

MORPHISMS OF SCHEMES

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1. Introduction

In this chapter we introduce some types of morphisms of schemes. A basic reference is [DG67].

2. Closed immersions

In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \rightarrow X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X , (b) $i^\# : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\#$ is locally generated by sections, see Schemes, Definitions 10.2 and 4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

Lemma 2.1. *Let $i : Z \rightarrow X$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism i is a closed immersion.*
- (2) *For every affine open $\text{Spec}(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \text{Spec}(R/I)$ as schemes over $U = \text{Spec}(R)$.*
- (3) *There exists an affine open covering $X = \bigcup_{j \in J} U_j$, $U_j = \text{Spec}(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = \text{Spec}(R_j/I_j)$ as schemes over $U_j = \text{Spec}(R_j)$.*
- (4) *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\# : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective.*
- (5) *The morphism i induces a homeomorphism of Z with a closed subset of X , the map $i^\# : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective, and the kernel $\text{Ker}(i^\#) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*
- (6) *The morphism i induces a homeomorphism of Z with a closed subset of X , the map $i^\# : \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z$ is surjective, and the kernel $\text{Ker}(i^\#) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 4.1 and 10.2. So (6) \Leftrightarrow (1). We have (1) \Rightarrow (2) by Schemes, Lemma 10.1. Trivially (2) \Rightarrow (3).

Assume (3). Each of the morphisms $\text{Spec}(R_j/I_j) \rightarrow \text{Spec}(R_j)$ is a closed immersion, see Schemes, Example 8.1. Hence $i^{-1}(U_j) \rightarrow U_j$ is a homeomorphism onto its image and $i^\#|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\#$ is surjective since this may be checked locally. We conclude that (3) \Rightarrow (4).

The implication (4) \Rightarrow (1) is Schemes, Lemma 24.2. The implication (5) \Rightarrow (6) is trivial. And the implication (6) \Rightarrow (5) follows from Schemes, Lemma 10.1. \square

Lemma 2.2. *Let X be a scheme. Suppose $i : Z \rightarrow X$ and $i' : Z' \rightarrow X$ are closed immersions corresponding to the quasi-coherent ideal sheaves $\mathcal{I} = \text{Ker}(i^\#)$ and $\mathcal{I}' = \text{Ker}(i'^\#)$ of \mathcal{O}_X .*

- (1) *The morphism $i : Z \rightarrow X$ factors as $Z \rightarrow Z' \rightarrow X$ for some $a : Z \rightarrow Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.*
- (2) *We have $Z \cong Z'$ as schemes over X if and only if $\mathcal{I} = \mathcal{I}'$.*

Proof. This follows from our discussion of closed subspaces in Schemes, Section 4 especially Schemes, Lemma 4.6. It also follows in a straightforward way from characterization (3) in Lemma 2.1 above. \square

Lemma 2.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:*

- (1) *The sheaf of ideals \mathcal{I} is locally generated by sections as a sheaf of \mathcal{O}_X modules.*
- (2) *The sheaf of ideals \mathcal{I} is quasi-coherent as a sheaf of \mathcal{O}_X -modules.*
- (3) *There exists a closed immersion $i : Z \rightarrow X$ whose corresponding sheaf of ideals $\text{Ker}(i^\#)$ is equal to \mathcal{I} .*

Proof. In Schemes, Section 4 we constructed the closed subspace associated to a sheaf of ideals locally generated by sections. This closed subspace is a scheme by Schemes, Lemma 10.1. Hence we see that (1) \Rightarrow (3) by our definition of a closed immersion of schemes. By Lemma 2.1 above we see that (3) \Rightarrow (2). And of course (2) \Rightarrow (1). \square

Lemma 2.4. *The base change of a closed immersion is a closed immersion.*

Proof. See Schemes, Lemma 18.2. \square

Lemma 2.5. *A composition of closed immersions is a closed immersion.*

Proof. We have seen this in Schemes, Lemma 24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 2.1. Since if $I \subset R$ is an ideal, and $\bar{J} \subset R/I$ is an ideal, then $\bar{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\bar{J} = R/J$. \square

Lemma 2.6. *A closed immersion is quasi-compact.*

Proof. This lemma is a duplicate of Schemes, Lemma 19.5. \square

Lemma 2.7. *A closed immersion is separated.*

Proof. This lemma is a special case of Schemes, Lemma 23.7. \square

3. Immersions

In this section we collect some facts on immersions.

Lemma 3.1. *Let $Z \rightarrow Y \rightarrow X$ be morphisms of schemes.*

- (1) *If $Z \rightarrow X$ is an immersion, then $Z \rightarrow Y$ is an immersion.*
- (2) *If $Z \rightarrow X$ is a quasi-compact immersion and $Z \rightarrow Y$ is quasi-separated, then $Z \rightarrow Y$ is a quasi-compact immersion.*
- (3) *If $Z \rightarrow X$ is a closed immersion and $Y \rightarrow X$ is separated, then $Z \rightarrow Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{array}{ccccc}
 Z & \longrightarrow & Y \times_X Z & \longrightarrow & Z \\
 & \searrow & \downarrow & & \downarrow \\
 & & Y & \longrightarrow & X
 \end{array}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \rightarrow Z$, whence an immersion by Schemes, Lemma 21.12. The arrow $Y \times_X Z \rightarrow Y$ is a base change of $Z \rightarrow X$ hence an immersion (Schemes, Lemma 18.2). Finally, a composition of immersions is an immersion (Schemes, Lemma 24.3). This proves (1). The other two results are proved in exactly the same manner. \square

Lemma 3.2. *Let $h : Z \rightarrow X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \rightarrow \overline{Z}$ an open immersion and $i : \overline{Z} \rightarrow X$ a closed immersion.*

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 23.7). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of \mathcal{O}_X -modules by Schemes, Lemma 24.1. This implies that $\mathcal{I} = \text{Ker}(\mathcal{O}_X \rightarrow h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to \mathcal{I} , see Lemma 2.3. By Schemes, Lemma 4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \rightarrow X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U . Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \rightarrow U$. Hence we see that $Z = \overline{Z} \cap U$. \square

Lemma 3.3. *Let $h : Z \rightarrow X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \rightarrow \overline{Z}$ an open immersion and $i : \overline{Z} \rightarrow X$ a closed immersion.*

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 12.5. By Schemes, Lemma 12.6 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \rightarrow X$ the inclusion morphism and $j : Z \rightarrow \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U . Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. \square

Example 3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X =$

$\text{Spec}(k[x_1, x_2, x_3, \dots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \rightarrow X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \dots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \dots) \subset k[x_1, x_2, x_3, \dots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \dots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals \tilde{I}_n on $D(x_n)$ agree on $D(x_n x_m)$, namely $\tilde{I}_n|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to \mathcal{I} . Thus $Z \rightarrow X$ is an immersion.

We claim that we cannot factor $Z \rightarrow X$ as $Z \rightarrow \bar{Z} \rightarrow X$, where $\bar{Z} \rightarrow X$ is closed and $Z \rightarrow \bar{Z}$ is open. Namely, \bar{Z} would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \dots][1/x_n]$ such that $I_n = Ik[x_1, x_2, x_3, \dots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \dots][1/x_n]$ which ends up in all I_n is 0! Hence I does not exist.

4. Closed immersions and quasi-coherent sheaves

The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 6.1 does for abelian sheaves. See also the discussion in Modules, Section 13.

Lemma 4.1. *Let $i : Z \rightarrow X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z . The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent \mathcal{O}_X -modules \mathcal{G} such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 2.6 and 2.7. Hence Schemes, Lemma 24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X .

By Modules, Lemma 13.4 the functor i_* is fully faithful.

Now we turn to the description of the essential image of the functor i_* . It is clear that $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent \mathcal{O}_Z -module, for example by our local description above. Next, suppose that \mathcal{G} is any quasi-coherent \mathcal{O}_X -module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_* i^* \mathcal{G}$$

is an isomorphism. By exactly the same arguments as above we see that it suffices to prove the following algebraic statement: Given a ring R , an ideal I and an R -module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R -modules. Proof of this easy algebra fact is omitted. \square

Let $i : Z \rightarrow X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote \mathcal{F} the sheaf $i_*\mathcal{F}$ on X .

Lemma 4.2. *Let X be a scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $\mathcal{G} \subset \mathcal{F}$ be a \mathcal{O}_X -submodule. There exists a unique quasi-coherent \mathcal{O}_X -submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent \mathcal{O}_X -module \mathcal{H} the map*

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular \mathcal{G}' is the largest quasi-coherent \mathcal{O}_X -submodule of \mathcal{F} contained in \mathcal{G} .

Proof. Let \mathcal{G}_a , $a \in A$ be the set of quasi-coherent \mathcal{O}_X -submodules contained in \mathcal{G} . Then the image \mathcal{G}' of

$$\bigoplus_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 24. The module \mathcal{G}' is contained in \mathcal{G} . Hence this is the largest quasi-coherent \mathcal{O}_X -module contained in \mathcal{G} .

To prove the formula, let \mathcal{H} be a quasi-coherent \mathcal{O}_X -module and let $\alpha : \mathcal{H} \rightarrow \mathcal{G}$ be an \mathcal{O}_X -module map. The image of the composition $\mathcal{H} \rightarrow \mathcal{G} \rightarrow \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in \mathcal{G}' . Hence α factors through \mathcal{G}' as desired. \square

Lemma 4.3. *Let $i : Z \rightarrow X$ be a closed immersion of schemes. There is a functor¹ $i^! : QCoh(\mathcal{O}_X) \rightarrow QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_* . (Compare Modules, Lemma 6.3.)*

Proof. Given quasi-coherent \mathcal{O}_X -module \mathcal{G} we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of \mathcal{G} of local sections annihilated by \mathcal{I} . By Lemma 4.2 there is a canonical largest quasi-coherent \mathcal{O}_X -submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent \mathcal{O}_Z -module \mathcal{F} . Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. \square

5. Supports of modules

In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 5. On the other hand, the support of a module was defined in Algebra, Section 61. These match.

Lemma 5.1. *Let X be a scheme. Let \mathcal{F} be a quasi-coherent sheaf on X . Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

- (1) \mathfrak{p} is in the support of M , and
- (2) x is in the support of \mathcal{F} .

Proof. This follows from the equality $\mathcal{F}_x = M_{\mathfrak{p}}$, see Schemes, Lemma 5.4 and the definitions. \square

Lemma 5.2. *Let X be a scheme. Let \mathcal{F} be a quasi-coherent sheaf on X . The support of \mathcal{F} is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module \mathcal{F}_x . Hence the complement of $\mathrm{Supp}(\mathcal{F})$ is closed under generalization. \square

¹This is likely nonstandard notation.

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

Lemma 5.3. *Let \mathcal{F} be a finite type quasi-coherent module on a scheme X . Then*

- (1) *The support of \mathcal{F} is closed.*
- (2) *For $x \in X$ we have*

$$x \in \text{Supp}(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$

- (3) *For any morphism of schemes $f : Y \rightarrow X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $\text{Supp}(f^*\mathcal{F}) = f^{-1}(\text{Supp}(\mathcal{F}))$.*

Proof. Part (1) is a reformulation of Modules, Lemma 9.6. You can also combine Lemma 5.1, Properties, Lemma 16.1, and Algebra, Lemma 39.5 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 19.1. Let $f : Y \rightarrow X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$

see Sheaves, Lemma 26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \text{Supp}(\mathcal{F})$ if and only if $y \in \text{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). \square

Lemma 5.4. *Let \mathcal{F} be a finite type quasi-coherent module on a scheme X . There exists a smallest closed subscheme $i : Z \rightarrow X$ such that there exists a quasi-coherent \mathcal{O}_Z -module \mathcal{G} with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

- (1) *If $\text{Spec}(A) \subset X$ is any affine open, and $\mathcal{F}|_{\text{Spec}(A)} = \widetilde{M}$ then $Z \cap \text{Spec}(A) = \text{Spec}(A/I)$ where $I = \text{Ann}_A(M)$.*
- (2) *The quasi-coherent sheaf \mathcal{G} is unique up to unique isomorphism.*
- (3) *The quasi-coherent sheaf \mathcal{G} is of finite type.*
- (4) *The support of \mathcal{G} and of \mathcal{F} is Z .*

Proof. Suppose that $i' : Z' \rightarrow X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Lemma 4.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf \mathcal{G}' on Z' . Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 16.1 and Algebra, Lemma 5.5 it follows that \mathcal{G}' is of finite type. We have $\text{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 39.5. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Lemma 2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A -module, then $\text{Ann}_A(M)_f = \text{Ann}_{A_f}(M_f)$. We omit the proof. \square

Definition 5.5. Let X be a scheme. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. The *scheme theoretic support* of \mathcal{F} is the closed subscheme $Z \subset X$ constructed in Lemma 5.4.

In this situation we often think of \mathcal{F} as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 4.1).

6. Scheme theoretic image

Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

Lemma 6.1. *Let $f : X \rightarrow Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X)$. If \mathcal{I} is quasi-coherent then we just take Z to be the closed subscheme determined by \mathcal{I} , see Lemma 2.3. This works by Schemes, Lemma 4.6. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals \mathcal{I}' contained in \mathcal{I} . This follows from Lemma 4.2. \square

Definition 6.2. Let $f : X \rightarrow Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 6.1 above.

We often just denote $f : X \rightarrow Z$ the factorization of f . If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y . Namely, if f is the immersion $Z \rightarrow X$ of Example 3.4 above then the scheme theoretic image of $Z \rightarrow X$ is X . But clearly the scheme theoretic image of $Z = Z \cap U \rightarrow U$ is just Z .

Lemma 6.3. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f . If f is quasi-compact then*

- (1) *the sheaf of ideals $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X)$ is quasi-coherent,*
- (2) *the scheme theoretic image Z is the closed subscheme determined by \mathcal{I} ,*
- (3) *for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is equal to $Z \cap U$, and*
- (4) *the image $f(X) \subset Z$ is a dense subset of Z , in other words the morphism $X \rightarrow Z$ is dominant (see Definition 8.1).*

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of \mathcal{I} commutes with restriction to open subschemes of Y . And if (1) holds then in the proof of Lemma 6.1 we showed (2). Thus it suffices to prove that \mathcal{I} is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1, \dots, n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow \mathcal{O}_{X'})$. By Schemes, Lemma 24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y . Hence we win. \square

Example 6.4. If $A \rightarrow B$ is a ring map with kernel I , then the scheme theoretic image of $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is the closed subscheme $\text{Spec}(A/I)$ of $\text{Spec}(A)$. This follows from Lemma 6.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

Lemma 6.5. *Let $f : X \rightarrow Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f . Let $z \in Z$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \searrow \\ \mathrm{Spec}(A) & \longrightarrow & Z \longrightarrow Y \\ & & \downarrow \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z . In particular any point of Z is the specialization of a point of $f(X)$.

Proof. Let $z \in \mathrm{Spec}(R) = V \subset Y$ be an affine open neighbourhood of z . By Lemma 6.3 we have $Z \cap V$ is the scheme theoretic closure of $f^{-1}(V) \rightarrow V$, and hence we may replace Y by V and assume $Y = \mathrm{Spec}(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \dots \cup U_n$ is a finite affine open covering. Write $U_i = \mathrm{Spec}(A_i)$. Let $I = \mathrm{Ker}(R \rightarrow A_1 \times \dots \times A_n)$. By Lemma 6.3 again we see that Z corresponds to the closed subscheme $\mathrm{Spec}(R/I)$ of Y . If $\mathfrak{p} \subset R$ is the prime corresponding to z , then we see that $I_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 9.12) we see that $R_{\mathfrak{p}} \rightarrow (A_1)_{\mathfrak{p}} \times \dots \times (A_n)_{\mathfrak{p}}$ is not zero. Hence one of the rings $(A_i)_{\mathfrak{p}}$ is not zero. Hence there exists an i and a prime $\mathfrak{q}_i \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. By Algebra, Lemma 48.2 we can choose a valuation ring $A \subset K = f.f.(A_i/\mathfrak{q}_i)$ dominating the local ring $R_{\mathfrak{p}}/\mathfrak{p}_1 R_{\mathfrak{p}} \subset f.f.(A_i/\mathfrak{q}_i)$. This gives the desired diagram. Some details omitted. \square

Lemma 6.6. *Let $f_1 : X \rightarrow Y_1$ and $Y_1 \rightarrow Y_2$ be morphisms of schemes. Let $f_2 : X \rightarrow Y_2$ be the composition. Let $Z_i \subset Y_i$, $i = 1, 2$ be the scheme theoretic image of f_i . Then the morphism $Y_1 \rightarrow Y_2$ induces a morphism $Z_1 \rightarrow Z_2$ and a commutative diagram*

$$\begin{array}{ccccc} X & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ & \searrow & \downarrow & & \downarrow \\ & & Z_2 & \longrightarrow & Y_2 \end{array}$$

Proof. See Schemes, Lemma 4.6. \square

Lemma 6.7. *Let $f : X \rightarrow Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced induced scheme structure on $\overline{f(X)}$.*

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 12.6. \square

7. Scheme theoretic closure and density

We take the following definition from [DG67, IV, Definition 11.10.2].

Definition 7.1. Let X be a scheme. Let $U \subset X$ be an open subscheme.

- (1) The scheme theoretic image of the morphism $U \rightarrow X$ is called the *scheme theoretic closure of U in X* .
- (2) We say U is *scheme theoretically dense in X* if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V .

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X , see Example 7.2. This is somewhat inelegant; but see Lemmas 7.3 and 7.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U \cap V)$ is injective, see Lemma 7.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

Example 7.2. Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \dots]/(x^n z_n)$. Set $I = (z_1, z_2, \dots) \subset A$. Consider the affine scheme $X = \text{Spec}(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \rightarrow \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X . Consider the morphism $X \rightarrow \text{Spec}(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X .

Lemma 7.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \rightarrow X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X .*

Proof. Follows from Lemma 6.3 part (3). \square

Example 7.4. Let A be a ring and $X = \text{Spec}(A)$. Let $f_1, \dots, f_n \in A$ and let $U = D(f_1) \cup \dots \cup D(f_n)$. Let $I = \text{Ker}(A \rightarrow \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $\text{Spec}(A/I)$ of X . Note that $U \rightarrow X$ is quasi-compact. Hence by Lemma 7.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

Lemma 7.5. *Let $j : U \rightarrow X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X . Hence the scheme theoretic closure of $U \cap V$ in V is equal to V , see proof of Lemma 6.1. Conversely, suppose that the scheme theoretic closure of $U \cap V$ is equal to V for all opens V . Suppose that $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $\text{Spec}(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $\text{Spec}(A/(f))$ a contradiction. \square

Lemma 7.6. *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X , then so is $U \cap V$.*

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \rightarrow \mathcal{O}_X(W \cap V) \rightarrow \mathcal{O}_X(W \cap V \cap U)$. By Lemma 7.5 both maps are injective. Hence the composite is injective. Hence by Lemma 7.5 $U \cap V$ is scheme theoretically dense in X . \square

Lemma 7.7. *Let $h : Z \rightarrow X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\bar{Z} \subset X$ be the scheme theoretic image of h . Then the morphism $Z \rightarrow \bar{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of \bar{Z} . Moreover, Z is topologically dense in \bar{Z} .*

Proof. By Lemma 3.2 or Lemma 3.3 we can factor $Z \rightarrow X$ as $Z \rightarrow \bar{Z}_1 \rightarrow X$ with $Z \rightarrow \bar{Z}_1$ open and $\bar{Z}_1 \rightarrow X$ closed. On the other hand, let $Z \rightarrow \bar{Z} \subset X$ be the scheme theoretic closure of $Z \rightarrow X$. We conclude that $\bar{Z} \subset \bar{Z}_1$. Since Z

is an open subscheme of \overline{Z}_1 it follows that Z is an open subscheme of \overline{Z} as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of \overline{Z}_1 in the proof of Lemma 3.3. Hence \overline{Z}_1 and \overline{Z} have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \rightarrow X$ is quasi-compact we argue as follows: The assertion that Z is scheme theoretically dense in \overline{Z} follows from Lemma 6.3 part (3). The last assertion follows from Lemma 6.3 part (4). \square

Lemma 7.8. *Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent*

- (1) U is topologically dense in X ,
- (2) the scheme theoretic closure of U in X is X , and
- (3) U is scheme theoretically dense in X .

Proof. This follows from Lemma 7.7 and the fact that the a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. \square

Lemma 7.9. *Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent*

- (1) the scheme theoretic closure of U in X is X , and
- (2) U is scheme theoretically dense in X .

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 7.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 6.7. \square

Lemma 7.10. *Let S be a scheme. Let X, Y be schemes over S . Let $f, g : X \rightarrow Y$ be morphisms of schemes over S . Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \rightarrow S$ is separated, then $f = g$.*

Proof. Follows from the definitions and Schemes, Lemma 21.5. \square

8. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 8.1. A morphism $f : X \rightarrow S$ of schemes is called *dominant* if the image of f is a dense subset of S .

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \dots$ is a countable collection of elements of k , then the morphism

$$\coprod_{i=1,2,\dots} \text{Spec}(k) \longrightarrow \text{Spec}(k[x])$$

with i th factor mapping to the point $x = \lambda_i$ is dominant.

Lemma 8.2. *Let $f : X \rightarrow S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f , then f is dominant.*

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 11.1, and that every point of S is contained in an irreducible component of S , see Topology, Lemma 7.3. \square

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

Lemma 8.3. *Let $f : X \rightarrow S$ be a quasi-compact morphism of schemes. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f .*

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \dots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod_{i=1, \dots, n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V . Also, f is dominant if and only if f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 29.6. \square

Here is a slightly more useful variant of the lemma above.

Lemma 8.4. *Let $f : X \rightarrow S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S . If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.*

Proof. Let $Z \subset S$ be the scheme theoretic image of f . We have to show that $\eta \notin Z$. This follows from Lemma 6.5 but can also be seen as follows. By Lemma 6.3 the morphism $X \rightarrow Z$ is dominant, which by Lemma 8.3 means all the generic points of all irreducible components of Z are in the image of $X \rightarrow Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z . \square

There is another case where dominant is the same as having all generic points of irreducible components in the image.

Lemma 8.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f . If so, then S has finitely many irreducible components as well.*

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \dots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S . Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f . The lemma follows. \square

9. Birational morphisms

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 9.3. Here is the formal definition.

Definition 9.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \rightarrow Y$ is *birational* if

- (1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y , and
- (2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y, f(\eta)} \rightarrow \mathcal{O}_{X, \eta}$ is an isomorphism.

Lemma 9.2. Let $f : X \rightarrow Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.

Proof. Follows immediately from the definitions. \square

Example 9.3. Here is an example of a birational morphism which is not an isomorphism over any open of the target. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}] / ((x - \alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \rightarrow A$ setting all y_α equal to zero. Both the morphism $\text{Spec}(A) \rightarrow \text{Spec}(B)$ and the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ are birational but not an isomorphism over any open.

10. Rational maps

Let X be a scheme. Note that if U, V are dense open in X , then so is $U \cap V$.

Definition 10.1. Let X, Y be schemes.

- (1) Let $f : U \rightarrow Y, g : V \rightarrow Y$ be morphisms of schemes defined on dense open subsets U, V of X . We say that f is *equivalent* to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X .
- (2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
- (3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an *S -rational map from X to Y* if there exists a representative $f : U \rightarrow Y$ of the equivalence class which is an S -morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent.

Definition 10.2. Let X be a scheme. A *rational function on X* is a rational map from X to $\mathbf{A}_{\mathbf{Z}}^1$.

See Constructions, Definition 5.1 for the definition of the affine line \mathbf{A}^1 . Let X be a scheme over S . For any open $U \subset X$ a morphism $U \rightarrow \mathbf{A}_{\mathbf{Z}}^1$ is the same as a morphism $U \rightarrow \mathbf{A}_S^1$ over S . Hence a rational function is also the same as a S -rational map from X into \mathbf{A}_S^1 .

Recall that we have the canonical identification $\text{Mor}(T, \mathbf{A}_{\mathbf{Z}}^1) = \Gamma(T, \mathcal{O}_T)$ for any scheme T , see Schemes, Example 15.2. Hence $\mathbf{A}_{\mathbf{Z}}^1$ is a ring-object in the category

of schemes. More precisely, the morphisms

$$\begin{aligned} + : \mathbf{A}_{\mathbf{Z}}^1 \times \mathbf{A}_{\mathbf{Z}}^1 &\longrightarrow \mathbf{A}_{\mathbf{Z}}^1 \\ (f, g) &\longmapsto f + g \\ * : \mathbf{A}_{\mathbf{Z}}^1 \times \mathbf{A}_{\mathbf{Z}}^1 &\longrightarrow \mathbf{A}_{\mathbf{Z}}^1 \\ (f, g) &\longmapsto fg \end{aligned}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}_{\mathbf{Z}}^1$ has a natural ring structure.

Definition 10.3. Let X be a scheme. The *ring of rational functions on X* is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

Lemma 10.4. *Let X be an irreducible scheme. Let $\eta \in X$ be the generic point of X . There is a canonical identification $R(X) \cong \mathcal{O}_{X,\eta}$. If X is integral then $R(X) = \kappa(\eta) = \mathcal{O}_{X,\eta}$ is a field.*

Proof. Omitted. □

Definition 10.5. Let X be an integral scheme. The *function field*, or the *field of rational functions* of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$. We can use the notion of the function field to elucidate the separation condition on an integral scheme. Note that by Lemma 10.4 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

Lemma 10.6. *Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X . Let η_i be the generic point of Z_i . If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x , there exists a function regular in a neighborhood of x which is not in \mathcal{O}_{X,η_2} .*

Proof. First observe that under the assumption of X being separated, there is a unique map of schemes $\text{Spec}(\mathcal{O}_{X,\eta_2}) \rightarrow X$ over X such that the composition

$$\text{Spec}(R(X)) \longrightarrow \text{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $\text{Spec}(R(X)) \rightarrow X$. Namely, there is the canonical map $can : \text{Spec}(\mathcal{O}_{X,\eta_2}) \rightarrow X$, see Schemes, Equation (13.1.1). Given a second morphism a to X , we have that a agrees with can on the generic point of $\text{Spec}(\mathcal{O}_{X,\eta_2})$ by assumption. Now being X being separated guarantees that the subset in $\text{Spec}(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 21.5. Hence $a = can$ on all of $\text{Spec}(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$\text{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow \text{Spec}(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to can . This implies that η_2 specializes to η_1 (see Schemes, Lemma 13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

Definition 10.7. Let φ be a rational map between two schemes X and Y . We say φ is *defined in a point* $x \in X$ if there exists a representative (U, f) of φ with $x \in U$. The *domain of definition* of φ is the set of all points where φ is defined.

With this definition it isn't true in general that φ has a representative which is defined on all of the domain of definition.

Lemma 10.8. *Let X and Y be schemes. Assume X reduced and Y separated. Let φ be a rational map from X to Y with domain of definition $U \subset X$. Then there exists a unique morphism $f : U \rightarrow Y$ representing φ . If X and Y are schemes over a separated scheme S and if φ is an S -rational map, then f is a morphism over S .*

Proof. Let (V, g) and (V', g') be representatives of φ . Then g, g' agree on a dense open subscheme $W \subset V \cap V'$. On the other hand, the equalizer E of $g|_{V \cap V'}$ and $g'|_{V \cap V'}$ is a closed subscheme of $V \cap V'$ (Schemes, Lemma 21.5). Now $W \subset E$ implies that $E = V \cap V'$ set theoretically. As $V \cap V'$ is reduced we conclude $E = V \cap V'$ scheme theoretically, i.e., $g|_{V \cap V'} = g'|_{V \cap V'}$. It follows that we can glue the representatives $g : V \rightarrow Y$ of φ to a morphism $f : U \rightarrow Y$, see Schemes, Lemma 14.1. We omit the proof of the final statement. \square

In general it does not make sense to compose rational maps. The reason is that the image of a representative of the first rational map may have empty intersection with the domain of definition of the second. However, if we assume that our schemes are irreducible and we look at dominant rational maps, then we can compose rational maps.

Definition 10.9. Let X and Y be irreducible schemes. A rational map from X to Y is called *dominant* if any representative $f : U \rightarrow Y$ is a dominant morphism of schemes.

By Lemma 8.5 it is equivalent to require that the generic point $\eta \in X$ maps to the generic point ξ of Y , i.e., $f(\eta) = \xi$ for any representative $f : U \rightarrow Y$. We can compose a dominant rational map φ between irreducible schemes X and Y with an arbitrary rational map ψ from Y to Z . Namely, choose representatives $f : U \rightarrow Y$ with $U \subset X$ open dense and $g : V \rightarrow Z$ with $V \subset Y$ open dense. Then $W = f^{-1}(V) \subset X$ is open nonempty (because it contains the generic point of X) and we let $\psi \circ \varphi$ be the equivalence class of $g \circ f|_W : W \rightarrow Z$. We omit the verification that this is well defined.

