaget, 1975), and learners simply "remember" or "acquire" the information. Per Sfard's (1998) acquisition metaphor, students do not actively participate in developing the structure, but acquire the knowledge of the structure instead. A common example is based on the Gestalt perspective. When we see a person, we do not need to look at the eyes, and then the ears, and then nose, and so on before we can recognize the person. Instead, we can simply look at the face as a whole to identify the person. The idea of ungenerated wholes is central to the Gestalt perspective.

One portion of the definite integral that could be considered a Gestalt aspect is seeing the definite integral as the area under a curve, without constructing it from the structure of the limit of Riemann sums. Thus, the definite integral would not be a well-developed object, but instead would be only a pseudo-object (Sfard, 1991). This will be discussed in more detail in the data analysis section. Viewing the definite integral as the area under a curve is certainly something that we want our students to be able to do, but we also want them to be able to generate the structure of a Riemann sum in order to have a better conceptual understanding of the definite integral.

On the opposite side of the spectrum is the atomistic view that sees only a collection of individual elements which Piaget calls aggregates (1970). Within this view, the student does not see any of the relationships between the individual elements, nor does the student consider any of the operations that are performed on the elements. An example of this from Riemann sums could be a student looking at a collection of rectangles, without considering the area of the rectangles, or an example could be a student who can see the area of n rectangles, but cannot imagine the number of rectangles increasing infinitely to form the definite integral.

Clearly, there are ideas about the definite integral that fit into the Gestalt view or the atomistic view, but neither of these views allows us to have a fully developed concept of all that is involved in conceptualizing a definite integral. Piaget claims that structuralism is the solution (1970). Within structuralism, Piaget claims that students do construct knowledge, but the construction of knowledge takes place within a system that has its own laws (Piaget, 1970). A structure consists not only of elements or aggregates, but the structure also consists of the operations on these elements and the relationships between these elements. Specifically, structures are self-regulating and are subject to the laws of reversibility, transformation, and wholeness.

Methods/Subjects

This research was designed using a teaching experiment methodology (Simon, 1995). Thus, a hypothetical learning trajectory was created. Participants in the study were students in a calculus workshop in which the author was one of two research assistants. Students enrolled in the calculus workshop were concurrently enrolled in a traditional first semester college calculus class, and students generally reported one of two reasons for enrolling in the extra workshop. Approximately half of the students claimed they were not good at math and wanted extra help with their calculus class, and the other half reported that they loved math, and simply wanted to take another math class.

Students were videotaped as they worked in groups on activities relating to definite integrals (see Table 1), although the phrases “definite integral” and “Riemann sum” were not used by the instructors until after the activities were completed, or until the students introduced the terms themselves. All students were very familiar with Oehrtman’s (2004) approximation framework and had worked in several contexts where they were required to find approximations, both overestimates and underestimates, determine a bound for their error, and find approximations that were accurate to within a predetermined bound, epsilon. The problem solving sessions extended over two and a half one-hour class periods.

Water Problem Group 1A	A uniform pressure *P* applied across a surface area *A* creates a total force of *F=PA*. The density of water is 62 lb per cubic foot, so that under water the pressure varies according to depth, *d*, as *P*=62*d*. a) Draw and label a large picture of a dam 100 feet wide and extending 50 feet under water. b) Approximate the total force of the water exerted on this dam. c) Find an approximation accurate to within 1000 pounds. d) Write a formula indicating how to find an approximation with any pre-determined accuracy, ε.
Spring Problem Group 1B	For a constant force F to move an object a distance d requires an amount of energy equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance *x* from its resting length is equal to $F = kx$, where *k* is a constant that depends on the particular spring. a) Draw and label a large picture of a spring initially displaced 5 cm from its natural length then stretched to a displacement of 10 cm. b) Approximate the energy required to do this if the spring constant is *k* = .155 N/cm. c) Find an approximation accurate to within 1000 ergs (1 erg = 10^{-5} N·cm). d) Write a formula indicating how to find an approximation with any pre-determined accuracy, ε.

