Abstract

We study a R&D race with fixed experimentation intensity (a stopping game) and winner-take-all termination. Firms are uncertain and privately informed about the arrival rate of the invention, which measures the promise and feasibility of the line of research they are both pursuing. Due to the interdependent-value nature of the problem, the equilibrium displays a strong herding effect that distinguishes our framework from war-of-attrition models. Nonetheless, equilibrium expenditure in R&D is sub-optimal when players and the planner are sufficiently impatient. The more pessimistic firm may prematurely exit the race, so that R&D activity is inefficiently postponed. This result stands in stark contrast to the overinvestment in research, due to duplication costs, typical of R&D races without private information. We conclude that private incentives curb information sharing in R&D competitions; in turn, secrecy inefficiently slows down the pace of innovation.
1. Introduction

Research and Development activities produce essentially new information. Most of this information—about the promise, feasibility and interim experimental results of a project—is private to the researcher, who has no incentives to disclose it, but rather to carefully protect it from industrial espionage. When private corporations sponsor university research, as a norm they require the faculty and graduate students involved to sign non-disclosure and exclusive-licensing agreements. While (at least a noisy signal of) the research effort devoted by firms to individual projects may be publicly observable, this is only a very coarse statistic of the private information about its results.¹

Inspired by these observations, this paper investigates the positive and normative effects of private information on a research project on the outcome of R&D competition. To the best of our knowledge, this is the first attempt in this direction. Accordingly, our main normative result is novel in the literature on R&D races. If the social planner and the players are sufficiently impatient, a failure of information aggregation makes aggregate equilibrium expenditure in R&D on average too low with respect to the social optimum. This is thus our central message. Private incentives stifle information sharing in R&D competitions. Secrecy, in turn, tends to inefficiently slow down the pace of innovation.

Our finding of suboptimal investments in research provides a rationale for policy interventions. Indeed, investment in R&D is widely considered a powerful engine of economic growth. Electoral campaign platforms often include pledges to subsidize and promote R&D as a key form of investment for the future. There appears to be an implicit but widespread consensus that, left to market forces, the equilibrium amount of investment in R&D would be socially inadequate. Institutions such as patents arose precisely to address some of the sources of underinvestment, but further support is often called for. It would be hard to mention one public statement stigmatizing the “excess” investment in R&D and the consequent need to curb it.

The theoretical economics literature has emphasized different sources of distortions in R&D investment, but its conclusions do not seem to uniformly support this widely held belief. Paradoxically, partial equilibrium analysis systematically comes to the conclusion that equilibrium

¹The amount of resources invested by a company in R&D is public information in the US only at the aggregate firm level, for tax reasons, and not at the single project level. In Europe, not even firm-level information is always disclosed in a verifiable manner.
R&D investment is socially excessive, due to duplication costs: in equilibrium each firm over experiments in the attempt of beating competitors, and fails to internalize the negative externality imposed on the losing competitors. This literature has developed around the workhorse Poisson model of inventions originally proposed by Reinganum (1981, 1982). In a general equilibrium growth context, Aghion and Howitt (1992) isolate two non-strategic effects which may induce R&D underinvestment. First, in a patent system, the innovator only appropriates of the monopoly profits from the invention, which is less than the whole social surplus of innovation. Second, the innovator does not internalize the impact of his contribution on future research (see also Horstmann, MacDonald and Slivinski (1985)).

Our analysis identifies a novel, purely strategic force that may generate equilibrium under-investment in R&D. This force originates from imperfect aggregation of information on research projects, and particularly about their feasibility or “promise.” The intuition is simple. In some cases, an innovation has many authors, who reach independently and almost simultaneously the same conclusion, and competitive pressures and randomness are likely to lead to duplication costs and overinvestment in research. In many other circumstances, however, different researchers have different information or beliefs about the feasibility or “promise” of the same lines of research. Positive information tends to be concealed, because disclosing it would lure into the race competitors, and possibly imitators. Because “pessimistic” researchers cannot access the private information of more optimistic agents, they give up early in the race or even fail to join it. From the standpoint of an observer holding fully shared (complete) information, the more pessimistic

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2 In the strategic multi-armed bandit literature (Bolton and Harris 1999, Cripps, Rady and Keller 2003), equilibrium experimentation is sub-optimal, but this is only because players cannot conceal their findings from each other. Under-experimentation merely of the underprovision of a public good. This context hardly applies to R&D, whose results are carefully protected.

3 The remaining two effects identified in their general equilibrium analysis lead to excess R&D investment. In particular, the “creative destruction” effect is akin to the duplication cost argument of partial equilibrium analysis: a successful innovation displaces incumbent monopolists, making existing technology redundant, a negative externality ignored by the innovator.

4 As an example of how large corporations go about scouting promising projects, Pfizer, currently the largest pharmaceutical company in the world after recently absorbing Warner-Lambert/Parke-Davis, invested in 2002 $5.3 billion in R&D through its specialized arm Pfizer Global Research and Development, which employs 12,500 scientists. In addition to this massive structure, quoting from the company official website, “250 partners in academia and industry strengthen our position on the cutting edge of science and biotechnology by providing access to novel R&D tools and to key data on emerging trends.”

5 A famous and thrilling example, illustrated in a recent best-seller (Maddox 2002), is the discovery of the helicoidal structure of DNA by Crick and Watson in England. Their Nobel-winning publication, based to a large extent on Rosalind Franklin’s pioneering X-ray pictures of “wet” DNA filaments in Cambridge, preceded by a few weeks a likely identical discovery by Linus Pauling, who had been working independently for months on the same project in California. Arguably, ex post, Pauling’s enormous talent (testified by the two Nobel prizes he received) would have been better allocated elsewhere.
firms exit the race too early. Due to borrowing or talent constraints, the more optimistic firms may be limited in the amount of resources she can devote to the project. Thus, the resulting aggregate level of experimentation is too low, research activity is inefficiently postponed, and the pace of innovation is inefficiently slowed down. The intuition, and our formal analysis, extend to the (empirically plausible) case in which private information about the interim results of research activity accumulate during the race, providing clues to the future promise of the project.

To illustrate these points formally and to explore the role of private information in R&D races, we lay out a simple analytical framework. Two firms challenge each other in a research race with fixed experimentation intensity and winner-take-all termination. The prize can arrive independently to each player still in the race, with identical Poisson arrival processes of underlying unknown parameter (the “promise” of the project). At the beginning of the race, a private signal informs each player of the project’s promise.\(^6\) Over time, each firm decides whether to stay in the race, paying a flow cost, or whether to quit. Once a firm has quit, prohibitive sunk costs make re-entry economically infeasible.

As time goes by, and the innovation does not arrive, players become pessimistic about the innovation’s arrival rate, and eventually quit the race. In the unique symmetric monotonic equilibrium of this game, each firm selects a stopping time conditional on whether the opponent still being in the race or not, and increasing in her own private signal. We show that in both cases a firm optimally quits the race when the flow investment cost equals the expected flow benefit conditional on her information. Surprisingly, when both firms are still in the race, the possibility that the opponent wins the race in the next instant turns out to have no effect on a firm’s marginal value of remaining in the race, and hence on the equilibrium stopping time.

Due to the interdependent-value nature of the game, the equilibrium displays a rather extreme “winner’s curse” property, or more precisely a “survivor’s curse”. Specifically, if the two private signals are sufficiently close, the firms herd on each other’s participation in the race, rationally presuming that the opponents’ signal is larger than it actually is. Because of equilibrium monotonicity of stopping times in signals, when a firm quits, the “survivor” discovers the quitter’s actual signal and discontinuously revises her beliefs downwards. If the two private signals are sufficiently similar, this negative surprise will make the survivor immediately quit and

\(^6\)In Section 7, we allow also for gradually accumulating private information about the interim results of the research project. The results of the analysis are qualitatively the same, therefore we illustrate first the simpler case where private information accrues before starting the race.
regret not having quit the race earlier. This “curse” is more extreme than in standard or all-pay interdependent-value auctions and wars of attrition models (see Krishna and Morgan (1997) for a complete analysis). This fact underlines a key predictive distinction between our R&D game and wars of attrition. Contrary to prior claims (see, for instance the discussion in Taylor (1995)) that R&D races can be generally represented as wars of attrition, our results show that this is not the case when considering private information and interdependent values.

Our welfare analysis compares the aggregate discounted amounts of experimentation in equilibrium and in the efficient team solution. The winner-take-all assumption makes the two firms unwilling to share their private information, unless they are joined in a single team. The optimal team policy is to run both firms’ facilities in parallel, and then to stop them simultaneously when the joint flow cost for experimenting equals the expected joint marginal benefit.

In equilibrium, a firm prematurely drops out of the race when her private signal is sufficiently unfavorable relative to joint aggregate information. In the Poisson structure of our model, the survivor remains in the race until the equilibrium joint experimentation durations coincide with the team’s solution. But still the equilibrium discounted expenditure in R&D will be too low with respect to the social optimum, because some experimentation is inefficiently postponed to the future. If instead the firms’ signal are very close, then the “survivor’s curse” implies that both firms will suboptimally delay their exit from the race, and equilibrium over-experimentation takes place. When averaging out over instances of under-experimentation and over-experimentation, the postponement effect dominates the delayed exit effect as long as the discount rate is large enough, and the equilibrium features suboptimal R&D investment.

For the sake of clarity, we choose to illustrate these points in the simplest possible setup. Our key normative finding seems to be robust to a number of extensions of the model. We allow for private information to accumulate gradually over time as research results accrue. We consider private information of partially interdependent value, to capture the possibility of different approaches to solve the same research question. We consider the possibility that the prize arrival to a player induces a positive or negative externality on the other player. And we discuss the implications of allowing for variable but unobservable R&D investment.

Although the standard assumption of winner-take-all is inspired by the patent system, sponsored research tournaments play a somewhat similar role. Venture capitalists, for example, frequently run R&D tournaments when they allow only the best entrepreneur to go to the ini-
tial public offering (IPO) market. Other examples include research tournaments in which the firm with the best idea wins an exclusive right for commercializing it. For example, the Federal Communications Commission recently sponsored a tournament to develop the best technology for high-definition television (HDTV): the technology of the winner was chosen as the HDTV standard.\(^7\) R&D cooperatives are an increasingly popular arrangements between universities and corporations in the US, and among competing firms in Japan. Although their primary rationale appears to be pooling resources to overcome borrowing constraints and to share fixed costs, they also involve transfers of technology (see e.g. Adams et alii (2000)). In our perspective they might also be greatly beneficial to pool information about the promise and feasibility of a project.

Section 2 reviews the related theoretical literature, Section 3 lays out the model, Section 4 characterizes the team solution, Section 5 the unique symmetric monotonic equilibrium, Section 6 illustrates its normative properties, Section 7 concludes the paper and illustrates various extensions of the model, an Appendix contains the proofs.

2. Related Literature

Our work is related to several strands of literature, but presents important conceptual differences with respect to each of them.

First and foremost, the key benchmark are R&D races modeled either as differential or stopping games. The differential game approach is put forth in Reinganum (1981, 1982). At each moment in time \( t \), each player \( i \) selects an experimentation intensity \( u_i(t) \), paying a quadratic cost. The intensity affects linearly the arrival rate of the invention, which is \( u_i(t) \lambda \). Innovation arrivals are independent across players, and the first player to achieve the innovation wins the race. A simplified version of this differential game, where each player experiments with fixed intensity until it drops out of the race, can be understood as a stopping game. Choi (1985) takes this simplified route to extend the analysis to the case of uncertain \( \lambda \) with commonly known prior. This work is further extended by Malueg and Tsutsui (1999) in a full-featured differential game à la Reinganum. In these models with symmetric information, equilibrium R&D investment is

\(^7\)Sponsored R&D tournaments are discussed, for example, in Aoki (2001, Chapter 14), and studied in Taylor (1995) and Fullerton and McAfee (1999). We thank David Levine and Mike Baye for bringing these institutions to our attention.
socially excessive, due to duplication costs.\textsuperscript{8,9} We adopt the simplified stopping game approach to address the effects of private information about the unknown promise of the project. We identify both instances of overinvestment, due to the “survivor’s curse” effect, and of underinvestment, due to the postponement of research activity that follows inefficient information aggregation.\textsuperscript{10}

In order to analyze private information in the context of R&D races, this paper adapts and extends solution techniques from auction theory and timing games with interdependent values. But, while our game shares many elements of an all-pay ascending auction, or equivalently of a war of attrition with interdependent values,\textsuperscript{11} the models are not isomorphic. Specifically, the payoff specifications are different, and this induces a radically different equilibrium behavior.

To see this, consider as a benchmark the symmetric equilibrium strategy of a standard two-player common-value ascending second-price auctions (see Milgrom (1981) and Milgrom and Weber (1982)): a player with private signal $x$ quits the game at the time $\tau(x)$ when her bid (i.e. the cost) equals the expected value of winning the auctioned good (the benefit) conditional on \textit{both} players holding signal $x$. Unless $x$ is very large, in the monotonic symmetric equilibrium of a common-value war of attrition or all-pay auction, each player leaves the race much \textit{earlier} than this time $\tau(x)$.\textsuperscript{12} If in fact the opponent adopts strategy $\tau$, as the conjectured optimal

\textsuperscript{8}Before Reinganum (1981)’s dynamic analysis, the duplication costs effect was identified in the “static” models of Dasgupta and Stiglitz (1980), Loury (1979) and Lee and Wilde (1980). Baye and Hoppe (2003) show strategic equivalence between rent-seeking games, differential patent races with infinitely patient players, and games where firms choose how many research facilities to run, and the winner is the owner of the facility with the largest random outcome. Excessive investment in equilibrium is a common feature of all these classes of games.

\textsuperscript{9}An alternative approach to modeling R&D competition is the “tug-of-war”: firms take turns in making costly steps towards a “finish line.” In the absence of uncertainty, when solved by backward induction, these games predict a rather dramatic preemption effect: once a firm is known to be ahead in the race, the opponents drop out of the race, and the winner acts as if she faced no competition in the race (Fudenberg, Gilbert, Stiglitz and Tirole (1983), Harris and Vickers (1985)). As a result, the equilibrium displays no duplication costs, and it is indeed socially efficient. However, once we reintroduce uncertainty in the duration of each step (Harris and Vickers (1987)) the preemption effect vanishes, and equilibrium R&D investment is again socially excessive, due to duplication costs. Horner (2003) obtains similar results in a model with uncertainty but no finish line, and where a player’s payoff is larger when ahead in the race.

\textsuperscript{10}Reinganum (1981) identifies a source of under-investment in the following technological externality. Say that in the team’s problem, each project’s success rate increases with the expenditure in both facilities, because of knowledge sharing. As a result, fractioning expenditure across facilities increases the joint hazard rate of the innovation. In order to separate the welfare effect of information aggregation from this technological inefficiency, our model assumes constant returns to scale in R&D: because prize arrivals are independent across firms, the total arrival rate of an invention rises linearly in the number of active players. Positive (negative) correlation in prize arrivals is a form of (negative) positive spillovers, in the terminology of Reinganum (1981).

\textsuperscript{11}See Krishna and Morgan (1997) for a general treatment. Such models have been applied to several contexts in economics, biology and political science. For instance, the war of attrition has been used as a model for conflict among animals (Maynard Smith (1982), and Riley (1980)) and the struggle for survival among firms (Fudenberg and Tirole (1985, 1986)). Bulow and Klemperer (1999) solve the generalized war of attrition where the last $k$ out of $n$ players receive a prize. The all-pay auction has been used to model rent-seeking activity, such as lobbying (Hillman and Riley (1989), Baye, Kovenock and de Vries (1993)).

\textsuperscript{12}Krishna and Morgan (1997) find that the monotonic symmetric equilibrium bidding strategy $b(\cdot)$ of a 2-player
best-response own stopping time $\tau(x)$ is approaching, the player increasingly believes that the opponent’s signal is likely to be larger than $x$, and hence that the race is lost. As the expected benefit of staying in the race vanishes, the player anticipates exit to avoid paying the cost of attrition.

In our game, prize arrivals are independent across players, with unknown arrival rate. Hence the information that the opponent’s signal is larger than $x$ does not imply that the player will lose the race, it only conveys good news on the player’s prize arrival rate. As a result, in the monotonic symmetric equilibrium of our game, each player with signal $x$ postpone exit after the time when her flow cost equals the expected flow benefit (the value of the prize multiplied by the expected arrival rate) conditional on both players holding signal $x$. In a sense, the informational spillover that derives from common value and independent arrivals places our game on the opposite side of common-value wars of attritions, with common-value standard auctions in between.

Our game differs dramatically from the war of attrition or all-pay auction also from a normative viewpoint. The monotonic equilibrium of all-pay auctions (as well as that of standard first-price and second-price auction) without reserve price is Pareto-efficient: because of monotonicity, the prize is allocated to the player who values it most, and welfare analysis is mostly concerned with revenue maximization. In our R&D race with private information of interdependent values, inefficiencies arise in equilibrium due to the antagonistic effects of herding (which induces equilibrium over-experimentation in terms of duplication costs) and imperfect information aggregation (which induces equilibrium under-experimentation by postponing R&D activity).

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War of attrition is such that:

$$b(x) = \int_{-\infty}^{x} v(y, y) \lambda(y|y) dy,$$

where $v(y, y)$ is the expected value of the auctioned good conditional on both players’ signals being equal to $y$, and $\lambda(y|y)$ is the hazard rate that the opponent’s signal equals $y$ given that the player’s signal is $y$. Unless $x$ is large, $\lambda(y|y)$ is small enough that this integral is smaller than $v(x, x)$, the expected value of the good conditional on both players holding signal $x$.

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13 This postponement is akin to herding effects studied, for instance, by Smith and Sorensen (2000), but unlike herding models, we consider two long-run and forward-looking players, who act at endogenous times. Unlike our model and herding models, Bulow and Klemperer (1994) derive similar frenzies and crashes also in private-values environments.