In this way we obtain a category whose objects are irreducible schemes and whose morphisms are dominant rational maps. Given a base scheme S we can similarly define a category whose objects are irreducible schemes over S and whose morphisms are dominant S -rational maps.

Definition 10.10. Let X and Y be irreducible schemes.

- (1) We say X and Y are *birational* if X and Y are isomorphic in the category of irreducible schemes and dominant rational maps.
- (2) Assume X and Y are schemes over a base scheme S . We say X and Y are *S -birational* if X and Y are isomorphic in the category of irreducible schemes over S and dominant S -rational maps.

If X and Y are birational irreducible schemes, then the set of rational maps from X to Z is bijective with the set of rational map from Y to Z for all schemes

Z (functorially in Z). For “general” irreducible schemes this is just one possible definition. Another would be to require X and Y have isomorphic nonempty opens, or we could ask that X and Y have isomorphic rings of rational functions. For varieties all three conditions are equivalent (see insert future reference here).

Remark 10.11. There is a variant of Definition 10.1 where we consider only those morphism $U \rightarrow Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 7.6 to see that we obtain an equivalence relation. An equivalence class of these is called a *pseudo-morphism from X to Y* . If X is reduced the two notions coincide.

11. Surjective morphisms

Definition 11.1. A morphism of schemes is said to be *surjective* if it is surjective on underlying topological spaces.

Lemma 11.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

Lemma 11.3. *Let X and Y be schemes over a base scheme S . Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S .*

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 17.5. □

Lemma 11.4. *The base change of a surjective morphism is surjective.*

Proof. Let $f : X \rightarrow Y$ be a morphism of schemes over a base scheme S . If $S' \rightarrow S$ is a morphism of schemes, let $p : X_{S'} \rightarrow X$ and $q : Y_{S'} \rightarrow Y$ be the canonical projections. The commutative square

$$\begin{array}{ccc} X_{S'} & \xrightarrow{p} & X \\ f_{S'} \downarrow & & \downarrow f \\ Y_{S'} & \xrightarrow{q} & Y \end{array}$$

identifies $X_{S'}$ as a fibre product of $X \rightarrow Y$ and $Y_{S'} \rightarrow Y$. Let Z be a subset of the underlying topological space of X . Then $q^{-1}(f(Z)) = f_{S'}^{-1}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 11.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \rightarrow Y_{S'}$. □

Example 11.5. Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $\text{Spec}(\mathbf{C}) \rightarrow \text{Spec}(\mathbf{R})$. The base change $\text{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \rightarrow \text{Spec}(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$ (the decomposition comes from the idempotent $\frac{1 \otimes 1 + i \otimes i}{2}$) and hence $\text{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ has two points.

Lemma 11.6. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow p & \swarrow q \\ & & Z \end{array}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 11.7 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. \square

12. Radicial and universally injective morphisms

In this section we define what it means for a morphism of schemes to be *radicial* and what it means for a morphism of schemes to be *universally injective*. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

Definition 12.1. Let $f : X \rightarrow S$ be a morphism.

- (1) We say that f is *universally injective* if and only if for any morphism of schemes $S' \rightarrow S$ the base change $f' : X_{S'} \rightarrow S'$ is injective (on underlying topological spaces).
- (2) We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x) \supset \kappa(f(x))$ is purely inseparable.

Lemma 12.2. Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:

- (1) For every field K the induced map $\text{Mor}(\text{Spec}(K), X) \rightarrow \text{Mor}(\text{Spec}(K), S)$ is injective.
- (2) The morphism f is universally injective.
- (3) The morphism f is radicial.
- (4) The diagonal morphism $\Delta_{X/S} : X \rightarrow X \times_S X$ is surjective.

Proof. Let K be a field, and let $s : \text{Spec}(K) \rightarrow S$ be a morphism. Giving a morphism $x : \text{Spec}(K) \rightarrow X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = \text{Spec}(K) \times_S X \rightarrow \text{Spec}(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K . Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{array}{ccc} K & \longleftarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \kappa(x') & \longleftarrow & \kappa(s') \end{array}$$

of fields. By Schemes, Lemma 13.3 we get two morphisms $a, a' : \text{Spec}(K) \rightarrow X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \rightarrow X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(s) \subset \kappa(x)$ is not purely inseparable, then we may find a field extension $\kappa(s) \subset K$ such that $\kappa(x)$ has two $\kappa(s)$ -homomorphisms into K . By Schemes, Lemma 13.3 this implies that the map

$\text{Mor}(\text{Spec}(K), X) \rightarrow \text{Mor}(\text{Spec}(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radical, i.e., they imply (3).

Assume (3). By Schemes, Lemma 13.3 a morphism $\text{Spec}(K) \rightarrow X$ is given by a pair $(x, \kappa(x) \rightarrow K)$. Property (3) says exactly that associating to the pair $(x, \kappa(x) \rightarrow K)$ the pair $(s, \kappa(s) \rightarrow \kappa(x) \rightarrow K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \rightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one such prime ideal \mathfrak{p} , which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radical. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \rightarrow S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. \square

Lemma 12.3. *A universally injective morphism is separated.*

Proof. Combine Lemma 12.2 with the remark that $X \rightarrow S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 21.3 and the discussion following it. \square

Lemma 12.4. *A base change of a universally injective morphism is universally injective.*

Proof. This is formal. \square

Lemma 12.5. *A composition of radical morphisms is radical, and so the same holds for the equivalent condition of being universally injective.*

Proof. Omitted. \square

13. Affine morphisms

Definition 13.1. A morphism of schemes $f : X \rightarrow S$ is called *affine* if the inverse image of every affine open of S is an affine open of X .

Lemma 13.2. *An affine morphism is separated and quasi-compact.*

Proof. Let $f : X \rightarrow S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 19.2. We will show f is separated using Schemes, Lemma 21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s . By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. \square

Lemma 13.3. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is affine.*
- (2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.*

- (3) *There exists a quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{A} and an isomorphism $X \cong \underline{\text{Spec}}_S(\mathcal{A})$ of schemes over S . See Constructions, Section 4 for notation.*

Moreover, in this case $X = \underline{\text{Spec}}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 19.2 we see that f is quasi-compact. By Schemes, Lemma 21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of \mathcal{O}_S -algebras. Thus we have the scheme $g : Y = \underline{\text{Spec}}_S(\mathcal{A}) \rightarrow S$ over S . The identity map $\text{id} : \mathcal{A} = f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $\text{can} : X \rightarrow Y$ over S , see Constructions, Lemma 4.7. By assumption and the lemma just cited the restriction $\text{can}|_{f^{-1}(W_j)} : f^{-1}(W_j) \rightarrow g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. \square

Remark 13.4. We can also argue directly that (2) implies (1) in Lemma 13.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $\text{Spec}(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 6.4 there is a canonical morphism $\psi : f^{-1}(V) \rightarrow \text{Spec}(A)$ over $\text{Spec}(R) = V$. By Schemes, Lemma 11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1, \dots, n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \dots, n\} \rightarrow J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 16.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because \mathcal{A} is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i} . It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $\text{Spec}(A_{h_i})$ of $\text{Spec}(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $\text{Spec}(A) = \bigcup \text{Spec}(A_{h_i})$ we win.

Lemma 13.5. *Let S be a scheme. There is an anti-equivalence of categories*

$$\begin{array}{ccc} \text{Schemes affine} & \longleftrightarrow & \text{quasi-coherent sheaves} \\ \text{over } S & & \text{of } \mathcal{O}_S\text{-algebras} \end{array}$$

which associates to $f : X \rightarrow S$ the sheaf $f_\mathcal{O}_X$. Moreover, this equivalence is compatible with arbitrary base change.*

Proof. The functor from right to left is given by $\underline{\text{Spec}}_S$. The two functors are mutually inverse by Lemma 13.3 and Constructions, Lemma 4.6 part (3). The final statement is Constructions, Lemma 4.6 part (2). \square

Lemma 13.6. *Let $f : X \rightarrow S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{ \begin{array}{c} \text{category of quasi-coherent} \\ \mathcal{O}_X\text{-modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{category of quasi-coherent} \\ \mathcal{A}\text{-modules} \end{array} \right\}$$

Moreover, an \mathcal{A} -module is quasi-coherent as an \mathcal{O}_S -module if and only if it is quasi-coherent as an \mathcal{A} -module.

Proof. Omitted. □

Lemma 13.7. *The composition of affine morphisms is affine.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g . Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f . Hence $(g \circ f)^{-1}(U)$ is affine. □

Lemma 13.8. *The base change of an affine morphism is affine.*

Proof. Let $f : X \rightarrow S$ be an affine morphism. Let $S' \rightarrow S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \rightarrow S'$ the base change of f . For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 13.3 above. □

Lemma 13.9. *A closed immersion is affine.*

Proof. The first indication of this is Schemes, Lemma 8.2. See Schemes, Lemma 10.1 for a complete statement. □

Lemma 13.10. *Let X be a scheme. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \rightarrow X$ is affine.*

Proof. This follows from Properties, Lemma 24.4 and the definition. □

Lemma 13.11. *Suppose $g : X \rightarrow Y$ is a morphism of schemes over S .*

- (1) *If X is affine over S and $\Delta : Y \rightarrow Y \times_S Y$ is affine, then g is affine.*
- (2) *If X is affine over S and Y is separated over S , then g is affine.*
- (3) *A morphism from an affine scheme to a scheme with affine diagonal is affine.*
- (4) *A morphism from an affine scheme to a separated scheme is affine.*

Proof. Proof of (1). The base change $X \times_S Y \rightarrow Y$ is affine by Lemma 13.8. The morphism $(1, g) : X \rightarrow X \times_S Y$ is the base change of $Y \rightarrow Y \times_S Y$ by the morphism $X \times_S Y \rightarrow Y \times_S Y$. Hence it is affine by Lemma 13.8. The composition of affine morphisms is affine (see Lemma 13.7) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 13.9) and Y/S separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

Lemma 13.12. *A morphism between affine schemes is affine.*

Proof. Immediate from Lemma 13.11 with $S = \text{Spec}(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 13.3. □

Lemma 13.13. *Let S be a scheme. Let A be an Artinian ring. Any morphism $\text{Spec}(A) \rightarrow S$ is affine.*

Proof. Omitted. □

14. Quasi-affine morphisms

Recall that a scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 15.1.

Definition 14.1. A morphism of schemes $f : X \rightarrow S$ is called *quasi-affine* if the inverse image of every affine open of S is a quasi-affine scheme.

Lemma 14.2. *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \rightarrow S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 19.2. We will show f is separated using Schemes, Lemma 21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s . By assumption $f^{-1}(W)$ is isomorphic to an open subscheme of an affine scheme, say $f^{-1}(W) \rightarrow Y$ is such an open immersion. Choose affine open neighbourhoods $x_1 \in U \subset f^{-1}(W)$ and $x_2 \in V \subset f^{-1}(W)$. We may think of U and V as open subschemes of Y and hence we see that $U \cap V$ is affine and that $\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \rightarrow \mathcal{O}(U \cap V)$ is surjective (by the lemma cited above applied to U, V in Y). Hence by the lemma cited we conclude that f is separated. \square

Lemma 14.3. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is quasi-affine.*
- (2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.*
- (3) *There exists a quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{A} and a quasi-compact open immersion*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \text{Spec}_S(\mathcal{A}) \\ & \searrow & \swarrow \\ & S & \end{array}$$

over S .

- (4) *Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 4.7.*

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 19.2 we see that f is quasi-compact. By Schemes, Lemma 21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of \mathcal{O}_S -algebras. Thus we have the scheme $g : Y = \text{Spec}_S(\mathcal{A}) \rightarrow S$ over S . The identity map $\text{id} : \mathcal{A} = f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $\text{can} : X \rightarrow Y$ over S , see Constructions, Lemma 4.7. By assumption, the lemma just cited, and Properties, Lemma 15.4 the restriction $\text{can}|_{f^{-1}(W_j)} : f^{-1}(W_j) \rightarrow g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude

that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). \square

Lemma 14.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g . Let $j : g^{-1}(U) \rightarrow V$ be a quasi-compact open immersion into an affine scheme V . By Lemma 14.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative spectrum $\underline{\text{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$ -algebras \mathcal{A} . By Schemes, Lemma 24.1 the sheaf $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf of \mathcal{O}_V -algebras with the property that $j^*\mathcal{A}' = \mathcal{A}$. Hence we get a commutative diagram

$$\begin{array}{ccccc} f^{-1}(g^{-1}(U)) & \longrightarrow & \underline{\text{Spec}}_{g^{-1}(U)}(\mathcal{A}) & \longrightarrow & \underline{\text{Spec}}_V(\mathcal{A}') \\ & & \downarrow & & \downarrow \\ & & g^{-1}(U) & \xrightarrow{j} & V \end{array}$$

with the square being a fibre square, see Constructions, Lemma 4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. \square

Lemma 14.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \rightarrow S$ be a quasi-affine morphism. By Lemma 14.3 above we can find a quasi-coherent sheaf of \mathcal{O}_S -algebras \mathcal{A} and a quasi-compact open immersion $X \rightarrow \underline{\text{Spec}}_S(\mathcal{A})$ over S . Let $g : S' \rightarrow S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \rightarrow S'$ the base change of f . Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \rightarrow \underline{\text{Spec}}_{S'}(g^*\mathcal{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 19.3 and 18.2 and Constructions, Lemma 4.6). By Lemma 14.3 again we conclude that $X_{S'} \rightarrow S'$ is quasi-affine. \square

Lemma 14.6. *A quasi-compact immersion is quasi-affine.*

Proof. Let $X \rightarrow S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 7.7 the morphism $X \rightarrow S$ factors as $X \rightarrow Z \rightarrow S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 2.1 implies Z is affine. Hence we win. \square

Lemma 14.7. *Let S be a scheme. Let X be an affine scheme. A morphism $f : X \rightarrow S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X , hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \rightarrow S$ where X is affine and S quasi-separated follows from Schemes, Lemma 21.15 applied to $X \rightarrow S \rightarrow \text{Spec}(\mathbf{Z})$. \square

Lemma 14.8. *Suppose $g : X \rightarrow Y$ is a morphism of schemes over S . If X is quasi-affine over S and Y is quasi-separated over S , then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \rightarrow Y$ is quasi-affine by Lemma 14.5. The morphism $X \rightarrow X \times_S Y$ is a quasi-compact immersion as $Y \rightarrow S$ is quasi-separated, see Schemes, Lemma 21.12. A quasi-compact immersion is quasi-affine by Lemma 14.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 14.4). Thus we win. \square

15. Types of morphisms defined by properties of ring maps

In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

Definition 15.1. Let P be a property of ring maps.

- (1) We say that P is *local* if the following hold:
 - (a) For any ring map $R \rightarrow A$, and any $f \in R$ we have $P(R \rightarrow A) \Rightarrow P(R_f \rightarrow A_f)$.
 - (b) For any rings R, A , any $f \in R, a \in A$, and any ring map $R_f \rightarrow A$ we have $P(R_f \rightarrow A) \Rightarrow P(R \rightarrow A_a)$.
 - (c) For any ring map $R \rightarrow A$, and $a_i \in A$ such that $(a_1, \dots, a_n) = A$ then $\forall i, P(R \rightarrow A_{a_i}) \Rightarrow P(R \rightarrow A)$.
- (2) We say that P is *stable under base change* if for any ring maps $R \rightarrow A, R \rightarrow R'$ we have $P(R \rightarrow A) \Rightarrow P(R' \rightarrow R' \otimes_R A)$.
- (3) We say that P is *stable under composition* if for any ring maps $A \rightarrow B, B \rightarrow C$ we have $P(A \rightarrow B) \wedge P(B \rightarrow C) \Rightarrow P(A \rightarrow C)$.

Definition 15.2. Let P be a property of ring maps. Let $f : X \rightarrow S$ be a morphism of schemes. We say f is *locally of type P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ has property P .

This is not a “good” definition unless the property P is a local property. Even if P is a local property we will not automatically use this definition to say that a morphism is “locally of type P ” unless we also explicitly state the definition elsewhere.

Lemma 15.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X , and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U))$ holds.*

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \rightarrow \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u , see Schemes, Lemma 11.5. By Definition 15.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \rightarrow \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u , see Schemes, Lemma 11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \rightarrow \mathcal{O}_X(U'_u))$ by Definition 15.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 15.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \dots, u_n \in U$ such that

$$U = U_{u_1} \cup \dots \cup U_{u_n}.$$

By Definition 15.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U))$ holds. \square

Lemma 15.4. *Let P be a local property of ring maps. Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is locally of type P .*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U))$.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is locally of type P .*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $P(\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i))$ holds, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of type P then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is locally of type P .

Proof. This follows from Lemma 15.3 above. \square

Lemma 15.5. *Let P be a property of ring maps. Assume P is local and stable under composition. The composition of morphisms locally of type P is locally of type P .*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms locally of type P . Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood $U \subset f^{-1}(V)$ of x . By Lemma 15.4 the ring maps $\mathcal{O}_Z(W) \rightarrow \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$ satisfy P . Hence $\mathcal{O}_Z(W) \rightarrow \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. \square

Lemma 15.6. *Let P be a property of ring maps. Assume P is local and stable under base change. The base change of a morphism locally of type P is locally of type P .*

Proof. Let $f : X \rightarrow S$ be a morphism locally of type P . Let $S' \rightarrow S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \rightarrow S'$ the base change of f . For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 15.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U_i)$ all satisfy P . By the material in Schemes, Section 17 we see that $f'^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \rightarrow \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. \square

Lemma 15.7. *The following properties of a ring map $R \rightarrow A$ are local.*

- (1) *(Isomorphism on local rings.) For every prime \mathfrak{q} of A lying over $\mathfrak{p} \subset R$ the ring map $R \rightarrow A$ induces an isomorphism $R_{\mathfrak{p}} \rightarrow A_{\mathfrak{q}}$.*
- (2) *(Open immersion.) For every prime \mathfrak{q} of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \rightarrow A$ induces an isomorphism $R_f \rightarrow A_f$.*
- (3) *(Reduced fibres.) For every prime \mathfrak{p} of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
- (4) *(Fibres of dimension at most n .) For every prime \mathfrak{p} of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n .*
- (5) *(Locally Noetherian on the target.) The ring map $R \rightarrow A$ has the property that A is Noetherian.*

(6) Add more here as needed².

Proof. Omitted. \square

Lemma 15.8. *The following properties of ring maps are stable under base change.*

- (1) *(Isomorphism on local rings.) For every prime \mathfrak{q} of A lying over $\mathfrak{p} \subset R$ the ring map $R \rightarrow A$ induces an isomorphism $R_{\mathfrak{p}} \rightarrow A_{\mathfrak{q}}$.*
- (2) *(Open immersion.) For every prime \mathfrak{q} of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \rightarrow A$ induces an isomorphism $R_f \rightarrow A_f$.*
- (3) *(Reduced fibres.) For every prime \mathfrak{p} of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
- (4) *(Fibres of dimension at most n .) For every prime \mathfrak{p} of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n .*
- (5) Add more here as needed³.

Proof. Omitted. \square

Lemma 15.9. *The following properties of ring maps are stable under composition.*

- (1) *(Isomorphism on local rings.) For every prime \mathfrak{q} of A lying over $\mathfrak{p} \subset R$ the ring map $R \rightarrow A$ induces an isomorphism $R_{\mathfrak{p}} \rightarrow A_{\mathfrak{q}}$.*
- (2) *(Open immersion.) For every prime \mathfrak{q} of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \rightarrow A$ induces an isomorphism $R_f \rightarrow A_f$.*
- (3) *(Locally Noetherian on the target.) The ring map $R \rightarrow A$ has the property that A is Noetherian.*
- (4) Add more here as needed⁴.

Proof. Omitted. \square

16. Morphisms of finite type

Recall that a ring map $R \rightarrow A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \dots, x_n]$ as an R -algebra, see Algebra, Definition 6.1.

Definition 16.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is of *finite type at $x \in X$* if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and an affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is of finite type.
- (2) We say that f is *locally of finite type* if it is of finite type at every point of X .
- (3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

Lemma 16.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is locally of finite type.*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is of finite type.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is locally of finite type.*

²But only those properties that are not already dealt with separately elsewhere.

³But only those properties that are not already dealt with separately elsewhere.

⁴But only those properties that are not already dealt with separately elsewhere.

- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite type then for any open subschemes $U \subset X, V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is locally of finite type.

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is of finite type” is local. We check conditions (a), (b) and (c) of Definition 15.1. By Algebra, Lemma 13.2 being of finite type is stable under base change and hence we conclude (a) holds. By the same lemma being of finite type is stable under composition and trivially for any ring R the ring map $R \rightarrow R_f$ is of finite type. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 23.3. \square

Lemma 16.3. *The composition of two morphisms which are locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 16.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 19.4 we see that the composition of morphisms of finite type is of finite type. \square

Lemma 16.4. *The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 16.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 13.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 19.3 we see that the base change of a morphism of finite type is a morphism of finite type. \square

Lemma 16.5. *A closed immersion is of finite type. An immersion is locally of finite type.*

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. \square

Lemma 16.6. *Let $f : X \rightarrow S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 30.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) \square

Lemma 16.7. *Let $f : X \rightarrow S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.*

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 5.4 and Lemma 16.6 above. Then apply Schemes, Lemma 21.14 to conclude that f is quasi-separated. \square

Lemma 16.8. *Let $X \rightarrow Y$ be a morphism of schemes over a base scheme S . If X is locally of finite type over S , then $X \rightarrow Y$ is locally of finite type.*

Proof. Via Lemma 16.2 this translates into the following algebra fact: Given ring maps $A \rightarrow B \rightarrow C$ such that $A \rightarrow C$ is of finite type, then $B \rightarrow C$ is of finite type. (See Algebra, Lemma 6.2). \square

17. Points of finite type and Jacobson schemes

Let S be a scheme. A finite type point s of S is a point such that the morphism $\text{Spec}(\kappa(s)) \rightarrow S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

Lemma 17.1. *Let S be a scheme. Let k be a field. Let $f : \text{Spec}(k) \rightarrow S$ be a morphism. The following are equivalent:*

- (1) *The morphism f is of finite type.*
- (2) *The morphism f is locally of finite type.*
- (3) *There exists an affine open $U = \text{Spec}(R)$ of S such that f corresponds to a finite ring map $R \rightarrow k$.*
- (4) *There exists an affine open $U = \text{Spec}(R)$ of S such that the image of f consists of a closed point u in U and the field extension $\kappa(u) \subset k$ is finite.*

Proof. The equivalence of (1) and (2) is obvious as $\text{Spec}(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $\text{Spec}(R) = U \subset S$ such that the image of f is contained in U , and the ring map $R \rightarrow k$ is of finite type. Let $\mathfrak{p} \subset R$ be the kernel. Then $R/\mathfrak{p} \subset k$ is of finite type. By Algebra, Lemma 33.2 there exist a $\bar{f} \in R/\mathfrak{p}$ such that $(R/\mathfrak{p})_{\bar{f}}$ is a field and $(R/\mathfrak{p})_{\bar{f}} \rightarrow k$ is a finite field extension. If $f \in R$ is a lift of \bar{f} , then we see that k is a finite R_f -module. Thus (2) \Rightarrow (3).

Suppose that $\text{Spec}(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \rightarrow k$. Then f is locally of finite type by Lemma 16.2. Thus (3) \Rightarrow (2).

Suppose $R \rightarrow k$ is finite. The image of $R \rightarrow k$ is a field over which k is finite by Algebra, Lemma 35.16. Hence the kernel of $R \rightarrow k$ is a maximal ideal. Thus (3) \Rightarrow (4).

The implication (4) \Rightarrow (3) is immediate. \square

Lemma 17.2. *Let S be a scheme. Let A be an Artinian local ring with residue field κ . Let $f : \text{Spec}(A) \rightarrow S$ be a morphism of schemes. Then f is of finite type if and only if the composition $\text{Spec}(\kappa) \rightarrow \text{Spec}(A) \rightarrow S$ is of finite type.*

Proof. Since the morphism $\text{Spec}(\kappa) \rightarrow \text{Spec}(A)$ is of finite type it is clear that if f is of finite type so is the composition $\text{Spec}(\kappa) \rightarrow S$ (see Lemma 16.3). For the converse, note that $\text{Spec}(A) \rightarrow S$ maps into some affine open $U = \text{Spec}(B)$ of S as $\text{Spec}(A)$ has only one point. To finish apply Algebra, Lemma 52.3 to $B \rightarrow A$. \square

Recall that given a point s of a scheme S there is a canonical morphism $\mathrm{Spec}(\kappa(s)) \rightarrow S$, see Schemes, Section 13.

Definition 17.3. Let S be a scheme. Let us say that a point s of S is a *finite type point* if the canonical morphism $\mathrm{Spec}(\kappa(s)) \rightarrow S$ is of finite type. We denote $S_{\mathrm{ft-pts}}$ the set of finite type points of S .

We can describe the set of finite type points as follows.

Lemma 17.4. *Let S be a scheme. We have*

$$S_{\mathrm{ft-pts}} = \bigcup_{U \subset S \text{ open}} U_0$$

where U_0 is the set of closed points of U . Here we may let U range over all opens or over all affine opens of S .

Proof. Immediate from Lemma 17.1. □

Lemma 17.5. *Let $f : T \rightarrow S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{\mathrm{ft-pts}}) \subset S_{\mathrm{ft-pts}}$.*

Proof. If T is the spectrum of a field this is Lemma 17.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 16.3). □

Lemma 17.6. *Let $f : T \rightarrow S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{\mathrm{ft-pts}}) = S_{\mathrm{ft-pts}}$.*

Proof. We have $f(T_{\mathrm{ft-pts}}) \subset S_{\mathrm{ft-pts}}$ by Lemma 17.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = \mathrm{Spec}(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 17.4. Now $T_s \rightarrow T$ is a morphism of finite type as a base change of $s \rightarrow S$ (Lemma 16.4). Hence the image of t in T is a finite type point by Lemma 17.5 which maps to s by construction. □

Lemma 17.7. *Let S be a scheme. For any locally closed subset $T \subset S$ we have*

$$T \neq \emptyset \Rightarrow T \cap S_{\mathrm{ft-pts}} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{\mathrm{ft-pts}}$ is dense in T .

Proof. Note that T carries a scheme structure (see Schemes, Lemma 12.4) such that $T \rightarrow S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 16.5. Hence by Lemma 17.5 we see $T_{\mathrm{ft-pts}} \subset S_{\mathrm{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 17.4. □

It follows that most of the material from Topology, Section 17 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

Lemma 17.8. *Let S be a scheme. The following are equivalent:*

- (1) *For every finite type morphism $f : \mathrm{Spec}(k) \rightarrow S$ with k a field the image consists of a closed point of S . In the terminology introduced above: finite type points of S are closed points of S .*
- (2) *For every locally finite type morphism $T \rightarrow S$ closed points map to closed points.*

- (3) For every locally finite type morphism $f : T \rightarrow S$ any closed point $t \in T$ maps to a closed point $s \in S$ and $\kappa(s) \subset \kappa(t)$ is finite.
- (4) The scheme S is Jacobson.

Proof. We have trivially (3) \Rightarrow (2) \Rightarrow (1). The discussion above shows that (1) implies (4). Hence it suffices to show that (4) implies (3). Suppose that $T \rightarrow S$ is locally of finite type. Choose $t \in T$ with $s = f(t)$ as in (3). Choose affine open neighbourhoods $\text{Spec}(R) = U \subset S$ of s and $\text{Spec}(A) = V \subset T$ of t with $f(V) \subset U$. The induced ring map $R \rightarrow A$ is of finite type (see Lemma 16.2) and R is Jacobson by Properties, Lemma 6.3. Thus the result follows from Algebra, Proposition 34.18. \square

Lemma 17.9. *Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.*

Proof. This is clear from Algebra, Proposition 34.18 (and Properties, Lemma 6.3 and Lemma 16.2). \square

Lemma 17.10. *The following types of schemes are Jacobson.*

- (1) Any scheme locally of finite type over a field.
- (2) Any scheme locally of finite type over \mathbf{Z} .
- (3) Any scheme locally of finite type over a 1-dimensional Noetherian domain with infinitely many primes.
- (4) A scheme of the form $\text{Spec}(R) \setminus \{\mathfrak{m}\}$ where (R, \mathfrak{m}) is a Noetherian local ring. Also any scheme locally of finite type over it.