Table 1

Data Analysis

Near the beginning of the problem solving session, both groups attempted to set up a definite integral, but were unable to do so. When the students in group 1A initially wanted to use an integral, they were discouraged to do so by the professor. Instead they were asked to use the ideas of the approximation framework. Thus, they approximated the force using the average pressure on the dam and the entire area. After computing this approximation, the students decided to set up an integral to check their approximation. However, they were unsure of how to set up the appropriate integral. At first they set up the integral $\int_0^{50} 5000 \cdot 62d$. They did not include the *dd* (or *"dx"*), and did not discuss this. Much more importantly, they use 5000 as the area, which is the area of the entire dam. Instead, they needed to have the area of one strip with a width of 100 and an infinitely small height.

1A Student A: *[writes* $\int_0^{50}$ *]. Let's do this real quick. So P times A is just…it's just this [points to* $A_{contact} = 100 \cdot 50 = 5000\,ft^2$ *on whiteboard] times 62d, right? A constant…times d?*

1A Student B: *62 d A, isn't it?*

1A Student A: *But A is…oh yeah.*

1A Student B: *Oh yeah.*

1A Student A: *Isn't A constant?*

1A Student B: *A is constant. So it's 5000 times 62 d.*

1A Student A: *[continues writing. Now has* $\int_0^{50} 5000 \cdot 62d$ *]*

Since they had difficulty constructing an appropriate integral, it seems likely that they chose this method because the problem was similar to those they solved in their calculus classes using integrals, and not because they understood (at that point) the structure of accumulation and definite integrals. These students recognized that the solution included a product of two terms, but one of the terms in their product was incorrect. The students compared their answer with the approximation they found earlier using average force and realized that one of the two solutions must have been incorrect.

1A Student B: *Ok, that's nothing close to what—*
1A Student A: *we have written down. That's sad.*
1A Student B: *That's the answer.*
1A Student A: *Maybe we did it wrong. Maybe we set up the integral wrong*
1A Student B: *yeah, that's a possibility*
1A Student A: *Maybe the area. Cause we're (inaudible) it at each level [draws a thin horizontal strip on whiteboard] and the area eventually goes to zero.*
1A Student B: *It's very upsetting that we're wrong*
1A Student A: *(laughs) It's very sad.*

Notice in the above excerpt, student A begins to discuss a necessary part of the Riemann sum, but at this point, her thoughts are not well developed enough, indicating gaps in her understanding of the structure of the definite integral. Eventually, the group computed an approximation based on the structure of a Riemann sum, allowing the students to approach the problem from a more conceptual basis, describing the underlying product structure of the definite integral to determine the appropriate integrand. The important thing to note here is that the students could only set up the integral *after* they had explored the problem using Riemann sums, and developed an understanding of the underlying structure. The following paragraphs discuss their actions when computing an approximation based on the structure of a Riemann sum.

1A Student B: *couldn't you do a summation?*
1A Student A: *yeah, we could do a summation of them.*
1A Student B: *Like do like 10 intervals and do a summation of them.*

Next, the students determined which terms to use for the product. Also, note that the idea of breaking the dam into pieces and then adding the force on each piece does not seem to be a conceptual obstacle for the students in any way. The students computed two approximations using 50 subintervals. One approximation was an overestimate (using the pressure at the bottom of each slice) and the other approximation was an underestimate (using the pressure at the top of each slice). The students recognized that both an overestimate and an underestimate would be needed in order to be able to bound the error. Next, the students needed to find a way to make their approximation more accurate and quickly decided to use more subintervals on the dam, making their Δd smaller. Although the students did not mention the word limit, the students discussed that their approximation could be accurate to within any predetermined accuracy if they used small enough intervals.

Group 1B also attempted to initially set up an integral to solve the problem, but they were also unsuccessful. In this case, the students were unsure if they function they should integrate should be the formula for force, $F = kx$, or the formula for energy, $E = Fd$.

1B Student A: *That's what it's supposed to be [pointing to E = Fd]. Or is it this one? [pointing to F = kx] Is it this one [pointing to energy formula], or this one [pointing to force formula]?*
1B Student B: *it's of energy*
1B Student A: *It's of energy, so it's the integral of force times distance.*
RA 2: *But you're not using integrals.*
1B Student A: *yeah, I know. I'm just trying to remember which one that it is that you use.*
1B Student B: *You're trying to find energy.*
1B Student A: *But in the equation…*
1B Student C: *You need the force.*

Student B thought the formula should be the one for energy, but student C thought it should be force. Although student C's method would have led to a correct answer, he was never able to justify why this was the correct method. Instead, he often said, "that's just what it is".