14 On the other hand, bidders postpone their quitting time beyond $\tau(x)$ as in our game, i.e. they bid more aggressively than in the standard second-price auction, also when they are endowed with a share of the good auctioned off (e.g a stake of a take-over target company), see Bulow, Huang and Klemperer (1999). This is purely coincidental: a bidder who owns such a ‘toe-hold’ is willing to bid aggressively just because every price quoted is not only a bid for the remaining shares but also an ask for her own holdings.

15 Among the many cited examples of wars of attrition, a useful benchmark is the work by Fudenberg and Tirole (1985, 1986). They model a duopoly war with changing demand as a war of attrition with private information of private values (marginal costs). Similarly to our results, they show that the weakest firm may exit too early or too late with respect to social optimum, but this is purely coincidental. While in our R&D race problem the efficiency
Chamley and Gale (1994) [CG] study a discrete-time timing game of common interest, where players are privately informed on the “state of the economy” and may have investment opportunity, that can be irreversibly exercised at any period. As in our model, an irreversible timing decision is the only way to credibly communicate private information. This results in imperfect information aggregation. Three key differences distinguish our analysis from CG. First, in CG players face a coordination problem, where either all should invest or none; if it were feasible for them to communicate, they would have no reason to conceal their private information. We analyze a game of conflicting interests, with payoff congestion, where players have every incentive to conceal their private information. If they could persuade outsiders, players would always downplay the promise of the project. Second, CG allow for no information accumulation over time. So, their game ends almost immediately if time periods become very short. Inefficiency then results because players may quickly coordinate on a wrong decision (either invest in a good state, or underinvest in bad state). In our R&D game, instead, learning over time plays a central role, as the negative public information that the prize has not arrived, which accumulates at a rate that is proportional to the number of players still in the race, works against the good news that the opponent is still in the race. Third, from a substantive viewpoint, in the CG model herding delays investment, as each player would like to wait and see how many opponents choose to invest. But this incentive to wait and see is naturally counteracted in the context of R&D, by the incentive to ‘be the first to enter the race’ so as to gain some advantage over the opponents. Our model is appropriate to study environments where herding extends experimentation durations and works in favor of overinvestment, so that underinvestment arises from other sources.\textsuperscript{16}

\textsuperscript{16}Gul and Lundholm (1995) study a two-player continuous-time coordination timing game that shares many similarities with CG. Again, players would like to exchange information if they were allowed to, and the herding effect goes in the direction of delaying an irreversible decision with random consequences (i.e. delaying investment in CG’s terminology). As in our game, the first players who takes her timing decision reveals all her private information, leaving the opponent to act fully informed.
3. The Game

Two players, $A$ and $B$, play the following stopping game. A prize $b > 0$ arrives to player $i = A, B$ at a random time $t_i \geq 0$, according to a Poisson process of constant hazard rate $\lambda \geq 0$, c.d.f. $F(t_i|\lambda) = 1 - e^{-\lambda t_i}$ and density $f(t_i|\lambda) = \lambda e^{-\lambda t_i}$. Conditional on the common $\lambda$, the two arrival times $t_A, t_B$ are independent. In order to know $t_i$ and receive the prize, player $i$ must keep paying a flow cost $c > 0$. Stopping payments of such costs implies that the prize is abandoned irreversibly and that $t_i$ will never be learned. We make a winner-take-all assumption: the first player to receive the prize ends the game. Costs and prizes are discounted at rate $r$.

The common hazard rate of arrival of either prize, $\lambda \geq 0$, is drawn by Nature, unobserved by the players, from a Gamma distribution:

$$\pi(\lambda) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha \lambda / \beta}, \text{ for } \alpha > 0, \beta > 0.$$ 

Before starting to pay costs, each player $i$ observes a private signal $z_i \leq 0$ distributed according to a negative exponential distribution: for every $Z \leq 0$ and $z = z_i$,

$$H(Z|\lambda) = \Pr(z \leq Z|\lambda) = e^{\lambda Z} \text{ with density } H'(Z|\lambda) = h(Z|\lambda) = \lambda e^{\lambda Z}.$$

The two private signals $z_A, z_B$ are conditionally (on $\lambda$) independent.

We will refer to a “project” as the possibility of paying $c$ to activate the arrival of a prize. In our game each player has a “project”; based on the realization of the private signal, she decides whether to pursue it or not and, if so, when to stop it irreversibly conditional on the other player being still in the game or not. The canonical application of the model is as follows. Each of two firms might start an R&D project of the same nature. Before starting the project, each firm observes a private signal on its “promise” $\lambda$. This key parameter is common to both projects, because they revolve around the same question, device etc.; but, conditional on the promise, the actual winner is determined by luck and by willingness to continue investing resources in research.

Belief Updating. We now describe how different kinds of information are used by each player. Two posterior beliefs play a key role in our analysis. We consider player $A$’s updating, player $B$’s being symmetric. First, suppose that player $A$ is fully informed about the realizations of both players’ signals, $z_A = x, z_B = y$, and that project $A$ has not delivered a prize by time $t$ and
properties of the Gamma distribution, this is:

\[
\pi (\lambda | z_A = x, z_B = y, t_A \geq t, t_B \geq t') = \frac{\pi (\lambda) h (x|\lambda) h (y|\lambda) [1 - F (t|\lambda)] [1 - F (t'|\lambda)]}{\int_A \pi (\lambda') h (x|\lambda') h (y|\lambda') [1 - F (t|\lambda')] [1 - F (t'|\lambda')] d\lambda'}
\]

which is shown in the Appendix to be \( \text{Gamma}(\alpha - x - y + t + t', \beta + 2) \). We denote this density by \( \pi_{t,t'}(\lambda | x, y) \) and its c.d.f. by \( \Pi_{t,t'}(\lambda | x, y) \).

Second, suppose that player \( A \) is fully informed, as before, about her own private signal realization \( z_A = x \), and knows that project \( A \) has not delivered a prize by time \( t \) and project \( B \) by time \( t' \). But, now, player \( A \) only knows about her opponent’s signal \( z_B \) that it is not smaller than \( y \). Conditional on this information, the posterior belief density on \( \lambda \) is:

\[
\pi (\lambda | z_A = x, z_B \geq y, t_A \geq t, t_B \geq t') = \frac{\pi (\lambda) h (x|\lambda) [1 - H (y|\lambda)] [1 - F (t|\lambda)] [1 - F (t'|\lambda)]}{\int_A \pi (\lambda') h (x|\lambda') [1 - H (y|\lambda')] [1 - F (t|\lambda')] [1 - F (t'|\lambda')] d\lambda'}
\]

whose closed-form expression in terms of \( x, y, t, t' \) is derived in the Appendix. We denote this density by \( \pi_{t,t'}(\lambda | x, y+) \) and its c.d.f. by \( \Pi_{t,t'}(\lambda | x, y+) \).

We will show that the key statistic to determine optimal stopping in the model is the expected hazard rate of prize arrival. Conditional on “complete” information \( (x, y, t, t') \), using the properties of the Gamma distribution, this is:

\[
\mathbb{E}_{t,t'} [\lambda | x, y] = \int_A \lambda \pi_{t,t'}(\lambda | x, y) d\lambda = \frac{\beta + 2}{\alpha - x - y + t + t'}
\]

clearly increasing in \( x \) and \( y \) and decreasing in \( t, t' \). Conditional on the partial information that the opponent signal is above \( y \), using the expression for \( \pi_{t,t'}(\lambda | x, y+) \) derived in the Appendix:

\[
\mathbb{E}_{t,t'} [\lambda | x, y+] = \int_A \lambda \pi_{t,t'}(\lambda | x, y+) d\lambda = (\beta + 1) \frac{(\alpha + t + t' - x)^{-\beta - 2} - (\alpha + t + t' - x - y)^{-\beta - 2}}{(\alpha + t + t' - x)^{-\beta - 1} - (\alpha + t + t' - x - y)^{-\beta - 1}}
\]

**Lemma 1.** The posterior expected hazard rates of arrival of the prize \( \mathbb{E}_{t,t'} [\lambda | x, y], \mathbb{E}_{t,t'} [\lambda | x, y+] \), conditional on no prize \( A \) arrival by time \( t \), no prize \( B \) arrival by time \( t' \), the realization of the signal \( z_A = x \), and the realization of signal \( z_B \) being (respectively) exactly equal to or weakly larger than \( y \), are both strictly decreasing in \( t \) and in \( t' \), vanish as \( t \) grows unbounded, and are strictly increasing in \( x \) and in \( y \).

Finally, knowing that the opponent’s private signal realization equals \( y \) for sure is bad news compared to knowing only that it is larger than \( y \).
Lemma 2. For any \( t, t', x, \) and \( y < 0, \Pi_{t,t'}(\lambda|x, y) \prec_{FSD} \Pi_{t,t'}(\lambda|x, y+) \prec_{FSD} \Pi_{t,t'}(\lambda|x, 0), \)

\[
\mathbb{E}_{t,t'}[\lambda|x, y] < \mathbb{E}_{t,t'}[\lambda|x, y+] < \mathbb{E}_{t,t'}[\lambda|x, 0].
\]

4. The Team Solution

We begin our analysis by studying the first-best solution, in which players join forces in a single team, sharing four pieces of information: the two signal realizations \( x, y \), that project \( A \) has not delivered a prize by time \( t \) and project \( B \) by time \( t' \). In principle the team may choose to stop projects in sequence. We first consider the case in which one of the two projects (project “1”) has been irreversibly stopped at time \( T_1 \geq 0 \), and the other (project “2”) is still ongoing at time \( T_2 \geq T_1 \). This will allow us later to solve backwards the problem where both projects are still ongoing. Project 1 may be indifferently either project \( A \) or \( B \), as they have identical statistical properties.

4.1. Optimal Stopping of the Last Running Project by the Team

At time \( t \), conditional on a true value of the prize hazard rate \( \lambda \), unknown to the players, and on the fact that no prize has arrived to date, the expected value of planning to stop a single project at some future date \( T_2 \geq t \) equals

\[
U_{2,t}(T_2|\lambda) = \int_t^{T_2} \frac{f(s|\lambda)}{1 - F(t|\lambda)} \left[ e^{-r(v-t)} dv + e^{-r(s-t)} b \right] ds + \frac{1 - F(T_2|\lambda)}{1 - F(t|\lambda)} \int_t^{T_2} (-c) e^{-r(v-t)} dv,
\]

where the subscript “2” denotes the relevance of this value for the second project, the first project being already off line. The first term is the expected discounted return in case the prize arrives before the project is stopped. Here \( f(s|\lambda) / [1 - F(t|\lambda)] = \lambda e^{-\lambda(s-t)} \) is the density of the prize arrival time, conditioned on no prize having arrived so far. The second term is the expected discounted return in case the prize does not arrive by the planned quitting time \( T_2 \), premultiplied by the chance that this happens \([1 - F(T_2|\lambda)] / [1 - F(t|\lambda)] = e^{-\lambda(T_2-t)} \). The team, having stopped the first project at calendar time \( T_1 \), plans at a subsequent time \( t \geq T_1 \) to stop the second project at an even further time \( T_{2,t}^*(x, y, T_1) \geq t \) such that

\[
T_{2,t}^*(x, y, T_1) = \arg \max_{T_2 \geq t} \left\{ W_{2,t}(T_2|x, y, T_1) = \int_A U_{2,t}(T_2|\lambda) \pi_{t,T_1}(\lambda|x, y) d\lambda \right\}.
\]

4.1. Optimal Stopping of the Last Running Project by the Team

\[
(4.1)
\]

Recall that \( \pi_{t,T_1}(\lambda|x, y) \) denotes the density of posterior beliefs conditional on the signal realizations \( x, y \) being known exactly, and on the facts that neither prize had arrived by time \( T_1 \), when
the first project was stopped, and that the second project running alone did not arrive in \((T_1, t)\)
either.

In order to determine this optimal choice, we may differentiate the value \(W_{2,t}(T_2|x, y, T_1)\) with
respect to current time \(t\) and obtain a differential equation for the value, which is the continuous-
time Hamilton-Jacobi-Bellman equation for this problem. However, we choose to work directly
on the integral, non-recursive form of the value, as written above, for two reasons. First, the
integral form allows to solve for the value function without having to guess its functional form,
as is commonly done in dynamic programming or in solving differential equations. Second,
this constructive approach needs no indirect arguments based on the applicability of recursive
methods, which often lack sufficient conditions for an optimum.

By inspection, we see that the team’s expected value \(W_{2,t}(T_2|x, y, T_1)\) of stopping the second
project at time \(T_2\), after stopping the first at time \(T_1\), is \(C^2\) in \(T_2\) for every \(T_2 \geq t\) and every
\(x, y, t, T_1\). Therefore, to find the optimal \(T_2\) we can study the derivative of the expected value
function \((4.1)\). Since this type of manipulations will be used repeatedly in later omitted proofs,
it is instructive to go through them at least once:

\[
\frac{dW_{2,t}(T_2|x, y, T_1)}{dT_2} = \int_{\Lambda} \frac{d}{dT_2} U_{2,t}(T_2|\lambda) \pi_{t,T_1}(\lambda|x, y) d\lambda
\]

\[
= \int_{\Lambda} \left[ \frac{f(T_2|\lambda)}{1 - F(t|\lambda)} \left( \int_t^{T_2} (-c) e^{-r(v-t)} dv + e^{-r(T_2-t)} b \right) - \frac{f(T_2|\lambda)}{1 - F(t|\lambda)} \int_t^{T_2} (-c) e^{-r(v-t)} dv \right] \pi(\lambda) h(x|\lambda) h(y|\lambda) \left[ 1 - F(t|\lambda) \right] \left[ 1 - F(T_1|\lambda) \right] d\lambda
\]

\[
\propto \int_{\Lambda} \left[ \frac{f(T_2|\lambda)}{1 - F(T_2|\lambda)} \left( b - c \right) \right] \pi(\lambda) h(x|\lambda) h(y|\lambda) \left[ 1 - F(T_2|\lambda) \right] \left[ 1 - F(T_1|\lambda) \right] d\lambda
\]

\[
= bE_{T_2,T_1} f(T_2|\lambda) \left| x, y \right| - c = bE_{T_2,T_1} \left[ \lambda \right| x, y] - c
\]

where in the second line we simplify the first and third terms and we use the expression for
\(\pi_{t,T_1}(\lambda|x, y)\) from \((3.1)\), in the fourth line we multiply and divide the integrand by \([1 - F(T_2|\lambda)]\)
and the whole expression by a positive renormalizing factor independent of \(\lambda\), that we omit.
Therefore, the first-order condition simply equates the posterior expected hazard rate of the
remaining prize to the cost/benefit ratio:

\[
bE_{T_2,T_1} \left[ \lambda \right| x, y] = c. \tag{4.2}
\]

Intuitively, the marginal cost \(c\) of proceeding an extra instant must equal the marginal benefit,
which consists of the prize \(b\) multiplied by its expected hazard rate conditional on all available
information. Due to exponential discounting, this condition is independent of the planning time $t$; therefore an optimal stopping time $T_2 \in [T_1, \infty)$, if it exists, is time-consistent.

We now determine the optimal stopping time of the second project, $T^*_2(x, y, T_1)$ for any signal pair $x, y$, current calendar time $t$, and time $T_1$ when the first project was stopped. Using the explicit expression for $\mathbb{E}_{T_2, T_1} [\lambda | x, y]$ with respect to the Gamma distribution derived in the Appendix, and rearranging terms, the first-order condition (4.2) yields the unique solution:

$$T_2 = \frac{b}{c} (\beta + 2) + x + y - \alpha - T_1 \equiv T_2(x, y, T_1).$$

(4.3)

In the Appendix we prove that this first-order condition is also sufficient:

**Lemma 3.** For every pair of signals $x, y$, if the team has stopped the first project at time $T_1$, the optimal stopping time of the second project is

$$T^*_2(x, y, T_1) = \max\{T_1, T_2(x, y, T_1)\},$$

where $T_1 \geq T_2(x, y, T_1)$ if and only if $T_1 \geq \frac{1}{2} \left[ \frac{b}{c} (\beta + 2) + x + y - \alpha \right] \equiv T^*(x, y).$  

(4.4)

If $T_1 \geq T^*(x, y)$, the team carries on both projects for a long enough time, then it must optimally stop them simultaneously: $T^*_2(x, y, T_1) = T_1$.

The stopping time $T^*(x, y)$ has an intuitive expression (Cf. 4.3): it is increasing in the benefit/cost ratio $b/c$, in the signal realizations $x, y$, in the prior mean $\beta/\alpha$ and the prior variance $\beta^2/\alpha$ of beliefs about the hazard rate $\lambda$. The variance effect stems from a standard option value of information: stopping later means (in the language of Moscarini and Smith 2001) “experimenting” more, because one sacrifices payoffs today in the hope of a random return from new knowledge in the future.

**4.2. Optimal Stopping of the First Project by the Team**

We may now calculate by backward induction the optimal stopping time of the first project $T^*_1(x, y)$. Specifically, we will show that the team’s optimal policy always prescribes to stop both projects simultaneously, so that $T^*_1(x, y) = T^*(x, y) = T^*_2(x, y, T^*_1(x, y))$.

The team plans at time $t$ a stopping time

$$T^*_1(x, y) = \arg \max_{T_1 \geq t} \left\{ W_{1,t}(T_1 | x, y) = \int_{\Lambda} U_{1,t}(T_1 | \lambda) \pi_{t,t}(\lambda | x, y) d\lambda \right\}$$

13
where

\[
U_{1,t}(T_1|\lambda) = \int_t^{T_1} \frac{2f(s|\lambda)(1 - F(s|\lambda))}{(1 - F(t|\lambda))^2} \left[ \int_s^t (-2c) e^{-r(t-v)} dv + e^{-r(s-t)} b \right] ds \\
+ \frac{(1 - F(T_1|\lambda))^2}{(1 - F(t|\lambda))^2} \left[ \int_t^{T_1} (-2c) e^{-r(t-v)} dv + e^{-r(T_1-v)} U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) \right]
\]

is the expected value of stopping at time \(T_1\) conditional on a known value of \(\lambda\). Here \(2f(s|\lambda)(1 - F(s|\lambda))\)

is the density of arrival of the prize to either one of the two projects, namely the derivative of the corresponding c.d.f. \(1 - (1 - F(s|\lambda))^2\), the first line collects payoffs in case the prize arrives while both projects run together (before \(T_1\)), and the second line the cost of running two projects fruitlessly until \(T_1\) and then collecting the payoff of continuing optimally with one project from that moment forward.