Proof. We will use Lemma 17.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of \mathbf{Z} is Jacobson, see Algebra, Lemma 34.6. For (3) see Algebra, Lemma 60.4. For (4) see Properties, Lemma 6.4. \square

18. Universally catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \dots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 10.4. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 11.1.

Definition 18.1. Let S be a scheme. Assume S is locally Noetherian. We say S is *universally catenary* if for every morphism $X \rightarrow S$ locally of finite type the scheme X is catenary.

This is a “better” notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 15. Many schemes are universally catenary, see Lemma 18.4 below.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 101.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section

11. Recall that a ring A is called *universally catenary* if A is Noetherian and for every finite type ring map $A \rightarrow B$ the ring B is catenary. See Algebra, Definition 101.5. Many interesting rings which come up in algebraic geometry satisfy this property.

Lemma 18.2. *Let S be a locally Noetherian scheme. The following are equivalent*

- (1) S is universally catenary,
- (2) there exists an open covering of S all of whose members are universally catenary schemes,
- (3) for every affine open $\text{Spec}(R) = U \subset S$ the ring R is universally catenary, and
- (4) there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 16.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 16.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $\text{Spec}(R)$ is a universally catenary scheme, then every scheme $\text{Spec}(A)$ with A a finite type R -algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 101.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \rightarrow S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \rightarrow S$ maps into one of the U_i . By Lemma 16.2 the induced ring map $\mathcal{O}(U_i) \rightarrow \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 11.2. \square

Lemma 18.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

- (1) S is universally catenary, and
- (2) all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.

Proof. Assume that all local rings of S are universally catenary. Let $f : X \rightarrow S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversely, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 18.2. Say $S = \text{Spec}(R)$ and s corresponds to the prime ideal \mathfrak{p} . Any finite type $R_{\mathfrak{p}}$ -algebra A' is of the form $A_{\mathfrak{p}}$ for some finite type R -algebra A . By assumption (and Lemma 18.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_{\mathfrak{p}}$ is universally catenary. \square

Lemma 18.4. *The following types of schemes are universally catenary.*

- (1) Any scheme locally of finite type over a field.

- (2) Any scheme locally of finite type over a Cohen-Macaulay scheme.
- (3) Any scheme locally of finite type over \mathbf{Z} .
- (4) Any scheme locally of finite type over a 1-dimensional Noetherian domain.
- (5) And so on.

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 101.6. Also, use the last assertion of Lemma 18.2. Some details omitted. \square

19. Nagata schemes, reprise

See Properties, Section 13 for the definitions and basic properties of Nagata and universally Japanese schemes.

Lemma 19.1. *Let $f : X \rightarrow S$ be a morphism. If S is Nagata and f locally of finite type then X is Nagata. If S is universally Japanese and f locally of finite type then X is universally Japanese.*

Proof. For “universally Japanese” this follows from Algebra, Lemma 150.18. For “Nagata” this follows from Algebra, Proposition 150.30. \square

Lemma 19.2. *The following types of schemes are Nagata.*

- (1) Any scheme locally of finite type over a field.
- (2) Any scheme locally of finite type over a Noetherian complete local ring.
- (3) Any scheme locally of finite type over \mathbf{Z} .
- (4) Any scheme locally of finite type over a Dedekind ring of characteristic zero.
- (5) And so on.

Proof. By Lemma 19.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 150.31. \square

20. The singular locus, reprise

We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

Definition 20.1. Let X be a locally Noetherian scheme. We say X is *J-2* if for every morphism $Y \rightarrow X$ which is locally of finite type the regular locus $\text{Reg}(Y)$ is open in Y .

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 35.1.

Lemma 20.2. *Let X be a locally Noetherian scheme. The following are equivalent*

- (1) X is *J-2*,
- (2) there exists an open covering of X all of whose members are *J-2* schemes,
- (3) for every affine open $\text{Spec}(R) = U \subset X$ the ring R is *J-2*, and
- (4) there exists an affine open covering $S = \bigcup U_i$ such that each $\mathcal{O}(U_i)$ is *J-2* for all i .

Moreover, in this case any scheme locally of finite type over X is *J-2* as well.

Proof. By Lemma 16.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 16.3). Thus it

is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $\text{Spec}(R)$ is J-2, then for every finite type R -algebra A the regular locus of the scheme $\text{Spec}(A)$ is open. Hence R is J-2, by definition (see More on Algebra, Definition 35.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) \Rightarrow (2) and (3) \Rightarrow (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \rightarrow X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \rightarrow X$ maps into one of the U_i . By Lemma 16.2 the induced ring map $\mathcal{O}(U_i) \rightarrow \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = \text{Spec}(\mathcal{O}(V_j))$ is open. Since $\text{Reg}(Y) \cap V_j = \text{Reg}(V_j)$ we conclude that $\text{Reg}(Y)$ is open as desired. \square

Lemma 20.3. *The following types of schemes are J-2.*

- (1) Any scheme locally of finite type over a field.
- (2) Any scheme locally of finite type over a Noetherian complete local ring.
- (3) Any scheme locally of finite type over \mathbf{Z} .
- (4) Any scheme locally of finite type over a Dedekind ring of characteristic zero.
- (5) And so on.

Proof. By Lemma 20.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 36.6. \square

21. Quasi-finite morphisms

A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 118.

Recall that a finite type ring map $R \rightarrow A$ is quasi-finite at a prime \mathfrak{q} if \mathfrak{q} defines an isolated point of its fibre, see Algebra, Definition 118.3.

Definition 21.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *quasi-finite at a point* $x \in X$ if there exist an affine neighbourhood $\text{Spec}(A) = U \subset X$ of x and an affine open $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \rightarrow A$ is of finite type, and $R \rightarrow A$ is quasi-finite at the prime of A corresponding to x (see above).
- (2) We say f is *locally quasi-finite* if f is quasi-finite at every point x of X .
- (3) We say that f is *quasi-finite* if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 49 on Zariski's Main Theorem.

Lemma 21.2. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(s) \supset \kappa(x)$ is an algebraic field extension, then*

- (1) x is a closed point of its fibre, and
- (2) if in addition s is a closed point of S , then x is a closed point of X .

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 18.5 to prove the first statement we may replace X by X_s and S by $\text{Spec}(\kappa(s))$. Thus we may assume that $S = \text{Spec}(k)$ is the spectrum of a field. In this case, let $\text{Spec}(A) = U \subset X$ be any affine open containing x . The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $k \subset \kappa(\mathfrak{q})$ is an algebraic field extension. By Algebra, Lemma 34.9 we see that \mathfrak{q} is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. \square

The following lemma is a version of the Hilbert Nullstellensatz.

Lemma 21.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(s) \subset \kappa(x)$ is a finite field extension.*

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 21.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 16.2 the ring map $R \rightarrow A$ is of finite type. Let $\mathfrak{q} \subset A$, resp. $\mathfrak{p} \subset R$ be the prime ideal corresponding to x , resp. s . Consider the fibre ring $\bar{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\bar{\mathfrak{q}}$ be the prime of \bar{A} corresponding to \mathfrak{q} . The assumption that x is a closed point of its fibre implies that $\bar{\mathfrak{q}}$ is a maximal ideal of \bar{A} . Since \bar{A} is an algebra of finite type over the field $\kappa(\mathfrak{p})$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 33.1, that $\kappa(\bar{\mathfrak{q}})$ is a finite extension of $\kappa(\mathfrak{p})$. Since $\kappa(s) = \kappa(\mathfrak{p})$ and $\kappa(x) = \kappa(\mathfrak{q}) = \kappa(\bar{\mathfrak{q}})$ we win. \square

Lemma 21.4. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. Let $g : S' \rightarrow S$ be any morphism. Denote $f' : X' \rightarrow S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(x)}$ then x' is closed in $X'_{f'(x')}$.*

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 21.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. \square

Lemma 21.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x , then the residue field extension $\kappa(s) \subset \kappa(x)$ is finite.*

Proof. This is clear from Algebra, Definition 118.3. \square

Lemma 21.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s . Assume f is locally of finite type. The following are equivalent:*

- (1) *The morphism f is quasi-finite at x .*
- (2) *The point x is isolated in X_s .*
- (3) *The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x .*
- (4) *For any pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak{q} \subset A$ the ring map $R \rightarrow A$ is quasi-finite at \mathfrak{q} .*

Proof. Assume f is quasi-finite at x . By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s . Hence $\{x\} \subset U_s$ is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s .

Note that X_s is a Jacobson scheme by Lemma 17.10 (and Lemma 16.4). If x is isolated in X_s , i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s . It is clear that there is no point $x' \in X_s$, distinct from x , specializing to x .

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x , specializing to x . Consider a pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s . By Lemma 16.2 the ring map $R \rightarrow A$ is of finite type. Consider the fibre ring $\bar{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\bar{\mathfrak{q}}$ be the prime of \bar{A} corresponding to \mathfrak{q} . Since $\text{Spec}(\bar{A})$ is an open subscheme of the fibre X_s we see that $\bar{\mathfrak{q}}$ is a maximal ideal of \bar{A} and that there is no point of $\text{Spec}(\bar{A})$ specializing to $\bar{\mathfrak{q}}$. This implies that $\dim(\bar{A}_{\bar{\mathfrak{q}}}) = 0$. Hence by Algebra, Definition 118.3 we see that $R \rightarrow A$ is quasi-finite at \mathfrak{q} , i.e., $X \rightarrow S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) – (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. \square

Lemma 21.7. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $s \in S$. Assume that*

- (1) f is locally of finite type, and
- (2) $f^{-1}(\{s\})$ is a finite set.

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s .

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k , and (b) has finitely many points. Then Lemma 17.10 shows T is a Jacobson scheme. A finite Jacobson space is discrete, see Topology, Lemma 17.6. Apply this remark to the fibre X_s which is locally of finite type over $\text{Spec}(\kappa(s))$ to see the first statement. Finally, apply Lemma 21.6 to see the second. \square

Lemma 21.8. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent*

- (1) f is locally quasi-finite,
- (2) for every $s \in S$ the fibre X_s is a discrete topological space, and
- (3) for every morphism $\text{Spec}(k) \rightarrow S$ where k is a field the base change X_k has an underlying discrete topological space.

Proof. It is immediate that (3) implies (2). Lemma 21.6 shows that (2) is equivalent to (1). Assume (2) and let $\text{Spec}(k) \rightarrow S$ be as in (3). Denote $s \in S$ the image of $\text{Spec}(k) \rightarrow S$. Then X_k is the base change of X_s via $\text{Spec}(k) \rightarrow \text{Spec}(\kappa(s))$. Hence every point of X_k is closed by Lemma 21.4. As $X_k \rightarrow \text{Spec}(k)$ is locally of finite type (by Lemma 16.4), we may apply Lemma 21.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. \square

Lemma 21.9. *Let $f : X \rightarrow S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.*

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 16.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 21.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 21.6 we see that x is an isolated point of its fibre. The lemma is proved. \square

Lemma 21.10. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) f is quasi-finite, and
- (2) f is locally of finite type, quasi-compact, and has finite fibres.

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 21.7. Hence it is quasi-finite by Lemma 21.9. \square

Recall that a ring map $R \rightarrow A$ is quasi-finite if it is of finite type and quasi-finite at all primes of A , see Algebra, Definition 118.3.

Lemma 21.11. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) The morphism f is locally quasi-finite.
- (2) For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is quasi-finite.
- (3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is locally quasi-finite.
- (4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is locally quasi-finite.

Proof. For a ring map $R \rightarrow A$ let us define $P(R \rightarrow A)$ to mean “ $R \rightarrow A$ is quasi-finite” (see remark above lemma). We claim that P is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 15.1. In the proof of Lemma 16.2 we have seen that (a), (b) and (c) hold for the property of being “of finite type”. Note that, for a finite type ring map $R \rightarrow A$, the property $R \rightarrow A$ is quasi-finite at \mathfrak{q} depends only on the local ring $A_{\mathfrak{q}}$ as an algebra over $R_{\mathfrak{p}}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 15.1 follow immediately. For example, suppose $R \rightarrow A$ is a ring map such that all of the ring maps $R \rightarrow A_{a_i}$ are quasi-finite for $a_1, \dots, a_n \in A$ generating the unit ideal. We conclude that $R \rightarrow A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_{\mathfrak{q}}$ is isomorphic as an R -algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \rightarrow A$ is quasi-finite at \mathfrak{q} .

We conclude that Lemma 15.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P . Since $P(R \rightarrow A)$ is “ $R \rightarrow A$ is quasi-finite” which means $R \rightarrow A$ is quasi-finite at every prime of A , this follows from Lemma 21.6. \square

Lemma 21.12. *The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.*

Proof. In the proof of Lemma 21.11 we saw that $P =$ “quasi-finite” is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 15.2. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 118.7. By the above, Lemma 21.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 19.4 we see that the composition of quasi-finite morphisms is quasi-finite. \square

We will see later (Lemma 49.2) that the set U of the following lemma is open.

Lemma 21.13. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $g : S' \rightarrow S$ be a morphism of schemes. Denote $f' : X_{S'} \rightarrow S'$ the base change of f by g and denote $g' : X_{S'} \rightarrow X$ the projection. Assume X is locally of finite type over S .*

- (1) *Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U_{S'} = (g')^{-1}(U)$.*
- (2) *The base change of a locally quasi-finite morphism is locally quasi-finite.*
- (3) *The base change of a quasi-finite morphism is quasi-finite.*

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 118.8 (combined with the fact that f' is also locally of finite type by Lemma 16.4). By the above, Lemma 21.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 19.3 we see that the base change of a quasi-finite morphism is quasi-finite. \square

Lemma 21.14. *Let $f : X \rightarrow S$ be a morphism of schemes of finite type. Let $s \in S$. There are at most finitely many points of X lying over s at which f is quasi-finite.*

Proof. The fibre X_s is a scheme of finite type over a field, hence Noetherian (Lemma 16.6). Hence the topology on X_s is Noetherian (Properties, Lemma 5.5) and can have at most a finite number of isolated points (by elementary topology). Thus our lemma follows from Lemma 21.6. \square

Lemma 21.15. *Any immersion is locally quasi-finite.*

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. \square

Lemma 21.16. *Let $X \rightarrow Y$ be a morphism of schemes over a base scheme S . Let $x \in X$. If $X \rightarrow S$ is quasi-finite at x , then $X \rightarrow Y$ is quasi-finite at x . If X is locally quasi-finite over S , then $X \rightarrow Y$ is locally quasi-finite.*

Proof. Via Lemma 21.11 this translates into the following algebra fact: Given ring maps $A \rightarrow B \rightarrow C$ such that $A \rightarrow C$ is quasi-finite, then $B \rightarrow C$ is quasi-finite. This follows from Algebra, Lemma 118.6 with $R = A$, $S = S' = C$ and $R' = B$. \square

22. Morphisms of finite presentation

Recall that a ring map $R \rightarrow A$ is of finite presentation if A is isomorphic to $R[x_1, \dots, x_n]/(f_1, \dots, f_m)$ as an R -algebra for some n, m and some polynomials f_j , see Algebra, Definition 6.1.

Definition 22.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is of *finite presentation at* $x \in X$ if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is of finite presentation.
- (2) We say that f is *locally of finite presentation* if it is of finite presentation at every point of X .
- (3) We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$\text{colim Mor}_S(T_i, X) = \text{Mor}_S(\lim T_i, X)$$

for any directed system of affine schemes T_i over S . See Limits, Proposition 5.1. In Limits, Section 9 we show that, if $S = \lim_i S_i$ is a limit of affine schemes, any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i .

Lemma 22.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is locally of finite presentation.*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is of finite presentation.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is locally of finite presentation.*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is locally of finite presentation.

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is of finite presentation” is local. We check conditions (a), (b) and (c) of Definition 15.1. By Algebra, Lemma 13.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By the same lemma being of finite presentation is stable under composition and trivially for any ring R the ring map $R \rightarrow R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 23.3. \square

Lemma 22.3. *The composition of two morphisms which locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 22.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 19.4 and 21.13 we see that the composition of morphisms of finite presentation is of finite presentation. \square

Lemma 22.4. *The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 22.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 13.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 19.3 and 21.13 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. \square

Lemma 22.5. *Any open immersion is locally of finite presentation.*

Proof. This is true because an open immersion is a local isomorphism. \square

Lemma 22.6. *Any open immersion is of finite presentation if and only if it is quasi-compact.*

Proof. We have seen (Lemma 22.5) that an open immersion is locally of finite presentation. We have seen (Schemes, Lemma 23.7) that an immersion is separated and hence quasi-separated. From this and Definition 22.1 the lemma follows. \square

Lemma 22.7. *A closed immersion $i : Z \rightarrow X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \text{Ker}(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$ is of finite type (as an \mathcal{O}_X -module).*

Proof. On any affine open $\text{Spec}(R) \subset X$ we have $i^{-1}(\text{Spec}(R)) = \text{Spec}(R/I)$ and $\mathcal{I} = \tilde{I}$. Moreover, \mathcal{I} is of finite type if and only if I is a finite R -module for every such affine open (see Properties, Lemma 16.1). And R/I is of finite presentation over R if and only if I is a finite R -module. Hence we win. \square

Lemma 22.8. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Omitted. \square

Lemma 22.9. *Let $f : X \rightarrow S$ be a morphism.*

- (1) *If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
- (2) *If S is locally Noetherian and f of finite type then f is of finite presentation.*

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 30.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally

of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 16.7 (and Lemma 22.8). \square

Lemma 22.10. *Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S , then X is quasi-compact and quasi-separated.*

Proof. Omitted. \square

Lemma 22.11. *Let $f : X \rightarrow Y$ be a morphism of schemes over S .*

- (1) *If X is locally of finite presentation over S and Y is locally of finite type over S , then f is locally of finite presentation.*
- (2) *If X is of finite presentation over S and Y is quasi-separated and locally of finite type over S , then f is of finite presentation.*

Proof. Proof of (1). Via Lemma 22.2 this translates into the following algebra fact: Given ring maps $A \rightarrow B \rightarrow C$ such that $A \rightarrow C$ is of finite presentation and $A \rightarrow B$ is of finite type, then $B \rightarrow C$ is of finite type. See Algebra, Lemma 6.2.

Part (2) follows from (1) and Schemes, Lemmas 21.14 and 21.15. \square

Lemma 22.12. *Let $f : X \rightarrow Y$ be a morphism of schemes with diagonal $\Delta : X \rightarrow X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism of schemes over X (via the second projection $X \times_Y X \rightarrow X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is locally of finite type over X (by Lemma 16.4). Thus the first statement holds by Lemma 22.11. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is a monomorphism, hence separated (Schemes, Lemma 23.3). \square

23. Constructible sets

Constructible and locally constructible sets of schemes have been discussed in Properties, Section 2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

Lemma 23.1. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) constructible in Y , then $f^{-1}(E)$ is (locally) constructible in X .*

Proof. To show that the inverse image of every constructible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X , see Topology, Lemma 14.3. The significance of V being retrocompact in Y is just that the open immersion $V \rightarrow Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \rightarrow X$ is quasi-compact too, see Schemes, Lemma 19.3. Hence we see $f^{-1}(V)$ is retrocompact in X . Suppose E is locally constructible in Y . Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \rightarrow V$ as a morphism into V . By Properties, Lemma 2.1 we see that $E \cap V$ is constructible in V . By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U . Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. \square

Lemma 23.2. *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume*

- (1) *f is quasi-compact and locally of finite presentation, and*
- (2) *Y is quasi-compact and quasi-separated.*

Then the image of every constructible subset of X is constructible in Y .

Proof. By Properties, Lemma 2.5 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \dots \cup U_n$ with each U_i affine open in X . If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 14.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 28.9. \square

Theorem 23.3 (Chevalley's Theorem). *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible.*

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \dots \cup U_n$ with each U_i affine open in X . If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 28.9. \square

Lemma 23.4. *Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \rightsquigarrow x\} \subset E$, then E contains an open neighbourhood of x .*

Proof. Assume $\{x' \mid x' \rightsquigarrow x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 28.3 we can find a morphism of affine schemes $f : Y \rightarrow X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f . By Lemma 6.5 and the assumption $\{x' \mid x' \rightsquigarrow x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E . \square

24. Open morphisms

Definition 24.1. Let $f : X \rightarrow S$ be a morphism.

- (1) We say f is *open* if the map on underlying topological spaces is open.
- (2) We say f is *universally open* if for any morphism of schemes $S' \rightarrow S$ the base change $f' : X_{S'} \rightarrow S'$ is open.

According to Topology, Lemma 18.6 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 24.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 26.8). This sometimes in turn implies that the morphism is open.

Lemma 24.2. *Let $f : X \rightarrow S$ be a morphism.*

- (1) *If f is locally of finite presentation and generalizations lift along f , then f is open.*

- (2) If f is locally of finite presentation and generalizations lift along every base change of f , then f is universally open.

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 40.3 and Proposition 40.8. \square

See also Lemma 26.9 for the case of a morphism flat of finite presentation.

Lemma 24.3. *A composition of (universally) open morphisms is (universally) open.*

Proof. Omitted. \square

Lemma 24.4. *Let k be a field. Let X be a scheme over k . The structure morphism $X \rightarrow \text{Spec}(k)$ is universally open.*

Proof. Let $S \rightarrow \text{Spec}(k)$ be a morphism. We have to show that the base change $X_S \rightarrow S$ is open. The question is local on S and X , hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 40.10. \square

Lemma 24.5. *Let $\varphi : X \rightarrow Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.*

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y . Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 40.2 (combined with Algebra, Lemma 40.3). \square

Lemma 24.6. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $g : Y' \rightarrow Y$ be open and surjective such that the base change $f' : X' \rightarrow Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 11.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 11.7. Thus f is quasi-compact. \square

25. Submersive morphisms

Definition 25.1. Let $f : X \rightarrow Y$ be a morphism of schemes.

- (1) We say f is *submersive*⁵ if the continuous map of underlying topological spaces is submersive, see Topology, Definition 5.3.
- (2) We say f is *universally submersive* if for every morphism of schemes $Y' \rightarrow Y$ the base change $Y' \times_Y X \rightarrow Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

⁵This is very different from the notion of a submersion of differential manifolds.

26. Flat morphisms

Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 26.9. A very important class of results, namely criteria for flatness will be discussed (insert future reference here).

Recall that a module M over a ring R is *flat* if the functor $-\otimes_R M : \text{Mod}_R \rightarrow \text{Mod}_R$ is exact. A ring map $R \rightarrow A$ is said to be *flat* if A is flat as an R -module. See Algebra, Definition 38.1.

Definition 26.1. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules.

- (1) We say f is *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
- (2) We say that \mathcal{F} is *flat over S at a point* $x \in X$ if the stalk \mathcal{F}_x is a flat $\mathcal{O}_{S,f(x)}$ -module.
- (3) We say f is *flat* if f is flat at every point of X .
- (4) We say that \mathcal{F} is *flat over S* if \mathcal{F} is flat over S at every point x of X .

Thus we see that f is flat if and only if the structure sheaf \mathcal{O}_X is flat over S .

Lemma 26.2. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules. The following are equivalent

- (1) The sheaf \mathcal{F} is flat over S .
- (2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_S(V)$ -module $\mathcal{F}(U)$ is flat.
- (3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the modules $\mathcal{F}|_{U_i}$ is flat over V_j , for all $j \in J, i \in I_j$.
- (4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}_S(V_j)$ -module, for all $j \in J, i \in I_j$.

Moreover, if \mathcal{F} is flat over S then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is flat over V .

Proof. Let $R \rightarrow A$ be a ring map. Let M be an A -module. If M is R -flat, then for all primes \mathfrak{q} the module $M_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ with \mathfrak{p} the prime of R lying under \mathfrak{q} . Conversely, if $M_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ for all primes \mathfrak{q} of A , then M is flat over R . See Algebra, Lemma 38.19. This equivalence easily implies the statements of the lemma. \square

Lemma 26.3. Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent

- (1) The morphism f is flat.
- (2) For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is flat.
- (3) There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is flat.
- (4) There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is flat, for all $j \in J, i \in I_j$.

Moreover, if f is flat then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is flat.

Proof. This is a special case of Lemma 26.2 above. □

Lemma 26.4. *Let $X \rightarrow Y \rightarrow Z$ be morphisms of schemes. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. If \mathcal{F} is flat over Y , and Y is flat over Z , then \mathcal{F} is flat over Z .*

Proof. See Algebra, Lemma 38.3. □

Lemma 26.5. *The composition of flat morphisms is flat.*

Proof. This is a special case of Lemma 26.4. □

Lemma 26.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules. Let $g : S' \rightarrow S$ be a morphism of schemes. Denote $g' : X' = X_{S'} \rightarrow X$ the projection. Let $x' \in X'$ be a point with image $x = g(x') \in X$. If \mathcal{F} is flat over S at x , then $(g')^*\mathcal{F}$ is flat over S' at x' . In particular, if \mathcal{F} is flat over S , then $(g')^*\mathcal{F}$ is flat over S' .*

Proof. See Algebra, Lemma 38.6. □

Lemma 26.7. *The base change of a flat morphism is flat.*

Proof. This is a special case of Lemma 26.6. □

Lemma 26.8. *Let $f : X \rightarrow S$ be a flat morphism of schemes. Then generalizations lift along f , see Topology, Definition 18.3.*

Proof. See Algebra, Section 40. □

Lemma 26.9. *A flat morphism locally of finite presentation is universally open.*

Proof. This follows from Lemmas 26.8 and Lemma 24.2 above. We can also argue directly as follows.

Let $f : X \rightarrow S$ be flat locally of finite presentation. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \rightarrow S$ is open. By definition we may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \rightarrow \mathcal{O}_X(U_i)$ is of finite presentation. Thus $U_i \rightarrow V_i$ is open by Algebra, Proposition 40.8. The lemma follows. □

Lemma 26.10. *Let $f : X \rightarrow Y$ be a quasi-compact, surjective, flat morphism. A subset $T \subset Y$ is open (resp. closed) if and only if $f^{-1}(T)$ is open (resp. closed). In other words, f is a submersive morphism.*

Proof. The question is local on Y , hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \dots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \amalg \dots \amalg X_n \rightarrow Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \amalg \dots \amalg f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \rightarrow B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) =$

$V(I)$ for $I \subset A$ an ideal. Then $T = f(f^{-1}(T)) = f(V(I))$ is the image of $\text{Spec}(A/I) \rightarrow \text{Spec}(B)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 26.8). Hence by Topology, Lemma 18.5 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 18.2). Thus T is closed by Algebra, Lemma 40.5. \square

Lemma 26.11. *Let $h : X \rightarrow Y$ be a morphism of schemes over S . Let \mathcal{G} be a quasi-coherent sheaf on Y . Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x , then*

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then \mathcal{G} is flat over S , if and only if $h^\mathcal{G}$ is flat over S . If h is surjective and flat, and X is flat over S , then Y is flat over S .*

Proof. You can prove this by applying Algebra, Lemma 38.8. Here is a direct proof. Let $s \in S$ be the image of y . Consider the local ring maps $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 38.16. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 26.4. Let $M' \rightarrow M$ be an injection of $\mathcal{O}_{S,s}$ -modules. By the faithful flatness mentioned above we have

$$\begin{aligned} & \text{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \rightarrow M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ &= \text{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \rightarrow M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 38.4. \square

Lemma 26.12. *Let $f : Y \rightarrow X$ be a morphism of schemes. Let \mathcal{F} be a finite type quasi-coherent \mathcal{O}_X -module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of scheme theoretic support on affines as given in Lemma 5.4 we reduce to Algebra, Lemma 39.4. \square

Lemma 26.13. *Let $f : X \rightarrow Y$ be a flat morphism of schemes. Let $V \subset Y$ be a retrocompact open which is scheme theoretically dense. Then $f^{-1}V$ is scheme theoretically dense in X .*

Proof. We will use the characterization of Lemma 7.5. We have to show that for any open $U \subset X$ the map $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U \cap f^{-1}V)$ is injective. It suffices to prove this when U is an affine open which maps into an affine open $W \subset Y$. Say $W = \text{Spec}(A)$ and $U = \text{Spec}(B)$. Then $V \cap W = D(f_1) \cup \dots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 28.1. Thus we have to show that $B \rightarrow B_{f_1} \times \dots \times B_{f_n}$ is injective. We are given that $A \rightarrow A_{f_1} \times \dots \times A_{f_n}$ is injective and that $A \rightarrow B$ is flat. Since $B_{f_i} = A_{f_i} \otimes_A B$ we win. \square

Lemma 26.14. *Let $f : X \rightarrow Y$ be a flat morphism of schemes. Let $g : V \rightarrow Y$ be a quasi-compact morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \rightarrow X$. Then $Z' = f^{-1}Z$.*

Proof. Recall that Z is cut out by $\mathcal{I} = \text{Ker}(\mathcal{O}_Y \rightarrow g_*\mathcal{O}_V)$ and Z' is cut out by $\mathcal{I}' = \text{Ker}(\mathcal{O}_X \rightarrow (V \times_Y X \rightarrow X)_*\mathcal{O}_{V \times_Y X})$, see Lemma 6.3. Hence the question is local on X and Y and we may assume X and Y affine. Note that we may replace

V by $\coprod V_i$ where $V = V_1 \cup \dots \cup V_n$ is a finite affine open covering. Hence we may assume g is affine. In this case $(V \times_Y X \rightarrow X)_* \mathcal{O}_{V \times_Y X}$ is the pullback of $g_* \mathcal{O}_V$ by f . Since f is flat we conclude that $f^* \mathcal{I} = \mathcal{I}'$ and the lemma holds. \square

27. Flat closed immersions

Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

Lemma 27.1. *Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection*

$$\left\{ \begin{array}{l} \text{closed subschemes } Z \subset X \\ \text{such that } Z \rightarrow X \text{ is flat} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{closed subsets } Z \subset X \\ \text{closed under generalizations} \end{array} \right\}$$

Proof. The affine case is Algebra, Lemma 104.4. In general the lemma follows by covering X by affines and glueing. Details omitted. \square

Lemma 27.2. *A flat closed immersion of finite presentation is the open immersion of an open and closed subscheme.*

Proof. The affine case is Algebra, Lemma 104.5. In general the lemma follows by covering X by affines. Details omitted. \square

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

Definition 27.3. Let X be a scheme. Let $T \subset X$ be a connected component. The *canonical scheme structure on T* is the unique scheme structure on T such that the closed immersion $T \rightarrow X$ is flat, see Lemma 27.1.