The primary difference between groups 1A and 1B was the aspect of the problem on which each group focused. Group 1A focused mainly on the problem within its context (water pressure on a dam), while group 1B rarely discussed their context (force of a spring). Instead, pilot group 1B drew a graph and talked mainly of area under the curve. When they abandoned their efforts to set up an integral, they moved on to area. The students graphed the force, $F = .155x$ and discussed the energy in terms of area under this curve.

1B Student C: *That's what we were figuring out, it's force with respect to distance and the area under this is energy.*

When I asked the students to explain to me *why* area under the curve was equal to energy, they could not explain, and were never confident that they were correct in graphing force, instead of energy. Their only justification was that they had gotten confirmation from one of the research assistants that this was an acceptable method. When I pushed them to explain *why* this was an appropriate method, they were unable to do so.

I hypothesize that one of the reasons the students struggled with explaining area is because they did not understand the structure of the Riemann sum. Several times throughout the video, the students in this group incorrectly said that the "summation of forces equals energy". It is not just the summation of *forces* that equals energy, but it is the summation of the *products* of force and distance that equals energy. The students were attending to the summation layer of the definite integral, but did not include the product layer. The following excerpt indicates that the statement made by the students is not a case of metonymy, but in fact a conceptual error. The excerpt below also illustrates Oehrtman's (2002) collapse metaphor. The student seems to be visualizing one dimensional lines as the area under the curve, instead of two dimensional rectangles.

1B Student E: *That the summation of the forces equal energy.*
1B Student B: *There, the answer is true.*
1B Student E: *That's how I found that. I don't know if it was right or not. All I know is that's what I found.*
RA 2: *Is it the sum of just the force?*
1B Student E: *Yeah, 'cause if you sum all the forces up underneath the graph… like the integral… you should get the energy.*

As discussed earlier, group 1A focused on the context of the problem (water pressure on a dam) and did not graph a function or consider area under a curve. One reason for this may be the nature of their activity. The first question asked the students to draw and label a large picture of a dam. This picture seems to be more helpful for reasoning about the problem than in the context of the spring. To solve the problem, the students broke the dam into horizontal slices, and calculated the area of each section and an approximate pressure on each section. The picture of the dam is a nice representation of the area of each strip. Also, since the pressure depends on depth, the picture was helpful in determining the pressure on each strip.

Conclusion

Although the title "Riemann sums" was not stressed by the professor or any of the research assistants in the class, the students in group 1A seemed to have a good understanding of the concepts involved. The understanding of Riemann sums held by group 1B is questionable. Since they only referred to their problem in terms of area under a curve, it is unclear of their knowledge of Riemann sums. Of course, area under a curve and Riemann sums are mathematically equivalent, but it is appears that the students in pilot group 1B had only a pseudo-structural understanding. They may be proficient in dealing with area under a curve, but may not be able to solve other accumulation problems without thinking about area under a curve, or may not be able to relate the area under a curve to the structure of a Riemann sum.

In particular, context 1A (the water problem) proves much more difficult when trying to use area under a curve as a tool for solving the problem. The formulas shown in the description of the problem are $P=62d$ and $F=PA$, but the function that would need to be graphed in order to apply area under a curve is $f(d) = 62 \cdot 100d$. In contrast, the formula that would need to be graphed for problem 1B (the spring problem) in order to correctly use area under a curve as a tool is $F = kx$, which is the formula for force that is given in the statement of the problem.

I do not in any way claim that area under a curve is a bad representation or that it should not be taught. Instead, I claim that area under a curve is not *sufficient* for understanding the definite integral. It can be a powerful tool when the underlying structure of the definite integral is present, but the above example of group 1B illustrates what can happen when this structure is missing.

References

Artigue, M. (1991). Analysis. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 167-198). Boston: Kluwer.

Oehrtman, M. (2002). *Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning.* Unpublished Doctoral Dissertation, The University of Texas, Austin.

Oehrtman, M. (2004). *Approximation as a foundation for understanding limit concepts.* Paper presented at the PMENA.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14*(1), 1-18.

Piaget, J. (1970). *Structuralism*. New York: Basic Books, Inc.

Piaget, J. (1975). *The equilibration of cognitive structures* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*, 4-13.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *26*(2), 114-145.

Thompson, P., & Silverman, J. (2006). The concept of accumulation in calculus (pp. 18): Arizona State University.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.