To find \(T^*_1(x,y)\), as for the second project we differentiate the value function \(W_{1,t}\) with respect to \(T_1\), and after substantial manipulations that are omitted but available upon request,

\[
\frac{dW_{1,t}(T_1|x,y)}{dT_1} \propto -c + \mathbb{E}_{T_1,T_1}[\lambda(b - U_{2,T_1}(T_2^*(x,y,T_1)|\lambda))]|x,y].
\]

(4.5)

Intuitively, by delaying the stopping time \(T_1\) of the first project an extra instant, the team pays the flow cost \(c\) and receives the following expected marginal benefit: at hazard rate \(\lambda\), the prize arrives and the benefit \(b\) is obtained, but on the other hand the continuation value \(U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)\) of proceeding with only one project is lost.\(^{17}\) If both projects are stopped simultaneously, namely if \(T_1\) is such that \(T_2^*(x,y,T_1) = T_1\), then clearly the continuation value of the second project alone is zero: \(U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) = 0\). In this case the familiar expression \(dW_{1,t}(T_1|x,y)/dT_1 \propto (b\mathbb{E}_{T_1,T_1}[\lambda|x,y] - c)\) obtains.

Next, we show that the optimal stopping time \(T^*_1(x,y)\) of the first project cannot exceed the magnitude \(T^*(x,y)\) defined in (4.4). In fact, proceed by contradiction. By Lemma 3, if \(T_1 > T^*(x,y)\), then \(T_2^*(x,y,T_1) = T_1\), i.e. the team stops the second project at the same time as the first one, so the continuation value is zero for every value of \(\lambda\): \(U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) = 0\). Thus

\[
\frac{dW_{1,t}(T_1|x,y)}{dT_1} \propto -c + b\mathbb{E}_{T_1,T_1}[\lambda|x,y] < -c + b\mathbb{E}_{T^*(x,y),T^*(x,y)}[\lambda|x,y] = 0
\]

\(^{17}\)Notice that \(U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) < b\), as in the continuation the impatient team can earn at most \(b\), and not immediately a.s. Furthermore, the hazard rate \(\lambda\) of the prize and the continuation value conditional on \(\lambda\), namely \(U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)\), are multiplied within the posterior expectation: the team cares about their covariance induced by the common dependence on \(\lambda\).
where the first proportionality follows from (4.5), the inequality from the assumption $T_1 > T^*(x, y)$ and the monotonicity of the expected hazard rate, and the last equality from the definition of $T^*(x, y)$ as the solution to the first-order condition for the team. But then $T_1 = T^*_1(x, y)$ violates a necessary first-order condition and cannot be optimal.

If parameters are such that $T^*(x, y) = 0$, then using $0 \leq T^*_1(x, y) \leq T^*(x, y) = 0$ we get $T^*_2(x, y, T^*_1(x, y)) = T^*_1(x, y) = 0$ and no project is ever started. But, in the case that $T^*(x, y) > 0$, the key question is still whether it is best for the team to stop the two projects simultaneously at time $T^*(x, y)$, or to stop them sequentially, so that $T^*_1(x, y) < T^*_2(x, y, T^*_1(x, y))$. By Lemma 3 sequential stopping requires that $T^*_1(x, y) < T^*(x, y)$; the (long) proof of the following Lemma shows that this inequality is in fact impossible.

**Lemma 4.** For every pair of signals $x, y$ such that $T^*(x, y) > 0$, the optimal stopping time $T^*_1(x, y)$ of the first project cannot be strictly smaller than $T^*(x, y)$.

The results of this section are summarized in the following Proposition.

**Proposition 1. (The Team Solution)** For every pair of signals $x, y$ on the unobserved promise $\lambda$ of the two projects, the team optimally stops both projects simultaneously at time

$$
T^*_1(x, y) = T^*_2(x, y, T^*_1(x, y)) = \max \{0, \frac{1}{2}[(\beta + 2) \frac{b}{c} + x + y - \alpha]\}.
$$

The intuition behind this first result is as follows. Since the two prize arrivals are independent, keeping two projects open instead of one is equivalent to doubling both the intensity of experimentation and the experimentation costs. If there were no uncertainty, the team’s optimal policy would be ‘bang-bang’: the team would either pursue the prize at the maximal intensity available, by keeping both projects open, when the marginal trade-off between costs and values is positive, or shut down both projects when the marginal trade-off is negative. Without uncertainty the trade-off is time invariant, so the two projects stay open either for ever or never. With uncertain hazard rate $\lambda$, the team learns about $\lambda$ by experimenting, and becomes pessimistic as time goes by and prizes do not arrive. Most importantly, it learns faster by keeping both projects open, rather than only one. As the team is impatient and discounts the future, there is no reason to slow down the rate of learning by proceeding with one project only. Hence, whenever the marginal trade-off between flow costs and expected flow value is positive, the team chooses to pursue the prize at the maximal intensity available, and it switches both off when the marginal trade-off becomes negative.
5. Equilibrium in Semi-Separating Strategies

5.1. Definition

In the game, each player observes only her own private signal and whether her rival is still in the race. Owing to the winner-take-all assumption, no player would reveal her private signal truthfully to the opponent. Hence, private information may only be revealed through quitting decisions. A player draws information from elapsing time in two ways. First, she verifies that no prize has arrived (when one prize arrives, the game is over); second, she can see whether the opponent is still in the game or not. The first piece of information is always bad news, because posterior beliefs deteriorate in a first-order stochastic dominance sense by Lemma 2; the second is either bad or good news, depending on the monotonicity of the opponent’s quitting strategy in her private signal.

For each player $i = A, B$, a pure strategy in this game is a pair of functions $(\tau_1^i, \tau_2^i)$, describing stopping behavior.\(^{18}\) We focus on symmetric equilibria, hence we omit the superscript $i$ from the strategies.\(^{19}\) The stopping time function as “quitter no. 1”, given own private signal and that the opponent is still in the race, is denoted by $\tau_1 : \mathbb{R}_- \to \mathbb{R}_+$. Here, for any signal $x$, the stopping time $\tau_1(x)$ prescribes that the player stays in the race until time $\tau_1(x)$ unless observing that the opponent has left the race at any time $\hat{\tau} < \tau_1(x)$. The choice of the extended positive real numbers $\mathbb{R}_+$ as the range of $\tau_1$ is made to allow for the possibility that a player may decide to stay in the game and wait for the prize forever, given some signal realization $x$, and given that the opponent is not leaving the game either. Note that the stopping time $\tau_1(x) = 0$ prescribes that the player should not enter the race at all.\(^{20}\)

We restrict attention to strategies where the stopping $\tau_1$ as “first quitter” satisfies the following monotonicity requirement: there exist $\underline{x}, \bar{x} \leq 0$ such that $\tau_1(x) = 0$ if $x \leq \underline{x}$, $\tau_1(x)$ is positive and strictly increasing if $\underline{x} < x \leq \bar{x}$, and $\tau_1(x) = \infty$ if $x > \bar{x}$. If $\bar{x} \leq 0$, then the stopping

\(^{18}\)Without loss in generality, we can restrict attention to time-consistent stopping times chosen by the players, function of the private signal and of how many players are left in the game, but not of calendar time. In fact, consider (for the sake of illustration) player $A$; even if he revises his stopping decision as time goes by, all that matters to player $B$ is when player $A$ does quit, because player $B$ observes only player $A$’s actions, not his intentions. So any previous plans made by $A$ and later revised are immaterial: player $B$ cares about $A$’s decision to stop at time $\tau$ as planned at time $\tau$.

\(^{19}\)As will become evident, the restriction to symmetric equilibria is made mostly for computational ease, and our main welfare prediction of under-experimentation when signals disagree sufficiently holds a fortiori in asymmetric equilibria, were they to exist.

\(^{20}\)An alternative interpretation of this model is that each player is initially uninformed, and optimally chooses to enter the race and gather information, which comes in an instantaneous single signal $x$ for simplicity. Given $x$, she then decides whether to leave the race or continue.
time \( \tau_1(x) \) is finite for any \( x \) in the support. We denote by \( g \) the inverse function of \( \tau_1 \) on the domain \([\underline{x}, \bar{x}]\).

The stopping strategy as “quitter no. 2”, given own private signal and that the opponent has already left the game, is denoted by \( \tau_2 : [\underline{x}, 0] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \). Here \( \tau_2(x, \tau) \geq \tau \) describes the player’s stopping time when holding signal \( x \) and after the opponent has quit at time \( \tau \).  

Our analysis focuses on symmetric monotonic equilibrium: both players adopt the same equilibrium strategy \((\tau_1^*, \tau_2^*)\), which is a best response to itself, and \( \tau_1^* \) also satisfies the above monotonicity requirement with thresholds denoted by \( \underline{x}^*, \bar{x}^* \).

5.2. Equilibrium Play After the Opponent Quits

In a symmetric monotonic equilibrium, when a player observes a private signal \( z \), enters the game at time 0, and then quits first according to the equilibrium strategy at time \( \tau = \tau_1^*(z) > 0 \), the remaining player perfectly infers her opponent’s private signal \( g^*(\tau) \), and from that moment on solves a single-project decision problem equivalent to that of the team. It follows immediately that the optimal stopping time of the last player is given by the team’s stopping rule, conditional on the first player having left at \( \tau > 0 \), namely \( \tau_2^*(y, \tau) = T_2^*(y, g(\tau), \tau) \).

**Proposition 2. (Equilibrium Play after the Opponent Quits)** In any symmetric monotonic equilibrium, for any \( \tau > 0 \), the optimal stopping time of a player with signal \( x \), after the opponent quits at time \( \tau \) and reveals her private information \( g^*(\tau) \), equals the team’s optimal stopping time of the second project conditional on the same information:

\[
\tau_2^*(x, \tau) = \max\{\tau, \frac{b}{c} (\beta + 2) + x + g^*(\tau) - \alpha - \tau\}.
\]

As mentioned, if (say) player B even fails to join the game, then player A cannot perfectly infer player B’s signal realization \( z_B \), because the equilibrium strategy \( \tau_1^* \) is not invertible for \( z_B \leq \underline{x}^* \equiv g^*(0) \), but only learns that \( z_B \leq \underline{x}^* \).

---

21 When \( \tau_2(x, \tau) = \tau \), the player leaves the race immediately after seeing that the opponent left at \( \tau \). Formally, this continuous stopping time is derived from a finite-time approximation where each period lasts \( \Delta \), and \( \lim_{\Delta \rightarrow 0^+} \tau_2(x, \tau) = \dot{\tau} \). Metaphorically, in the moment that a player leaves the race, the clock is ‘stopped for an instant’ and the remaining player is left to choose whether to continue or follow suit. For a general treatment on how to construct stopping time strategies in continuous time games and on their interpretation, see Simon and Stinchcombe (1989).

22 The equilibrium is semi-separating, because not entering the game reveals to the opponent only an upper bound \( \underline{x}^* \) to the observed private signal.
Proposition 3. (Equilibrium Play after the Opponent Fails to Join the Game) In any symmetric monotonic equilibrium, if the opponent fails to join the game, a player with private signal realization \( x \) optimally stops at time

\[
\tau^*_2(x, 0-) = \max\{0, \frac{b}{c} (\beta + 1) + x + x^* - \alpha\}.
\]

Since entering the game for an arbitrarily small length of time, and then quitting, perfectly reveals own private information \( x \), while not joining the game at all only reveals an upper bound \( x^* \) to \( x \), there is a natural discontinuity in the equilibrium strategy as a second quitter \( \tau^*_2(x, \hat{\tau}) \) at \( \hat{\tau} = 0 \). In fact, for any \( x > \alpha - (\beta + 2) b/c - x^* \), so that entering the game for some time is optimal,

\[
\lim_{\hat{\tau} \to 0} \tau^*_2(x, \hat{\tau}) = \frac{b}{c} (\beta + 2) + x + x^* - \alpha > \max\left\{0, \frac{b}{c} (\beta + 1) + x + x^* - \alpha\right\} = \tau^*_2(x, 0-).
\]

For future reference, we let \( x^{**} \equiv \inf\{x : \tau^*_2(x, 0) > 0\} \), the lowest signal for which a player is willing to stay in the race, upon seeing that the opponent did not enter the game. To summarize:

- if \( x > x^{**} \) then the player enters and stays in for some time no matter what the opponent does;
- if \( x^* \leq x \leq x^{**} \) then the player enters and quits right away if the opponent failed to join;
- if \( x^* < x \) then the player does not enter at all.

5.3. Equilibrium Play Before the Opponent Quits

The most complex part of the equilibrium characterization concerns the earlier phase of the game, when both players are still in the game. Each player must plan an optimal stopping time based on the hypothesis that the opponent will quit later, and on the resulting information about the opponent’s private information.

The Value Function. We first determine the value function of a player at any time \( t > 0 \) for quitting at time \( \tau \geq t \), conditional on the facts that opponent has not quit yet at time \( \tau \) and is adopting a monotonic strategy \((\tau_1, \tau_2)\), with associated inverse \( g = \tau_1^{-1} \). We consider the problem of player A who contemplates stopping first at time \( \tau^A \), the other player’s calculations being symmetric. We can write the expected value at time \( t \) for planning at time \( t \) to stop at some time \( \tau > t \), conditional on \( \lambda \). Using the subscript “1” to denote the expected value of the
first quitter:

\[
Q_{1,t} (\tau | \lambda) = \int_t^\tau \frac{f(s|\lambda)}{1 - F(t|\lambda)} 1 - F(s|\lambda) 1 - H(g(s)|\lambda) \left[ \int_s^\tau -ce^{-r(v-t)} dv + e^{-r(s-t)} b \right] ds \\
+ \int_t^\tau \frac{f(s|\lambda)}{1 - F(t|\lambda)} \left[ 1 - F(s|\lambda) 1 - H(g(s)|\lambda) \right] \left[ \int_s^\tau -ce^{-r(v-t)} dv \right] ds \\
+ \int_t^\tau \left( \frac{1 - F(s|\lambda)}{1 - F(t|\lambda)} \right)^\frac{2}{3} h(g(s)|\lambda) g'(s) \left[ \int_s^\tau -ce^{-r(v-t)} dv \right] ds \\
+ \left( \frac{1 - F(\tau|\lambda)}{1 - F(t|\lambda)} \right)^\frac{2}{3} 1 - H(g(\tau)|\lambda) \int_t^\tau -ce^{-r(v-t)} dv
\]

(5.1)

Each one of the four lines, with associated probabilities and discounted payoffs, corresponds to one of the four possible and exhaustive events that can take place at any time \( s \) between the current time \( t \) and any future date \( \tau \) at which the player plans to quit first (i.e. provided the opponent has not quit by then). We go through the four events and lines in order.

First, player \( A \)'s prize arrives at \( t_A \in [t, \tau^A) \), before (the prize arrives to the rival at time) \( t_B \) and before (the opponent quits first at time) \( \tau^B \). Conditional on the true arrival rate \( \lambda \), the density of this event for \( t_A = s \in [t, \tau^A) \) is

\[
\frac{f(s|\lambda)}{1 - F(t|\lambda)} 1 - F(s|\lambda) 1 - H(g(s)|\lambda)
\]

as shown in the first line of (5.1). In this case, \( A \) wins and takes all, pays costs up to that time \( t_A \) and collects the prize \( b \). Second, player \( B \)'s prize arrives at \( t_B \in [t, \tau^A) \), before \( A \)'s prize arrives at \( t_A \) and before \( A \) quits at \( \tau^A \). As a result, \( B \) wins and takes all, the game is over at time \( t_B \), player \( A \) just pays costs. The density of this event for \( t_B = s \in [t, \tau^A) \) is the same as the density of the arrival of the prize to player \( A \). Third, player \( B \) quits at \( \tau^B \in [t, \tau^A) \) first, i.e. before either prize arrives. Then the signal \( z_B \) is revealed to \( A \) by inverting \( z_B = g(\tau^B) \). The density of this event for \( t_A = s \in [t, \tau^A) \) is in the third line of (5.1). Player \( A \) pays costs and collects \( W_{2,s}(\tau^*_2(x,s)|x,g(s),s) \), the continuation value of going on alone optimally with the new information. Fourth and last, nothing happens in the time interval \([t, \tau^A)\); no one quits and no prize arrives. In this case player \( A \) quits at \( \tau^A \) and just pays costs. The probability of this event is in the fourth line of (5.1).