It turns out that we can determine when every finite flat \mathcal{O}_X -module is finite locally free using the previous lemma.

Lemma 27.4. *Let X be a scheme. The following are equivalent*

- (1) *every finite flat quasi-coherent \mathcal{O}_X -module is finite locally free, and*
- (2) *every closed subset $Z \subset X$ which is closed under generalizations is open.*

Proof. In the affine case this is Algebra, Lemma 104.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \rightarrow X$ such that i is flat. Then $i_* \mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \text{Supp}(i_* \mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) \Rightarrow (2) follows from the characterization of flat closed immersions in Lemma 27.1.

For the converse assume that X satisfies (2). Let \mathcal{F} be a finite flat quasi-coherent \mathcal{O}_X -module. The support $Z = \text{Supp}(\mathcal{F})$ of \mathcal{F} is closed, see Modules, Lemma 9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 75.4 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of \mathcal{F} is open and closed. The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \dots$ are finite flat quasi-coherent \mathcal{O}_X -modules also,

see Modules, Section 18. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \coprod U_1 \coprod U_2 \coprod \dots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \dots$ for all i . Let x be a point of X , and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r . By Nakayama's lemma, see Algebra, Lemma 19.1 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \dots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \dots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent \mathcal{O}_U -module whose support is all of U . By the above it is generated by a single element, namely $s_1 \wedge \dots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals \mathcal{I} such that $\mathcal{O}_U/\mathcal{I}$ is flat over \mathcal{O}_U and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 27.1. Thus $s_1 \wedge \dots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that \mathcal{F} is finite locally free as desired. \square

28. Generic flatness

A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 114 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 28.2 which shows the result holds if we only assume the base is reduced.

Proposition 28.1 (Generic flatness). *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules. Assume*

- (1) S is integral,
- (2) f is of finite type, and
- (3) \mathcal{F} is a finite type \mathcal{O}_X -module.

Then there exists an open dense subscheme $U \subset S$ such that $X_U \rightarrow U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over \mathcal{O}_{X_U} .

Proof. As S is integral it is irreducible (see Properties, Lemma 3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = \text{Spec}(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1, \dots, n} X_i$. Write $X_i = \text{Spec}(B_i)$. Then B_i is a finite type A -algebra, see Lemma 16.2. Moreover there are finite type B_i -modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i -module M_i , see Properties, Lemma 16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \text{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i -module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i -module.

At this point we apply Algebra, Lemma 114.3 the pairs $(A \rightarrow B_i, M_{ij})$ and to the pairs $(A \rightarrow B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a) $A_{f_{ij}} \rightarrow B_{i, f_{ij}}$ is flat and of finite presentation and $M_{ij, f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation

over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i} . Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f , i.e., perform the base change by $U = \text{Spec}(A_f) \rightarrow S$. After this base change we see that each of $A \rightarrow B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i . This already proves that $X \rightarrow S$ is quasi-compact, locally of finite presentation, flat, and that \mathcal{F} is flat over S and of finite presentation over \mathcal{O}_X , see Lemma 22.2 and Properties, Lemma 16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i , see Algebra, Lemma 39.7. Hence we see that $X \rightarrow S$ is quasi-separated by Schemes, Lemma 21.7. This proves the proposition. \square

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

Proposition 28.2 (Generic flatness, reduced case). *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules. Assume*

- (1) S is reduced,
- (2) f is of finite type, and
- (3) \mathcal{F} is a finite type \mathcal{O}_X -module.

Then there exists an open dense subscheme $U \subset S$ such that $X_U \rightarrow U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over \mathcal{O}_{X_U} .

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 28.1 using Algebra, Lemma 114.7 instead of Algebra, Lemma 114.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 26.2 and 22.2, we may work affine locally on S . Thus we may assume that $S = \text{Spec}(A)$, where A is a reduced ring (see Properties, Lemma 3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1, \dots, n} X_i$. Write $X_i = \text{Spec}(B_i)$. Then B_i is a finite type A -algebra, see Lemma 16.2. Moreover there are finite type B_i -modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i -module M_i , see Properties, Lemma 16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \text{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i -module. Then $V(I_{ij}) = \text{Supp}(M_{ij})$ and M_{ij} is a finite B_i -module.

At this point we apply Algebra, Lemma 114.7 the pairs $(A \rightarrow B_i, M_{ij})$ and to the pairs $(A \rightarrow B_i, M_i)$. Thus we obtain dense opens $U(A \rightarrow B_i, M_{ij}) \subset S$ and dense opens $U(A \rightarrow B_i, M_i) \subset S$ with notation as in Algebra, Equation (114.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap_{i,j} U(A \rightarrow B_i, M_{ij}) \quad \cap \quad \bigcap_i U(A \rightarrow B_i, M_i)$$

is open and dense in S . We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \rightarrow B_i, M_{ij})$ and $U(A \rightarrow B_i, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \rightarrow B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of

finite presentation over B_{i,f_i} . Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \rightarrow S$ is flat and of finite presentation over $D(f)$ and that \mathcal{F} restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f , i.e., perform the base change by $\text{Spec}(A_f) \rightarrow S$. After this base change we see that each of $A \rightarrow B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i . This already proves that $X \rightarrow S$ is quasi-compact, locally of finite presentation, flat, and that \mathcal{F} is flat over S and of finite presentation over \mathcal{O}_X , see Lemma 22.2 and Properties, Lemma 16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \text{Supp}(M_{ij})$ is a quasi-compact open of X_i , see Algebra, Lemma 39.7. Hence we see that $X \rightarrow S$ is quasi-separated by Schemes, Lemma 21.7. This proves the proposition. \square

Remark 28.3. The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [GR71] by Raynaud and Gruson contains many wonderful results in this direction.

29. Morphisms and dimensions of fibres

Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X . See Topology, Definition 9.1.

Lemma 29.1. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then*

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s, x}) + \text{trdeg}_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 112.3. \square

Lemma 29.2. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then*

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s, x}$ is flat over $\mathcal{O}_{Y_s, y}$, which holds for example if $\mathcal{O}_{X, x}$ is flat over $\mathcal{O}_{Y, y}$.

Proof. Note that $\text{trdeg}_{\kappa(s)}(\kappa(x)) = \text{trdeg}_{\kappa(y)}(\kappa(x)) + \text{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 29.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s, x}) \leq \dim(\mathcal{O}_{X_y, x}) + \dim(\mathcal{O}_{Y_s, y}).$$

For this see Algebra, Lemma 108.6. For the flat case see Algebra, Lemma 108.7. \square

Lemma 29.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then $\dim_x(X_s) = \dim_{x'}(X'_{s'})$.

Proof. Follows immediately from Algebra, Lemma 112.6. \square

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

Lemma 29.4. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X .

Proof. This is immediate from Algebra, Lemma 121.6 \square

Lemma 29.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 29.4 is retrocompact in X . (See Topology, Definition 11.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the induced morphism $f|_U : U \rightarrow S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U . Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 121.8 (and Lemma 22.2). \square

Lemma 29.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then*

- (1) $\dim_x(X_s) \leq \dim_{x'}(X_s)$,
- (2) $\text{trdeg}_{\kappa(s)}(\kappa(x)) > \text{trdeg}_{\kappa(s)}(\kappa(x'))$, and
- (3) $\dim(\mathcal{O}_{X_s, x}) < \dim(\mathcal{O}_{X_s, x'})$.

Proof. The first inequality follows from Lemma 29.4. The third inequality follows since $\mathcal{O}_{X_s, x}$ is a localization of $\mathcal{O}_{X_s, x'}$ at a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s, x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s, x'}$. The second inequality follows from Algebra, Lemma 112.2. \square

30. Morphisms of given relative dimension

In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

Definition 30.1. Let $f : X \rightarrow S$ be a morphism of schemes. Assume f is locally of finite type.

- (1) We say f is of *relative dimension $\leq d$ at x* if $\dim_x(X_{f(x)}) \leq d$.
- (2) We say f is of *relative dimension $\leq d$* if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
- (3) We say f is of *relative dimension d* if all nonempty fibres X_s are equidimensional of dimension d .

This is not a particularly well behaved notion, but it works well in a number of situations.

Lemma 30.2. *Let $f : X \rightarrow S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d , then so does any base change of f . Same for relative dimension $\leq d$.*

Proof. This is immediate from Lemma 29.3. \square

Lemma 30.3. *Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If*

- (1) f has relative dimension d ,
- (2) g has relative dimension e , and
- (3) f is flat,

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 29.2. \square

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

Lemma 30.4. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume that*

- (1) f is flat,
- (2) f is locally of finite presentation, and
- (3) for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 8.1)

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \rightarrow S$ has relative dimension d .

Proof. This is immediate from Algebra, Lemma 126.8. \square

Lemma 30.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.*

Proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 21.5) and x is isolated in X_s (by Lemma 21.6), hence $\dim_x(X_s) = 0$ by Lemma 29.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 29.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x . Hence x is closed in its fibre by Lemma 21.2 and by Lemma 21.6 (3) we conclude that f is quasi-finite at x . \square

Lemma 30.6. *Let $f : X \rightarrow Y$ be a morphism of locally Noetherian schemes which is flat, locally of finite type and of relative dimension d . For every point x in X with image y in Y we have $\dim_x(X) = \dim_y(Y) + d$.*

Proof. After shrinking X and Y to open neighborhoods of x and y , we can assume that $\dim(X) = \dim_x(X)$ and $\dim(Y) = \dim_y(Y)$, by definition of the dimension of a scheme at a point (Properties, Definition 10.1). The morphism f is open by Lemmas 22.9 and 26.9. Hence we can shrink Y to arrange that f is surjective. It remains to show that $\dim(X) = \dim(Y) + d$.

Let a be a point in X with image b in Y . By Algebra, Lemma 108.7,

$$\dim(\mathcal{O}_{X,a}) = \dim(\mathcal{O}_{Y,b}) + \dim(\mathcal{O}_{X_b,a}).$$

Taking the supremum over all points a in X , it follows that $\dim(X) = \dim(Y) + d$, as we want, see Properties, Lemma 10.2. \square

31. The dimension formula

For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X .

Lemma 31.1. *Let S be a scheme. Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume*

- (1) S is locally Noetherian,
- (2) f is locally of finite type,
- (3) X and S integral, and
- (4) f dominant.

We have

$$(31.1.1) \quad \dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + \operatorname{trdeg}_{R(S)} R(X) - \operatorname{trdeg}_{\kappa(s)} \kappa(x).$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 109.1. \square

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 19.1.

Lemma 31.2. *Let S be a universally catenary scheme. Let $\delta : S \rightarrow \mathbf{Z}$ be a dimension function. Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite type. Then the map*

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + \operatorname{trdeg}_{\kappa(f(x))} \kappa(x) \end{aligned}$$

is a dimension function on X .

Proof. Let $f : X \rightarrow S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X . We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x .

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y . We may clearly replace X by U and S by V . Thus we may assume that $X = \operatorname{Spec}(A)$ and $S = \operatorname{Spec}(R)$ and that f is given by a ring map $R \rightarrow A$. The ring R is universally catenary (Lemma 18.2) and the map $R \rightarrow A$ is of finite type (Lemma 16.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = \operatorname{Spec}(R/\mathfrak{p}) \subset S$ is a dimension function. The ring R/\mathfrak{p} is universally catenary. The restriction of $\delta_{X/S}$ to $X' = \operatorname{Spec}(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ' . Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes.

Note that $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$ at a non-maximal prime (Schemes, Lemma 13.2). Hence $\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,y})$ and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,y}) - 1$ if y is an immediate specialization of x .

Write $s = f(x) \neq f(y) = s'$. We see, using equality in (31.1.1), that

$$\begin{aligned} \delta_{X/S}(x) - \delta_{X/S}(y) &= \delta(s) - \delta(s') \\ &\quad + \dim(\mathcal{O}_{S,s}) - \dim(\mathcal{O}_{S,s'}) \\ &\quad - \dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y}). \end{aligned}$$

Since δ is a dimension function on the scheme S the difference $\delta(s) - \delta(s')$ is equal to $\text{codim}(\overline{\{s'\}}, \overline{\{s\}})$ by Topology, Lemma 19.2. As S is integral, catenary this is equal to $\text{codim}(\overline{\{s'\}}, S) - \text{codim}(\overline{\{s\}}, S)$ (Topology, Lemma 10.6). And this in turn is equal to $\dim(\mathcal{O}_{S,s'}) - \dim(\mathcal{O}_{S,s})$ by Properties, Lemma 11.4. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = -\dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y})$$

and hence the lemma follows from our remarks about the dimensions of these local rings above. \square

Another application of the dimension formula is that the dimension does not change under “alterations” (to be defined later).

Lemma 31.3. *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume that*

- (1) *Y is locally Noetherian,*
- (2) *X and Y are integral schemes,*
- (3) *f is dominant, and*
- (4) *f is locally of finite type.*

Then we have

$$\dim(X) \leq \dim(Y) + \text{trdeg}_{R(Y)} R(X).$$

If f is closed⁶ then equality holds.

Proof. Let $f : X \rightarrow Y$ be as in the lemma. Let $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \dots \rightsquigarrow \xi_e$ be a sequence of specializations in X . We may assume that $x = \xi_e$ is a closed point of X , see Properties, Lemma 5.8. In particular, setting $y = f(x)$, we see x is a closed point of its fibre X_y . By the Hilbert Nullstellensatz we see that $\kappa(x)$ is a finite extension of $\kappa(y)$, see Lemma 21.3. By the dimension formula, Lemma 31.1, we see that

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}) + \text{trdeg}_{R(Y)} R(X)$$

Hence we conclude that $e \leq \dim(Y) + \text{trdeg}_{R(Y)} R(X)$ as desired.

Next, assume f is also closed. Say $\bar{\xi}_0 \rightsquigarrow \bar{\xi}_1 \rightsquigarrow \dots \rightsquigarrow \bar{\xi}_d$ is a sequence of specializations in Y . We want to show that $\dim(X) \geq d + r$. We may assume that $\bar{\xi}_0 = \eta$ is the generic point of Y . The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is nonempty as f is dominant. Hence by Lemma 17.10 it is a Jacobson scheme. Thus by Lemma 17.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y . Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 31.1 applied to the scheme X_η over the universally catenary scheme $\text{Spec}(\kappa(\eta))$ (see Lemma 18.4) and the point ξ_0 . This means that we can find $\xi_{-r} \rightsquigarrow \dots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0$ in X . On the other hand, as f is closed specializations lift along f , see Topology, Lemma 18.6. Thus, as ξ_0 lies over $\eta = \bar{\xi}_0$ we can find specializations $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \dots \rightsquigarrow \xi_d$ lying over $\bar{\xi}_0 \rightsquigarrow \bar{\xi}_1 \rightsquigarrow \dots \rightsquigarrow \bar{\xi}_d$. In other words we have

$$\xi_{-r} \rightsquigarrow \dots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \dots \rightsquigarrow \xi_d$$

⁶For example if f is proper, see Definition 42.1.

which means that $\dim(X) \geq d + r$ as desired. \square

32. Syntomic morphisms

An algebra A over a field k is called a *global complete intersection over k* if $A \cong k[x_1, \dots, x_n]/(f_1, \dots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $\text{Spec}(A)$ can be covered by standard opens each of which are global complete intersections over k . See Algebra, Section 130. Recall that a ring map $R \rightarrow A$ is *syntomic* if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 131.1.

Definition 32.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *syntomic at $x \in X$* if there exists a affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is syntomic.
- (2) We say that f is *syntomic* if it is syntomic at every point of X .
- (3) If $S = \text{Spec}(k)$ and f is syntomic, then we say that X is a *local complete intersection over k* .
- (4) A morphism of affine schemes $f : X \rightarrow S$ is called *standard syntomic* if there exists a global relative complete intersection $R \rightarrow R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ (see Algebra, Definition 131.5) such that $X \rightarrow S$ is isomorphic to

$$\text{Spec}(R[x_1, \dots, x_n]/(f_1, \dots, f_c)) \rightarrow \text{Spec}(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 42 we will consider morphisms $X \rightarrow S$ which locally are of the form

$$\text{Spec}(R[x_1, \dots, x_n]/(f_1, \dots, f_c)) \rightarrow \text{Spec}(R).$$

for some Koszul-regular sequence f_1, \dots, f_r in $R[x_1, \dots, x_n]$. Such a morphism will be called a *local complete intersection morphism*. One we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

Lemma 32.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is syntomic.*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is syntomic.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is syntomic.*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is syntomic, for all $j \in J, i \in I_j$.*

Moreover, if f is syntomic then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is syntomic.

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is syntomic” is local. We check conditions (a), (b) and (c) of Definition 15.1. By Algebra, Lemma 131.3 being syntomic is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 131.17 being syntomic is stable under composition and trivially for any ring R the ring map $R \rightarrow R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 131.4. \square

Lemma 32.3. *The composition of two morphisms which are syntomic is syntomic.*

Proof. In the proof of Lemma 32.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 131.17. \square

Lemma 32.4. *The base change of a morphism which is syntomic is syntomic.*

Proof. In the proof of Lemma 32.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 15.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 131.3. \square

Lemma 32.5. *Any open immersion is syntomic.*

Proof. This is true because an open immersion is a local isomorphism. \square

Lemma 32.6. *A syntomic morphism is locally of finite presentation.*

Proof. True because a syntomic ring map is of finite presentation by definition. \square

Lemma 32.7. *A syntomic morphism is flat.*

Proof. True because a syntomic ring map is flat by definition. \square

Lemma 32.8. *A syntomic morphism is universally open.*

Proof. Combine Lemmas 32.6, 32.7, and 26.9. \square

Let k be a field. Let A be a local k -algebra essentially of finite type over k . Recall that A is called a *complete intersection over k* if we can write $A \cong R/(f_1, \dots, f_c)$ where R is a regular local ring essentially of finite type over k , and f_1, \dots, f_c is a regular sequence in R , see Algebra, Definition 130.5.

Lemma 32.9. *Let k be a field. Let X be a scheme locally of finite type over k . The following are equivalent:*

- (1) X is a local complete intersection over k ,
- (2) for every $x \in X$ there exists an affine open $U = \text{Spec}(R) \subset X$ neighbourhood of x such that $R \cong k[x_1, \dots, x_n]/(f_1, \dots, f_c)$ is a global complete intersection over k , and
- (3) for every $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a complete intersection over k .

Proof. The corresponding algebra results can be found in Algebra, Lemmas 130.8 and 130.9. \square

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a *local model* for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

Lemma 32.10. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite presentation. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

- (1) *The morphism f is syntomic at x .*
- (2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard syntomic.*
- (3) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat and $\mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x}$ is a complete intersection over $\kappa(s)$ (see Algebra, Definition 130.5).*

Proof. Follows from the definitions and Algebra, Lemma 131.15. □

Lemma 32.11. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are local complete intersections, then f is syntomic.*

Proof. Clear from Lemmas 32.9 and 32.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

Lemma 32.12. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \rightarrow S$, consider the base change $f' : X' \rightarrow S'$ of f and the projection $g' : X' \rightarrow X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is syntomic.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \text{Spec}(\kappa(s')) \times_{\text{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 130.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 32.10. □

Lemma 32.13. *Let R be a ring. Let $R \rightarrow A = R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ be a relative global complete intersection. Set $S = \text{Spec}(R)$ and $X = \text{Spec}(A)$. Consider the morphism $f : X \rightarrow S$ associated to the ring map $R \rightarrow A$. The function $x \mapsto \dim_x(X_{f(x)})$ is constant with value $n - c$.*

Proof. By Algebra, Definition 131.5 $R \rightarrow A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime \mathfrak{p} of R the fibre ring $\kappa(\mathfrak{p})[x_1, \dots, x_n]/(\bar{f}_1, \dots, \bar{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 130.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. □

Lemma 32.14. *Let $f : X \rightarrow S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X .*

Proof. By Lemma 32.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 32.13. \square

Lemma 32.14 says that the following definition makes sense.

Definition 32.15. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \rightarrow S$ is *syntomic of relative dimension d* if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d .

Lemma 32.16. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

- (1) f is surjective and syntomic,
- (2) p is syntomic, and
- (3) q is locally of finite presentation⁷.

Then q is syntomic.

Proof. By Lemma 26.11 we see that q is flat. Hence it suffices to show that the fibres of $Y \rightarrow S$ are local complete intersections, see Lemma 32.11. Let $s \in S$. Consider the morphism $X_s \rightarrow Y_s$. This is a base change of the morphism $X \rightarrow Y$ and hence surjective, and syntomic (Lemma 32.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 16.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = \text{Spec}(k)$. Let $y \in Y$. Choose an affine open $\text{Spec}(A) \subset Y$ neighbourhood of y . Let $\text{Spec}(B) \subset X$ be an affine open such that $f(\text{Spec}(B)) \subset \text{Spec}(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \dots, x_n] \rightarrow A$ with kernel I . Choose a surjection $A[y_1, \dots, y_m] \rightarrow B$, which gives rise in turn to a surjection $k[x_i, y_j] \rightarrow B$ with kernel J . Let $\mathfrak{q} \subset k[x_i, y_j]$ be the prime corresponding to $y \in \text{Spec}(B)$ and let $\mathfrak{p} \subset k[x_i]$ the prime corresponding to $x \in \text{Spec}(A)$. Since x maps to y we have $\mathfrak{p} = \mathfrak{q} \cap k[x_i]$. Consider the following commutative diagram of local rings:

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & \xlongequal{\quad} & B_{\mathfrak{q}} & \longleftarrow & k[x_1, \dots, x_n, y_1, \dots, y_m]_{\mathfrak{q}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & \xlongequal{\quad} & A_{\mathfrak{p}} & \longleftarrow & k[x_1, \dots, x_n]_{\mathfrak{p}} \end{array}$$

We claim that the hypotheses of Algebra, Lemma 130.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \rightarrow Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 32.10. The output of Algebra, Lemma 130.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 32.10 again we see that Y is syntomic over k at y as desired. \square

⁷In fact, if f is surjective, flat, and of finite presentation and p is syntomic, then both q and f are syntomic, see Descent, Lemma 10.7.

33. Conormal sheaf of an immersion

Let $i : Z \rightarrow X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \rightarrow \mathcal{I}^2 \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow 0$$

of quasi-coherent sheaves on X . Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by \mathcal{I} it corresponds to a sheaf on Z by Lemma 4.1. This quasi-coherent \mathcal{O}_Z -module is called the *conormal sheaf of Z in X* and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 4.

In case $i : Z \rightarrow X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \rightarrow X \setminus \partial Z$, where $\partial Z = \bar{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where \mathcal{I} is the ideal sheaf of the closed immersion $i : Z \rightarrow X \setminus \partial Z$.

Definition 33.1. Let $i : Z \rightarrow X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent \mathcal{O}_Z -module $\mathcal{I}/\mathcal{I}^2$ described above.

In [DG67, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 33.2. *Let $i : Z \rightarrow X$ be an immersion. The conormal sheaf of i has the following properties:*

- (1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U . Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*

$$\mathcal{C}_{Z/X} = i^*\mathcal{I} = i^{-1}(\mathcal{I}/\mathcal{I}^2)$$

- (2) *For any affine open $\text{Spec}(R) = U \subset X$ such that $Z \cap U = \text{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. \square

Lemma 33.3. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & X \\ f \downarrow & & \downarrow g \\ Z' & \xrightarrow{\quad} & X' \end{array}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of \mathcal{O}_Z -modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(\text{Spec}(R) = U \subset X, \text{Spec}(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = \text{Spec}(R/I)$ and $Z' \cap U' = \text{Spec}(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $f^\sharp : R' \rightarrow R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X . Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus (g^{-1}(\partial Z') \cup \partial Z)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into \mathcal{I} under the canonical map $g^*\mathcal{I}' \rightarrow \mathcal{O}_X$, see Schemes, Lemmas 4.6 and 4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \rightarrow \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \rightarrow i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \rightarrow I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). \square

Lemma 33.4. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & X \\ f \downarrow & i & \downarrow g \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \rightarrow \mathcal{C}_{Z/X}$ of Lemma 33.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \rightarrow R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \rightarrow I/I^2$ is surjective. If $R' \rightarrow R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. \square

Lemma 33.5. *Let $Z \rightarrow Y \rightarrow X$ be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \rightarrow \mathcal{C}_{Z/X} \rightarrow \mathcal{C}_{Z/Y} \rightarrow 0$$

where the maps come from Lemma 33.3 and $i : Z \rightarrow Y$ is the first morphism.

Proof. Via Lemma 33.3 this translates into the following algebra fact. Suppose that $C \rightarrow B \rightarrow A$ are surjective ring maps. Let $I = \text{Ker}(B \rightarrow A)$, $J = \text{Ker}(C \rightarrow A)$ and $K = \text{Ker}(C \rightarrow B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \rightarrow J/J^2 \rightarrow I/I^2 \rightarrow 0.$$

This follows immediately from the observation that $I = J/K$. \square

34. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 127) and the corresponding section in the chapter on sheaves of modules (Modules, Section 23).

Definition 34.1. Let $f : X \rightarrow S$ be a morphism of schemes. The *sheaf of differentials* $\Omega_{X/S}$ of X over S is the sheaf of differentials of f viewed as a morphism of ringed spaces (Modules, Definition 23.10) equipped with its *universal S -derivation*

$$d_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

It turns out that $\Omega_{X/S}$ is a quasi-coherent \mathcal{O}_X -module for example as it is isomorphic to the conormal sheaf of the diagonal morphism $\Delta : X \rightarrow X \times_S X$ (Lemma 34.5). We have defined the module of differentials of X over S using a universal property, namely as the receptacle of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above. For convenience we restate the universal property here.

Lemma 34.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow \mathrm{Der}_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ d_{X/S}$$

is an isomorphism of functors $\mathrm{Mod}(\mathcal{O}_X) \rightarrow \mathrm{Sets}$.

Proof. This is just a restatement of the definition. \square

Lemma 34.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $U \subset X$, $V \subset S$ be open subschemes such that $f(U) \subset V$. Then there is a unique isomorphism $\Omega_{X/S}|_U = \Omega_{U/V}$ of \mathcal{O}_U -modules such that $d_{X/S}|_U = d_{U/V}$.*

Proof. This is a special case of Modules, Lemma 23.5 if we use the canonical identification $f^{-1}\mathcal{O}_S|_U = (f|_U)^{-1}\mathcal{O}_V$. \square

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U .

Lemma 34.4. *Let $R \rightarrow A$ be a ring map. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules on $X = \mathrm{Spec}(A)$. Set $S = \mathrm{Spec}(R)$. The rule which associates to an S -derivation on \mathcal{F} its action on global sections defines a bijection between the set of S -derivations of \mathcal{F} and the set of R -derivations on $M = \Gamma(X, \mathcal{F})$.*

Proof. Let $D : A \rightarrow M$ be an R -derivation. We have to show there exists a unique S -derivation on \mathcal{F} which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n}D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. \square

Lemma 34.5. *Let $f : X \rightarrow S$ be a morphism of schemes. For any pair of affine opens $\text{Spec}(A) = U \subset X$, $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ there is a unique isomorphism*

$$\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}.$$

compatible with $d_{X/S}$ and $d : A \rightarrow \Omega_{A/R}$.

Proof. We claim that the A -module $M = \Gamma(U, \Omega_{X/S}) = \Gamma(U, \Omega_{U/V})$ together with $d_{X/S} = d_{U/V} : A \rightarrow M$ is a universal R -derivation of A . This follows by combining Lemmas 34.4 and 34.2 above. The universal property of $d : A \rightarrow \Omega_{A/R}$ (see Algebra, Lemma 127.3) and the Yoneda lemma (Categories, Lemma 3.5) imply there is a unique isomorphism of A -modules $M \cong \Omega_{A/R}$ compatible with derivations. \square

Remark 34.6. The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \rightarrow S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S . These form a basis for the topology on X . Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U . We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 34.5 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

The following lemma gives yet another way to define the sheaf of differentials and it in particular shows that $\Omega_{X/S}$ is quasi-coherent if X and S are schemes.

Lemma 34.7. *Let $f : X \rightarrow S$ be a morphism of schemes. There is a canonical isomorphism between $\Omega_{X/S}$ and the conormal sheaf of the diagonal morphism $\Delta_{X/S} : X \rightarrow X \times_S X$.*

Proof. We first establish the existence of a couple of “global” sheaves and global maps of sheaves, and further down we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \rightarrow X \times_S X$ is an immersion, see Schemes, Lemma 21.2. Let \mathcal{J} be the ideal sheaf of the immersion which lives over some open subscheme W of $X \times_S X$ such that $\Delta(X) \subset W$ is closed. Let us take the one that was found in the proof of Schemes, Lemma 21.2. Note that the sheaf of rings $\mathcal{O}_W/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \mathcal{O}_W/\mathcal{J}^2 \rightarrow \Delta_*\mathcal{O}_X \rightarrow 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$ -algebras with kernel the conormal sheaf of Δ (see Definition 33.1 and Lemma 33.2).

$$0 \rightarrow \mathcal{C}_{X/X \times_S X} \rightarrow \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \rightarrow \mathcal{O}_X \rightarrow 0$$

This places us in the situation of Modules, Lemma 23.11. The projection morphisms $p_i : X \times_S X \rightarrow X$, $i = 1, 2$ induce maps of sheaves of rings $(p_i)^\# : (p_i)^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{X \times_S X}$. We may restrict to W and quotient by \mathcal{J}^2 to get $(p_i)^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_W/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \rightarrow \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \rightarrow \mathcal{O}_X$, as in Modules, Lemma 23.11. Thus we get an S -derivation $d = s_2 - s_1 : \mathcal{O}_X \rightarrow \mathcal{C}_{X/X \times_S X}$. By the universal property of the module of differentials we find a unique \mathcal{O}_X -linear map

$$\Omega_{X/S} \rightarrow \mathcal{C}_{X/X \times_S X}, \quad fdg \mapsto fs_2(g) - fs_1(g)$$

To see the map is an isomorphism, let us work this out over suitable affine opens. We can cover X by affine opens $\text{Spec}(A) = U \subset X$ whose image is contained in an affine open $\text{Spec}(R) = V \subset S$. According to the proof of Schemes, Lemma 21.2 $U \times_V U \subset X \times_S X$ is an affine open contained in the open W mentioned above. Also $U \times_V U = \text{Spec}(A \otimes_R A)$. The sheaf \mathcal{J} corresponds to the ideal $J = \text{Ker}(A \otimes_R A \rightarrow A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$ -modules

$$0 \rightarrow J/J^2 \rightarrow (A \otimes_R A)/J^2 \rightarrow A \rightarrow 0$$

The sections s_i correspond to the ring maps

$$A \rightarrow (A \otimes_R A)/J^2, \quad s_1 : a \mapsto a \otimes 1, \quad s_2 : a \mapsto 1 \otimes a.$$

By Lemma 33.2 we have $\Gamma(U, \mathcal{C}_{X/X \times_S X}) = J/J^2$ and by Lemma 34.5 we have $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$. The map above is the map $adb \mapsto a \otimes b - ab \otimes 1$ which is shown to be an isomorphism in Algebra, Lemma 127.13. \square

Lemma 34.8. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ \downarrow & \searrow f & \downarrow \\ S' & \xrightarrow{\quad} & S \end{array}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \rightarrow f_*\Omega_{X'/S'}$ is a S -derivation. Hence we obtain a canonical map of \mathcal{O}_X -modules $\Omega_{X/S} \rightarrow f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$ -module homomorphism

$$c_f : f^*\Omega_{X/S} \rightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*d_{X/S}(h)$ maps to $d_{X'/S'}(f^*h)$ for any local section h of \mathcal{O}_X .

Proof. This is a special case of Modules, Lemma 23.12. In the case of schemes we can also use the functoriality of the conormal sheaves (see Lemma 33.3) and Lemma 34.7 to define c_f . Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (127.5.1)). \square

Lemma 34.9. *Let $f : X \rightarrow Y$, $g : Y \rightarrow S$ be morphisms of schemes. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0$$

where the maps come from applications of Lemma 34.8.

Proof. This is the sheafified version of Algebra, Lemma 127.7. \square

Lemma 34.10. *Let $X \rightarrow S$ be a morphism of schemes. Let $g : S' \rightarrow S$ be a morphism of schemes. Let $X' = X_{S'}$ be the base change of X . Denote $g' : X' \rightarrow X$ the projection. Then the map*

$$(g')^*\Omega_{X/S} \rightarrow \Omega_{X'/S'}$$

of Lemma 34.8 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 127.12. \square

Lemma 34.11. *Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \rightarrow X$ and $q : X \times_S Y \rightarrow Y$ be the projection morphisms. The maps from Lemma 34.8*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 34.10 the composition $p^*\Omega_{X/S} \rightarrow \Omega_{X \times_S Y/S} \rightarrow \Omega_{X \times_S Y/Y}$ is an isomorphism, and similarly for q . Moreover, the cokernel of $p^*\Omega_{X/S} \rightarrow \Omega_{X \times_S Y/S}$ is $\Omega_{X \times_S Y/X}$ by Lemma 34.9. The result follows. \square

Lemma 34.12. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is a finite type \mathcal{O}_X -module.*

Proof. Immediate from Algebra, Lemma 127.16, Lemma 34.5, Lemma 16.2, and Properties, Lemma 16.1. \square

Lemma 34.13. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is an \mathcal{O}_X -module of finite presentation.*

Proof. Immediate from Algebra, Lemma 127.15, Lemma 34.5, Lemma 22.2, and Properties, Lemma 16.2. \square

Lemma 34.14. *If $X \rightarrow S$ is an immersion, or more generally a monomorphism, then $\Omega_{X/S}$ is zero.*

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. Hence $\Omega_{X/S} = 0$ by Lemma 34.7. The algebraic version is Algebra, Lemma 127.5. \square

Lemma 34.15. *Let $i : Z \rightarrow X$ be an immersion of schemes over S . There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \rightarrow i^*\Omega_{X/S} \rightarrow \Omega_{Z/S} \rightarrow 0$$

where the first arrow is induced by $d_{X/S}$ and the second arrow comes from Lemma 34.8.

Proof. This is the sheafified version of Algebra, Lemma 127.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of “induced by $d_{X/S}$ ” here. Namely, we may assume that i is a closed immersion by shrinking X . Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $d_{X/S} : \mathcal{I} \rightarrow \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$ -linear. By Lemma 4.1 this corresponds to a map $\mathcal{C}_{Z/X} \rightarrow i^*\Omega_{X/S}$ as desired. \square

Lemma 34.16. *Let $i : Z \rightarrow X$ be an immersion of schemes over S , and assume i (locally) has a left inverse. Then the canonical sequence*

$$0 \rightarrow \mathcal{C}_{Z/X} \rightarrow i^*\Omega_{X/S} \rightarrow \Omega_{Z/S} \rightarrow 0$$

of Lemma 34.15 is (locally) split exact. In particular, if $s : S \rightarrow X$ is a section of the structure morphism $X \rightarrow S$ then the map $\mathcal{C}_{S/X} \rightarrow s^\Omega_{X/S}$ induced by $d_{X/S}$ is an isomorphism.*

Proof. Follows from Algebra, Lemma 127.10. Clarification: if $g : X \rightarrow Z$ is a left inverse of i , then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \rightarrow \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \rightarrow X$ over S (see Schemes, Lemma 21.12) and in that case $\Omega_{Z/S} = 0$. \square

Remark 34.17. Let $X \rightarrow S$ be a morphism of schemes. According to Lemma 34.11 we have

$$\Omega_{X \times_S X/S} = \text{pr}_1^*\Omega_{X/S} \oplus \text{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \rightarrow X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 34.16 we obtain a canonical short exact sequence

$$0 \rightarrow \mathcal{C}_{X/X \times_S X} \rightarrow \Omega_{X/S} \oplus \Omega_{X/S} \rightarrow \Omega_{X/S} \rightarrow 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by Lemma 34.7 we have an identification $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $d_{X/S}(f) = s_2(f) - s_1(f)$ in this identification it turns out that the left arrow is the map $(-1, 1)$ ⁸.

Lemma 34.18. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & X \\ & \searrow i & \downarrow \\ & & Y \\ & \swarrow j & \\ & & \end{array}$$

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \rightarrow \mathcal{C}_{Z/X} \rightarrow i^*\Omega_{X/Y} \rightarrow 0$$

where the first arrow comes from Lemma 33.3 and the second from Lemma 34.15.

Proof. The algebraic version of this is Algebra, Lemma 129.7. \square

35. Smooth morphisms

Let $f : X \rightarrow Y$ be a map of topological spaces. Consider the following condition:

- (*) For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) = V$ and such that there is an isomorphism

$$\begin{array}{ccccc} V \times B_d(0, 1) & \xrightarrow{\cong} & U & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \xlongequal{\quad} & V & \longrightarrow & Y \end{array}$$

where $B_d(0, 1) \subset \mathbf{R}^d$ is a ball of radius 1 around 0.

Smooth morphisms are the analogue of such morphisms in the category of schemes. See Lemma 35.11 and Lemma 37.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \rightarrow A$ is a *smooth ring map* if A is of finite presentation over R and if the naive cotangent complex of

⁸Namely, the local section $d_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $d_{X \times_S X/X}$ to the local section $1 \otimes 1 \otimes f - 1 \otimes f \otimes 1 - 1 \otimes 1 \otimes f + f \otimes 1 \otimes 1 = \text{pr}_2^*d_{X/S}(f) - \text{pr}_1^*d_{X/S}(f)$.

A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 132.1.

Definition 35.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *smooth at* $x \in X$ if there exists an affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is smooth.
- (2) We say that f is *smooth* if it is smooth at every point of X .
- (3) A morphism of affine schemes $f : X \rightarrow S$ is called *standard smooth* if there exists a standard smooth ring map $R \rightarrow R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ (see Algebra, Definition 132.6) such that $X \rightarrow S$ is isomorphic to

$$\text{Spec}(R[x_1, \dots, x_n]/(f_1, \dots, f_c)) \rightarrow \text{Spec}(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

Lemma 35.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is smooth.*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is smooth.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is smooth.*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.*

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is smooth.

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is smooth” is local. We check conditions (a), (b) and (c) of Definition 15.1. By Algebra, Lemma 132.4 being smooth is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 132.14 being smooth is stable under composition and for any ring R the ring map $R \rightarrow R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 132.13. \square

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 15.

Lemma 35.3. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.*

Proof. Follows from Algebra, Lemma 132.16. \square

Lemma 35.4. *The composition of two morphisms which are smooth is smooth.*

Proof. In the proof of Lemma 35.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 132.14. \square

Lemma 35.5. *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 35.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 15.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 132.4. \square

Lemma 35.6. *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. \square

Lemma 35.7. *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 132.10. \square

Lemma 35.8. *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. \square

Lemma 35.9. *A smooth morphism is flat.*

Proof. Combine Lemmas 32.7 and 35.7. \square

Lemma 35.10. *A smooth morphism is universally open.*

Proof. Combine Lemmas 35.9, 35.8, and 26.9. Or alternatively, combine Lemmas 35.7, 32.8. \square

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a *local model* for a smooth morphism.

Lemma 35.11. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

- (1) *The morphism f is smooth at x .*
- (2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 132.7 and 132.10. \square

Lemma 35.12. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$\text{rank}_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S . By Lemma 35.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 132.7 (and the definition of a relative global complete intersection, see Algebra, Definition 131.5). \square

Lemma 35.12 says that the following definition makes sense.

Definition 35.13. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \rightarrow S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d .

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d . By Lemma 35.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d , and (d) $\Omega_{X/S}$ finite locally free of rank d . It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d . A counter example is given by $\text{Spec}(\mathbf{F}_p[t]) \rightarrow \text{Spec}(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 35.14. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

- (1) *The morphism f is smooth at x .*
- (2) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$ -module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.*
- (3) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$ -vector space*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

can be generated by at most $\dim_x(X_{f(x)})$ elements.

- (4) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard smooth.*
- (5) *There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x_1, \dots, x_n]/(f_1, \dots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \dots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \dots & \partial f_c/\partial x_2 \\ \dots & \dots & \dots & \dots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \dots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an element of A not in \mathfrak{q} .

Proof. Note that if f is smooth at x , then we see from Lemma 35.11 that (4) holds, and (5) is a slightly weakened version of (4). Moreover, this implies that the ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat (see Lemma 35.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 35.12). This implies (2) and (3).

By Lemma 34.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S . Hence the displayed equality in part (3) of the lemma. By Lemma 34.12 these modules are of finite type. Hence the minimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$ -vector space by Nakayama's Lemma (Algebra, Lemma 19.1). This in particular shows that (2) and (3) are equivalent.

Combining Algebra, Lemmas 132.16 and 135.3 shows that (2) and (3) imply (1). Finally, (5) implies (4) see for example Algebra, Example 132.8. \square

Lemma 35.15. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \rightarrow S$, consider the base change $f' : X' \rightarrow S'$ of f and the projection $g' : X' \rightarrow X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is smooth.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \text{Spec}(\kappa(s')) \times_{\text{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 132.18. The second part follows from the first because in that case T is the (open) set of points where f is smooth according to Lemma 35.3. \square

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

Lemma 35.16. *Let $f : X \rightarrow Y$, $g : Y \rightarrow S$ be morphisms of schemes. Assume f is smooth. Then*

$$0 \rightarrow f^* \Omega_{Y/S} \rightarrow \Omega_{X/S} \rightarrow \Omega_{X/Y} \rightarrow 0$$

(see Lemma 34.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \rightarrow B \rightarrow C$ with $B \rightarrow C$ smooth, then the sequence

$$0 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0$$

of Algebra, Lemma 127.7 is exact. This is Algebra, Lemma 134.1. \square

Lemma 35.17. *Let $i : Z \rightarrow X$ be an immersion of schemes over S . Assume that Z is smooth over S . Then the canonical exact sequence*

$$0 \rightarrow \mathcal{C}_{Z/X} \rightarrow i^* \Omega_{X/S} \rightarrow \Omega_{Z/S} \rightarrow 0$$

of Lemma 34.15 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \rightarrow B \rightarrow C$ with $A \rightarrow C$ smooth and $B \rightarrow C$ surjective with kernel J , then the sequence

$$0 \rightarrow J/J^2 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow 0$$

of Algebra, Lemma 127.9 is exact. This is Algebra, Lemma 134.2. \square

Lemma 35.18. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow j & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are immersions and $X \rightarrow Y$ is smooth. Then the canonical exact sequence

$$0 \rightarrow \mathcal{C}_{Z/Y} \rightarrow \mathcal{C}_{Z/X} \rightarrow i^* \Omega_{X/Y} \rightarrow 0$$

of Lemma 34.18 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \rightarrow B \rightarrow C$ with $A \rightarrow C$ surjective and $A \rightarrow B$ smooth, then the sequence

$$0 \rightarrow I/I^2 \rightarrow J/J^2 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow 0$$

of Algebra, Lemma 129.7 is exact. This is Algebra, Lemma 134.3. \square

Lemma 35.19. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

- (1) f is surjective, and smooth,
- (2) p is smooth, and
- (3) q is locally of finite presentation⁹.

Then q is smooth.

Proof. By Lemma 26.11 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y . Suppose f has relative dimension a at x and p has relative dimension b at x . By Lemma 35.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a . By the short exact sequence of Lemma 35.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 29.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \rightarrow S$ is smooth at y by Lemma 35.14 part (2). \square

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 14.7.

Lemma 35.20. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $\sigma : S \rightarrow X$ be a section of f . Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $\text{Spec}(A) = U \subset S$ of s and $\text{Spec}(B) = V \subset X$ of x such that*

- (1) $f(V) \subset U$ and $\sigma(U) \subset V$,
- (2) with $I = \text{Ker}(\sigma^\# : B \rightarrow A)$ the module I/I^2 is a free A -module, and
- (3) $B^\wedge \cong A[[x_1, \dots, x_d]]$ as A -algebras where B^\wedge denotes the completion of B with respect to I .

Proof. Pick an affine open $U \subset S$ containing s . Pick an affine open $V \subset f^{-1}(U)$ containing x . Pick an affine open $U' \subset \sigma^{-1}(U)$ containing s . Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = \text{Spec}(A')$ and $V' = \text{Spec}(B')$. By Algebra, Lemma 134.4 the module $I'/(I')^2$ is finite locally free as a A' -module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 134.4. \square

⁹In fact this is implied by (1) and (2), see Descent, Lemma 10.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 10.5.

The dimension of a scheme X at a point x (Properties, Definition 10.1) is just the dimension of X at x as a topological space, see Topology, Definition 9.1. This is not the dimension of the local ring $\mathcal{O}_{X,x}$, in general.

Lemma 35.21. *Let $f : X \rightarrow Y$ be a smooth morphism of locally Noetherian schemes. For every point x in X with image y in Y ,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y),$$

where X_y denotes the fiber over y .

Proof. After replacing X by an open neighborhood of x , there is a natural number d such that all fibers of $X \rightarrow Y$ have dimension d at every point, see Lemma 35.12. Then f is flat (Lemma 35.9), locally of finite type (Lemma 35.8), and of relative dimension d . Hence the result follows from Lemma 30.6. \square

36. Unramified morphisms

We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \rightarrow A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [Ray70]). A ring map $R \rightarrow A$ is called *G-unramified* if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [DG67]). See Algebra, Definition 144.1.

Definition 36.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *unramified at $x \in X$* if there exists a affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is unramified.
- (2) We say that f is *G-unramified at $x \in X$* if there exists a affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is G-unramified.
- (3) We say that f is *unramified* if it is unramified at every point of X .
- (4) We say that f is *G-unramified* if it is G-unramified at every point of X .

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 36.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

Lemma 36.2. *Let $f : X \rightarrow S$ be a morphism of schemes. Then*

- (1) *f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and*
- (2) *f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.*

Proof. By definition a ring map $R \rightarrow A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 34.5. \square

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

Lemma 36.3. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is unramified (resp. G -unramified).*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is unramified (resp. G -unramified).*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is unramified (resp. G -unramified).*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is unramified (resp. G -unramified), for all $j \in J, i \in I_j$.*

Moreover, if f is unramified (resp. G -unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is unramified (resp. G -unramified).

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is unramified” is local. We check conditions (a), (b) and (c) of Definition 15.1. These properties are proved in Algebra, Lemma 144.3. \square

Lemma 36.4. *The composition of two morphisms which are unramified is unramified. The same holds for G -unramified morphisms.*

Proof. The proof of Lemma 36.3 shows that being unramified (resp. G -unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being unramified (resp. G -unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 144.3. \square

Lemma 36.5. *The base change of a morphism which is unramified is unramified. The same holds for G -unramified morphisms.*

Proof. The proof of Lemma 36.3 shows that being unramified (resp. G -unramified) is a local property of ring maps. Hence the lemma follows from Lemma 15.5 combined with the fact that being unramified (resp. G -unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 144.3. \square

Lemma 36.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G -unramified.*

Proof. Follows from the definitions and Lemma 22.9. \square

Lemma 36.7. *Any open immersion is G -unramified.*

Proof. This is true because an open immersion is a local isomorphism. \square

Lemma 36.8. *A closed immersion $i : Z \rightarrow X$ is unramified. It is G -unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = \text{Ker}(\mathcal{O}_X \rightarrow i_*\mathcal{O}_Z)$ is of finite type (as an \mathcal{O}_X -module).*

Proof. Follows from Lemma 22.7 and Algebra, Lemma 144.3. \square

Lemma 36.9. *An unramified morphism is locally of finite type. A G -unramified morphism is locally of finite presentation.*

Proof. An unramified ring map is of finite type by definition. A G -unramified ring map is of finite presentation by definition. \square

Lemma 36.10. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x . In particular, an unramified morphism is locally quasi-finite.*

Proof. See Algebra, Lemma 144.6. \square

Lemma 36.11. *Fibres of unramified morphisms.*

- (1) *Let X be a scheme over a field k . The structure morphism $X \rightarrow \text{Spec}(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k .*
- (2) *If $f : X \rightarrow S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. Part (2) follows from part (1) and Lemma 36.5. Let us prove part (1). We first use Algebra, Lemma 144.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \rightarrow \text{Spec}(k)$ is unramified. Conversely, suppose that $X \rightarrow \text{Spec}(k)$ is unramified. By Algebra, Lemma 144.5 for every $x \in X$ the residue field extension $k \subset \kappa(x)$ is finite separable. Hence all points of X are closed points (see Lemma 21.2 for example). Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 144.5 again these local rings are fields, and we win. \square

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

Lemma 36.12. *Let $f : X \rightarrow S$ be a morphism of schemes.*

- (1) *If f is unramified then for any $x \in X$ the field extension $\kappa(f(x)) \subset \kappa(x)$ is finite separable.*
- (2) *If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.*
- (3) *If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G -unramified.*

Proof. Follows from Algebra, Lemmas 144.5 and 144.7. \square

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

Lemma 36.13. *Let $f : X \rightarrow S$ be a morphism.*

- (1) *If f is unramified, then the diagonal morphism $\Delta : X \rightarrow X \times_S X$ is an open immersion.*
- (2) *If f is locally of finite type and Δ is an open immersion, then f is unramified.*
- (3) *If f is locally of finite presentation and Δ is an open immersion, then f is G -unramified.*

Proof. The first statement follows from Algebra, Lemma 144.4. The second statement from the fact that $\Omega_{X/S}$ is the conormal sheaf of the diagonal morphism (Lemma 34.7) and hence clearly zero if Δ is an open immersion. \square

Lemma 36.14. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:*

- (1) *The morphism f is unramified (resp. G -unramified) at x .*
- (2) *The fibre X_s is unramified over $\kappa(s)$ at x .*
- (3) *The $\mathcal{O}_{X,x}$ -module $\Omega_{X/S,x}$ is zero.*
- (4) *The $\mathcal{O}_{X_s,x}$ -module $\Omega_{X_s/s,x}$ is zero.*
- (5) *The $\kappa(x)$ -vector space*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is zero.

- (6) *We have $\mathfrak{m}_s \mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*

Proof. Note that if f is unramified at x , then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 34. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 34.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S . This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 34.12 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$ -vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 19.1). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X , see Modules, Lemma 9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x , which implies (1). The equivalence of (1) and (3) applied to $X_s \rightarrow s$ implies the equivalence of (2) and (4). At this point we have seen that (1) – (5) are equivalent.

Alternatively you can use Algebra, Lemma 144.3 to see the equivalence of (1) – (5) more directly.

The equivalence of (1) and (6) follows from Lemma 36.12. It also follows more directly from Algebra, Lemmas 144.5 and 144.7. \square

Lemma 36.15. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set*

$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$

commutes with arbitrary base change: For any morphism $g : S' \rightarrow S$, consider the base change $f' : X' \rightarrow S'$ of f and the projection $g' : X' \rightarrow X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G -unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \text{Spec}(\kappa(s')) \times_{\text{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'}/s',x'}} \kappa(x')$$

see Lemma 34.10. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 36.14. The second part follows from the first because in that case T is the (open) set of points where f is G -unramified according to Lemma 36.14. \square

Lemma 36.16. *Let $f : X \rightarrow Y$ be a morphism of schemes over S .*

- (1) *If X is unramified over S , then f is unramified.*
- (2) *If X is G -unramified over S and Y of finite type over S , then f is G -unramified.*

Proof. Assume that X is unramified over S . By Lemma 16.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 34.9. Thus f is unramified. If X is G -unramified over S and Y of finite type over S , then by Lemma 22.11 we see that f is locally of finite presentation and we conclude that f is G -unramified. \square

Lemma 36.17. *Let S be a scheme. Let X, Y be schemes over S . Let $f, g : X \rightarrow Y$ be morphisms over S . Let $x \in X$. Assume that*

- (1) *the structure morphism $Y \rightarrow S$ is unramified,*
- (2) *$f(x) = g(x)$ in Y , say $y = f(x) = g(x)$, and*
- (3) *the induced maps $f^\#, g^\# : \kappa(y) \rightarrow \kappa(x)$ are equal.*

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \rightarrow Y \times_S Y$. By assumption (1) and Lemma 36.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X . And assumptions (2) and (3) imply that x is in this open subset. \square

37. Étale morphisms

The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over \mathbf{C} . It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over \mathbf{C} this topology gives rise to the “correct” Betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \rightarrow Y$ is an étale morphism of varieties over \mathbf{C} , and if x is a closed point of X , then f induces an isomorphism $\mathcal{O}_{Y, f(x)}^\wedge \rightarrow \mathcal{O}_{X, x}^\wedge$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \rightarrow A$ is said to be *étale* if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 138.1.

Definition 37.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *étale at $x \in X$* if there exists a affine open neighbourhood $\text{Spec}(A) = U \subset X$ of x and affine open $\text{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \rightarrow A$ is étale.
- (2) We say that f is *étale* if it is étale at every point of X .
- (3) A morphism of affine schemes $f : X \rightarrow S$ is called *standard étale* if $X \rightarrow S$ is isomorphic to

$$\text{Spec}(R[x]_g/(f)) \rightarrow \text{Spec}(R)$$

where $R \rightarrow R[x]_g/(f)$ is a standard étale ring map, see Algebra, Definition 138.14, i.e., f is monic and f' invertible in $R[x]_g/(f)$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 35.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

Lemma 37.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *The morphism f is étale.*
- (2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \rightarrow \mathcal{O}_X(U)$ is étale.*
- (3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \rightarrow V_j$, $j \in J, i \in I_j$ is étale.*
- (4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \rightarrow \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.*

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \rightarrow V$ is étale.

Proof. This follows from Lemma 15.3 if we show that the property “ $R \rightarrow A$ is étale” is local. We check conditions (a), (b) and (c) of Definition 15.1. These all follow from Algebra, Lemma 138.3. \square

Lemma 37.3. *The composition of two morphisms which are étale is étale.*

Proof. In the proof of Lemma 37.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 15.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 138.3. \square

Lemma 37.4. *The base change of a morphism which is étale is étale.*

Proof. In the proof of Lemma 37.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 15.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 138.3. \square

Lemma 37.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x .*

Proof. This follows immediately from the definitions. \square

Lemma 37.6. *An étale morphism is locally quasi-finite.*

Proof. By Lemma 37.5 an étale morphism is unramified. By Lemma 36.10 an unramified morphism is locally quasi-finite. \square

Lemma 37.7. *Fibres of étale morphisms.*

- (1) *Let X be a scheme over a field k . The structure morphism $X \rightarrow \text{Spec}(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k .*

- (2) If $f : X \rightarrow S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.

Proof. You can deduce this from Lemma 36.11 via Lemma 37.5 above. Here is a direct proof.

We will use Algebra, Lemma 138.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \rightarrow \text{Spec}(k)$ is étale. Conversely, suppose that $X \rightarrow \text{Spec}(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k . Hence all points of X are closed points (see Lemma 21.2 for example). Thus X is a discrete space and we win. \square

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with “étale fibres”.

Lemma 37.8. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.*

Proof. You can deduce this from Algebra, Lemma 138.7. Here is another proof.

By Lemma 37.7 a fibre X_s is étale and hence smooth over s . By Lemma 35.3 we see that $X \rightarrow S$ is smooth. By Lemma 36.12 we see that f is unramified. We conclude by Lemma 37.5. \square

Lemma 37.9. *Any open immersion is étale.*

Proof. This is true because an open immersion is a local isomorphism. \square

Lemma 37.10. *An étale morphism is syntomic.*

Proof. See Algebra, Lemma 132.10 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. \square

Lemma 37.11. *An étale morphism is locally of finite presentation.*

Proof. True because an étale ring map is of finite presentation by definition. \square

Lemma 37.12. *An étale morphism is flat.*

Proof. Combine Lemmas 32.7 and 37.10. \square

Lemma 37.13. *An étale morphism is open.*

Proof. Combine Lemmas 37.12, 37.11, and 26.9. \square

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a *local model* for a general étale morphism.