At any time \( t > 0 \), given that the opponent adopts a monotonic strategy \( \tau^*_1 \), with inverse \( g^* \), quitting first at time \( \tau_1 = \tau^*_1(x) \) after observing private signal \( x = g^*(\tau_1) \), each player chooses
the following optimal stopping time as first quitter:

$$\tau^*_1(t) = \arg \max_{\tau \geq t} \left\{ V_{1,t}(\tau|x) = \int_{\Lambda} Q_{1,t}(\tau|\lambda) \pi_{t,t}(\lambda|x, g(t) + d\lambda) \right\}.$$ 

**The First-Order Condition: Necessity and Sufficiency.** In order to find the optimal stopping time of a player before the opponent quits, we differentiate the expected value $V_{1,t}(\tau|x)$ with respect to the stopping time $\tau$. After substantial manipulations that we omit but make available upon request, we obtain:

$$\frac{dV_{1,t}(\tau|x)}{d\tau} = -c + \mathbb{E}_{r,\tau} [\lambda|x, g(\tau) + b + W_{2,r}(\tau^*_2(x, \tau)|x, g(\tau), \tau) \mathbb{E}_{r,\tau} \left[ \frac{h(g(\tau)|\lambda)g'(\tau)}{1 - H(g(\tau)|\lambda)} x, g(\tau) + \right]] \tag{5.2}$$

This marginal value equals minus the flow cost $-c$ plus two flow expected benefit terms, the expected hazard rate $\mathbb{E}_{r,\tau} [\lambda|x, g(\tau) + b$ of prize arrival times the prize value $b$, and the expected hazard rate $\mathbb{E}_{r,\tau} \left[ \frac{h(g(\tau)|\lambda)g'(\tau)}{1 - H(g(\tau)|\lambda)} x, g(\tau) + \right]$ of the opponent leaving the game times the continuation value $W_{2,r}(\tau^*_2(x, \tau)|x, g(\tau), \tau)$ of remaining alone. Since RHS of (5.2) is independent of current time $t$, any stopping time $\tau_1(x)$ satisfying the first-order condition $dV_{1,t}(\tau|x)/d\tau = 0$ must be independent of $t$, and hence is time consistent.

Remarkably, the possibility that the opponent may receive the prize and win the race does not enter the marginal value of waiting to quit the game first. This surprising property has a simple intuition. The derivative $dV_{1,t}(\tau|x)/d\tau$ captures the difference in value at $\tau$ between, on the one hand, staying in the race for an extra small $\Delta \tau$ and then leaving at time $\tau + \Delta \tau$, and on the other hand leaving immediately at $\tau$. In this period of time $\Delta \tau$, the cost $c\Delta \tau$ is paid up-front and sunk, and either one of the two prizes of size $b$ or $W_{2,r}(\tau^*_2(x, \tau)|x, g(\tau), \tau)$ may arrive. But if nothing arrives, in which case the player quits, or if the opponent wins in the meantime, then either way the payoff is zero. It is immaterial to this cost-benefit analysis that the prize could also arrive to the opponent: the only effect of this is to end to game before $\tau + \Delta \tau$ at no further cost, nor benefit to the player. Since the player is anyway ending the game at time $\tau + \Delta \tau$, the arrival of the prize to the opponent at any given $t \in (\tau, \tau + \Delta \tau)$ bears no change in the player’s marginal value for waiting as $\Delta \tau$ vanishes.

The next Lemma establishes two “corner properties” of any equilibrium strategy. First, never leaving the game as long as the other player stays in cannot be a best response to itself for any signal realization $x \leq 0$. In other words, there are no symmetric monotonic equilibria where the players herd on each other’s experimentation so much that they remain in the race forever.
Conversely, a player stays in the game for a positive amount of time, when her opponent enters
the game and stays in, provided that her signal is good enough.

**Lemma 5.** Suppose that player $B$ plays a monotonic strategy $\tau_1$, of quitting first at time $\tau_1(y)$
after privately observing $y$, with inverse $g$. For any signal realization $x$ observed by player $A$,
there exists $\hat{\tau} > 0$ large enough that player A’s marginal value of waiting $V'_{1,t}(\tau|x)$ is negative for
any $\tau \geq \hat{\tau}$. For any signal realization $x$ such that $b \mathbb{E}_{0,0}[\lambda|x, g(0) +] > c$, there exists $\tau > 0$ small
enough that $V'_{1,t}(\tau|x) > 0$ for any $t \leq \tau \leq \tau$.

**Equilibrium Characterization.** The key result of this subsection is a “survivor’s curse,” that
we identify in any monotonic equilibrium of this class of optimal-stopping games of conflicting
interests. Suppose that (say) player $B$ adopts a monotonic strategy $g$, and that player $A$
edowed with signal $x$ plans, in the event that $B$ remains in the race, to quit first at a time $\tau$ which satisfies
the first-order condition $V'_{1,t}(\tau|x) = 0$. Then, if $B$ quits first at any time $\hat{\tau}$ earlier than but close
enough to $\tau$, then $A$ must also immediately leave the race, regretting not having left earlier:
formally $T_2(x, g(\hat{\tau}), \hat{\tau}) < \hat{\tau}$ and hence $\tau^*_x(x, \hat{\tau}) = \hat{\tau}$. The intuition behind this result is simple.
When player $A$ plans to remain in the race at any time $t < \tau$, and to leave exactly at $\tau$, she
conditions on player $B$ still being in the race and hence on the expectation $\mathbb{E}[z_B|z_B \geq g(\tau)]$ with
respect to $B$’s signal. If in fact $B$ quits first at a time $\hat{\tau}$ when $A$ is about to do so at $\tau$, namely $\hat{\tau}$ is
close to but smaller than $\tau$, then $A$ at $\hat{\tau}$ suddenly realizes that $B$ had observed signal $z_B = g(\hat{\tau})$,
which is much smaller than $\mathbb{E}[z_B|z_B \geq g(\hat{\tau})]$ and hence smaller also than $\mathbb{E}[z_B|z_B \geq g(\tau)]$. This
induces a sudden pessimistic revision of $A$’s beliefs with respect to the promise of the project;
accordingly, $A$ quits immediately after $B$, regretting her previous over-optimistic expectation of
the rival’s assessment of the project’s feasibility.\footnote{As a mirror image of this survivors’ curse, a ‘quitter’s curse’ arises if player $B$ leaves the race at a time $\hat{\tau}$
much earlier than $\tau$, the planned stopping time of player $A$. In this case, player $A$ remains in the race after $\hat{\tau}$, and
player $B$ regrets having left.}

**Lemma 6.** Suppose that player $B$ plays a monotonic strategy $(\tau_1, \tau_2)$ with inverse $g = \tau_1^{-1}$.
For any signal realization $x \leq 0$, if player $A$ is planning to quit first at time $\tau > 0$ such that
$\frac{dV'_{1,t}(\tau|x)}{d\tau} = 0$, then for any $\hat{\tau} < \tau$ but close enough to $\tau$ player A’s optimal stopping strategy
after $B$ quits at $\hat{\tau}$ is to follow suit and gain a nil continuation value:

$$\tau^*_x(x, \hat{\tau}) = \hat{\tau} > T_2(x, g(\hat{\tau}), \hat{\tau}) = \frac{b}{c}(\beta + 2) + x + g(\hat{\tau}) - \hat{\tau} - \alpha.$$
Proof. By definition \( \tau^*_2(x, \hat{\tau}) = \max\{\hat{\tau}, T_2(x, g(\hat{\tau}), \hat{\tau})\} \), so we only need to show that \( \hat{\tau} > T_2(x, g(\hat{\tau}), \hat{\tau}) \), where \( T = T_2(x, g(\hat{\tau}), \hat{\tau}) \) solves the first-order condition

\[
c = bE_{T, \hat{\tau}} [\lambda | x, g(\hat{\tau})] = b \frac{\beta + 2}{\alpha - \hat{x} - g(\hat{\tau}) + \hat{\tau} + T}.
\]

By continuity and because the RHS is decreasing in \( T \), we only need to show that \( c > bE_{x, \tau} [\lambda | x, g(\tau)] \); the flow cost dominates the expected benefits for staying in the race, if the opponent leaves at \( \tau \).

Since \( \tau \) solves the first-order condition \( 0 = V_{1,t}^\prime(\tau | x) \), using equation (5.2), we obtain:

\[
0 = bE_{x, \tau} [\lambda | x, g(\tau) +] + W_{2, \tau} (\tau^*_2(x, \tau) | x, g(\tau), \tau) E_{x, \tau} \left[ \frac{h(g(\tau) | \lambda) g'(\tau)}{1 - H(g(\tau) | \lambda)} \right] x; g(\tau) + c
\]

\[
\geq bE_{x, \tau} [\lambda | x, g(\tau) +] - c > bE_{x, \tau} [\lambda | x, g(\tau)] - c,
\]

where the first inequality follows because \( \frac{h(g(\tau) | \lambda) g'(\tau)}{1 - H(g(\tau) | \lambda)} > 0 \) and \( W_{2, \tau} (\tau^*_2(x, \tau) | x, g(\tau), \tau) \geq 0 \), whereas the second inequality follows from Lemma 2: knowing that the opponent’s signal is exactly \( g(\tau) \) is bad news with respect to knowing that it is at least \( g(\tau) \).

In light of the Lemma 6, for any \( x \), the First-Order Condition \( dV_{1,t}(\tau | x)/d\tau = 0 \) can be rewritten very simply as:

\[
c = bE_{x, \tau} [\lambda | x, g(\tau) +],
\]

(5.3)

which says that a player quits first at the time \( \tau \) when the flow cost equals the expected flow benefit consisting of the value of the prize \( b \) times expected hazard rate \( \lambda \) of arrival of the prize, conditional on own private information \( x \), on the opponent’s private signal \( y \) being larger than \( g(\tau) \), and on neither prize having arrived by time \( \tau \). In any symmetric monotonic equilibrium \((\tau^*_1, \tau^*_2)\), symmetry implies that \( g^*(\tau^*_1(x)) = x \) (and in particular \( g^*(0) = x^* \)), so that Equation (5.3) is further simplified as

\[
c = bE_{x, \tau} [\lambda | x, x+] = b(\beta + 1) \frac{(\alpha + 2\tau - x)^{-\beta - 2} - (\alpha + 2\tau - 2x)^{-\beta - 2}}{(\alpha + 2\tau - x)^{-\beta - 1} - (\alpha + 2\tau - 2x)^{-\beta - 1}},
\]

(5.4)

By Lemma 1, the RHS of this equation is strictly decreasing in \( \tau \) and strictly increasing in \( x \). Therefore, this equation has a unique positive and increasing solution \( \tau_1(x) \) for any \( x \) such that \( bE_{0,0} [\lambda | x, g^*(0) +] = bE_{0,0} [\lambda | x, x^* +] \geq c \), i.e. \( x \leq x^* \), and no solutions otherwise. In particular, \( x^* = g^*(0) \) is uniquely determined by

\[
c = b(\beta + 1) \frac{(\alpha - x^*)^{-\beta - 2} - (\alpha - 2x^*)^{-\beta - 2}}{(\alpha - x^*)^{-\beta - 1} - (\alpha - 2x^*)^{-\beta - 1}}.
\]

(5.5)
So far, our analysis has singled out as the unique candidate symmetric monotonic equilibrium stopping strategy the monotone function $\tau^*_1$ such that: $\tau^*_1(x)$ is the unique solution of the implicit function (5.4) if $bE_r, \lambda |x, x^*| > c$, and $\tau^*_1(x) = 0$ otherwise. Lemma 5 in the Appendix has shown that if $bE_r, \lambda |x, x^*| > c$ and the opponent adopts the stopping strategy $\tau^*_1$, a player endowed with signal $x$ who is in the race at time $t$ finds it optimal to quit it at time $\tau^*_1(x) > 0$.

In order to conclude that the strategy $\tau^*_1$ is the unique symmetric monotonic equilibrium, we are only left to determine the optimal decision at the very beginning of the game, i.e. at time $t = 0$. We verify in the Appendix (Lemma A.1) that if the opponent enters the game if and only if $x > \bar{x}^*$, it is optimal to enter the race if and only if $x > \bar{x}^*$. Lemma 2 immediately implies that $\bar{x}^{**} > \bar{x}^*$: upon knowing that the opponent failed to join the race, a player becomes more pessimistic than if the opponent entered for a whatsoever small amount of time, hence, whenever $\bar{x}^* < x < \bar{x}^{**}$, the player will initially join the race and then exit immediately after if the opponent did not join. The following Proposition summarizes our findings for equilibrium play $\tau^*_1$ before the opponent quits and, together with Proposition 2 for the strategy $\tau^*_2$ after the opponent has quit, fully characterizes the unique symmetric monotonic equilibrium $\{\tau^*_1, \tau^*_2\}$.

**Proposition 4. (Equilibrium Play Before the Opponent Has Quit)** The unique symmetric monotonic equilibrium quitting time, conditional on a private signal $x$ and on the opponent still being in the game, is

$$
\tau^*_1(x) = \begin{cases} 
0 & \text{if } x < \bar{x}^* \\
\tau_1(x) & \text{if } x \geq \bar{x}^*, 
\end{cases}
$$

where $\tau_1(x)$ is the unique increasing solution of (5.4) and $\bar{x}^* < 0$ is the unique root of $\tau_1(x) = 0$, or equivalently (5.5).\(^{24}\)

6. Welfare Comparison

We are finally in a position to investigate the welfare properties of the unique symmetric monotonic equilibrium, by comparing its outcome to the team solution. Without loss of generality, we let $x \leq y$, so that player $A$ is more pessimistic than player $B$. To avoid triviality, we assume that $\alpha < (\beta + 2) b/c$; if not $T^*(x, y) = \max\{0, [(\beta + 2)b/c + x + y - \alpha]/2\} = 0$ and

\(^{24}\)When more than two firms are involved in the race, in a symmetric equilibrium, the state variable of the game is the number of firms $n$ still present in the race. The equilibrium again displays our “survivor’s curse” property when signals are close enough. Inspecting the stopping-time equation conditional on all the firms still being in the race: $bE[\lambda |z_i = x, z_j \geq x, \forall j \neq i; T_k > \tau, \forall k \neq c, \text{ we conclude that equilibrium stopping times are decreasing in } n$, because beliefs become pessimistic faster conditional on no prize arrival to any of $n$ firms, the larger is $n$. 

23
\(\tau_1(x) = \tau_2(x, \hat{\tau}) = \tau_1(y) = \tau_2(y, \hat{\tau}) = 0\) for any signals \((x, y) \in \mathbb{R}^2\) and time \(\hat{\tau} \in \mathbb{R}_+\): neither the team, nor any of the two players ever participate in the game.\(^{25}\)

First, suppose that \(y \geq x > x^*\), so that both players join the game. When player \(A\) quits, she reveals her private signal \(x\) to \(B\). If player \(B\)'s signal \(y\) is sufficiently close to \(x\), our "survivor's curse" implies that player \(B\) learns that the players have been herding on each other into remaining too long in the race, and immediately follows \(A\) suit, regretting not having left before. When regret occurs, there is clearly excessive experimentation in equilibrium because both projects are stopped too late.

**Lemma 7.** Suppose that \(y \geq x > x^*\). If

\[
\tau_1^*(x) > T^*(x, y) = \frac{1}{2} \left[ (\beta + 2) \frac{b}{c} - \alpha + x + y \right]
\]

(6.1)

then both players’ equilibrium stopping times are higher than the efficient common stopping time of both projects:

\[
\tau_2^*(y, \tau_1^*(x)) = \tau_1^*(x) > T^*(x, y).
\]

Since the function \(T^*(x, y)\) is increasing in both \(x\) and \(y\), given that \(y \geq x\), the inequality in Equation (6.1) holds if \(x\) and \(y\) are sufficiently close. An extreme case is when \(T^*(x, y) = 0\), i.e. \(x + y < \alpha - (\beta + 2) \frac{b}{c}\) and \(y \geq x > x^*\): both players enter the game whereas the team would not participate. This may happen with positive probability because \(2x^* < \alpha - (\beta + 2) \frac{b}{c}\).\(^{26}\) The source of the inefficiency is a form of herding, as the first player to quit puts too much weight on the public information that the opponent is still in the game, and too little weight on her own private information.

Second, a more complex case arises when \(y > x > x^*\), \(x\) and \(y\) are sufficiently large that \(T^*(x, y) > 0\), so that the team plays, but \(x\) is sufficiently small with respect to \(y\) that \(\tau_1^*(x) < T^*(x, y)\): player \(A\), when quitting at time \(\tau_1^*(x)\), underestimates the opponent’s signal \(y\) and quits too soon with respect to the team’s solution. Player \(B\) stays in the race until time

\[
\tau_2^*(y, \tau_1^*(x)) = \frac{b}{c} (\beta + 2) + g(\tau) + y - \alpha - \tau_1^*(x).
\]

\(^{25}\)If \(\alpha \geq (\beta + 2) \frac{b}{c}\), then \(T^*(0, 0) = 0\) and hence \(dV_{1,\tau}(\tau|x)/d\tau = bE_{\tau,\tau}[\lambda|x, g(\tau)+] - c \leq bE_{0,0}[\lambda|0,0] \leq 0\), so that also \(\tau_1(x) = \tau_2(x, \hat{\tau}) = \tau_1(y) = \tau_2(y, \hat{\tau}) = 0\) for any signals \((x, y) \in \mathbb{R}^2\) and time \(\hat{\tau} \in \mathbb{R}_+\).