Lemma 37.14. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

- (1) *The morphism f is étale at x .*
- (2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard étale (see Definition 37.1).*

Proof. Follows from the definitions and Algebra, Proposition 138.17. \square

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 37.15. *Let $f : X \rightarrow S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

- (1) *The morphism f is étale at x .*
- (2) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$ -module $\Omega_{X/S,x}$ is zero.*
- (3) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$ -vector space*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is zero.

- (4) *The local ring map $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s \mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*
- (5) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard smooth of relative dimension 0.*
- (6) *There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x_1, \dots, x_n]/(f_1, \dots, f_n)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \dots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \dots & \partial f_n/\partial x_2 \\ \dots & \dots & \dots & \dots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \dots & \partial f_n/\partial x_n \end{pmatrix}$$

mapping to an element of A not in \mathfrak{q} .

- (7) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \rightarrow V$ is standard étale.*
- (8) *There exist affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$

with $P, Q \in R[x]$, P monic and $P' = dP/dx$ mapping to an element of A not in \mathfrak{q} .

Proof. Use Lemma 37.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) – (7) combine Lemmas 35.14 and 36.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 37.5. Some details omitted. \square

Lemma 37.16. *A morphism is étale at a point if and only if it is flat and G -unramified at that point. A morphism is étale if and only if it is flat and G -unramified.*

Proof. This is clear from Lemmas 37.15 and 36.14. \square

Lemma 37.17. *Let $f : X \rightarrow S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is étale over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \rightarrow S$, consider the base change $f' : X' \rightarrow S'$ of f and the projection $g' : X' \rightarrow X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed locally of finite presentation and flat then the same holds for the open set of points where f is étale.

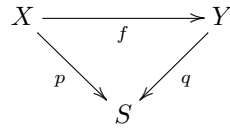
Proof. Combine Lemmas 37.16 and 36.15. □

Our proof of the following lemma is somewhat complicated. It uses the “Critère de platitude par fibres” to see that a morphism $X \rightarrow Y$ over S between schemes étale over S is automatically flat. The details are in the chapter on commutative algebra.

Lemma 37.18. *Let $f : X \rightarrow Y$ be a morphism of schemes over S . If X and Y are étale over S , then f is étale.*

Proof. See Algebra, Lemma 138.9. □

Lemma 37.19. *Let*



be a commutative diagram of morphisms of schemes. Assume that

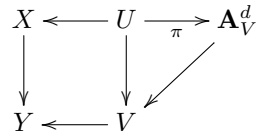
- (1) f is surjective, and étale,
- (2) p is étale, and
- (3) q is locally of finite presentation¹⁰.

Then q is étale.

Proof. By Lemma 35.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 29.2 and the fact that f and p have relative dimension 0. □

A final characterization of smooth morphisms is that a smooth morphism $f : X \rightarrow S$ is locally the composition of an étale morphism by a projection $\mathbf{A}_S^d \rightarrow S$.

Lemma 37.20. *Let $\varphi : X \rightarrow Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x , then there exist an integer $d \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*



where π is étale.

¹⁰In fact this is implied by (1) and (2), see Descent, Lemma 10.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 10.5.

Proof. By Lemma 35.11 we can find affine opens U and V as in the lemma such that $\varphi|_U : U \rightarrow V$ is standard smooth. Write $U = \text{Spec}(A)$ and $V = \text{Spec}(R)$ so that we can write

$$A = R[x_1, \dots, x_n]/(f_1, \dots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \dots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \dots & \partial f_c/\partial x_2 \\ \dots & \dots & \dots & \dots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \dots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A . Then it is clear that $R[x_{c+1}, \dots, x_n] \rightarrow A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \dots, x_n] \rightarrow A$ is étale. As $\mathbf{A}_V^{n-c} = \text{Spec}(R[x_{c+1}, \dots, x_n])$ the lemma with $d = n - c$. \square

38. Relatively ample sheaves

Let X be a scheme and \mathcal{L} an invertible sheaf on X . Then \mathcal{L} is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s , where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 24.1. We relativize this as follows.

Definition 38.1. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. We say \mathcal{L} is *relatively ample*, or *f -relatively ample*, or *ample on X/S* , or *f -ample* if $f : X \rightarrow S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of \mathcal{L} to the open subscheme $f^{-1}(V)$ of X is ample.

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \rightarrow S$ to be of finite type.

Lemma 38.2. *Let $X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $n \geq 1$. Then \mathcal{L} is f -ample if and only if $\mathcal{L}^{\otimes n}$ is f -ample.*

Proof. This follows from Properties, Lemma 24.2. \square

Lemma 38.3. *Let $f : X \rightarrow S$ be a morphism of schemes. If there exists an f -ample invertible sheaf, then f is separated.*

Proof. Being separated is local on the base (see Schemes, Lemma 21.8 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 24.9. \square

There are many ways to characterize relatively ample invertible sheaves, by relativizing any of the list of equivalent conditions in Properties, Proposition 24.14. We will add these here as needed.

Lemma 38.4. *Let $f : X \rightarrow S$ be a quasi-compact morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . The following are equivalent:*

- (1) *The invertible sheaf \mathcal{L} is f -ample.*
- (2) *There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \rightarrow V_i$.*
- (3) *There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.*

- (4) There exists a quasi-coherent graded \mathcal{O}_S -algebra \mathcal{A} and a map of graded \mathcal{O}_X -algebras $\psi : f^*\mathcal{A} \rightarrow \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and

$$r_{\mathcal{L}, \psi} : X \longrightarrow \underline{\text{Proj}}_S(\mathcal{A})$$

is an open immersion (see Constructions, Lemma 19.1 for notation).

- (5) The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.
- (6) Same as (4) but just requiring $r_{\mathcal{L}, \psi}$ to be an immersion.

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (see Properties, Lemma 24.12 and Constructions, Lemma 8.8). Hence f is separated. By Schemes, Lemma 24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded \mathcal{O}_S -algebra. Denote $\psi : f^*\mathcal{A} \rightarrow \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 19 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 19.1 part (3) shows that the restriction of $r_{\mathcal{L}, \psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 24.10 which is an open immersion according to Properties, Lemma 24.12. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\text{Proj}}_S(\mathcal{A}) \rightarrow S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 16.7 we see that $\pi^{-1}(V) \subset \underline{\text{Proj}}_S(\mathcal{A})$ is equal to $\text{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L}, \psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\text{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\text{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 19.1 and the final statement of Constructions, Lemma 14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 24.13 and 24.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S . Hence we may assume that S is affine, and we have to show that \mathcal{L} is ample on X . In this case the morphism $r_{\mathcal{L}, \psi}$ is identified with the morphism, also denoted $r_{\mathcal{L}, \psi} : X \rightarrow \text{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \rightarrow \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\text{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \text{Proj}(A)$. By Constructions, Lemma 10.6 (see also Properties, Lemma 24.13) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 24.3 we conclude that the pullback of $\mathcal{O}_Y^{d'}$ is ample. Combining these results with Properties, Lemma 24.2 we conclude that \mathcal{L} is ample as desired. \square

Lemma 38.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume S affine. Then \mathcal{L} is f -relatively ample if and only if \mathcal{L} is ample on X .*

Proof. Immediate from Lemma 38.4 and the definitions. \square

Lemma 38.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Then f is quasi-affine if and only if \mathcal{O}_X is f -relatively ample.*

Proof. Follows from Properties, Lemma 25.1 and the definitions. \square

Lemma 38.7. *Let $f : X \rightarrow Y$ be a morphism of schemes, \mathcal{M} an invertible \mathcal{O}_Y -module, and \mathcal{L} an invertible \mathcal{O}_X -module.*

- (1) *If \mathcal{L} is f -ample and \mathcal{M} is ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample for $a \gg 0$.*
- (2) *If \mathcal{M} is ample and f quasi-affine, then $f^*\mathcal{M}$ is ample.*

Proof. Assume \mathcal{L} is f -ample and \mathcal{M} ample. By assumption Y and f are quasi-compact (see Definition 38.1 and Properties, Definition 24.1). Hence X is quasi-compact. Pick $x \in X$. We can choose $m \geq 1$ and $t \in \Gamma(Y, \mathcal{M}^{\otimes m})$ such that Y_t is affine and $f(x) \in Y_t$. Since \mathcal{L} restricts to an ample invertible sheaf on $f^{-1}(Y_t) = X_{f^*t}$ we can choose $n \geq 1$ and $s \in \Gamma(X_{f^*t}, \mathcal{L}^{\otimes n})$ with $x \in (X_{f^*t})_s$ with $(X_{f^*t})_s$ affine. By Properties, Lemma 24.6 there exists an integer $e \geq 1$ and a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes em})$ which restricts to $s(f^*t)^e$ on X_{f^*t} . For any $b > 0$ consider the section $s'' = s'(f^*t)^b$ of $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$. Then $X_{s''} = (X_{f^*t})_s$ is an affine open of X containing x . Picking b such that n divides $e + b$ we see $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$ is the n th power of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ for some a and we can get any a divisible by m and big enough. Since X is quasi-compact a finite number of these affine opens cover X . We conclude that for some a sufficiently divisible and large enough the invertible sheaf $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on X . On the other hand, we know that $\mathcal{M}^{\otimes c}$ (and hence its pullback to X) is globally generated for all $c \gg 0$ by Properties, Proposition 24.14. Thus $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a+c}$ is ample (Properties, Lemma 24.5) for $c \gg 0$ and (1) is proved.

Part (2) follows from Lemma 38.6, Properties, Lemma 24.2, and part (1). \square

Lemma 38.8. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Let $S' \rightarrow S$ be a morphism of schemes. Let $f' : X' \rightarrow S'$ be the base change of f and denote \mathcal{L}' the pullback of \mathcal{L} to X' . If \mathcal{L} is f -ample, then \mathcal{L}' is f' -ample.*

Proof. By Lemma 38.4 it suffices to find an affine open covering $S' = \bigcup U'_i$ such that \mathcal{L}' restricts to an ample invertible sheaf on $(f')^{-1}(U'_i)$ for all i . We may choose U'_i mapping into an affine open $U_i \subset S$. In this case the morphism $(f')^{-1}(U'_i) \rightarrow f^{-1}(U_i)$ is affine as a base change of the affine morphism $U'_i \rightarrow U_i$ (Lemma 13.8). Thus $\mathcal{L}'|_{(f')^{-1}(U'_i)}$ is ample by Lemma 38.7. \square

39. Very ample sheaves

Recall that given a quasi-coherent sheaf \mathcal{E} on a scheme S the *projective bundle* associated to \mathcal{E} is the morphism $\mathbf{P}(\mathcal{E}) \rightarrow S$, where $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\text{Sym}(\mathcal{E}))$, see Constructions, Definition 21.1.

Definition 39.1. Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. We say \mathcal{L} is *relatively very ample* or more precisely *f -relatively very ample*, or *very ample on X/S* , or *f -very ample* if there exist a quasi-coherent \mathcal{O}_S -module \mathcal{E} and an immersion $i : X \rightarrow \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

Lemma 39.2. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. If f is quasi-compact and \mathcal{L} is a relatively very ample invertible sheaf, then \mathcal{L} is a relatively ample invertible sheaf.*

Proof. By definition there exists quasi-coherent \mathcal{O}_S -module \mathcal{E} and an immersion $i : X \rightarrow \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \text{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\text{Proj}}_S(\mathcal{A})$ by definition. The graded \mathcal{O}_S -algebra \mathcal{A} comes equipped with a map

$$\psi : \mathcal{A} \rightarrow \bigoplus_{n \geq 0} \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \rightarrow \bigoplus_{n \geq 0} f_* \mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^* \mathcal{A} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L}, \psi}$ associated to this map is exactly the immersion i . Hence the result follows from part (6) of Lemma 38.4. \square

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf \mathcal{L} there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

- (1) Even if S is affine, it can happen that no finite integer n works because $X \rightarrow S$ is not of finite type, see Example 39.4.
- (2) The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the d th power. See Example 39.5. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $\text{Spec}(k)$ gives an example.

To see our version of the converse take a look at Lemma 40.5 below. We will do some preliminary work before proving it.

Example 39.3. Let S be a scheme. Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_S -algebra generated by \mathcal{A}_1 over \mathcal{A}_0 . Set $X = \underline{\text{Proj}}_S(\mathcal{A})$. In this case $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X . Namely, the morphism associated to the graded \mathcal{O}_S -algebra map

$$\text{Sym}_{\mathcal{O}_X}^*(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \rightarrow \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 18.5.

Example 39.4. Let k be a field. Consider the graded k -algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \dots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \dots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \text{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f -ample for the morphism $f : X \rightarrow \text{Spec}(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f -relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A . Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k . On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \dots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \dots)$ which is not of finite type over k .

Example 39.5. Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be pairwise distinct elements of k^* . (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k -algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod_{i=1}^{2d} (U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \text{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d , and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

Lemma 39.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . If \mathcal{L} is relatively very ample on X/S then f is separated.*

Proof. Being separated is local on the base (see Schemes, Section 21). An immersion is separated (see Schemes, Lemma 23.7). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 8.8. \square

Lemma 39.7. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . Assume f is quasi-compact. The following are equivalent*

- (1) \mathcal{L} is relatively very ample on X/S ,
- (2) there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j ,
- (3) there exists a quasi-coherent sheaf of graded \mathcal{O}_S -algebras \mathcal{A} generated in degree 1 over \mathcal{O}_S and a map of graded \mathcal{O}_X -algebras $\psi : f^*\mathcal{A} \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \rightarrow \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L}, \psi} : X \rightarrow \text{Proj}_S(\mathcal{A})$ is an immersion, and
- (4) f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \rightarrow \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L}, \psi} : X \rightarrow \mathbf{P}(f_*\mathcal{L})$ is an immersion.

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-separation in (4) is used to guarantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \rightarrow V_j$ the restriction of f . We see that f is separated by Lemma 39.6 (as being separated is local on the base). Consider the map $\psi : f^*f_*\mathcal{L} \rightarrow \mathcal{L}$. On each V_j there exists a quasi-coherent sheaf \mathcal{E}_j and an embedding $i : X_j \rightarrow \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}_{X_j} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. In other words there is a map $\mathcal{E}_j \rightarrow (f_*\mathcal{L})|_{X_j}$ such that the composition

$$f_j^*\mathcal{E}_j \rightarrow (f^*f_*\mathcal{L})|_{X_j} \rightarrow \mathcal{L}|_{X_j}$$

is surjective. Hence we conclude that ψ is surjective. Let $r_{\mathcal{L}, \psi} : X \rightarrow \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. Using the maps $\mathcal{E}_j \rightarrow (f_*\mathcal{L})|_{X_j}$ we see that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L}, \psi}} \mathbf{P}(f_*\mathcal{L})|_{V_j} \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

which shows that $r_{\mathcal{L}, \psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let \mathcal{A} be a quasi-coherent sheaf of graded \mathcal{O}_S -algebras generated in degree 1 over \mathcal{O}_S . Consider the map of graded \mathcal{O}_S -algebras

$\mathrm{Sym}(\mathcal{A}_1) \rightarrow \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see Constructions, Lemma 18.5. Hence it is clear that (3) implies (1). \square

40. Ample and very ample sheaves relative to finite type morphisms

In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

Lemma 40.1. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . Assume that*

- (1) *the invertible sheaf \mathcal{L} is very ample on X/S ,*
- (2) *the morphism $X \rightarrow S$ is of finite type, and*
- (3) *S is affine.*

Then there exists an $n \geq 0$ and an immersion $i : X \rightarrow \mathbf{P}_S^n$ over S such that $\mathcal{L} \cong i^ \mathcal{O}_{\mathbf{P}_S^n}(1)$.*

Proof. Assume (1), (2) and (3). Condition (3) means $S = \mathrm{Spec}(R)$ for some ring R . Condition (1) means by definition there exists a quasi-coherent \mathcal{O}_S -module \mathcal{E} and an immersion $\alpha : X \rightarrow \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R -module M . Thus we have

$$\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\mathrm{Proj}(\mathrm{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \mathrm{Sym}_R^d(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \mathrm{Sym}_R^d(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \rightarrow D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \mathrm{Sym}_R^{d_j}(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = \mathrm{Spec}(A_j)$. Condition (2) also implies that A_j is of finite type over R , see Lemma 16.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R -algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \mathrm{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \dots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \mathrm{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R . Consider the graded ring map

$$\psi : R[Y_0, \dots, Y_n] \longrightarrow \mathrm{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote $F_j, F_{j,k}$ the elements of $R[Y_0, \dots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 11.1 we obtain an open subscheme

$$U(\psi) \subset \mathrm{Proj}(\mathrm{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \rightarrow \mathbf{P}_R^n$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}_R^n$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^* \mathcal{O}_{\mathbf{P}_R^n}(1) \rightarrow \mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^* \mathcal{O}_{\text{Proj}(\text{Sym}_R(M))}(1)$ we win. \square

Lemma 40.2. *Let $\pi : X \rightarrow S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \rightarrow \mathbf{A}_S^n$ over S .*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $\text{Spec}(A)$, see Properties, Lemma 15.4. Moreover, the set of opens X_f , for those $f \in A$ such that X_f is affine, forms a basis for the topology of X , see the proof of Properties, Lemma 15.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \dots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 16.2 the ring maps $\mathcal{O}(V_j) \rightarrow \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \dots, a_N \in A$ such that the elements $a_1, \dots, a_N, f_1, \dots, f_m, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j . Take $n = N + m$ and let

$$i : X \longrightarrow \mathbf{A}_S^n$$

be the morphism given by the global sections $a_1, \dots, a_N, f_1, \dots, f_m$ of the structure sheaf of X . Let $D(x_j) \subset \mathbf{A}_S^n$ be the open subscheme where the j th coordinate function is nonzero. Then it is clear that $i^{-1}(D(x_j))$ is X_{f_j} and that the induced morphism $X_{f_j} \rightarrow D(x_j)$ factors through the affine open $\text{Spec}(\mathcal{O}(V_j)[x_1, \dots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \dots, x_n, 1/x_j] \rightarrow A_{f_j}$ is surjective by construction we conclude that the restriction of i to X_{f_j} is an immersion as desired. \square

Lemma 40.3. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . Assume that*

- (1) *the invertible sheaf \mathcal{L} is ample on X , and*
- (2) *the morphism $X \rightarrow S$ is locally of finite type.*

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exists an $n \geq 0$ and an immersion $i : X \rightarrow \mathbf{P}_S^n$ over S such that $\mathcal{L}^{\otimes d} \cong i^ \mathcal{O}_{\mathbf{P}_S^n}(1)$.*

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 24.14 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X . Hence we can find finitely many such elements $a_0, \dots, a_n \in A_+$ such that

- (1) we have $X = \bigcup_{i=0, \dots, n} X_{a_i}$,
- (2) each X_{a_i} is affine, and
- (3) each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 16.2 we see that the ring maps $\mathcal{O}_S(V_i) \rightarrow \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \dots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$ -algebra. By Properties, Lemma 24.6 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer

which is a common multiple of all the degrees of the elements a_i, a_{ij} . Consider the elements

$$a_i^{N/\deg(a_i)}, a_{ij} a_i^{(N/\deg(a_i)) - e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X . Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}_S^m \quad \text{with } m = n + \sum n_i$$

over S , see Constructions, Lemma 13.1 and Definition 13.2. Moreover, $j^* \mathcal{O}_{\mathbf{P}_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates T_0, \dots, T_n, T_{ij} instead of T_0, \dots, T_m . For $i = 0, \dots, n$ we have $i^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via j^\sharp we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}_{V_i}^m$ of \mathbf{P}_S^m . Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \rightarrow \mathbf{P}_S^m$ given by global sections $s'_0, \dots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 24.14 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0 . Namely, given an integer $d \geq d_0$ we may choose $s''_1, \dots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d-d_2})$ which generate $\mathcal{L}^{\otimes d-d_2}$ over X . Set $n = (m+1)t$ and denote s_0, \dots, s_n the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \dots, m$, $\beta = 1, \dots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}_S^n$$

such that $i^* \mathcal{O}_{\mathbf{P}_S^n}(1) \cong \mathcal{L}^{\otimes d}$. We omit the verification that since j was an immersion also the morphism i so obtained is an immersion also. (Hint: Segre embedding.) \square

Lemma 40.4. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume S affine and f of finite type. The following are equivalent*

- (1) \mathcal{L} is ample on X ,
- (2) \mathcal{L} is f -ample,
- (3) $\mathcal{L}^{\otimes d}$ is f -very ample for some $d \geq 1$,
- (4) $\mathcal{L}^{\otimes d}$ is f -very ample for all $d \gg 1$,
- (5) for some $d \geq 1$ there exist $n \geq 1$ and an immersion $i : X \rightarrow \mathbf{P}_S^n$ such that $\mathcal{L}^{\otimes d} \cong i^* \mathcal{O}_{\mathbf{P}_S^n}(1)$, and
- (6) for all $d \gg 1$ there exist $n \geq 1$ and an immersion $i : X \rightarrow \mathbf{P}_S^n$ such that $\mathcal{L}^{\otimes d} \cong i^* \mathcal{O}_{\mathbf{P}_S^n}(1)$.

Proof. The equivalence of (1) and (2) is Lemma 38.5. The implication (2) \Rightarrow (6) is Lemma 40.3. Trivially (6) implies (5). As \mathbf{P}_S^n is a projective bundle over S (see Constructions, Lemma 21.4) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 39.2 and Lemma 38.2. \square

Lemma 40.5. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible \mathcal{O}_X -module. Assume S quasi-compact and f of finite type. The following are equivalent*

- (1) \mathcal{L} is f -ample,

- (2) $\mathcal{L}^{\otimes d}$ is f -very ample for some $d \geq 1$,
- (3) $\mathcal{L}^{\otimes d}$ is f -very ample for all $d \gg 1$.

Proof. Trivially (3) implies (2). Lemma 39.2 guarantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that \mathcal{L} is f -ample. Choose a finite affine open covering $S = V_1 \cup \dots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 40.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 39.7. \square

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

Lemma 40.6. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . Assume f is of finite type. The following are equivalent:*

- (1) \mathcal{L} is f -relatively very ample, and
- (2) there exist an open covering $S = \bigcup V_j$, for each j an integer n_j , and immersions

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}_{V_j}^{n_j}$$

over V_j such that $\mathcal{L}|_{X_j} \cong i_j^* \mathcal{O}_{\mathbf{P}_{V_j}^{n_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 40.1 to each of the restrictions of f and \mathcal{L} . We see that (2) implies (1) by Lemma 39.7. \square

Lemma 40.7. *Let $f : X \rightarrow S$ be a morphism of schemes. Let \mathcal{L} be an invertible sheaf on X . Assume f is of finite type. The following are equivalent:*

- (1) \mathcal{L} is f -relatively ample, and
- (2) there exist an open covering $S = \bigcup V_j$, for each j an integers $d_j \geq 1$, $n_j \geq 0$, and immersions

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}_{V_j}^{n_j}$$

over V_j such that $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^* \mathcal{O}_{\mathbf{P}_{V_j}^{n_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 40.4 to each of the restrictions of f and \mathcal{L} . We see that (2) implies (1) by Lemma 38.4. \square

41. Quasi-projective morphisms

The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [DG67]. The version with the letter ‘‘H’’ is the definition in [Har77].

Definition 41.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say f is *quasi-projective* if f is of finite type and there exists an f -relatively ample invertible \mathcal{O}_X -module.

- (2) We say f is *H-quasi-projective* if f if there exists a quasi-compact immersion $X \rightarrow \mathbf{P}_S^n$ over S for some n .¹¹
- (3) We say f is *locally quasi-projective* if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \rightarrow V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S .

Lemma 41.2. *Let $f : X \rightarrow S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.*

Proof. Omitted. □

Lemma 41.3. *A H-quasi-projective morphism is quasi-projective.*

Proof. Omitted. □

Lemma 41.4. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is locally quasi-projective.*
- (2) *There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \rightarrow V_j$ is H-quasi-projective.*

Proof. By Lemma 41.3 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and \mathcal{L} is a relatively ample invertible sheaf on X/S . By Lemma 40.4 we may assume \mathcal{L} is ample on X . By Lemma 40.3 we see that there exists an immersion of X into a projective space over S , i.e., X is H-quasi-projective over S as desired. □

42. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}_S^n \rightarrow S$ of projective n -space, and this is in fact the motivating example leading to the definition.

Definition 42.1. Let $f : X \rightarrow S$ be a morphism of schemes. We say f is *proper* if f is separated, finite type, and universally closed.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

Lemma 42.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is universally closed.*
- (2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \rightarrow V_j$ is universally closed for all indices j .*

Proof. This is clear from the definition. □

¹¹This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 43.1), which does not imply f is quasi-compact. See Lemma 43.11 for the implication in the other direction.

Lemma 42.3. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is proper.*
- (2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \rightarrow V_j$ is proper for all indices j .*

Proof. Omitted. □

Lemma 42.4. *The composition of proper morphisms is proper. The same is true for universally closed morphisms.*

Proof. A composition of closed morphisms is closed. If $X \rightarrow Y \rightarrow Z$ are universally closed morphisms and $Z' \rightarrow Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \rightarrow Z' \times_Z Y$ is closed and $Z' \times_Z Y \rightarrow Z'$ is closed. Hence the result for universally closed morphisms. We have seen that “separated” and “finite type” are preserved under compositions (Schemes, Lemma 21.13 and Lemma 16.3). Hence the result for proper morphisms. □

Lemma 42.5. *The base change of a proper morphism is proper. The same is true for universally closed morphisms.*

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 21.13). It is true for morphisms of finite type (Lemma 16.4). Hence it is true for proper morphisms. □

Lemma 42.6. *A closed immersion is proper, hence a fortiori universally closed.*

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 23.7). A closed immersion is of finite type (Lemma 16.5). Hence a closed immersion is proper. □

Lemma 42.7. *Suppose given a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

with Y separated over S .

- (1) *If $X \rightarrow S$ is universally closed, then the morphism $X \rightarrow Y$ is universally closed.*
- (2) *If X proper over S , then the morphism $X \rightarrow Y$ is proper.*

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \rightarrow S$ is universally closed (resp. proper). We factor the morphism as $X \rightarrow X \times_S Y \rightarrow Y$. The first morphism is a closed immersion, see Schemes, Lemma 21.11. Hence the first morphism is proper (Lemma 42.6). The projection $X \times_S Y \rightarrow Y$ is the base change of a universally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 42.5. Thus $X \rightarrow Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 42.4). □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

Lemma 42.8. *Let S be a scheme. Let $f : X \rightarrow Y$ be a morphism of schemes over S . If X is universally closed over S and f is surjective then Y is universally closed over S . In particular, if also Y is separated and of finite type over S , then Y is proper over S .*

Proof. Assume X is universally closed and f surjective. Denote $p : X \rightarrow S$, $q : Y \rightarrow S$ the structure morphisms. Let $S' \rightarrow S$ be a morphism of schemes. The base change $f' : X_{S'} \rightarrow Y_{S'}$ is surjective (Lemma 11.4), and the base change $p' : X_{S'} \rightarrow S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. \square

Lemma 42.9. *Suppose given a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{\quad h \quad} & Y \\ & \searrow f & \swarrow g \\ & & S \end{array}$$

Assume

- (1) $X \rightarrow S$ is a proper morphism, and
- (2) $Y \rightarrow S$ is separated and locally of finite type.

Then the scheme theoretic image $Z \subset Y$ of h is proper over S and $X \rightarrow Z$ is surjective.

Proof. The scheme theoretic image of h is constructed in Section 6. Observe that h is quasi-compact (Schemes, Lemma 21.15) hence $h(X) \subset Z$ is dense (Lemma 6.3). On the other hand $h(X)$ is closed in Y (Lemma 42.7) hence $X \rightarrow Z$ is surjective. Thus $Z \rightarrow S$ is a proper (Lemma 42.8). \square

The proof of the following lemma is due to Bjorn Poonen, see this location.

Lemma 42.10. *A universally closed morphism of schemes is quasi-compact.*

Proof. Let $f : X \rightarrow S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \rightarrow V$ is not universally closed, hence we may assume that $S = \text{Spec}(A)$ for some ring A .

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X . Let $T = \text{Spec}(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) - \bigcup_{i \in I} (X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \rightarrow T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i . Then $t \in T_i$ for all i , and the fiber of $Z_s \rightarrow T_s$ above t is isomorphic to $(X - \bigcup_{i \in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T . Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible

on some neighborhood U of s . Let J be the finite set of $j \in I$ such that y_j appears in g . Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j \in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U , we can find a point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = \text{Spec}(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $\text{Spec}(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption “ $f_T(Z)$ closed” is wrong and we conclude indeed that f_T is not closed, as desired. \square

The target of a separated scheme under a surjective universally closed morphism is separated.

Lemma 42.11. *Let S be a scheme. Let $f : X \rightarrow Y$ be a surjective universally closed morphism of schemes over S .*

- (1) *If X is quasi-separated, then Y is quasi-separated.*
- (2) *If X is separated, then Y is separated.*
- (3) *If X is quasi-separated over S , then Y is quasi-separated over S .*
- (4) *If X is separated over S , then Y is separated over S .*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = \text{Spec}(\mathbf{Z})$ (see Schemes, Definition 21.3). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/S}} & X \times_S X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_S X \rightarrow X \times_S Y \rightarrow Y \times_S Y$. Hence it is also quasi-compact, see Lemma 42.10.

Assume X is quasi-separated over S , i.e., $\Delta_{X/S}$ is quasi-compact. If $V \subset Y \times_S Y$ is a quasi-compact open, then $V \times_{Y \times_S Y} X \rightarrow \Delta_{Y/S}^{-1}(V)$ is surjective and $V \times_{Y \times_S Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/S}$ is quasi-compact, i.e., Y is quasi-separated over S .

Assume X is separated over S , i.e., $\Delta_{X/S}$ is a closed immersion. Then $X \rightarrow Y \times_S Y$ is closed as a composition of closed morphisms. Since $X \rightarrow Y$ is surjective, it follows that $\Delta_{Y/S}(Y)$ is closed in $Y \times_S Y$. Hence Y is separated over S by the discussion following Schemes, Definition 21.3. \square

Lemma 42.12. *Let $f : X \rightarrow S$ and $h : U \rightarrow X$ be morphisms of schemes. Assume that f and h are quasi-compact and that $h(U)$ is dense in X . If given any commutative solid diagram*

$$\begin{array}{ccccc} \text{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & \nearrow \text{dashed} & & \downarrow \\ \text{Spec}(A) & \longrightarrow & & \longrightarrow & S \end{array}$$

where A is a valuation ring with field of fractions K , there exists a unique dotted arrow making the diagram commute, then f is universally closed.

Proof. We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Schemes, Proposition 20.6. To do this, consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a valuation ring and K is the fraction field of A . Note that since valuation rings and fields are reduced, we may replace U , X , and S by their respective reductions by Schemes, Lemma 12.6. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \rightarrow X$ is X , see Lemma 6.7. We may also replace S by an affine open through which the morphism $\mathrm{Spec}(A) \rightarrow S$ factors. Thus we may assume that $S = \mathrm{Spec}(R)$.

Let $\mathrm{Spec}(B) \subset X$ be an affine open through which the morphism $\mathrm{Spec}(K) \rightarrow X$ factors. Choose a polynomial algebra P over B and a B -algebra surjection $P \rightarrow K$. Then $\mathrm{Spec}(P) \rightarrow X$ is flat. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \rightarrow \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 26.14. By Lemma 6.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B -algebra map $\varphi : K \rightarrow A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A''/\mathfrak{m}_{A''}$ (Algebra, Lemma 48.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 48.8. As C is an R -algebra with fraction field K' , we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow \text{dotted} & \downarrow \\ \mathrm{Spec}(C) & \longrightarrow & & & S \end{array}$$

as in the statement of the lemma. Thus a dotted arrow fitting into the diagram as indicated. By the uniqueness assumption of the lemma the composition $\mathrm{Spec}(A') \rightarrow \mathrm{Spec}(C) \rightarrow X$ agrees with the given morphism $\mathrm{Spec}(A') \rightarrow \mathrm{Spec}(P) \rightarrow \mathrm{Spec}(B) \subset X$. Hence the restriction of the morphism to the spectrum of $C/\mathfrak{m}_{A'} = A''$ induces the given morphism $\mathrm{Spec}(K'') = \mathrm{Spec}(A''/\mathfrak{m}_{A''}) \rightarrow \mathrm{Spec}(K) \rightarrow X$. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(A'') \rightarrow X$. The image of the induced ring map $\mathcal{O}_{X,x} \rightarrow A''$ is a local subring which is contained in $K \subset K''$. Since A is maximal for the relation of domination in K and since $A \subset A''$, we have $A = K \cap A''$. We

conclude that $\mathcal{O}_{X,x} \rightarrow A''$ factors through $A \subset A''$. In this way we obtain our desired arrow $\text{Spec}(A) \rightarrow X$. \square

Remark 42.13. The assumption on uniqueness of the dotted arrows in Lemma 42.12 is necessary (details omitted). The uniqueness is guaranteed if f is separated (Schemes, Lemma 22.1). Conversely, if h and f satisfy the assumptions of the lemma and f is quasi-separated, then f is separated as well as universally closed (details omitted; apply the lemma to the diagonal morphism as in the proof of Schemes, Lemma 22.2).

43. Projective morphisms

We will use the definition of a projective morphism from [DG67]. The version of the definition with the “H” is the one from [Har77]. The resulting definitions are different. Both are useful.

Definition 43.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say f is *projective* if X is isomorphic as an S -scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type \mathcal{O}_S -module \mathcal{E} .
- (2) We say f is *H-projective* if there exists an integer n and a closed immersion $X \rightarrow \mathbf{P}_S^n$ over S .
- (3) We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \rightarrow U_i$ is projective.

As expected, a projective morphism is quasi-projective, see Lemma 43.10. Conversely, quasi-projective morphisms are often compositions of open immersions and projective morphisms, see Lemma 43.12.

Example 43.2. Let S be a scheme. Let \mathcal{A} be a quasi-coherent graded \mathcal{O}_S -algebra generated by \mathcal{A}_1 over \mathcal{A}_0 . Assume furthermore that \mathcal{A}_1 is of finite type over \mathcal{O}_S . Set $X = \text{Proj}_S(\mathcal{A})$. In this case $X \rightarrow S$ is projective. Namely, the morphism associated to the graded \mathcal{O}_S -algebra map

$$\text{Sym}_{\mathcal{O}_X}^*(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \rightarrow \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 18.5.

Lemma 43.3. *An H-projective morphism is H-quasi-projective. An H-projective morphism is projective.*

Proof. The first statement is immediate from the definitions. The second holds as \mathbf{P}_S^n is a projective bundle over S , see Constructions, Lemma 21.4. \square

Lemma 43.4. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is locally projective.*
- (2) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \rightarrow U_i$ is H-projective.*

Proof. By Lemma 43.3 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $\text{Spec}(R) = U \subset S$ an affine open neighbourhood of s such

that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of \mathcal{O}_U -modules \mathcal{E} . Write $\mathcal{E} = \widetilde{M}$ for some finite type R -module M (see Properties, Lemma 16.1). Choose generators $x_0, \dots, x_n \in M$ of M as an R -module. Consider the surjective graded R -algebra map

$$R[X_0, \dots, X_n] \longrightarrow \text{Sym}_R(M).$$

According to Constructions, Lemma 11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}_R^n$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of \mathbf{P}_U^n (as a scheme over U). In other words: (2) holds. \square

Lemma 43.5. *A locally projective morphism is proper.*

Proof. Let $f : X \rightarrow S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 42.3. Hence, by Lemma 43.4 above we may assume there exists a closed immersion $X \rightarrow \mathbf{P}_S^n$. By Lemmas 42.4 and 42.6 it suffices to prove that $\mathbf{P}_S^n \rightarrow S$ is proper. Since $\mathbf{P}_S^n \rightarrow S$ is the base change of $\mathbf{P}_{\mathbf{Z}}^n \rightarrow \text{Spec}(\mathbf{Z})$ it suffices to show that $\mathbf{P}_{\mathbf{Z}}^n \rightarrow \text{Spec}(\mathbf{Z})$ is proper, see Lemma 42.5. By Constructions, Lemma 8.8 the scheme $\mathbf{P}_{\mathbf{Z}}^n$ is separated. By Constructions, Lemma 8.9 the scheme $\mathbf{P}_{\mathbf{Z}}^n$ is quasi-compact. It is clear that $\mathbf{P}_{\mathbf{Z}}^n \rightarrow \text{Spec}(\mathbf{Z})$ is locally of finite type since $\mathbf{P}_{\mathbf{Z}}^n$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type \mathbf{Z} -algebra

$$\mathbf{Z}[X_0/X_i, \dots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}_{\mathbf{Z}}^n \rightarrow \text{Spec}(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 8.11 and the valuative criterion (see Schemes, Proposition 20.6). \square

Lemma 43.6. *Let S be a scheme. There exists a closed immersion*

$$\mathbf{P}_S^n \times_S \mathbf{P}_S^m \longrightarrow \mathbf{P}_S^{nm+n+m}$$

called the Segre embedding.

Proof. It suffices to prove this when $S = \text{Spec}(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \text{Proj}(\mathbf{Z}[X_0, \dots, X_n])$, $\mathbf{P}^m = \text{Proj}(\mathbf{Z}[Y_0, \dots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \text{Proj}(\mathbf{Z}[Z_0, \dots, Z_{nm+n+m}])$. In order to map into \mathbf{P}^{nm+n+m} we have to write down an invertible sheaf \mathcal{L} on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Constructions, Lemma 13.1. The invertible sheaf we take is

$$\mathcal{L} = \text{pr}_1^* \mathcal{O}_{\mathbf{P}^n}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0 Y_0, s_1 = X_1 Y_0, \dots, s_n = X_n Y_0, s_{n+1} = X_0 Y_1, \dots, s_{nm+n+m} = X_n Y_m.$$

These generate \mathcal{L} since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i, j) = (0, 0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \dots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \dots, X_n/X_0, Y_1/Y_0, \dots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0 . Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion. \square

Lemma 43.7. *A composition of H-projective morphisms is H-projective.*

Proof. Suppose $X \rightarrow Y$ and $Y \rightarrow Z$ are H-projective. Then there exist closed immersions $X \rightarrow \mathbf{P}_Y^n$ over Y , and $Y \rightarrow \mathbf{P}_Z^m$ over Z . Consider the following diagram

$$\begin{array}{ccccccc}
 X & \longrightarrow & \mathbf{P}_Y^n & \longrightarrow & \mathbf{P}_{\mathbf{P}_Z^m}^n & \xlongequal{\quad} & \mathbf{P}_Z^n \times_Z \mathbf{P}_Z^m & \longrightarrow & \mathbf{P}_Z^{nm+n+m} \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & & \downarrow & \searrow & \downarrow \\
 Y & \longrightarrow & \mathbf{P}_Z^m & & & & & & \\
 \downarrow & \swarrow & \downarrow & \swarrow & & & & & \downarrow \\
 Z & & & & & & & &
 \end{array}$$

Here the rightmost top horizontal arrow is the Segre embedding, see Lemma 43.6. The diagram identifies X as a closed subscheme of \mathbf{P}_Z^{nm+n+m} as desired. \square

Lemma 43.8. *A base change of a H-projective morphism is H-projective.*

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 18.2. \square

Lemma 43.9. *A base change of a (locally) projective morphism is (locally) projective.*

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type \mathcal{O} -module is of finite type (Modules, Lemma 9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 18.2. Some details omitted. \square

Lemma 43.10. *A projective morphism is quasi-projective.*

Proof. Let $f : X \rightarrow S$ be a projective morphism. Choose a closed immersion $i : X \rightarrow \mathbf{P}(\mathcal{E})$ where \mathcal{E} is a quasi-coherent, finite type \mathcal{O}_S -module. Then $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is f -very ample. Since f is proper (Lemma 43.5) it is quasi-compact. Hence Lemma 39.2 implies that \mathcal{L} is f -ample. Since f is proper it is of finite type. Thus we've checked all the defining properties of quasi-projective holds and we win. \square

Lemma 43.11. *Let $f : X \rightarrow S$ be a H-quasi-projective morphism. Then f factors as $X \rightarrow X' \rightarrow S$ where $X \rightarrow X'$ is an open immersion and $X' \rightarrow S$ is H-projective.*

Proof. By definition we can factor f as a quasi-compact immersion $i : X \rightarrow \mathbf{P}_S^n$ followed by the projection $\mathbf{P}_S^n \rightarrow S$. By Lemma 7.7 there exists a closed subscheme $X' \subset \mathbf{P}_S^n$ such that i factors through an open immersion $X \rightarrow X'$. The lemma follows. \square

Lemma 43.12. *Let $f : X \rightarrow S$ be a quasi-projective morphism with S quasi-compact and quasi-separated. Then f factors as $X \rightarrow X' \rightarrow S$ where $X \rightarrow X'$ is an open immersion and $X' \rightarrow S$ is projective.*

Proof. Let \mathcal{L} be f -ample. Since f is of finite type and S is quasi-compact $\mathcal{L}^{\otimes n}$ is f -very ample for some $n > 0$, see Lemma 40.5. Replace \mathcal{L} by $\mathcal{L}^{\otimes n}$. Write $\mathcal{F} = f_*\mathcal{L}$. This is a quasi-coherent \mathcal{O}_S -module by Schemes, Lemma 24.1 (quasi-projective morphisms are quasi-compact and separated, see Lemma 41.2). By Properties, Lemma 20.6 we can find a directed partially ordered set I and a system of finite type quasi-coherent \mathcal{O}_S -modules \mathcal{E}_i over I such that $\mathcal{F} = \text{colim } \mathcal{E}_i$. Consider the compositions $\psi_i : f^*\mathcal{E}_i \rightarrow f^*\mathcal{F} \rightarrow \mathcal{L}$. Choose a finite affine open covering $S = \bigcup_{j=1, \dots, m} V_j$. For each j we can choose sections

$$s_{j,0}, \dots, s_{j,n_j} \in \Gamma(f^{-1}(V_j), \mathcal{L}) = f_*\mathcal{L}(V_j) = \mathcal{F}(V_j)$$

which generate \mathcal{L} over $f^{-1}V_j$ and define an immersion

$$f^{-1}V_j \longrightarrow \mathbf{P}_{V_j}^{n_j},$$

see Lemma 40.1. Choose i such that there exist sections $e_{j,t} \in \mathcal{E}_i(V_j)$ mapping to $s_{j,t}$ in \mathcal{F} for all $j = 1, \dots, m$ and $t = 1, \dots, n_j$. Then the map ψ_i is surjective as the sections $f^*e_{j,t}$ have the same image as the sections $s_{j,t}$ which generate $\mathcal{L}|_{f^{-1}V_j}$. Whence we obtain a morphism

$$r_{\mathcal{L}, \psi_i} : X \longrightarrow \mathbf{P}(\mathcal{E}_i)$$

over S such that over V_j we have a factorization

$$f^{-1}V_j \rightarrow \mathbf{P}(\mathcal{E}_i)|_{V_j} \rightarrow \mathbf{P}_{V_j}^{n_j}$$

of the immersion given above. It follows that $r_{\mathcal{L}, \psi_i}|_{V_j}$ is an immersion, see Lemma 3.1. Since $S = \bigcup V_j$ we conclude that $r_{\mathcal{L}, \psi_i}$ is an immersion. Note that $r_{\mathcal{L}, \psi_i}$ is quasi-compact as $X \rightarrow S$ is quasi-compact and $\mathbf{P}(\mathcal{E}_i) \rightarrow S$ is separated (see Schemes, Lemma 21.15). By Lemma 7.7 there exists a closed subscheme $X' \subset \mathbf{P}(\mathcal{E}_i)$ such that i factors through an open immersion $X \rightarrow X'$. Then $X' \rightarrow S$ is projective by definition and we win. \square

Lemma 43.13. *Let S be a scheme which admits an ample invertible sheaf. Then*

- (1) *any projective morphism $X \rightarrow S$ is H-projective, and*
- (2) *any quasi-projective morphism $X \rightarrow S$ is H-quasi-projective.*

Proof. The assumptions on S imply that S is quasi-compact and separated, see Properties, Definition 24.1 and Lemma 24.12 and Constructions, Lemma 8.8. Hence Lemma 43.12 applies and we see that (1) implies (2). Let \mathcal{E} be a finite type quasi-coherent \mathcal{O}_S -module. By our definition of projective morphisms it suffices to show that $\mathbf{P}(\mathcal{E}) \rightarrow S$ is H-projective. If \mathcal{E} is generated by finitely many global sections, then the corresponding surjection $\mathcal{O}_S^{\oplus n} \rightarrow \mathcal{E}$ induces a closed immersion

$$\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}(\mathcal{O}_S^{\oplus n}) = \mathbf{P}_S^n$$

as desired. In general, let \mathcal{L} be an invertible sheaf on S . By Properties, Proposition 24.14 there exists an integer n such that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n}$ is globally generated by finitely many sections. Since $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n})$ by Constructions, Lemma 20.1 this finishes the proof. \square

44. Integral and finite morphisms

Recall that a ring map $R \rightarrow A$ is said to be *integral* if every element of A satisfies a monic equation with coefficients in R . Recall that a ring map $R \rightarrow A$ is said to be finite if A is *finite* as an R -module. See Algebra, Definition 35.1.

Definition 44.1. Let $f : X \rightarrow S$ be a morphism of schemes.

- (1) We say that f is *integral* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \rightarrow A$ is integral.
- (2) We say that f is *finite* if f is affine and if for every affine open $\text{Spec}(R) = V \subset S$ with inverse image $\text{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \rightarrow A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 44.7 at the end of the section.

Lemma 44.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is integral.*
- (2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \rightarrow \mathcal{O}_X(f^{-1}(U_i))$ is integral.*
- (3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \rightarrow U_i$ is integral.*

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \rightarrow U$ is integral.

Proof. See Algebra, Lemma 35.12. Some details omitted. □

Lemma 44.3. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) *The morphism f is finite.*
- (2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \rightarrow \mathcal{O}_X(f^{-1}(U_i))$ is finite.*
- (3) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \rightarrow U_i$ is finite.*

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \rightarrow U$ is finite.

Proof. See Algebra, Lemma 35.12. Some details omitted. □

Lemma 44.4. *A finite morphism is integral. An integral morphism which is locally of finite type is finite.*

Proof. See Algebra, Lemma 35.3 and Lemma 35.5. □

Lemma 44.5. *A composition of finite morphisms is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemmas 7.3 and 35.6. □

Lemma 44.6. *A base change of a finite morphism is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemma 35.11. □

Lemma 44.7. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) *f is integral, and*
- (2) *f is affine and universally closed.*

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 35.20 and 40.6.

Assume (2). We may assume f is the morphism $f : \text{Spec}(A) \rightarrow \text{Spec}(R)$ coming from a ring map $R \rightarrow A$. Let a be an element of A . We have to show that a is integral over R , i.e. that in the kernel I of the map $R[x] \rightarrow A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax - 1)$ and let J be the kernel of the composition $R[x] \rightarrow A[x] \rightarrow B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax - 1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum_{i \geq 0} f_i x^{n-i} = \sum_{i \geq 0} (aq_{i-1} - q_i)x^{n-i} = (a - x) \sum_{i \geq 0} q_i x^{n-i-1}$$

is clearly in I ; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $\text{Spec}(R[x]/(J + (x)))$ is empty.

Since f is universally closed the base change $\text{Spec}(A[x]) \rightarrow \text{Spec}(R[x])$ is closed. Hence the image of the closed subset $\text{Spec}(B) \subset \text{Spec}(A[x])$ is the closed subset $\text{Spec}(R[x]/J) \subset \text{Spec}(R[x])$, see Example 6.4 and Lemma 6.3. In particular $\text{Spec}(B) \rightarrow \text{Spec}(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{array}{ccccc} \text{Spec}(B) & \xrightarrow{g} & \text{Spec}(R[x]/J) & \longrightarrow & \text{Spec}(R[x]) \\ \uparrow & & \uparrow & & \uparrow 0 \\ \emptyset & \longrightarrow & \text{Spec}(R[x]/(J + (x))) & \longrightarrow & \text{Spec}(R) \end{array}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \rightarrow B$ sends x to an invertible element and $R[x] \rightarrow R$ sends x to 0. Since g is surjective this implies $\text{Spec}(R[x]/(J + (x)))$ is empty, as we wanted to show. □

Lemma 44.8. *Let $f : X \rightarrow S$ be an integral morphism. Then every point of X is closed in its fibre.*

Proof. See Algebra, Lemma 35.18. □

Lemma 44.9. *A finite morphism is quasi-finite.*

Proof. This is implied by Algebra, Lemma 118.4 and Lemma 21.9. Alternatively, all points in fibres are closed points by Lemma 44.8 (and the fact that a finite morphism is integral) and use Lemma 21.6 (3) to see that f is quasi-finite at x for all $x \in X$. □

Lemma 44.10. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent*

- (1) f is finite, and
- (2) f is affine and proper.

Proof. This follows formally from Lemma 44.7, the fact that a finite morphism is integral and separated, the fact that a proper morphism is the same thing as a finite type, separated, universally closed morphism, and the fact that an integral morphism of finite type is finite (Lemma 44.4). \square

Lemma 44.11. *A closed immersion is finite (and a fortiori integral).*

Proof. True because a closed immersion is affine (Lemma 13.9) and a surjective ring map is finite and integral. \square

Lemma 44.12. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms.*

- (1) *If $g \circ f$ is finite and g separated then f is finite.*
- (2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \rightarrow Y$ is finite (resp. integral) by Lemma 44.6. The morphism $X \rightarrow X \times_Z Y$ is a closed immersion as $Y \rightarrow Z$ is separated, see Schemes, Lemma 21.12. A closed immersion is finite (resp. integral), see Lemma 44.11. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 44.5. Thus we win. \square

Lemma 44.13. *Let $f : X \rightarrow Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.*

Proof. This reduces to Algebra, Lemma 103.6. \square

45. Universal homeomorphisms

The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 45.3.

Definition 45.1. A morphism $f : X \rightarrow Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \rightarrow Y'$ is a homeomorphism for every morphism $Y' \rightarrow Y$.

Lemma 45.2. *Let $f : X \rightarrow Y$ be a morphism of schemes. If f is a homeomorphism onto a closed subset of Y then f is affine.*

Proof. Let $y \in Y$ be a point. If $y \notin f(X)$, then there exists an affine neighbourhood of y which is disjoint from $f(X)$. If $y \in f(X)$, let $x \in X$ be the unique point of X mapping to y . Let $y \in V$ be an affine open neighbourhood. Let $U \subset X$ be an affine open neighbourhood of x which maps into V . Since $f(U) \subset V \cap f(X)$ is open in the induced topology by our assumption on f we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap f(X) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U . Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 13.3. \square

Lemma 45.3. *Let $f : X \rightarrow Y$ be a morphism of schemes. The following are equivalent:*

- (1) f is a universal homeomorphism, and
- (2) f is integral, universally injective and surjective.

Proof. Assume f is a universal homeomorphism. By Lemma 45.2 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 44.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 44.7 f is universally closed. Since it is also universally bijective (see Lemma 11.4) we see that it is a universal homeomorphism. \square

Lemma 45.4. *Let X be a scheme. The canonical closed immersion $X_{red} \rightarrow X$ (see Schemes, Definition 12.5) is a universal homeomorphism.*

Proof. Omitted. \square

Lemma 45.5. *Let $f : X \rightarrow S$ and $S' \rightarrow S$ be morphisms of schemes. Assume*

- (1) $S' \rightarrow S$ is a closed immersion,
- (2) $S' \rightarrow S$ is bijective on points,
- (3) $X \times_S S' \rightarrow S'$ is a closed immersion, and
- (4) $X \rightarrow S$ is of finite type or $S' \rightarrow S$ is of finite presentation.

Then $f : X \rightarrow S$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $S' \rightarrow S$ is a universal homeomorphism (for example because $S_{red} = S'_{red}$ and using Lemma 45.4). Hence (3) implies that $X \rightarrow S$ is homeomorphism onto a closed subset of S . Then $X \rightarrow S$ is affine by Lemma 45.2. Let $U \subset S$ be an affine open, say $U = \text{Spec}(A)$. Then $S' = \text{Spec}(A/I)$ by (1) for a locally nilpotent ideal I by (2). As f is affine we see that $f^{-1}(U) = \text{Spec}(B)$. Assumption (4) tells us B is a finite type A -algebra (Lemma 16.2) or that I is finitely generated (Lemma 22.7). Assumption (3) is that $A/I \rightarrow B/IB$ is surjective. From Algebra, Lemma 12.8 if $A \rightarrow B$ is of finite type or Algebra, Lemma 19.1 if I is finitely generated and hence nilpotent we deduce that $A \rightarrow B$ is surjective. This means that f is a closed immersion, see Lemma 2.1. \square

46. Finite locally free morphisms

In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 4.3.

Definition 46.1. Let $f : X \rightarrow S$ be a morphism of schemes. We say f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free \mathcal{O}_S -module. In this case we say f has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d .

Note that if $f : X \rightarrow S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \rightarrow S_d$ is finite locally free of degree d .

Lemma 46.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent:*

- (1) f is finite locally free,
- (2) f is finite, flat, and locally of finite presentation.

If S is locally Noetherian these are also equivalent to

(3) f is finite and flat.

Proof. See Algebra, Lemma 75.2. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 30.4. \square

Lemma 46.3. A composition of finite locally free morphisms is finite locally free.

Proof. Omitted. \square

Lemma 46.4. A base change of a finite locally free morphism is finite locally free.

Proof. Omitted. \square

Lemma 46.5. Let $f : X \rightarrow S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \rightarrow S_d$ of degree d .

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omitted. \square

Lemma 46.6. Let $f : Y \rightarrow X$ be a finite morphism with X affine. There exists a diagram

$$\begin{array}{ccccc} Z' & \longleftarrow & Y' & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow \\ & & X' & \longrightarrow & X \end{array}$$

where

- (1) $Y' \rightarrow Y$ and $X' \rightarrow X$ are surjective finite locally free,
- (2) $Y' = X' \times_X Y$,
- (3) $i : Y' \rightarrow Z'$ is a closed immersion,
- (4) $Z' \rightarrow X'$ is finite locally free, and
- (5) $Z' = \bigcup_{j=1, \dots, m} Z'_j$ is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to X' .

Proof. Write $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. See also More on Algebra, Section 14. Let $x_1, \dots, x_n \in B$ be generators of B over A . For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B . By Algebra, Lemma 131.9 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod_{k=1, \dots, d_i} (T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \dots, T_n] / (P_1(T_1), \dots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \rightarrow B'$, $T_i \mapsto 1 \otimes x_i$ is an A' -algebra surjection. Setting $X' = \text{Spec}(A')$, $Y' = \text{Spec}(B')$ and $Z' = \text{Spec}(C)$ we see that (1) – (4) hold. Part (5) holds because set theoretically $\text{Spec}(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \dots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. \square

The following lemma is stated in the correct generality in Lemma 49.4 below.

Lemma 46.7. *Let $f : Y \rightarrow X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y . Then $f(T) \subset X$ is a closed nowhere dense subset of X .*

Proof. By Lemma 44.10 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y , we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 20.4.

Assume X is affine. Choose a diagram

$$\begin{array}{ccccc} Z' & \longleftarrow & Y' & \xrightarrow{a} & Y \\ & \searrow i & \downarrow f' & & \downarrow f \\ & & X' & \xrightarrow{b} & X \end{array}$$

as in Lemma 46.6. The morphisms a, b are open since they are finite locally free (Lemmas 46.2 and 26.9). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 20.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 20.6. As i is a closed immersion, by Topology, Lemma 20.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1, \dots, n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X . Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i , then each $f(T_i)$ is nowhere dense in X as $Y_i \rightarrow X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 20.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V . In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2 . In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2 . Repeat sequentially with $i = 3, \dots, n$. The result is a disjoint union decomposition

$$\{1, \dots, n\} = I_1 \amalg I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \rightarrow X$ is an isomorphism. Then

$$f^{-1}(U) = V \amalg \bigcup_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y , as desired. \square

47. Generically finite morphisms

In this section we characterize maps between schemes which are locally of finite type and which are “generically finite” in some sense.