\(^{26}\)If this were not the case, Lemma 2 would induce the contradiction \(0 = bE_{0,0}[\lambda|x^*, x^*+] - c > bE_{0,0}[\lambda|x^*, x^*] - c \geq 0\).
In our Gamma-Poisson model, B’s equilibrium stopping time \( \tau_2^*(y, \hat{\tau}) \) is linear in signal \( y \) and in A’s exit time \( \hat{\tau} \). As a consequence, player B’s expected duration in the race (conditional on no prize arrival) exactly offsets the premature quitting by player A:

\[
\tau_2^*(y, \tau_1^*(x)) + \tau_1^*(x) = \frac{b}{c} (\beta + 2) + x + y - \alpha = 2T^*(x, y).
\]

Although the sum of equilibrium experimentation durations coincides with the team’s aggregate duration, a key inefficiency remains. The team solution dictates that the two projects should be synchronized and stopped simultaneously. In equilibrium, exit is sequential: player A exits too soon and her experimentation duration \( T^*(x, y) - \tau_1^*(x) \) is “postponed” by player B after she has completed her optimal duration \( T^*(x, y) \). Since the team is impatient, this corresponds to suboptimally slowing down the joint rate of innovation arrival, which implies suboptimal equilibrium experimentation.\(^{27}\)

To make this statement more formal, we decompose the expression for time-0 expected team’s welfare, conditional on stopping the first project at time \( T_1 \) and the second one at time \( T_2 \),

\[
W(T_1, T_2|x, y) = B(T_1, T_2|x, y) - K(T_1, T_2|x, y)
\]

as the difference between expected discounted reward, including all possible cases of arrival or non-arrival of the prize

\[
B(T_1, T_2|x, y) \equiv b \int \Lambda \left[ \int_0^{T_1} 2f(s|\lambda) (1 - F(s|\lambda)) e^{-rs} ds + (1 - F(T_1|\lambda)) \int_{T_1}^{T_2} f(s|\lambda) e^{-rs} ds \right] \pi_{0,0}(\lambda|x, y) d\lambda
\]

and the expected present discounted value of costs

\[
K(T_1, T_2|x, y) = \int \Lambda C(T_1, T_2|\lambda) \pi_{0,0}(\lambda|x, y) d\lambda, \text{ with }
\]

\[
C(T_1, T_2|\lambda) \equiv c \left[ \int_0^{T_1} 2f(s|\lambda) (1 - F(s|\lambda)) \int_{s}^{\lambda} 2e^{-rv} dv ds + (1 - F(T_1|\lambda))^2 \int_{T_1}^{T_2} 2e^{-rv} dv 
+ (1 - F(T_1|\lambda)) \int_{T_1}^{T_2} f(s|\lambda) \int_{T_1}^{s} e^{-rv} dv ds + (1 - F(T_1|\lambda)) (1 - F(T_2|\lambda)) \int_{T_1}^{T_2} e^{-rv} dv \right].
\]

We prove in the Appendix:

**Lemma 8.** If \( y > x > \frac{\alpha}{2} \), and \( 0 < \tau_1^*(x) < T^*(x, y) \), then the time-0 expected present discounted value of the research costs and of the prize are smaller in equilibrium than in the team solution:

\(^{27}\)This postponement effect extends to (monotonic) asymmetric equilibria, where they to exist, because also in such equilibria firms almost-surely leave the race sequentially, instead of simultaneously, and after the most pessimistic firm quits the race, her opponent acts fully informed mimicking the team’s optimal solution for a suboptimally constrained termination of the first project.
\[ K(\tau_1(x), \tau_2(y, \tau_1(x))|x, y) < K(T^*(x, y), T^*(x, y)|x, y) \text{ and hence } B(\tau_1(x), \tau_2(y, \tau_1(x))|x, y) < B(T^*(x, y), T^*(x, y)|x, y). \]

The source of this key result is a failure of information aggregation. Private information can only be revealed credibly by quitting decisions, therefore (when playing monotonic strategies) it may flow only from the initially more pessimistic player to her opponent, and never vice versa.

It remains to consider instances where one of the two players has such a low signal that it does not even enter the race. While these instances are in some sense less interesting, we nevertheless report them for the sake of completeness.

First, when player A fails to join the race, player B only learns that \( x \leq \underline{x}^* \) but cannot precisely figure out \( x \). Hence she quits at:

\[
\tau_2^*(y, 0) = \frac{b}{c} (\beta + 1) + y + \underline{x}^* - \alpha.
\]

Player B stays in the game more than a negligible amount of time if and only if \( y > \underline{x}^* \) where simple algebraic manipulations imply that \( T^*(\underline{x}^*, \underline{x}^*) > 0 \). Hence, when neither player stays in the race for more than an instant (i.e. \( x \leq \underline{x}^* \), \( y \leq \underline{x}^* \)), there are two possibilities: if \( T^*(x, y) = 0 \), the outcome is trivially efficient because also the team would not participate, whereas if \( T^*(x, y) > 0 \) we have a strong but somewhat special instance of under-experimentation in which the team would pursue the project and the players give up immediately.

Second, when \( x \leq \underline{x}^* \) but \( y > \underline{x}^* \), only player B joins, and remains for a significant amount of time in the race: Under-experimentation occurs if (i) \( \tau_2^*(y, 0) < 2T^*(x, y) \), i.e. \( x > \underline{x}^* - \frac{b}{c} \), using their respective expressions derived earlier. Due to imperfect equilibrium information transmission, when the most pessimistic player does not even enter the race both players end up stopping too soon in equilibrium. When (ii) \( \tau_2^*(y, 0) > 2T^*(x, y) \), i.e. \( x < \underline{x}^* - \frac{b}{c} \), and \( T^*(x, y) > 0 \), by the same reasoning as in Lemma 8, the equilibrium features under-experimentation if \( \tau_2^*(y, 0) \) is close enough to \( 2T^*(x, y) \) (\( x \) is large enough, but still below \( \underline{x}^* - \frac{b}{c} \)) and equilibrium over-experimentation otherwise. When (iii) \( T^*(x, y) = 0 \), instead, the players should not enter the race in the first place, and a somewhat special instance of over-experimentation arises.

The welfare analysis conditional on the signal pairs \((x, y)\) is summarized in Proposition 5 and depicted in Figure 6.1.

**Proposition 5.** *(Equilibrium Under- and Over-Experimentation Conditional on Signal Realizations)* Given the unique symmetric monotonic equilibrium \((\tau_1^*, \tau_2^*)\) with entry and
exit signal thresholds $\bar{x}^*$ and $\bar{x}^{**}$, and the team’s optimal simultaneous stopping time $T^*(x, y)$, there exist a function $\zeta: [\bar{x}^{**}, 0] \times \mathbb{R}_+ \to [\alpha - (\beta + 2) b/c, \bar{x}^* - b/c]$, decreasing in both arguments, and an increasing continuous function $\xi: (\bar{x}^*, 0) \to (\bar{x}^*, 0]$, implicitly defined by

$$
\tau_1^*(x) = T^*(x, \xi(x)) = \frac{1}{2} \left[ (\beta + 2) \frac{b}{c} - \alpha + x + \xi(x) \right],
$$

such that $\xi(x) > x$ for any $x < 0$, and that for any $y \geq x$,

1. (Equilibrium Under-Experimentation) if $x + y > \alpha - (\beta + 2) \frac{b}{c}$ (i.e. the team would participate in the game) and either $y > \xi(x)$ and $x > \bar{x}^*$ (both players join the game but the two private signals disagree sufficiently), or $y \leq \bar{x}^{**}$ and $x \leq \bar{x}^*$ (both players fail to join the game for more than an instant), or finally $y > \bar{x}^{**}$ and $\zeta(y) < x$ (player $B$ runs the race and player $A$ does not, but is close to do so), then equilibrium experimentation is too low. Both the total time-0 expected discounted experimentation costs and the total time-0 expected discounted benefit are smaller in equilibrium than in the team’s solution.

2. (Equilibrium Over-Experimentation) If $x > \bar{x}^*$ (both players enter the game) and either $y < \xi(x)$ and $x + y > \alpha - (\beta + 2) \frac{b}{c}$ (the team would participate in the game and the two private signals agree sufficiently), or $(x + y < \alpha - (\beta + 2) \frac{b}{c}$ (the team would not participate in the game), then equilibrium experimentation is excessive. The total time-0 expected discounted experimentation costs are larger in equilibrium than in the team’s solution. The same holds if $y > \bar{x}^{**}$ and $x < \zeta(y, r)$ (only player $B$ runs the race and player $A$’s signal is very low).

Thus far, we have characterized the normative properties of the equilibrium conditional on the realizations of the private signals. Our interpretation of the results is as follows. If the social planner has any reason to believe that players disagree in their beliefs about the promise of the project, then she would expect that under-experimentation occurs in equilibrium, whereas if the social planner deems that the players’ beliefs are in agreement, then she would expect equilibrium over-experimentation.

The next natural question is whether under-experimentation or over-experimentation is to be expected conditional only on prior beliefs about the promise of the project ($\alpha$ and $\beta$), and before the players observe their private signals. This is the perspective of a social planner who does not enjoy any informational advantage over any players. We establish that, as long as the
social planner is sufficiently impatient and deems that the “promise” of the project (the prior expectation of $\lambda$) is high enough, then she expects under-experimentation in equilibrium. This is the key result of our analysis and, to our knowledge, the first formal rationale for subsidies to R&D in Poisson patent races.

Intuitively, when the expected promise of the promise is high enough, the social planner is mostly concerned with signal outcomes $x, y$ such that $T^*(x,y) > 0$: it would be optimal that both players enter the game and exit simultaneously. In equilibrium, players jointly over-experiment if their maximum duration $\tau_1^*(x) = \tau_2(y, \tau_1^*(x))$ is larger than $T^*(x,y)$, i.e. the more pessimistic player overshoots the team solution; they jointly under-experiment if $\tau_1^*(x) < T^*(x,y)$ so that more pessimistic player exits too soon, and her shortfall $T^*(x,y) - \tau_1^*(x)$ is compensated by the opponent but postponed after $T^*(x,y)$. If the social planner is sufficiently impatient, then she is relatively more concerned ex-ante about the risk of premature exit and postponement than about the risk of delayed exit.

Formally, we let

$$D(\alpha, \beta; r) = \int_{\Lambda} \int_{\mathbb{R}_+^2} [C(T^*(x,y), T^*(x,y)|\lambda) - C(\tau_1(x), \tau_2(y, \tau_1(x)) |\lambda)] h(x|\lambda) h(y|\lambda) d\pi(\lambda)$$
denote the difference between equilibrium experimentation costs and team costs on the basis of $\alpha$ and $\beta$ only.

**Proposition 6. (Unconditional Welfare Properties of the Equilibrium)** Given the unique symmetric monotonic equilibrium $(\tau_1^*, \tau_2^*)$ with entry and exit signal thresholds $\pi^*$ and $\pi^{**}$ and the team’s solution $T^*$, if the ex-ante expectation of the project’s promise $\lambda$ is sufficiently large, i.e. if the project is ex-ante valuable enough, and if the discount rate $r$ is sufficiently large, i.e. the social planner is impatient enough, then $D(\alpha, \beta; r) < 0$: equilibrium under-experimentation occurs in an ex-ante sense. If either the project is not valuable ex-ante, or if it is very valuable but the social planner is very patient, then $D(\alpha, \beta; r) > 0$ and ex-ante equilibrium over-experimentation takes place.

### 7. Conclusion and Extensions

This paper studies a Poisson R&D race with fixed experimentation intensity and winner-take-all termination, where firms are privately informed about the promise and feasibility of their common line of research. We have calculated the unique symmetric monotonic equilibrium, and compared it with the optimal policy of a team that shares the firm’s private information. Due to the interdependent-value nature of the problem, the equilibrium displays a strong herding effect that distinguishes our framework from war-of-attrition models. Earlier models of R&D races without private information predicted equilibrium overinvestment due to duplication costs. Despite our herding effect, we have found that equilibrium expenditure in R&D is sub-optimally low whenever the social planner is sufficiently impatient. This is because the more pessimistic firm prematurely exits the race when information is sufficiently heterogeneous, so that R&D activity is inefficiently postponed. Our conclusions are that private incentives curb information sharing in R&D, and secrecy, in turn, inefficiently slows down the pace of innovation.

Our construction can be extended in several directions. We shall now discuss some of the most interesting and promising ones, to show that, by and large, our conclusions appear to be robust. It would go beyond the scope of this single paper to cover all cases thoroughly. While we have some formal arguments, whose proofs are available upon request, to validate robustness, we also present some intuitive reasons to expect our results to survive qualitatively intact. We also trace a map for future research.
Deterministic Accumulation of Knowledge over Time. Our model inherits from differential game models of R&D races the assumption that innovation arrival is governed by a Poisson process of parameter $\lambda$. This is equivalent to say that the unknown innovation hazard rate $L(t, \lambda) \equiv f(t|\lambda) / [1 - F(t|\lambda)]$ is constant over time (and equal to $\lambda$). Then, it is natural to postulate conjugate Gamma prior and negative exponentially distributed signals. It is not too difficult, however, to extend our equilibrium construction when the innovation hazard rates $L(t, \lambda)$ of unknown parameter $\lambda$ is not constant over time.\footnote{It is also easy to extend our analysis for the possibility that the value of the prize $b$ changes over time, for example growing at the economy’s rate.}

The engine of our equilibrium characterization is Theorem A.1 in the Appendix. Its power goes well beyond the Gamma-Poisson specification of our model.

Make the following mild regularity assumptions: the distribution of $\lambda$ has connected support $\Lambda$ (with $\underline{\lambda} = \inf \Lambda$ and $\overline{\lambda} = \sup \Lambda$), the hazard rate of the prize $L(t, \lambda)$ is continuous in $t, \lambda$, and strictly increasing in $\lambda$ for every $t$, with $L(t, \overline{\lambda}) > c/b > L(t, \underline{\lambda})$, and the density $h(x|\lambda)$ of private signals $x$ is differentiable, with support $X \subseteq \mathbb{R}$, and $\lim_{x \to \sup X} h(x|\lambda) < \infty$. In this environment, we can prove that our equilibrium characterization presented in Propositions 2 and 4 holds if the following two substantive restrictions are met. First, the signal density $h(x|\lambda)$ is log-supermodular, i.e. the ratio $h'(x|\lambda)/h(x|\lambda)$ is strictly increasing in $\lambda$. Second, the expected hazard rates $\mathbb{E}_{t,t'} [L(t, \lambda)|x, y]$ and $\mathbb{E}_{t,t'} [L(t, \lambda)|x, y+]$ are strictly decreasing in $t$ for any fixed signals $x, y$ and $t'$; and they both attain the lower bound $L(t, \underline{\lambda})$ in the limit as $t \to \infty$. The first condition is reminiscent of log-supermodularity conditions for pure strategy equilibrium existence derived in Athey (1999) and in Reny (1999). The second condition requires that for any fixed pair of signals $x, y$, as time goes by and the innovation does not arrive, one becomes more and more pessimistic about the promise of the project, both in the case that one knows precisely both signal realizations $x$ and $y$, or that one only knows one signal realization to be precisely $x$ while that the other signal realization is larger than $y$. We underline that this is not unduly strong. Among other things, it does not require that the actual innovation hazard rate $L(t, \lambda)$ be decreasing over time. As knowledge accumulate throughout the R&D process, $L(t, \lambda)$ is likely to be increasing over time; but it is also quite likely that one becomes less optimistic about its feasibility as time goes by and the innovation does not materialize.\footnote{Doraszelski (2002) develops computational techniques to simulate equilibrium investment paths in a differential R&D game with knowledge accumulation. He finds that under under some conditions, the firm that is behind in the race engages in catch-up behavior.}

\[30\]
Random Accumulation of Private Information over Time. Knowledge about the project may also arrive randomly as the race unfolds, and therefore remain private. A simple way to capture this phenomenon within our Gamma-Poisson model is to assume that each player $i$ who remains in the race may observe “good news” as time goes by. Each good news is the arrival of a known event at uncertain times, following a Poisson process of parameter $k\lambda$, where $k > 1$. Therefore, the more promising the project, the higher $\lambda$ and the more frequently good news accrue, typically before the innovation itself because $k > 1$. Ruling out initial private signals for simplicity, all private information at any time $t$ is represented by the number $n$ of good-news accrued to date $t$. For any $t$ and $n$, the player’s beliefs over $\lambda$ are again represented by a Gamma distribution. The calculation of the marginal value of waiting to exit is carried out in analogous way as in the present model. The relevant events are (i) the arrival of prize, (ii) the exit of the opponent, (iii) the arrival of good news. Each of these arrivals corresponds to a flow benefit that one weighs against the flow cost of remaining in the race.

Just like in our model, a symmetric monotonic equilibrium strategy consists of two stopping time functions. For number of good news $n$, the first stopping time function $\tau_1$ prescribes to leave the race at time $\tau_1(n)$ as long as the opponent is still in the race, whereas given that the opponent left the race at time $\hat{\tau}$, equilibrium exit is prescribed by stopping time function $\tau_2(n, \hat{\tau})$. Monotonicity requires that both these stopping functions are increasing in $n$. Each player inverts her opponent’s good-news index $g(\hat{\tau})$ upon observing her exit at time $\hat{\tau}$. If the opponent is still in the race, the only information available is that her good-news index $n$ is at least as large as $g(\hat{\tau})$. Hence as in our model, the equilibrium displays a survivor’s curse property: players herd on each other as long as they both stay in the race. Unlike our model, the equilibrium is fully-separating.

Partially Independent and Partially Public Values. Our model is easily extended to allow for interdependent values or partially public values, instead of pure common value. The two players address the same research project, each choosing a different approach to find a solution. The arrival rate of invention $i = A, B$ is $\lambda_i = \lambda + \varepsilon_i$, where $\lambda \sim Ga(\alpha, \beta)$ and $\varepsilon_i \sim Ga(\alpha, \beta_i)$, with $\varepsilon_A$ and $\varepsilon_B$ independent: $\lambda$ measures the “promise” of the project, and the idiosyncratic components $\varepsilon_i$ measure the specific promise of player $i$’s approach. If $\beta = 0$ then $\lambda = 0$ a.s.

\footnote{While on the equilibrium path firms may only exit at a countable set of times, there is no practical issue with beliefs off path, because of the assumption of irreversible exit.}
and the two projects are independent; if $\beta_A = \beta_B = 0$ then $\lambda_i = \lambda$ a.s. and we are back to our common value case, in general $\lambda_i \sim Ga(\alpha, \beta + \beta_i)$. Because each private signal $z_i \leq 0$ is drawn from the distribution $e^{\lambda z_i}$, this structure captures interdependent value. The extension to the case where information is not entirely private is addressed in a similar manner. Say that there are three i.i.d. signals $z_A, z_B$ and $z_0$ drawn from the distribution $e^{\lambda z}$. Signal $z_0$ is observable and hence $\lambda \sim Ga(\alpha + z_0, \beta + 1)$. Our current analysis extends after appropriate reparameterizations.

**Prize Arrival Externalities.** While the winner-take-all assumption is a natural benchmark, our framework can also accommodate prize arrival externalities, both negative (e.g. reduced market share, lost competitive edge), or positive (e.g. spillover effects, imitation, innovation dissemination, industry-wide benefits) on the other player. Inspection of the equilibrium equations characterizing the stopping time function $\tau^*_i$, reveals two possibilities. If the externality is borne by a player only while in the race, then positive (negative) externalities exacerbe (dampen) the herding effect: players postpone (anticipate) their exit because of the benefits (losses) obtained if the prize arrives to the opponent. If instead the externality is borne regardless of whether the player is still in the race or not, then exactly the opposite implication holds: Negative (positive) externalities exacerbe (dampen) herding, because players postpone (anticipate) their exit to reduce the probability that the prize arrives to the opponent (to save on flow cost and shift the burden of experimentation on the opponent).

Ruling out the (fairly implausible) case of negative externalities borne only if in the race, the team optimal solution still consists in running both projects and simultaneously stopping them when marginal benefits (which now internalize externality effects) are offset by flow costs. In fact, when externalities are borne regardless of whether in the race or not, they do not affect the team’s stopping policy, and positive externalities borne only if in the race, make the team prize value larger when both projects are open. In equilibrium, one player prematurely exits if private information disagrees sufficiently, and this results in under-experimentation if the team is impatient enough.

**Re-Entry and Sequential Entry.** For reasons of tractability, the rules of our game allow for the possibility to enter the game only at time 0. This assumption implicitly rules out the possibility of strategic sequential entry and re-entry, and allows us to focus on the equilibrium information aggregation properties induced by the quitting decision. While re-entry is most
plausibly costless in auctions (see Izmalkov (2003) for a thorough analysis of its consequences), in R&D environments it is more plausible that re-entry entails large sunk costs, following the establishment of appropriate research facilities and researchers teams. It is easily established that there exists a sunk cost level that makes re-entry unprofitable in our equilibrium. Free re-entry is even less natural as a rule of the game in research prize contexts.

As far as the quitting decision—our main focus—is concerned, sunk entry costs that make re-entry prohibitive are equivalent to just assuming irreversible exit. These entry costs, however, raise a quite different issue if we allow for sequential and strategic entry at the very beginning. The entry decision in our game of conflict would play a similar role as in the CG’s coordination game. In CG, players are privately informed on the promise of the project and have investment opportunity (entering the race and paying a sunk cost) that can be irreversibly exercised at any moment in time, revealing their information. Their results suggest that with a strategic entry decision in our game herding would delay investment, as each player would like to wait and see whether the opponent is sufficiently optimistic to enter the race, before paying the sunk entry cost. Following this reasoning, our main substantive result (under-experimentation) is likely strengthened when extending our game to allow for (costly) entry at any period. Also, because the issue of strategic entry has been clarified by CG, we concentrate on the less explored exit decision. Furthermore, this incentive to ‘wait and see at time 0’ is naturally counteracted in the context of R&D, by the incentive to ‘be the first to enter the race’ so as to gain some advantage over the opponent in the race. As is well known, this incentive to ‘being ahead’ is ruled out in Poisson R&D races. Ideally, a more general model would not put any restrictions on the timing of entry decisions and would explicitly model the advantage for starting early and being ahead in the race. We postpone the formulation and analysis of such a full-fledged model of entry in the race to future research and focus this paper exclusively on the previously unexplored effect of exit decisions.

Variable Experimentation Intensity. This requires abandoning the relatively comfortable environment of stopping games, to venture in the more technically involved class of differential games. Following Reinganum (1981), assume that at each moment in time $t$ each firm $i$ pays cost $C(u^i(t))$ to select flow experimentation intensity $u^i(t)$ and enjoy an innovation hazard rate
\( u^i(t) \lambda \), where \( C \) is an increasing and convex function.\(^{31}\) Again assuming that exit is irreversible, we would say that each firm may only observe if the opponent is engaging in the R&D race, but cannot observe precisely how many resources she is allocating to it.

As in our stopping game environment, the value of experimentation is larger, the larger is the expected hazard rate of the prize; but also higher-order moments of beliefs with respect to the unknown hazard rate are likely to play a role. For example, the higher the variance, the larger is the marginal value of gathering information, and hence the larger the optimal experimentation flow. This complicates proving the existence of a monotonic equilibrium, where experimentation intensity is increasing in the signals and decreasing in time. If such an equilibrium exists, however, we see no reason to believe that its information aggregation properties would differ from those of our stopping game equilibrium, including the survivor’s curse.

Turning to welfare analysis, we expect that in such a monotonic equilibrium, under-experimentation would still occur when signals disagree sufficiently. In that case, the most pessimistic firm exits much too prematurely, and her contribution to aggregate experimentation is not significant. Once it has left the race, the remaining firm acts fully informed. Because of cost convexity (decreasing returns to scale), it chooses experimentation intensities that are smaller than the team’s optimal solution in the same contingencies. The least costly team’s experimentation policy requires to equalize experimentation intensity across the two facilities, and it is costlier for a single firm to achieve the same aggregate experimentation intensity.

A. Appendix. Omitted Proofs.

Posterior Beliefs. We show the expressions for the posterior beliefs in (3.1) and (3.2) as well as the resulting expected hazard rates in terms of parameters

\[
\pi_{t,t'}(\lambda|x,y) = \frac{e^{-\alpha \lambda e^{\lambda x} \lambda e^{\lambda y} e^{-\lambda t} e^{-\lambda' t'}}}{\int e^{-\alpha \lambda e^{\lambda x} \lambda e^{\lambda y} e^{-\lambda t} e^{-\lambda' t'}} d\lambda'} = \frac{e^{-\lambda(\alpha-x-y+t+t')} \lambda^{\beta+1}}{\Gamma(\beta+2) (\alpha-x-y+t+t')^{-\beta-2}}. \tag{A.1}
\]

\(^{31}\) The closest contribution appearing in the literature is Malueg and Tsutsui (1999) who allow for unknown hazard rate \( \lambda \) but not for private information. Their calculation techniques hinge on the belief over \( \lambda \) having a Gamma distribution. This is incompatible with equilibrium beliefs for the continuous-signal case, since inference under asymmetric information yields a left-truncated distribution. As is typically the case in these continuous action games of private information, resorting to a discrete signal construction is likely to disturb pure-strategy existence.
\[ \pi_{t,t'}(x,y+) = \frac{e^{-\alpha \lambda} \lambda^{\beta-1} e^{\lambda x} (1 - e^{\lambda y}) e^{-\lambda t} e^{-\lambda t'} \lambda^{\beta} \left[ e^{-\lambda(\alpha+t+t'-x)} - e^{-\lambda(\alpha+t+t'-x-y)} \right]} {\Gamma(\beta+1) \left( (\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1} \right)} \]

We begin our proofs with a very useful technical result.

**Theorem A.1.** Let \( q(\lambda, \theta) : \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) differentiable in \( \theta \) and integrable in \( \lambda \) with \( \int_{\Lambda} q(\lambda, \theta) \, d\lambda \in (0, \infty) \). Then the c.d.f. defined by:

\[ \varphi(L, \theta) = \frac{\int_{\Delta} q(\lambda, \theta) \, d\lambda}{\int_{\Lambda} q(\lambda', \theta) \, d\lambda'} \]

for every \( L \in (\Delta, \bar{\lambda}) \) is stochastically strictly increasing in \( \theta_i \in \theta \) if \( q(\lambda, \theta) \) is log-supermodular in \( (\lambda, \theta_i) \), i.e. if

\[ \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} \]

is strictly increasing in \( \lambda \).

**Proof.** We want to show that for every \( L \in (\Delta, \bar{\lambda}) \)

\[ 0 > \frac{\partial}{\partial \theta_i} \left( \frac{\int_{\Delta} q(\lambda, \theta) \, d\lambda}{\int_{\Lambda} q(\lambda', \theta) \, d\lambda'} \right) = \frac{\int_{\Delta} \frac{\partial q(\lambda, \theta)}{\partial \theta_i} \, d\lambda}{\int_{\Lambda} q(\lambda, \theta) \, d\lambda} - \frac{\int_{\Delta} q(\lambda, \theta) \, d\lambda \int_{\Delta} \frac{\partial q(\lambda, \theta)}{\partial \theta_i} \, d\lambda}{\left[ \int_{\Lambda} q(\lambda, \theta) \, d\lambda \right]^2} \]

using

\[ \int \frac{\partial q(\lambda, \theta)}{\partial \theta_i} \, d\lambda = \int \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} q(\lambda, \theta) \, d\lambda \]

the claim reads

\[ \int_{\Delta} \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} q(\lambda, \theta) \, d\lambda > \int_{\Delta} \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} q(\lambda, \theta) \, d\lambda \]

A sufficient condition for the latter inequality is that the RHS be strictly increasing in \( L \). Since the RHS is differentiable in \( L \), it suffices that

\[ 0 < \frac{\partial}{\partial L} \left[ \int_{\Delta} \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} q(\lambda, \theta) \, d\lambda \right] \]

\[ = \frac{\partial \log q(L, \theta)}{\partial \theta_i} \frac{q(L, \theta)}{\int_{\Delta} q(\lambda', \theta) \, d\lambda'} - q(L, \theta) \frac{\int_{\Delta} \frac{\partial \log q(\lambda, \theta)}{\partial \theta_i} \, d\lambda}{\left[ \int_{\Delta} q(\lambda', \theta) \, d\lambda' \right]^2} \]
or
\[
\frac{\partial \log q \left(L, \tilde{\theta} \right)}{\partial \theta_i} > \int_{\Delta} \frac{\partial \log q \left(\lambda, \tilde{\theta} \right)}{\partial \theta_i} \frac{q \left(\lambda, \tilde{\theta} \right)}{\int_{\Delta} q \left(\lambda', \tilde{\theta} \right) \, d\lambda'} \, d\lambda.
\]

But, since \( \int_{\Delta} \frac{q \left(\lambda, \tilde{\theta} \right)}{\int_{\Delta} q \left(\lambda', \tilde{\theta} \right) \, d\lambda'} \, d\lambda = 1 \), this follows from the assumption that the LHS is strictly increasing in \( \lambda \). \( \blacksquare \)

**Proof of Lemma 1.** It suffices to show that for every \( x, y, t, t' \), the c.d.f. associated with the posterior beliefs \( \pi_{t,t'} (\lambda | x, y) \) are stochastically strictly decreasing in \( t \) and in \( t' \) and strictly increasing in \( x \) and in \( y \). We prove all these results as corollaries of Theorem A.1. Let \( \tilde{\theta} = (x, y, t, t') \), and

\[
q \left(\lambda, \tilde{\theta} \right) = \pi \left(\lambda \right) h \left(x | \lambda \right) \left[1 - H \left( y | \lambda \right) \right] \left[1 - F \left( t | \lambda \right) \right] \left[1 - F \left( t' | \lambda \right) \right],
\]

so that

\[
\varphi \left(L, \tilde{\theta} \right) = \int_{\Delta} \pi_{t,t'} \left(\lambda | x, y \right) \, d\lambda.
\]

Since

\[
\frac{\partial \log q \left(\lambda, \tilde{\theta} \right)}{\partial t} = - \frac{f \left(t | \lambda \right)}{1 - F \left(t | \lambda \right)} = \lambda, \quad \frac{\partial \log q \left(\lambda, \tilde{\theta} \right)}{\partial t'} = - \frac{f \left(t' | \lambda \right)}{1 - F \left(t' | \lambda \right)} = -\lambda
\]

and

\[
\frac{\partial \log q \left(\lambda, \tilde{\theta} \right)}{\partial x} \frac{h' \left(x | \lambda \right)}{h \left(x | \lambda \right)} = \lambda, \quad \frac{\partial \log q \left(\lambda, \tilde{\theta} \right)}{\partial y} \frac{h \left(y | \lambda \right)}{1 - H \left(y | \lambda \right)} = \frac{\lambda e^{\lambda x}}{1 - e^{\lambda x}}
\]

are strictly increasing in \( \lambda \), all monotonicity results follow from Theorem A.1. The result that \( \lim_{\tau \to \infty} E_{t,t'} \left[\lambda | x, y \right] = 0 \) follows from:

\[
\lim_{\tau \to \infty} \pi_{t,t} \left(\lambda > \varepsilon | x, y \right) = \lim_{\tau \to \infty} \int_{\varepsilon}^{\infty} e^{-\alpha \lambda \beta \lambda^{-1} \lambda e^{\lambda x}} \left(1 - e^{\lambda y} \right) e^{-\lambda \tau} \, d\lambda = 0. \quad \blacksquare
\]

**Proof of Lemma 2.** To show the first claim, we need to show that for any \( L \in (\Delta, \lambda) \),

\[
\frac{\int_{\Delta} \pi \left(\lambda \right) h \left(x | \lambda \right) \left[1 - H \left(y | \lambda \right) \right] \left[1 - F \left(t | \lambda \right) \right] \left[1 - F \left(t' | \lambda \right) \right] \, d\lambda}{\int_{\Delta} \pi \left(\lambda' \right) h \left(x | \lambda' \right) \left[1 - H \left(y | \lambda' \right) \right] \left[1 - F \left(t | \lambda' \right) \right] \left[1 - F \left(t' | \lambda' \right) \right] \, d\lambda'} < \frac{\int_{\Delta} \pi \left(\lambda \right) h \left(x | \lambda \right) h \left(y | \lambda \right) \left[1 - F \left(t | \lambda \right) \right] \left[1 - F \left(t' | \lambda \right) \right] \, d\lambda}{\int_{\Delta} \pi \left(\lambda' \right) h \left(x | \lambda' \right) h \left(y | \lambda' \right) \left[1 - F \left(t | \lambda' \right) \right] \left[1 - F \left(t' | \lambda' \right) \right] \, d\lambda'}
\]
this follows from
\[
\int_{\lambda}^{L} \frac{h(y|\lambda)}{1 - H(y|\lambda)} \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)] d\lambda
\]
\[
= \mathbb{E}_{t,t'} \left[ \frac{\lambda e^{\lambda y}}{1 - e^{\lambda y}} \right] \left[ t_0 \geq t, t_1 \geq t', x_0 = x, x_1 \geq y, \lambda \leq L \right]
\]
\[
> \mathbb{E}_{t,t'} \left[ \frac{\lambda e^{\lambda y}}{1 - e^{\lambda y}} \right] \left[ t_0 \geq t, t_1 \geq t', x_0 = x, x_1 \geq y \right]
\]
\[
= \int_{\lambda}^{L} \frac{h(y|\lambda)}{1 - H(y|\lambda)} \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)] d\lambda',
\]
where the inequality follows since
\[
\frac{\partial}{\partial \lambda} \left( \frac{\lambda e^{\lambda y}}{1 - e^{\lambda y}} \right) = \frac{e^{\lambda y} (1 - e^{\lambda y} + \lambda y)}{(1 - e^{\lambda y})^2} < 0 \text{ for any } \lambda \text{ and } y.
\]
The proof of the second inequality is analogous. ■

**Proof of Lemma 3.** We verify a sufficient condition. For every \( x, y, t, T_1 \),
\[
\frac{dW_{2,t}(T_2|x,y,T_1)}{dT_2} \propto b \mathbb{E}_{T_2,T_1} \left[ \lambda|x,y \right] - c = b \frac{\beta + 2}{\alpha - x - y + T_1 + T_2} - c,
\]
this immediately implies:
\[
T_2 < (>) b \frac{\beta + 2}{c} + x + y - \alpha - T_1 \iff \frac{dW_{2,t}(T_2|x,y,T_1)}{dT_2} > (<) 0.
\]
Therefore \( T_2(x,y,T_1) \) if positive, is the unique local and thus global maximum of the value function \( W_{2,t}(T_2|x,y,T_1) \), independently of \( t \). It follows that if \( t < T_2(x,y,T_1) \), then the second project is kept open at time \( t \), whereas if \( t \geq T_2(x,y,T_1) \), it is optimal to turn the second project off. This in turn implies that the optimal stopping time is
\[
T^*_2(x,y,T_1) = \max\{T_1, T_2(x,y,T_1)\}.
\]
■

**Proof of Lemma 4.** First we calculate
\[
U_{2,t}(T|\lambda) = \int_{t}^{T} \frac{f(s|\lambda)}{1 - F(t|\lambda)} \left[ \int_{t}^{s} (-c) e^{-r(v-t)} dv + e^{-r(s-t)} b \right] ds + \frac{1 - F(T|\lambda)}{1 - F(t|\lambda)} \int_{t}^{T} (-c) e^{-r(v-t)} dv.
\]
\[
= \int_{t}^{T} \lambda e^{-\lambda(s-t)} \left[ \int_{t}^{s} (-c) e^{-r(v-t)} dv + e^{-r(s-t)} b \right] ds - e^{-\lambda(T-t)} \int_{t}^{T} (-c) e^{-r(v-t)} dv.
\]
\[
= (\lambda b - c) \frac{1 - e^{-(\lambda+r)(T-t)}}{\lambda + r}.
\]
Using this expression in (4.5), for every \( T_1 \in (t, T^*(x,y)] \) we obtain:
\[
\frac{dW_{1,t}(T_1|x,y)}{dT_1} \propto -c + \mathbb{E}_{T_1,T_1} \left[ \lambda \left( b - (\lambda b - c) \frac{1 - e^{-(\lambda+r)(T_2^*(x,y,T_1)-T_1)}}{\lambda + r} \right) \right] |x,y|
\]
using $c = E_{T_1,T_1} \left[c|x, y\right]$, some algebra and $T^*(x, y) > 0$,

$$
\begin{align*}
&= E_{T_1,T_1} \left[ (\lambda b - c) \frac{r + \lambda e^{-(\lambda + r)(T_2^*(x,y,T_1)-T_1)}}{\lambda + r} \right]_{x, y} \\
&= \int_A \left( \lambda b - c \right) \frac{r + \lambda e^{-(\lambda + r) \frac{1}{2}(\beta + 2) + x + y - \alpha - 2 T_1}}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1}}{\Gamma (\beta + 2)(\alpha - x - y + 2 T_1)^{-\beta - 2} d\lambda} \\
&\propto S(T_1) \equiv \int_A \left( \lambda b - c \right) \frac{r + \lambda e^{-(\lambda + r) \frac{1}{2}(\beta + 2) + x + y - \alpha - 2 T_1}}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda},
\end{align*}
$$

where note that $S(T^*(x,y)) = 0$ because if $T_1 = T^*(x,y)$ then $T_2^*(x,y,T_1) - T_1 = 0$.