Lemma 47.1. *Let X, Y be schemes. Let $f : X \rightarrow Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y . The following are equivalent:*

- (1) *the set $f^{-1}(\{\eta\})$ is finite,*
- (2) *there exist affine opens $U_i \subset X$, $i = 1, \dots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \rightarrow V$ is finite.*

If f is quasi-separated, then these are also equivalent to

- (3) *there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \rightarrow V$ is finite.*

If f is quasi-compact and quasi-separated, then these are also equivalent to

- (4) *there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \rightarrow V$ is finite.*

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η , and we may and do assume throughout the proof below that Y is affine, say $Y = \text{Spec}(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \dots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 118.9 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \rightarrow Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume $f^{-1}(\{\eta\}) = \{\xi_1, \dots, \xi_n\}$ and assume that f is quasi-separated. Since Y is affine this implies that X is quasi-separated. Since each ξ_i maps to a generic point of an irreducible component of Y , we see that each ξ_i is a generic point of an irreducible component of X . By Properties, Lemma 27.1 we can find an affine open $U \subset X$ containing each ξ_i . By Algebra, Lemma 118.9 we see that after replacing Y by a standard open in Y the morphisms $U \rightarrow Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) – (3) with no further assumptions on f . Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) – (3) imply (4). Let U, V be as in (3). Replace Y by V . Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \rightarrow Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 8.4 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \rightarrow V'$ is finite. \square

Example 47.2. Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field \mathbf{F}_2 . Thus the topology on $X = \text{Spec}(A)$ is totally disconnected and quasi-compact. The projection maps $A \rightarrow \mathbf{F}_2$ define open points of $\text{Spec}(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 14.3. The morphism $f : Y \rightarrow X$ is quasi-compact, finite type and has finite fibres but is not quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 47.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse

image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separate the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

Remark 47.3. An alternative to Lemma 47.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Section 31.

Lemma 47.4. *Let X, Y be integral schemes. Let $f : X \rightarrow Y$ be locally of finite type. Assume f is dominant. The following are equivalent:*

- (1) *the extension $R(Y) \subset R(X)$ has transcendence degree 0,*
- (2) *the extension $R(Y) \subset R(X)$ is finite,*
- (3) *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \rightarrow V$ is finite, and*
- (4) *the generic point of X is the only point of X mapping to the generic point of Y .*

If f is separated, or if f is quasi-compact, then these are also equivalent to

- (5) *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \rightarrow V$ is finite.*

Proof. Choose any affine opens $\text{Spec}(A) = U \subset X$ and $\text{Spec}(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \rightarrow A$ is of finite type (Lemma 16.2). By Lemma 8.5 the generic point of X maps to the generic point of Y hence $R \rightarrow A$ is injective. Let $K = f.f.(R) = R(Y)$ and $L = f.f.(A) = R(X)$. Then $K \subset L$ is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \dots, x_n \in A$ be generators of A over R . By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i . Then we conclude that $R_{f_1 \dots f_n} \rightarrow A_{f_1 \dots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X , and let $\eta' \in Y$ be the generic point of Y . Assume (4). Then $\dim_\eta(X_\eta) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 29.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η' . As x is a specialization of η , this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 35.17. Hence $x = \eta$. Thus we see that (1) – (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f . What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) – (4) imply (5).

Assume U, V as in (3) and assume f is separated. Then $U \rightarrow f^{-1}(V)$ is a morphism from a scheme proper over V (Lemma 44.10) into a scheme separated over V . Hence $U \subset f^{-1}(V)$ is closed Lemma 42.7. Since X is irreducible we conclude $U = f^{-1}(V)$. This proves (5).

Assume f is quasi-compact. Let U, V be as in (3). Then $f^{-1}(V)$ is quasi-compact. Consider the closed subset $Z = f^{-1}(V) \setminus U$. Since Z does not contain the generic point of X we see that the quasi-compact morphism $f : Z \rightarrow V$ is not dominant

by Lemma 8.3. Hence after shrinking V we may assume that $f^{-1}(V) = U$ which implies that (5) holds. \square

Definition 47.5. Let X and Y be integral schemes. Let $f : X \rightarrow Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) – (4) of Lemma 47.4. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the *degree of X over Y* .

It is possible to extend this notion to a morphism $f : X \rightarrow Y$ if (a) Y is integral with generic point η , (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. In this case we can define

$$\deg(X/Y) = \sum_{\xi \in X, f(\xi) = \eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 10.4) the dimensions above are finite by Lemma 47.1 above. However, for most applications the definition given above is the right one.

Lemma 47.6. *Let X, Y, Z be integral schemes. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then*

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 7.6. \square

Remark 47.7. Let $f : X \rightarrow Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define *generically finite* morphisms. These correspond to whether you want the property to be local on the source or local on the target:

- (1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 47.1.
- (2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \rightarrow Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y , but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

Definition 47.8. Let X be an integral scheme. A *modification of X* is a birational proper morphism $f : X' \rightarrow X$ with X' integral.

Let $f : X' \rightarrow X$ be a modification as in the definition. By Lemma 47.4 there exists a nonempty $U \subset X$ such that $f^{-1}(U) \rightarrow U$ is finite. By generic flatness (Proposition 28.1) we may assume $f^{-1}(U) \rightarrow U$ is flat and of finite presentation. So $f^{-1}(U) \rightarrow U$ is finite locally free (Lemma 46.2). Since f is birational, the degree of X' over X is 1. Hence $f^{-1}(U) \rightarrow U$ is finite locally free of degree 1, in other words it is an isomorphism. Thus we can *redefine* a modification to be a proper morphism

$f : X' \rightarrow X$ of integral schemes such that $f^{-1}(U) \rightarrow U$ is an isomorphism for some nonempty open $U \subset X$.

Definition 47.9. Let X be an integral scheme. An *alteration of X* is a proper dominant morphism $f : Y \rightarrow X$ with Y integral such that $f^{-1}(U) \rightarrow U$ is finite for some nonempty open $U \subset X$.

This is the definition as given in [dJ96], except that here we do not require X and Y to be Noetherian. Arguing as above we see that an alteration is a proper dominant morphism $f : Y \rightarrow X$ of integral schemes which induces a finite extension of function fields, i.e., such that the equivalent conditions of Lemma 47.4 hold.

48. Normalization

In this section we construct the *normalization*, and the *normalization of one scheme in another*.

Lemma 48.1. Let X be a scheme. Let \mathcal{A} be a quasi-coherent sheaf of \mathcal{O}_X -algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule

$$U \mapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \text{ integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\}$$

is a quasi-coherent \mathcal{O}_X -algebra, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. This is a subsheaf by the local nature of the conditions. It is an \mathcal{O}_X -algebra by Algebra, Lemma 35.7. Let $U \subset X$ be an affine open. Say $U = \text{Spec}(R)$ and say \mathcal{A} is the quasi-coherent sheaf associated to the R -algebra A . Then according to Algebra, Lemma 35.10 the value of \mathcal{A}' over U is given by the integral closure A' of R in A . This proves the last assertion of the lemma. To prove that \mathcal{A}' is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 35.9. \square

Definition 48.2. Let X be a scheme. Let \mathcal{A} be a quasi-coherent sheaf of \mathcal{O}_X -algebras. The *integral closure of \mathcal{O}_X in \mathcal{A}* is the quasi-coherent \mathcal{O}_X -subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 48.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{array}{ccc} Y = \underline{\text{Spec}}_X(\mathcal{A}) & \longrightarrow & X' = \underline{\text{Spec}}_X(\mathcal{A}') \\ & \searrow & \swarrow \\ & X & \end{array}$$

see Lemma 13.5. The scheme $X' \rightarrow X$ will be the normalization of X in the scheme Y . Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \rightarrow X$ of schemes. In this case the sheaf of \mathcal{O}_X -algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of \mathcal{O}_X -algebras whose relative spectrum is the normalization of X in Y . Here is the formal definition.

Definition 48.3. Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of schemes. Let \mathcal{O}' be the integral closure of \mathcal{O}_X in $f_*\mathcal{O}_Y$. The *normalization*

of X in Y is the scheme¹²

$$\nu : X' = \underline{\text{Spec}}_X(\mathcal{O}') \rightarrow X$$

over X . It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f .

The factorization is the composition of the canonical morphism $Y \rightarrow \underline{\text{Spec}}(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \rightarrow f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 48.4. *Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \rightarrow X$ is the normalization of X in Y is characterized by the following two properties:*

- (1) *the morphism ν is integral, and*
- (2) *for any factorization $f = \pi \circ g$, with $\pi : Z \rightarrow X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & Z \\ f' \downarrow & \nearrow h & \downarrow \pi \\ X' & \xrightarrow{\quad \nu} & X \end{array}$$

for some unique morphism $h : X' \rightarrow Z$.

Moreover, in (2) the morphism $h : X' \rightarrow Z$ is the normalization of Z in Y .

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of \mathcal{O}_X as in Definition 48.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \rightarrow X$ integral as in (2). By Definition 44.1 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of \mathcal{O}_X -algebras \mathcal{B} . The morphism $g : X \rightarrow Z$ corresponds to a map of \mathcal{O}_X -algebras $\chi : \mathcal{B} \rightarrow f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 44.1) we see from Lemma 48.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 13.5 this provides us with a unique morphism $h : X' \rightarrow Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 44.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \rightarrow Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \rightarrow Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \rightarrow Z$ which gives the last statement of the lemma. \square

Lemma 48.5. *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ f_2 \downarrow & & \downarrow f_1 \\ X_2 & \longrightarrow & X_1 \end{array}$$

¹²The scheme X' need not be normal, for example if $Y = X$ and $f = \text{id}_X$, then $X' = X$.

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 quasi-compact and quasi-separated. Let $f_i = \nu_i \circ f'_i$, $i = 1, 2$ be the canonical factorizations, where $\nu_i : X'_i \rightarrow X_i$ is the normalization of X_i in Y_i . Then there exists a canonical commutative diagram

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ f'_2 \downarrow & & \downarrow f'_1 \\ X'_2 & \longrightarrow & X'_1 \\ \nu_2 \downarrow & & \downarrow \nu_1 \\ X_2 & \longrightarrow & X_1 \end{array}$$

Proof. By Lemmas 48.4 (1) and 44.6 the base change $X_2 \times_{X_1} X'_1 \rightarrow X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a canonical morphism $X'_2 \rightarrow X_2 \times_{X_1} X'_1$ from Lemma 48.4 (2). This gives the middle horizontal arrow in the last diagram. \square

Lemma 48.6. *Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of schemes. Let $U \subset X$ be an open subscheme and set $V = f^{-1}(U)$. Then the normalization of U in V is the inverse image of U in the normalization of X in Y .*

Proof. Clear from the construction. \square

Lemma 48.7. *Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two schemes. Write $f_i = f|_{Y_i}$. Let X'_i be the normalization of X in Y_i . Then $X'_1 \amalg X'_2$ is the normalization of X in Y .*

Proof. In terms of integral closures this corresponds to the following fact: Let $A \rightarrow B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A'_i be the integral closure of A in B_i . Then $A'_1 \times A'_2$ is the integral closure of A in B . The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A . Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A'_i as described above (some details omitted). \square

Lemma 48.8. *Let $f : Y \rightarrow X$ be an integral morphism. Then the integral closure of X in Y is equal to Y .*

Proof. Omitted. \square

The following lemma is a generalization of the preceding one.

Lemma 48.9. *Let $f : X \rightarrow S$ be a quasi-compact, quasi-separated and universally closed morphism of schemes. Then $f_*\mathcal{O}_X$ is integral over \mathcal{O}_S . In other words, the normalization of S in X is equal to the factorization*

$$X \longrightarrow \underline{\text{Spec}}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 4.7.

Proof. The question is local on S , hence we may assume $S = \text{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R .

Think of h as a morphism as in the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & \mathbf{A}_S^1 \\ & \searrow f & \swarrow \\ & & S \end{array}$$

Let $Z \subset \mathbf{A}_S^1$ be the scheme theoretic image of h , see Definition 6.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}_S^1 \rightarrow S$ is separated, see Schemes, Lemma 21.15. By Lemma 6.3 the morphism $X \rightarrow Z$ is dominant. By Lemma 42.7 the morphism $X \rightarrow Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 42.8 to conclude that $Z \rightarrow S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}_S^1$, we see that $Z \rightarrow S$ is affine and proper, hence integral by Lemma 44.7. Writing $\mathbf{A}_S^1 = \text{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. \square

Lemma 48.10. *Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of schemes. Assume*

- (1) Y is a normal scheme,
- (2) any quasi-compact open $V \subset Y$ has a finite number of irreducible components.

Then the normalization X' of X in Y is a normal scheme. Moreover, the morphism $Y \rightarrow X'$ is dominant and induces a bijection of irreducible components.

Proof. We first prove that X' is normal. Let $U \subset X$ be an affine open. It suffices to prove that the inverse image of U in X' is normal (see Properties, Lemma 7.2). By Lemma 48.6 we may replace X by U , and hence we may assume $X = \text{Spec}(A)$ affine. In this case Y is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $Y = \coprod_{i=1, \dots, n} Y_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 7.5. By Lemma 48.7 we see that $X' = \coprod_{i=1, \dots, n} X'_i$, where X'_i is the normalization of X in Y_i . By Properties, Lemma 7.9 we see that $B_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$ is a normal domain. Note that $X'_i = \text{Spec}(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i , see Lemma 48.1. By Algebra, Lemma 36.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $X' = \coprod X'_i$ is a finite union of normal schemes and hence is normal.

It is clear from the description of X' above that $Y \rightarrow X'$ is dominant and induces a bijection on irreducible components if X is affine. The result in general follows from this by a topological argument (omitted). \square

Lemma 48.11. *Let $f : X \rightarrow S$ be a morphism. Assume that*

- (1) S is a Nagata scheme,
- (2) f is of finite type¹³, and
- (3) X is reduced.

Then the normalization $\nu : S' \rightarrow S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = \text{Spec}(R)$ where R is a Nagata ring. In this case we have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R . Since f is of finite type we can write $X = \bigcup_{i=1, \dots, n} U_i$ with each U_i affine. Say $U_i = \text{Spec}(B_i)$. Each B_i is a reduced ring of finite type

¹³The proof shows that the lemma holds if f is quasi-compact and “essentially of finite type”.

over R (Lemma 16.2). Moreover, $\Gamma(X, \mathcal{O}_X) \subset B = B_1 \times \dots \times B_n$. So A is contained in the integral closure A' of R in B . Note that B is a reduced finite type R -algebra. Since R is Noetherian it suffices to prove that A' is finite over R . This is Algebra, Lemma 150.16. \square

Next, we come to the normalization of a scheme X . We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X . Let

$$(48.11.1) \quad f : Y = \coprod_{\eta \in X^{(0)}} \text{Spec}(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 21).

Definition 48.12. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *normalization* of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

which is the normalization of X in the morphism $f : Y \rightarrow X$ (48.11.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X .

Lemma 48.13. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \rightarrow X_{red}$ is the normalization of X_{red} .*

Proof. Let $f : Y \rightarrow X$ be the morphism (48.11.1). We get a factorization $Y \rightarrow X_{red} \rightarrow X$ of f from Schemes, Lemma 12.6. By Lemma 48.4 we obtain a canonical morphism $X^\nu \rightarrow X_{red}$ and that X^ν is the normalization of X_{red} in Y . The lemma follows as $Y \rightarrow X_{red}$ is identical to the morphism (48.11.1) constructed for X_{red} . \square

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of \mathcal{O}_X in the sheaf of meromorphic functions \mathcal{K}_X (see Divisors, Section 15). Namely, $\mathcal{K}_X = f_* \mathcal{O}_Y$ in this case, see Divisors, Lemma 15.8 and its proof. We describe this here explicitly.

Lemma 48.14. *Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $\text{Spec}(A) = U \subset X$ be an affine open. Then*

- (1) A has finitely many minimal primes $\mathfrak{q}_1, \dots, \mathfrak{q}_t$,
- (2) the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \dots \times Q(A/\mathfrak{q}_t)$,
- (3) the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains A/\mathfrak{q}_i in the fields $Q(A/\mathfrak{q}_i)$, and
- (4) $\nu^{-1}(U)$ is identified with the spectrum of A' .

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 25.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 16.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 24.4 and 24.1. This proves (2). Part (3) follows from Algebra, Lemma 36.14, or Lemma 48.7. Part (4) holds because it is clear that $f^{-1}(U) \rightarrow U$ is the morphism

$$\mathrm{Spec}\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow \mathrm{Spec}(A)$$

where $f : Y \rightarrow X$ is the morphism (48.11.1). \square

Lemma 48.15. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components.*

- (1) *The normalization X^ν is a normal scheme.*
- (2) *The morphism $\nu : X^\nu \rightarrow X$ is integral, surjective, and induces a bijection on irreducible components.*
- (3) *For any integral, birational¹⁴ morphism $X' \rightarrow X$ there exists a factorization $X^\nu \rightarrow X' \rightarrow X$ and $X^\nu \rightarrow X'$ is the normalization of X' .*
- (4) *For any morphism $Z \rightarrow X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \rightarrow X^\nu \rightarrow X$.*

Proof. Let $f : Y \rightarrow X$ be as in (48.11.1). Part (1) follows from Lemma 48.10 and the fact that Y is normal. It also follows from the description of the affine opens in Lemma 48.14.

The morphism ν is integral by Lemma 48.4. By Lemma 48.10 the morphism $Y \rightarrow X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \rightarrow X$ induces a bijection on irreducible components. By construction $f : Y \rightarrow X$ is dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 44.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \rightarrow X$ is integral and birational. Any quasi-compact open U' of X' maps to a quasi-compact open of X , hence we see that U' has only finitely many irreducible components. Let $f' : Y' \rightarrow X'$ be the morphism (48.11.1) constructed starting with X' . As α is birational it is clear that $Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \rightarrow X' \rightarrow X$ exists and $X^\nu \rightarrow X'$ is the normalization of X' by Lemma 48.4. This proves (3).

Let $g : Z \rightarrow X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X . By Lemma 48.13 we have $X^\nu = X_{red}^\nu$ and by Schemes, Lemma 12.6 $Z \rightarrow X$ factors through X_{red} . Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z . Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = \mathrm{Spec}(A)$ and $W = \mathrm{Spec}(B)$, with $g|_W$ given by $\varphi : A \rightarrow B$. We will use the results of Lemma 48.14 freely. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the minimal primes of A . As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak{q} \subset B$ we have that $\varphi^{-1}(\mathfrak{q})$ is a minimal prime of A . Hence if $x \in A$ is a nonzerodivisor, i.e., $x \notin \bigcup \mathfrak{p}_i$, then $\varphi(x)$ is a nonzerodivisor in B . Thus we obtain a canonical ring map $Q(A) \rightarrow Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$

¹⁴It suffices if $X'_{red} \rightarrow X_{red}$ is birational.

(see Algebra, Lemma 36.11). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \rightarrow Q(B)$. Since $\nu^{-1}(U) = \text{Spec}(A')$ this gives the canonical factorization $W \rightarrow \nu^{-1}(U) \rightarrow U$ of $\nu|_W$. We omit the verification that it is unique. \square

Lemma 48.16. *A finite (or even integral) birational morphism of normal integral schemes is an isomorphism.*

Proof. Let $f : X \rightarrow Y$ be such a morphism. Let $V \subset Y$ be an affine open with inverse image $U \subset X$ which is an affine open too. Since f is a birational morphism of integral schemes, the homomorphism $\mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(U)$ is an injective map of domains which induces an isomorphism of fraction fields. As Y is normal, the ring $\mathcal{O}_Y(V)$ is integrally closed in the fraction field. Since f is finite (or integral) every element of $\mathcal{O}_X(U)$ is integral over $\mathcal{O}_Y(V)$. We conclude that $\mathcal{O}_Y(V) = \mathcal{O}_X(U)$. This proves that f is an isomorphism as desired. \square

Lemma 48.17. *Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \rightarrow X$ is a finite morphism.*

Proof. Follows from the definitions and Lemma 48.14. Namely, in this case the lemma says that $\nu^{-1}(\text{Spec}(A))$ is the spectrum of the integral closure of A in its field of fractions. \square

Lemma 48.18. *Let X be a Nagata scheme. The normalization $\nu : X^\nu \rightarrow X$ is a finite morphism.*

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 48.12 does apply. Write $X^\nu \rightarrow X$ as the composition $X^\nu \rightarrow X_{red} \rightarrow X$. As $X_{red} \rightarrow X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 44.5). Let $\text{Spec}(A) = U \subset X$ be an affine open. By Lemma 48.14 we have $\nu^{-1}(U) = \text{Spec}(\prod A'_i)$ where A'_i is the integral closure of A/\mathfrak{q}_i in its fraction field. As A is a Nagata ring (see Properties, Lemma 13.6) each of the ring extensions $A/\mathfrak{q}_i \subset A'_i$ are finite. Hence $A \rightarrow \prod A'_i$ is a finite ring map and we win. \square

49. Zariski's Main Theorem (algebraic version)

This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Cohomology of Schemes, Section 19 and More on Morphisms, Section 31.

Theorem 49.1 (Algebraic version of Zariski's Main Theorem). *Let $f : Y \rightarrow X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X in Y . Picture:*

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X' \\ & \searrow f & \swarrow \nu \\ & & X \end{array}$$

Then there exists an open subscheme $U' \subset X'$ such that

- (1) $(f')^{-1}(U') \rightarrow U'$ is an isomorphism, and
- (2) $(f')^{-1}(U') \subset Y$ is the set of points at which f is quasi-finite.

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = \text{Spec}(R)$ and $Y = \text{Spec}(A)$. Then $X' = \text{Spec}(A')$, where A' is the integral closure of R in A , see Definitions 48.2 and 48.3. By Algebra, Theorem 119.13 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \rightarrow U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \rightarrow X'_x$ is an isomorphism in a neighbourhood of y . Hence there is no point of Y_x which specializes to y , since this is true for $f'(y)$ in X'_x , see Lemma 44.8. By Lemma 21.6 part (3) this implies f is quasi-finite at y . \square

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

Lemma 49.2. *Let $f : X \rightarrow S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \rightarrow S$ is locally quasi-finite.*

Proof. Suppose f is quasi-finite at x . Let $x \in U = \text{Spec}(R) \subset X$, $V = \text{Spec}(A) \subset S$ be affine opens as in Definition 21.1. By either Theorem 49.1 above or Algebra, Lemma 119.14, the set of primes \mathfrak{q} at which $R \rightarrow A$ is quasi-finite is open in $\text{Spec}(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. \square

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 31.3.

Lemma 49.3. *Let $f : Y \rightarrow X$ be a morphism of schemes. Assume*

- (1) X and Y are affine, and
- (2) f is quasi-finite.

Then there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & \searrow f & \swarrow \pi \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 119.15 reformulated in the language of schemes. \square

Lemma 49.4. *Let $f : Y \rightarrow X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y . Then $f(T) \subset X$ is a nowhere dense subset of X .*

Proof. As in the proof of Lemma 46.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1, \dots, n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 20.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X . This reduces us to the case

where both X and Y are affine. At this point we apply Lemma 49.3 above to get a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & \searrow f & \swarrow \pi \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion. Set $\bar{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 20.3 we see \bar{T} is nowhere dense in Z . Since $f(T) \subset \pi(\bar{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 46.7. \square

50. Universally bounded fibres

Let X be a scheme over a field k . If X is finite over k , then $X = \text{Spec}(A)$ where A is a finite k -algebra. Another way to say this is that X is finite locally free over $\text{Spec}(k)$, see Definition 46.1. Hence $X \rightarrow \text{Spec}(k)$ has a *degree* which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the *degree* of the (finite) scheme X over k .

Definition 50.1. Let $f : X \rightarrow Y$ be a morphism of schemes.

- (1) We say the integer n *bounds the degrees of the fibres of f* if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
- (2) We say the *fibres of f are universally bounded*¹⁵ if there exists an integer n which bounds the degrees of the fibres of f .

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

Lemma 50.2. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:*

- (1) *the integer n bounds the degrees of the fibres of f , and*
- (2) *for every morphism $\text{Spec}(k) \rightarrow Y$, where k is a field, the fibre product $X_k = \text{Spec}(k) \times_Y X$ is finite over k of degree $\leq n$.*

In this case f is universally bounded and the schemes X_k have at most n points.

Proof. The implication (2) \Rightarrow (1) is trivial. The other implication holds because if the image of $\text{Spec}(k) \rightarrow Y$ is y , then $X_k = \text{Spec}(k) \times_{\text{Spec}(\kappa(y))} X_y$. \square

Lemma 50.3. *A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \rightarrow Y$ and m bounds the degrees of $g : Y \rightarrow Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \rightarrow Z$.*

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $\text{Spec}(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \rightarrow Y_z$ are the fibres of f at the corresponding points of Y , which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma

¹⁵This is probably nonstandard notation.

27.1 applied to the set of all points of X_z). Write $X_z = \text{Spec}(A)$, $Y_z = \text{Spec}(B)$, and $k = \kappa(z)$. Then $k \rightarrow B \rightarrow A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak{m} \subset B$ we have $\dim_{\kappa(\mathfrak{m})}(A/\mathfrak{m}A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_{\mathfrak{m}}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 51.2 and 51.6. So we see that $\dim_k(B) = \sum_{\mathfrak{m}} \dim_k(B_{\mathfrak{m}})$ and $\dim_k(A) = \sum_{\mathfrak{m}} \dim_k(A_{\mathfrak{m}})$ where in both cases \mathfrak{m} runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 19.1) we see that each $A_{\mathfrak{m}}$ is a quotient of $B_{\mathfrak{m}}^{\oplus n}$ as a $B_{\mathfrak{m}}$ -module. Hence $\dim_k(A_{\mathfrak{m}}) \leq n \dim_k(B_{\mathfrak{m}})$. Putting everything together we see that

$$\dim_k(A) = \sum_{\mathfrak{m}} \dim_k(A_{\mathfrak{m}}) \leq \sum_{\mathfrak{m}} n \dim_k(B_{\mathfrak{m}}) = n \dim_k(B) \leq nm$$

as desired. \square

Lemma 50.4. *A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \rightarrow Y$ and $Y' \rightarrow Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \rightarrow Y'$ is also bounded by n .*

Proof. This is clear from the result of Lemma 50.2. \square

Lemma 50.5. *Let $f : X \rightarrow Y$ be a morphism of schemes. Let $Y' \rightarrow Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \rightarrow Y'$ be the base change of f . If $Y' \rightarrow Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f' , then also n bounds the degrees of the fibres of f .*

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f' . We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = \text{Spec}(\kappa(y')) \times_{\text{Spec}(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) \square

Lemma 50.6. *An immersion has universally bounded fibres.*

Proof. The integer $n = 1$ works in the definition. \square

Lemma 50.7. *Let $f : X \rightarrow Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent*

- (1) *the integer n bounds the degrees of the fibres,*
- (2) *for every field k and morphism $\text{Spec}(k) \rightarrow Y$ the base change $X_k = \text{Spec}(k) \times_Y X$ has at most n points, and*
- (3) *for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{\text{sep}}$ the scheme $X_{\kappa(y)^{\text{sep}}}$ has at most n points.*

Proof. This follows from Lemma 50.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 37.7. \square

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

Lemma 50.8. *Let $f : X \rightarrow Y$ be a morphism of schemes. Assume that*

- (1) *f is locally quasi-finite, and*
- (2) *X is quasi-compact.*

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1, \dots, n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \dots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of “local quasi-finiteness” (see Lemma 21.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \rightarrow V_i$ are locally quasi-finite morphisms of affines, hence quasi-finite, see Lemma 21.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \rightarrow V_i$, then $\sum n_i$ works for f).

Assume $f : X \rightarrow Y$ is a quasi-finite morphism of affines. By Lemma 49.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & \searrow f & \swarrow \pi \\ & & Y \end{array}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 50.6) this reduces us to showing that π has universally bounded fibres (Lemma 50.3).

This reduces us to a morphism of the form $\text{Spec}(B) \rightarrow \text{Spec}(A)$ where $A \rightarrow B$ is finite. Say B is generated by x_1, \dots, x_n over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 35.3). With these notations it is clear that

$$\bigoplus_{0 \leq e_i < d_i, i=1, \dots, n} A \longrightarrow B, \quad (a_{(e_1, \dots, e_n)}) \longmapsto \sum a_{(e_1, \dots, e_n)} x_1^{e_1} \dots x_n^{e_n}$$

is a surjective A -module map. Thus for any prime $\mathfrak{p} \subset A$ this induces a surjective map $\kappa(\mathfrak{p})$ -vector spaces

$$\kappa(\mathfrak{p})^{\oplus d_1 \dots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak{p})$$

In other words, the integer $d_1 \dots d_n$ works in the definition of a morphism with universally bounded fibres. □

Lemma 50.9. *Consider a commutative diagram of morphisms of schemes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \swarrow h \\ & & Z \end{array}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g , then also n bounds the degree of the fibres of h .

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h . Let $z \in Z$. Consider the morphism of schemes $X_z \rightarrow Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an

Artinian ring (by Algebra, Lemma 51.2). By Lemma 13.13 the morphism $X_z \rightarrow Y_z$ is affine in particular quasi-compact. It follows from Lemma 26.10 that Y_z is a finite discrete as this holds for X_z . Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 27.1 applied to the set of all points of Y_z). Write $Y_z = \text{Spec}(B)$ and $X_z = \text{Spec}(A)$. Then A is faithfully flat over B , so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. \square

51. Other chapters

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