Differentiating $S(T_1)$ we obtain:

$$
\begin{align*}
S'(T_1) &= \frac{d}{dT_1} \left[ \int_A \left( \lambda b - c \right) \frac{r + \lambda e^{-(\lambda + r) \frac{1}{2}(\beta + 2) + x + y - \alpha - 2 T_1}}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda} \right] \\
&= \int_A \left( \lambda b - c \right) \frac{-2r \lambda + 2r \lambda e^{-(\lambda + r) \frac{1}{2}(\beta + 2) + x + y - \alpha - 2 T_1}}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda} \\
&= 2r \int_A \left( \lambda b - c \right) \frac{r + \lambda e^{-(\lambda + r) \frac{1}{2}(\beta + 2) + x + y - \alpha - 2 T_1}}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda} \\
&\quad + 2r \int_A \left( \lambda b - c \right) \frac{-r}{\lambda + r} \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda} \\
&= 2r S(T_1) - 2r \int_A \left( \lambda b - c \right) \frac{e^{-\lambda (\alpha - x - y + 2 T_1)} \lambda^{\beta + 1} d\lambda} \\
&= 2r S(T_1) - 2r \frac{\Gamma (\beta + 2)}{(\alpha - x - y + 2 T_1)^{\beta + 2}} \left[ \frac{(\beta + 2) b}{\lambda + r} \right]_{x, y}.
\end{align*}
$$

So, finally we obtain the differential equation

$$
S'(T_1) = 2r S(T_1) - 2r \frac{\Gamma (\beta + 2)}{(\alpha - x - y + 2 T_1)^{\beta + 3}} 2(T^*(x,y) - T_1) \quad (A.2)
$$

where clearly $(\alpha - x - y + 2 T_1)^{\beta + 3} > 0$. We use (A.2) to prove the claim. First, we exclude an optimal interior (positive) $T_1^*(x,y) \in (0, T^*(x,y))$. By contradiction: if there is an optimal interior $T_1 = T_1^*(x,y) \in (0, T^*(x,y))$, it must satisfy the NFOC

$$
\frac{dW_{1,t}(T_1|x,y)}{dT_1} = 0 \Rightarrow S(T_1) = 0.
$$

But then, $S(T_1^*(x,y)) = 0$ and $T_1^*(x,y) < T^*(x,y)$ in (A.2) together imply $S'(T_1^*(x,y)) < 0$. By continuity there exists $\varepsilon > 0$ such that $S(T_1^*(x,y) + \varepsilon) < 0$. Then by smoothness of $S$ (many times differentiable) and $S(T^*(x,y)) = 0$ there exists a $T_1^* \in (T_1^*(x,y), T^*(x,y))$ such that $S'(T_1^*) = 0 > S(T_1^*)$; but from (A.2) $T_1^* < T^*(x,y)$ and $S(T_1^*) < 0$ imply $S'(T_1^*) < 0$ a contradiction. Second, we exclude a corner solution at $T_1 = T_1^*(x,y) = 0$. For this, we would require

$$
\frac{dW_{1,t}(0|x,y)}{dT_1} \leq 0 \Rightarrow S(0) \leq 0
$$

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which, together with \( T^*(x,y) > 0 = T_1^*(x,y) \) and (A.2), imply \( S'(0) < 0 \). But then the function \( S(t) \) keeps declining at increasing rate from the initial value \( S(0) \leq 0 \) (formally the differential equation for \( S \) is exploding downward) as \( t \) rises from 0, contradicting \( S(T^*(x,y)) = 0 \). 

**Proof of Proposition 3.** We consider the problem of player A who contemplates stopping second at time \( \tau^A \), player B’s problem being symmetric. In the same manner as in the proof of Proposition 3, we compute

\[
\begin{align*}
\mathbb{E} [\lambda | z_A = x, z_B \leq y, t_A \geq t, t_B \geq t'] &= \int \lambda \frac{\pi (\lambda) h (x|\lambda) H (y|\lambda) [1 - F (t|\lambda)] [1 - F (t'|\lambda)]}{\int \lambda \frac{e^{-\alpha \lambda} \lambda^{\beta - 1} \lambda e^{\lambda x} e^{\lambda y} e^{-\lambda t} e^{-\lambda t'} d\lambda} d\lambda'
\end{align*}
\]

\[
= \int \lambda \frac{e^{-\alpha \lambda} \lambda^{\beta - 1} \lambda e^{\lambda x} e^{\lambda y} e^{-\lambda t} e^{-\lambda t'} d\lambda}{\int \lambda \frac{\lambda^{\beta + 1} e^{-\lambda (\alpha + t + t' - x - y)} d\lambda}}.
\]

If the first-order condition has a positive solution, this is unique and a global maximum of the value function. Therefore, the desired stopping time is the maximum of 0 and the solution \( \tau^A \) to \( c = b \mathbb{E} [\lambda | z_A = x, z_B \leq g (0), t_A \geq \tau^A, t_B \geq 0] \).

**Proof of Lemma 5.** From Proposition 2, we have

\[
\tau^*_2 (x, \tau) = \max \{ \tau, \frac{b}{c} (\beta + 2) + x + g (\tau) - \alpha - \tau \}.
\]

Since \( g (\tau) \leq 0 \), it follows that \( \frac{b}{c} (\beta + 2) + x + g (\tau) - \alpha - \tau \leq \frac{b}{c} (\beta + 2) + x - \alpha - \tau \). For any \( x \), there is \( \bar{\tau} \) large enough such that for any \( \tau \geq \bar{\tau}, \frac{b}{c} (\beta + 2) + x - \alpha - \tau \leq \tau \); hence \( \tau^*_2 (x, \tau) = \tau \), \( W_{2, \tau} (\tau^*_2 (x, \tau)|x, g (\tau), \tau) = 0 \) and

\[
V_{1,t} (\tau|x) = b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, g (\tau) + \right] + W_{2, \tau} (\tau^*_2 (x, \tau)|x, g (\tau), \tau) \mathbb{E}_{\tau, \tau} \left[ \frac{h (g (\tau)|\lambda) g' (\tau)}{1 - H (g (\tau)|\lambda)} | x, g (\tau) + \right] - c
\]

\[
= b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, g (\tau) + \right] - c \leq b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, 0 \right] = \frac{\beta + 2}{\alpha - x + 2 \tau} \to 0 \text{ for } \tau \to \infty.
\]

where the inequality follows by Lemma 2.

Since

\[
V_{1,t} (\tau|x) = b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, g (\tau) + \right] + W_{2, \tau} (\tau^*_2 (x, \tau)|x, g (\tau), \tau) \mathbb{E}_{\tau, \tau} \left[ \frac{h (g (\tau)|\lambda) g' (\tau)}{1 - H (g (\tau)|\lambda)} | x, g (\tau) + \right] - c
\]

\[
\geq b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, g (\tau) + \right] - c,
\]

for any \( x \) such that \( b \mathbb{E}_{\tau, \tau} \left[ \lambda | x, g (0) + \right] > c \), and any \( t \leq \tau \) small enough, it must be that \( V_{1,t} (\tau|x) > 0 \) by continuity of \( V_{1,t} \).

**Lemma A.1.** Suppose that player B plays the first-quitter stopping strategy \( \tau^*_2 \). If player A holds a signal \( x \leq x^* \), then at time 0 she optimally chooses not to enter the game. Whereas if \( x > x^* \), then player A enters the game at time 0, and optimally selects the time-consistent stopping time \( \tau^*_1 (x) \).
Proof. Consider the choice at time $t = 0$ of player $A$ with a signal $x$. If she chooses not to enter the game and set $\tau = 0$, her payoff is $V_{1,0}(0|x) = 0$. If she chooses to enter the game, and set $\tau > 0$, then she will observe whether $B$ enters the game or not. This allows us to write $A$’s expected payoff for playing any $\tau > 0$ as:

$$V_{1,0}(\tau|x) = \Pr(z_B \leq x|x) \lim_{t \to 0} W_{2,t}(\tau^*_2(x,0)|z_A = x, z_B \leq x, t_B \geq 0) + [1 - \Pr(z_B \leq x|x)] \lim_{t \to 0} V_{1,t}(\tau|x)$$

Suppose first that $x$ is such that $bE_{\tau,\tau}[\lambda|x,x+] \leq c$; and hence that the equilibrium prescription is $\tau_1(x) = 0$. Since this implies that for any $\tau \geq t > 0$, $V'_1(\tau|x) < 0$; it follows that for any $\tau > 0$, $\lim_{t \to 0} V'_{1,t}(\tau|x) < 0$. Since

$$b \frac{\beta + 1}{\alpha - x - x + 2\tau} < b \frac{\beta + 2}{\alpha - x - x + 2\tau} = bE_{\tau,\tau}[\lambda|x,x] < bE_{\tau,\tau}[\lambda|x,x+]$$

where the last inequality is by from Lemma 2, it follows that $\tau^*_2(x,0) = 0$ by Proposition 3, and hence $\lim_{t \to 0} W_{2,t}(\tau^*_2(x,0)|z_A = x, z_B \leq x, t_B \geq 0) = 0$. So the player optimally chooses to follow the equilibrium prescription $\tau_1(x) = 0$. Second, suppose that $x$ is such that $bE_{\tau,\tau}[\lambda|x,x+] > c$; the player will comply with the equilibrium prescription $\tau_1(x) > 0$ because $\lim_{t \to 0} W_{2,t}(\tau^*_2(x,0)|z_A = x, z_B \leq x, t_B \geq 0) \geq 0$ and for any $\tau$ small enough $\lim_{t \to 0} V'_{1,t}(\tau|x) > 0$. ■

Proof of Lemma 8. Consider the time-0 team’s expected costs for adopting stopping times $T_1 \leq T_2$ (such that $T_1 + T_2 = 2T^*$) and conditional on any $\lambda$:

$$C(T_1, T_2|\lambda) = c \left[ \int_0^{T_1} 2f(s|\lambda) (1 - F(s|\lambda)) \int_0^s 2e^{-rv}dvds + (1 - F(T_1|\lambda))^2 \int_0^{T_1} 2e^{-rv}dv \right.$$ 

$$+ (1 - F(T_1|\lambda)) \int_0^{T_2} f(s|\lambda) \int_0^s e^{-rv}dvds + (1 - F(T_1|\lambda)) (1 - F(T_2|\lambda)) \int_0^{T_2} e^{-rv}dv \right]$$

$$= c \left[ \int_0^{T_1} 2\lambda e^{-\lambda s}e^{-\lambda s} \int_0^s 2e^{-rv}dvds + e^{-2\lambda T_1} \left( \int_0^{T_1} 2e^{-rv}dv \right) \right.$$ 

$$+ e^{-\lambda T_1} \int_0^{T_2} \lambda e^{-\lambda s} \int_0^s e^{-rv}dvds + e^{-2\lambda T_1} \int_0^{T_1} e^{-rv}dvds \right]$$

$$= c \left[ \frac{2 - e^{-2\lambda T_1}}{r} - \frac{4\lambda}{(2\lambda + r) r} \left(1 - e^{-(2\lambda+r)T_1}\right) + \frac{2}{r} \left(e^{-2\lambda T_1} - e^{-(2\lambda+r)T_1}\right) \right.$$ 

$$+ \frac{1}{r} \left(e^{-\lambda T_1} \left(e^{-(\lambda+r)T_1} - e^{-\lambda T_2+r T_1}\right) - e^{-\lambda T_1} \frac{\lambda}{\lambda + r} \left(e^{-(\lambda+r)T_1} - e^{-(\lambda+r)T_2}\right) \right) \right.$$ 

$$+ \frac{1}{r} \left(e^{-\lambda T_2-(\lambda+r)T_1} - e^{-(\lambda+r)T_2-\lambda T_1}\right) \right]$$

multiplying by $r/c$, simplifying terms and substituting $2T^* - T_1$ for $T_2$,

$$\frac{rC(T_1, 2T^* - T_1|\lambda)}{c} = \frac{2r}{2\lambda + r} - e^{-(2\lambda+r)T_1} \frac{r^2}{(\lambda + r) (2\lambda + r)} - \frac{r}{\lambda + r} e^{-(\lambda+r)2T^*+r T_1}$$

Taking a derivative with respect to $T_1$,

$$\frac{\partial C(T_1, 2T^* - T_1|\lambda)}{\partial T_1} \propto e^{-(2\lambda+r)T_1} \frac{r^2}{\lambda + r} - e^{-(\lambda+r)2T^*+r T_1} = e^{-(2\lambda+r)T_1} \frac{r^2}{\lambda + r} \left(1 - e^{-(\lambda+r)2(T^* - T_1)}\right).$$
This quantity is strictly positive if and only if \( T_1 < T^* \), thereby implying that if \( T_1 + T_2 = 2T^* \), then \( C(T_1, T_2 | \lambda) \) is maximized by setting \( T_1 = T^* = T_2 \).

If \( y \geq x > z \) and \( \tau_1^*(x) < T^*(x, y) \), then, since \( \tau_2'(y, \tau_1^*(x)) = 2T^*(x, y) - \tau_1^*(x) \),

\[
C(\tau_1^*(x), \tau_2'(y, \tau_1^*(x)) | \lambda) < C(T^*(x, y), T^*(x, y)| \lambda) \text{ for every } \lambda,
\]

and hence

\[
\int_\Lambda C(\tau_1^*(x), \tau_2'(y, \tau_1^*(x)) | \lambda) \pi_{0,0}(\lambda|x, y) d\lambda < \int_\Lambda C(T^*(x, y), T^*(x, y)| \lambda) \pi_{0,0}(\lambda|x, y) d\lambda.
\]

This inequality, together with the inequality

\[
W(\tau_1^*(x), \tau_2'(y, \tau_1^*(x)) | x, y) < W(T^*(x, y), T^*(x, y)| x, y)
\]

implied by the team’s optimal stopping times \( T_1 = T_2 = T^*(x, y) \), shows that

\[
B(\tau_1^*(x), \tau_2'(y, \tau_1^*(x)) | x, y) < B(T^*(x, y), T^*(x, y)| x, y).
\]

**Proof of Proposition 6.** For any signal pair \((x, y)\), we let \( z = x + y \), \( \mathcal{H}(z) \) denote the unconditional distribution of \( z \), and \( \mathcal{H}(x|z) \) the conditional distribution of \( x \) given \( z \). From the convolution formula, the density of \( z \) conditional on the arrival rate \( \lambda \) equals

\[
\int_z^0 \lambda e^{\lambda(z-x)} \lambda e^{\lambda x} dx = \lambda^2 e^{\lambda z} \int_z^0 dx = -z\lambda^2 e^{\lambda z}
\]

(recall \( y \leq 0 \), so \( z \leq x \)). The c.d.f. is then

\[
\mathcal{H}(z) = \int_0^\infty -z\lambda^2 e^{\lambda z} \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha \lambda} \lambda^{\beta-1} d\lambda = -z \frac{\alpha^\beta}{(\alpha - z)^{\beta+1}} \beta (\beta + 1).
\]

Next, notice from their expressions in (resp.) (4.4) and (A.1) that both the team’s optimal stopping time \( T^*(x, y) \) and the posterior beliefs about the arrival rate after observing both signals \( x, y \) but before beginning the race \( \pi_{0,0}(\lambda|x, y) \) depend on \( x, y \) only through their sum \( z = x + y \). Therefore, with a slight abuse of notation we let \( T^*(z) = T^*(x, z - x) \) for any \( 0 \geq x \geq z \), and \( \pi_{0,0}(\lambda|z) = \pi_{0,0}(\lambda|x, y) \).

Without loss of generality normalize \( c = 1 \) and let \( \mathcal{C}(x, z|\lambda) = C(\tau_1^*(x), \tau_2^*(z - x, \tau_1^*(x)) | \lambda) \) denote the total expected discounted cost of experimentation given \( \lambda \) and \( x \geq z \) in equilibrium, and \( \mathcal{C}^*(z|\lambda) = C(T^*(z), T^*(z)| \lambda) \) the analogous magnitude given \( \lambda \) and \( z \) when following the team’s optimal policy. Conditioning on the sum of signals \( z \), then on \( x \) and finally on \( \lambda \), the difference in expected PDV of experimentation costs is:

\[
D(\alpha, \beta; r) = \int_0^\infty \int_0^z \int_\Lambda [\mathcal{C}(x, z|\lambda) - \mathcal{C}^*(z|\lambda)] d\pi_{0,0}(\lambda|z) d\mathcal{H}(x|z) d\mathcal{H}(z).
\]

Split this integral in two parts, one for \( T^*(z) > 0 \), i.e. \( z > \alpha - (\beta + 2) \frac{z}{c} \), and the residual. This second piece, conditional on the team not experimenting, is

\[
\int_{-\infty}^{\alpha - (\beta + 2) \frac{z}{c}} \int_z^0 \int_\Lambda [\mathcal{C}(x, z|\lambda) - \mathcal{C}^*(z|\lambda)] d\pi_{0,0}(\lambda|z) d\mathcal{H}(x|z) d\mathcal{H}(z).
\]
The probability of this second event equals
\[ \Pr \left( z \leq \alpha - (\beta + 2) \frac{b}{c} \right) = \min \left\{ \left[ (\beta + 2) \frac{b}{c} - \alpha \right] \frac{\alpha^\beta}{(\alpha - z)^{\beta+2}} \beta (\beta + 1), 1 \right\} \]
This chance converges to zero both for \( \alpha \to 0 \) (by simple inspection) and for \( \beta \to \infty \): in fact, \( \alpha / (\alpha - z) < 1 \) for any \( z < 0 \), and for large \( \beta \) the geometric term \( \alpha^\beta / (\alpha - z)^{\beta+2} = \alpha^{-2} [\alpha / (\alpha - z)]^{\beta+2} \) vanishes faster than the exploding polynomial terms in \( \beta \).

Therefore, to study the sign of \( D(\alpha, \beta; r) \) as \( \beta / \alpha \to \infty \) we can concentrate on the term conditional on the team experimenting.

\[
D(\alpha, \beta, z; r) = \int_0^0 \int_{\lambda} \left[ C(x, z|\lambda) - C^*(z|\lambda) \right] d\mathcal{H}(x|z) d\Pi_0(\lambda|z) = 2 \int_{\lambda} \left\{ \int_z \left[ C^*(z|\lambda) - C(x, z|\lambda) \right] d\mathcal{H}(x|z) + \int_{z/2}^z \left[ C(x, z|\lambda) - C^*(z|\lambda) \right] d\mathcal{H}(x|z) \right\} d\Pi_0(\lambda|z),
\]
where we use the symmetry between \( x \) and \( y = z - x \) (which makes both \( C(x, z|\lambda) \) and \( d\mathcal{H}(x|z) \) symmetric in \( x \) and \( z - x \)), and calculate only the case that \( x < y = z - x \) (which entails \( x \leq z/2 \)); and where the function \( \tilde{\xi} : (\alpha - (\beta + 2) \frac{b}{c}, 0] \to (\alpha - (\beta + 2) \frac{b}{c}, 0] \) is implicitly defined by \( \tau_1^*(\tilde{\xi}(x)) = T^*(z) \), so that \( \tau_1^*(x) \to r_{\mathcal{I}}(z) \) if \( x > r_{\mathcal{I}}(z) \) and \( x \leq z/2 \) and the first player certainly over- (under-) experiments (since \( \xi(x) > x \), we know that \( \tilde{\xi}(z) < z/2 \). Conditional on \( \lambda \) and given discounting, we know from Lemma 8 that this implies that equilibrium features over- (under-) experimentation and \( C(x, z|\lambda) \to r_{\mathcal{I}}(z|\lambda) \). Hence both inner integrals are strictly positive, and their difference as written above has an ambiguous sign. The question we address now is which effect dominates on average.

Since \( \tau_2^*(y, \tau_1^*(x)) = \max\{\tau_1^*(x), 2T^*(x, y) - \tau_1^*(x)\} \), we can write:

\[
C(x, z|\lambda) = \int_0^{\tau_1^*(x)} 2f(s|\lambda)(1 - F(s|\lambda)) \int_s^z 2e^{-rv} dv ds + (1 - F(\tau_1^*(x)|\lambda))^2 \int_0^{\tau_1^*(x)} 2e^{-rv} dv \]
\[+ (1 - F(\tau_1^*(x)|\lambda)) \int_{\tau_1^*(x)}^{\max(\tau_1^*(x), 2T^*(z) - \tau_1^*(x))} f(s|\lambda) \int_s^{\tau_1^*(x)} e^{-rv} dv ds \]
\[+ (1 - F(\tau_1^*(x)|\lambda)) (1 - F(\max(\tau_1^*(x), 2T^*(z) - \tau_1^*(x))|\lambda)) \int_{\tau_1^*(x)}^{\max(\tau_1^*(x), 2T^*(z) - \tau_1^*(x))} e^{-rv} dv. \]

\[
C^*(z|\lambda) = \int_0^{T^*(z)} 2f(s|\lambda)(1 - F(s|\lambda)) \int_s^z 2e^{-rv} dv ds + (1 - F(T^*(z)|\lambda))^2 \int_0^{T^*(z)} 2e^{-rv} dv. \]

Now, if \( \tilde{\xi}(z) < x \leq z/2 \), then \( \tau_1^*(x) > T^*(z) \), and, simplifying notation by omitting the dependence on \( x \) and \( z \) in \( \tau_1^*(x) \) and \( T^*(z) \)

\[
C(x, z|\lambda) - C^*(z|\lambda) = \int_{T^*}^{\tau_1^*(x)} 2f(s|\lambda)(1 - F(s|\lambda)) \int_s^z 2e^{-rv} dv ds \]
\[+ (1 - F(\tau_1^*(x)|\lambda))^2 \int_0^{\tau_1^*(x)} 2e^{-rv} dv - (1 - F(T^*(z)|\lambda))^2 \int_0^{T^*(z)} 2e^{-rv} dv \]
\[= \int_{T^*}^{\tau_1^*(x)} 2\lambda e^{-2\lambda s} \int_s^z 2e^{-rv} dv ds + e^{-2\lambda \tau_1^*(x)} \int_0^{\tau_1^*(x)} 2e^{-rv} dv - e^{-2\lambda T^*} \int_0^{T^*(z)} 2e^{-rv} dv \]
\[= \frac{2}{r} \left( 1 - \frac{2\lambda}{2\lambda + r} \right) \left( e^{-(2\lambda + r)T^*} - e^{-2\lambda + r}\tau_1^*(x) \right). \]
If instead \( x < \tilde{\xi} (z) \leq z/2 \), then \( \tau^*_1 (x) < T^*(z) \), and, again simplifying notation

\[
C^* (z|\lambda) - C (x, z|\lambda) = \int_{\tau^*_1}^{T^*} 2f (s|\lambda) (1 - F (s|\lambda)) \int_0^s 2e^{-rv} dv ds + (1 - F (T^*|\lambda))^2 \int_0^{T^*} 2e^{-rv} dv
\]

\[
- (1 - F (\tau^*_1|\lambda))^2 \int_0^{\tau^*_1} 2e^{-rv} dv - (1 - F (\tau^*_1|\lambda)) \int_{\tau^*_1}^{2T^*-\tau^*_1} f (s|\lambda) \int_0^s e^{-rv} dv ds
\]

\[
- (1 - F (\tau^*_1|\lambda)) (1 - F (2T^* - \tau^*_1|\lambda)) \int_{\tau^*_1}^{2T^*-\tau^*_1} e^{-rv} dv
\]

\[
= \int_{\tau^*_1}^{T^*} 2\lambda e^{-\lambda s} e^{-\lambda s} \int_0^s 2e^{-rv} dv ds + e^{-\lambda T} \int_0^{T^*} 2e^{-rv} dv - e^{-2\lambda \tau_1} \int_0^{\tau^*_1} 2e^{-rv} dv
\]

\[
- e^{-\lambda \tau_1} \int_{\tau^*_1}^{2T^*-\tau^*_1} \lambda e^{-\lambda s} \int_0^s e^{-rv} dv ds - e^{-\lambda \tau_1} e^{-\lambda (2T^* - \tau^*_1)} \int_{\tau^*_1}^{2T^*-\tau^*_1} e^{-rv} dv
\]

\[
= \frac{2}{r} \left( 1 - \frac{2\lambda}{2\lambda + r} \right) \left( e^{-(2\lambda + r)\tau^*_1(u)} - e^{-(2\lambda + r)T^*} \right)
\]

\[
- \frac{1}{r} \left[ e^{-(2\lambda + r)\tau^*_1(u)} - \frac{2\lambda}{2\lambda + r} \right] \left( e^{-\lambda T} - \frac{\lambda}{\lambda + r} \left( e^{-2\lambda \tau_1} - e^{-\lambda 2T^*} \right) - e^{-2\lambda T^* - r(2T^* - \tau^*_1)} \right)
\]

For any fixed \( z \), and \( \lambda \), pick an arbitrary pair of signals \( (x, u) \) such that \( z \leq u < \tilde{\xi} (z) < x \leq z/2 \). We study the ratio:

\[
C^* (z|\lambda) - C (x, u|\lambda)
\]

\[
= \frac{2}{r} \left( 1 - \frac{2\lambda}{2\lambda + r} \right) \left( e^{-(2\lambda + r)\tau^*_1(u)} - e^{-(2\lambda + r)T^*} \right)
\]

\[
- \frac{1}{r} \left[ e^{-(2\lambda + r)\tau^*_1(u)} - \frac{2\lambda}{2\lambda + r} \right] \left( e^{-\lambda T} - \frac{\lambda}{\lambda + r} \left( e^{-2\lambda \tau_1} - e^{-\lambda 2T^*} \right) - e^{-2\lambda T^* - r(2T^* - \tau^*_1)} \right)
\]

\[
\leq \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-(2\lambda + r)T^*}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}} - \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-2\lambda T^* - r(2T^* - \tau^*_1(u))}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}}
\]

\[
= \frac{\lambda}{2\lambda + r} \left( e^{-2\lambda \tau_1} - e^{-\lambda 2T^*} \right)
\]

\[
\geq \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-(2\lambda + r)T^*}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}} - \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-2\lambda T^* - r(2T^* - \tau^*_1(u))}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}}
\]

Where the inequality follows because \( \tau^*_1 (x) > T^* > \tau^*_1 (u) \).

As the social planner becomes more and more impatient:

\[
\lim_{r \to \infty} \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-(2\lambda + r)T^*}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}} - \frac{\lambda}{2\lambda + r} \left( e^{-2\lambda \tau_1} - e^{-\lambda 2T^*} - e^{-(2\lambda + r)\tau^*_1(x)} \right)
\]

\[
= \lim_{r \to \infty} \frac{e^{-(2\lambda + r)\tau^*_1(u)} - e^{-2\lambda T^* - r(2T^* - \tau^*_1(u))}}{e^{-(2\lambda + r)T^*} - e^{-(2\lambda + r)\tau^*_1(x)}} - \frac{\lambda}{2\lambda + r} \left( e^{-2\lambda \tau_1} - e^{-\lambda 2T^*} - e^{-(2\lambda + r)\tau^*_1(x)} \right)
\]

\[
= \frac{1}{1 - e^{-\lambda T}} - \frac{1}{1 - e^{-\lambda T}} = +\infty,
\]
because \( \lim_{r \to \infty} e^{-(2\lambda+r)(T^*-\tau^*_1(u))} = T^* > \tau^*_1(u) \).

Since the argument holds for any arbitrary \( z > \alpha - (\beta + 2 \frac{h}{\xi}) \), any \( \lambda \) and any arbitrary pair of signals \( (x, u) \) such that \( u < \xi(z) < x \), this concludes that, for any \( z \),

\[ D(\alpha, \beta; z; r) < 0 \text{ for } r \text{ large enough}. \]

References


Omitted Algebraic Derivations Not Intended for Publication

Derivation of Equation (4.5). First we expand the derivative as in the analysis determining $T_2^s(x, y, T_1)$:

$$
\frac{dW_{1, t}(T_1|x, y)}{dT_1} = \int_{\Lambda} \frac{d}{dT_1} \hat{U}_t(T_1|\lambda) \pi_{t, t}(\lambda|x, y)d\lambda
$$

$$
= \int_{\Lambda} \left\{ \frac{2f(T_1|\lambda) (1 - F(T_1|\lambda))}{(1 - F(t|\lambda))^2} \left[ \int_{t}^{T_1} (-2c) e^{-r(v-t)}dv + e^{-r(T_1-t)b} \right] - \frac{2f(T_1|\lambda) (1 - F(T_1|\lambda))}{(1 - F(t|\lambda))^2} \left[ \int_{t}^{T_1} (-2c) e^{-r(v-t)}dv + e^{-r(T_1-t)}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) \right] + \frac{(1 - F(T_1|\lambda))^2}{(1 - F(t|\lambda))^2} \left[ (-2c) e^{-r(T_1-t)} + \frac{d}{dT_1}(e^{-r(T_1-t)}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1)) \right] \right\} \pi_{t, t}(\lambda|x, y)d\lambda
$$

simplifying the term in $2c$ and collecting the discounting term

$$
= e^{-r(T_1-t)} \int_{\Lambda} \left\{ \frac{2f(T_1|\lambda) (1 - F(T_1|\lambda))}{(1 - F(t|\lambda))^2} \left[ b - W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) \right] + \frac{(1 - F(T_1|\lambda))^2}{(1 - F(t|\lambda))^2} \cdot \left[ -2c - rW_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) + \frac{d}{dT_1}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) \right] \right\} \pi_{t, t}(\lambda|x, y)d\lambda
$$

using the definitions of $\pi_{t, t}(\lambda|x, y)$, $\pi_{t,T_1}(\lambda|x, y)$

$$
\frac{d}{dT_1}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) \propto \int_{\Lambda} \left[ \frac{f(T_1|\lambda)}{(1 - F(T_1|\lambda))} (b - W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1)) \right] \pi_{t,T_1}(\lambda|x, y)
$$

$$
+ \int_{\Lambda} \left[ -\frac{r}{2}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) + \frac{1}{2} \frac{d}{dT_1}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) - c \right] \pi_{t,T_1}(\lambda|x, y)d\lambda
$$

$$
\propto b\mathbb{E}_{T_1,T_1}[\lambda|x, y] - c - W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) \left( \frac{r}{2} + \mathbb{E}_{T_1,T_1}[\lambda|x, y] \right) + \frac{1}{2} \frac{d}{dT_1}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1).
$$

Using the expression for $W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1)$, and the envelope theorem to ignore the effect of $T_1$ on this value through $T_2^s(x, y, T_1)$,

$$
\frac{d}{dT_1}W_2,T_1(T_2^s(x, y, T_1)|x, y, T_1) = \frac{d}{dT_1}\mathbb{E}_{T_1,T_1}[U_2,T_1(T_2^s(x, y, T_1)|\lambda)|x, y]
$$

$$
= \frac{d}{dT_1} \int_{\Lambda} \int_{T_1}^{T_2^s(x,y,T_1)} \frac{f(s|\lambda)}{1 - F(T_1|\lambda)} \left[ \int_{T_1}^{s} (-c) e^{-r(v-t)}dv + e^{-r(s-T_1)b} \right] ds
$$

$$
+ \frac{1 - F(T_2^s(x, y, T_1)|\lambda)}{1 - F(T_1|\lambda)} \int_{T_1}^{T_2^s(x,y,T_1)} (-c) e^{-r(v-T_1)}dv \right] \pi_{T_1,T_1}(\lambda|x, y)d\lambda
$$

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Using the expression for $dV_1$ and replacing into (A.3) 

Recommending this expression for $dV_1$ into (A.3) and collecting terms

Using the expression for $\pi_{T_1, T_1}(\lambda | x, y)$ and after some algebra

and replacing into (A.3)

**Derivation of Equation (5.2).**

$$
\frac{dV_{1,t}(\tau|x)}{d\tau} = \int_\Lambda \left[ \frac{f(\tau|x)}{1 - F(t|\lambda)} - H(g(\tau)|\lambda) \left( \int_t^\tau e^{-r(v-t)}dv + e^{-r(\tau-t)}b \right) \right. \\
+ \left( \frac{1 - F(\tau|\lambda)}{1 - F(t|\lambda)} \right)^2 h(g(\tau)|\lambda)g'(\tau) \left( \int_t^\tau e^{-r(v-t)}dv + e^{-r(\tau-t)}W_{2,t}(\tau|x, g(\tau), \tau) \right) \\
+ \frac{2(1 - F(\tau|\lambda)) f(\tau|\lambda) [1 - H(g(\tau)|\lambda)] + [1 - F(\tau|\lambda)]^2 h(g(\tau)|\lambda)g'(\tau)}{[1 - F(t|\lambda)]^2 [1 - H(g(t)|\lambda)]} \int_t^\tau e^{-r(v-t)}dv \\
+ \left( \frac{1 - F(\tau|\lambda)}{1 - F(t|\lambda)} \right)^2 \left[ 1 - H(g(\tau)|\lambda) \right] \left( -c e^{-r(\tau-t)} \right) \pi_{t,t}(\lambda|x, g(t) + ) \right) d\lambda
$$
\[
\begin{align*}
&= \int_{\Lambda} \left[ \frac{f(\tau|\lambda)}{1 - F(t|\lambda)} \frac{1 - F(t|\lambda)}{1 - F(t|\lambda)} 1 - H(g(\tau)|\lambda) e^{-r(\tau-t)}b \\
&\quad+ \left( \frac{1 - F(\tau|\lambda)}{1 - F(t|\lambda)} \right)^2 \frac{h(g(\tau)|\lambda)g'(\tau)}{1 - H(g(\tau)|\lambda)} e^{-r(\tau-t)}W_{2,\tau}(x,\tau|x, g(\tau), \tau) \\
&\quad+ \left( \frac{1 - F(\tau|\lambda)}{1 - F(t|\lambda)} \right)^2 \left[ \frac{1 - H(g(\tau)|\lambda)}{1 - H(g(t)|\lambda)} \right] (-c) e^{-r(\tau-t)} \right] \pi_{t,t}(\lambda|x, g(t) +) d\lambda \\
&= \Sigma_{t,t}(\tau, x, g(\tau) +) \int_{\Lambda} \left\{ \frac{f(\tau|\lambda)}{1 - F(\tau|\lambda)} b + \frac{h(g(\tau)|\lambda)g'(\tau)}{1 - H(g(\tau)|\lambda)} W_{2,\tau}(x, \tau|x, g(\tau), \tau) - c \right\} \pi_{t,t}(\lambda|x, g(\tau) +) d\lambda \\
&\propto b\mathbb{E}_{\tau, t}[\lambda|x, g(\tau) +] + W_{2,\tau}(x, \tau|x, g(\tau), \tau)\mathbb{E}_{r, \tau} \left[ \frac{h(g(\tau)|\lambda)g'(\tau)}{1 - H(g(\tau)|\lambda)} \right] x, g(\tau) + - c
\end{align*}
\]

where we introduce a normalizing factor

\[
\Sigma_{t,t}(\tau, x, g(\tau) +) \equiv e^{-r(\tau-t)} \int_{\Lambda} \pi(\lambda) h(x|\lambda) [1 - H(g(\tau)|\lambda)] [1 - F(\tau|\lambda)]^2 d\lambda
\]

\[
\int_{\Lambda} \pi(\lambda) h(x|\lambda) [1 - H(g(t)|\lambda)] [1 - F(t|\lambda)]^2 d\lambda' > 0.
\]