CONTENT KNOWLEDGE FOR TEACHING: WHAT MAKES IT SPECIAL?

Deborah Loewenberg Ball, Mark Hoover Thames, and Geoffrey Phelps

While teacher content knowledge is crucially important to the improvement of teaching and learning, attention to its development and study has been uneven. Historically, researchers have focused on many aspects of teaching, but more often than not scant attention has been given to how teachers need to understand the subjects they teach. Further, when researchers, educators and policy makers have turned attention to teacher subject matter knowledge the assumption has often been that advanced study in the subject is what matters. Debates have focused on how much preparation teachers need in the content strands rather than on what type of content they need to learn.

In the mid-1980s, a major breakthrough initiated a new wave of interest in the conceptualization of teacher content knowledge. In his 1985 AERA presidential address, Lee Shulman identified a special domain of teacher knowledge, which he referred to as *pedagogical content knowledge*. He distinguished between content as it is studied and learned in disciplinary settings and the "special amalgam of content and pedagogy" needed for teaching the subject. These ideas had a major impact on the research community, immediately focusing attention on the foundational importance of content knowledge in teaching and on pedagogical content knowledge in particular.

This paper provides a brief overview of research on content knowledge and pedagogical content knowledge, describes how we have approached the problem, and reports on our efforts to define the domain of mathematical knowledge for teaching and to refine its sub-domains.

Content Knowledge and Its Role in Establishing Teaching as a Profession

A central contribution of the work of Shulman and his colleagues was to reframe the study of teacher knowledge in ways that included direct attention to the role of *content* in teaching. This was a radical departure from research of the day, which focused almost exclusively on general aspects of teaching such as classroom management, time allocation, or planning. A second contribution of the work was to leverage content knowledge as technical knowledge key to the establishment of teaching as a profession. Shulman and his colleagues argued that high quality instruction requires a sophisticated professional knowledge that goes beyond simple rules such as how long to wait for students to respond. To characterize professional knowledge for teaching, they developed typologies. Although the specific boundaries and names of categories varied across publications, one of the more complete articulations is reproduced below (Figure 1).

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter
- Knowledge of learners and their characteristics
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds
- Content knowledge
- Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding

(Shulman, 1987, p.8)

Figure 1. Shulman's Major Categories of Teacher Knowledge

These categories were meant to highlight the important role of content knowledge and to situate content-based knowledge in the larger landscape of professional knowledge for teaching. The first four categories address general dimensions of teacher knowledge that were the mainstay of teacher education programs at the time. For Shulman, they acted as placeholders in a broader conception of teacher knowledge that emphasized content knowledge. The remaining three categories defined content-specific dimensions and together comprised what Shulman referred to as the missing paradigm in research on teaching — "a blind spot with respect to content that characterizes most research on teaching, and as a consequence, most of our state-level programs of teacher evaluation and teacher certification" (Shulman, 1986b, p. 8). At the same time, however, Shulman made it clear that the general categories were crucial and that an emphasis placed on content dimensions of teacher knowledge was not intended to denigrate the importance of pedagogical understanding and skill: "Mere content knowledge is likely to be as useless pedagogically as content free skill" (Shulman, 1986b, p. 8).

The first of the three, *content knowledge*, includes knowledge of the subject and its organizing structures (Grossman, Wilson, & Shulman, 1989; Shulman, 1986b, 1987; Wilson, Shulman, & Richert, 1987). The second category, *curricular knowledge*, is "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986b, p.10). The last, and arguably most influential, of the three content-related categories is pedagogical content knowledge. Shulman defined pedagogical content knowledge as:

> … the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations — in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others…. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (Shulman, 1986b, p.7).

Interest in these ideas was immediate and widespread. In the two decades since these ideas were first presented, Shulman's presidential address (1986) and the related Harvard Education Review article (1987) have been cited in over 1200 refereed journal articles. This interest has been sustained with no less than fifty citations to these two articles in every year since 1990. Perhaps most remarkable is the reach of this work, with citations appearing in 125 different journals representing professions ranging from law to nursing to business and addressing knowledge for teaching preschool through doctoral studies. Much of the interest has focused directly on pedagogical content knowledge. Thousands of articles, book chapters, and reports make use of or claim to study the notion of pedagogical content knowledge in a wide variety of subject areas: science, mathematics, social studies, English, physical education, communication, religion, chemistry, engineering, music, special education, English language learning, higher education, and others. And, such studies show no signs of abating. Rarely does an idea — or a term — catch on at such a scale.

The continuing appeal of the notion of pedagogical content knowledge is that it bridges content knowledge and the practice of teaching, assuring that discussions of content are relevant to teaching and that discussions of teaching retain attention to content. As such, it is the unique province of teachers — a content-based form of professional knowledge. However, after two decades of work, the nature of this bridge remains inadequately understood and the "coherent theoretical framework," called for by Shulman (1986, p. 9) remains underdeveloped.

Two points are worth making here. First, researchers have failed to establish precise or agreed-upon definitions. Throughout the past twenty years, for example, researchers have used the term "pedagogical content knowledge" to refer to a wide range of aspects of subject matter knowledge and aspects of the teaching of subject matter. It is often unclear how ideas in one subject area relate to those in another subject area, or even whether findings within the same subject take similar or different views of teacher subject matter knowledge. Somewhat ironically, nearly one-third of the articles that cite pedagogical content knowledge do so without direct attention to a specific content area — the very emphasis of the notion — instead making general claims about teacher knowledge, teacher education, or policy.

Second, while the work of Shulman and his colleagues was developed from extensive observation of classroom teaching, most subsequent research takes particular domains of knowledge, such as pedagogical content knowledge, as given or uses only logical arguments to substantiate claims about the existence and the role of these domains. Few studies test whether there are, indeed, distinct bodies of identifiable content knowledge that matter for teaching. In particular, the field has done little to develop measures of such knowledge and to use these measures to test definitions and our understanding of the nature and the effects of content knowledge on teaching and learning. Overall, the literature uses the idea as though its theoretical foundations, conceptual distinctions, and empirical testing were fait accompli.

Lacking adequate definition and empirical testing, the ideas are bound to play a limited role in revamping the curriculum for teacher content preparation, in informing policies about certification and professional development, or in furthering our understanding of the relationships among teacher knowledge, teaching, and student learning. Without such work, the ideas remain, as they were twenty years ago, promising hypotheses based on logical and ad hoc arguments about the content people *think* teachers need.

Our Approach to Studying Mathematical Knowledge for Teaching

What do teachers need to know and be able to do to effectively carry out the work of teaching mathematics (Ball, Hill, & Bass, 2005)? Our research group chose to investigate this question in a way that might best be characterized as working "bottom up," beginning with practice. Because it seemed obvious that teachers need to know the topics and procedures that they teach — multiplication, equivalent fractions, and so on — we decided to focus specifically on <u>how</u> teachers need to know that content. In addition, we wanted to ask <u>what else</u> do teachers need to know about mathematics and how and where might teachers <u>use</u> such mathematical knowledge in practice?

Hence, we decided to focus on the "work of teaching." What do teachers *do* in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill? Instead of starting with the curriculum, or with standards for student learning, we study teachers' work. We seek to unearth the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching. We see this approach as a kind of "job analysis," similar to analyses done of other mathematically intensive occupations that range from nursing and engineering physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997) to carpentry and waiting tables. Our analyses lay the foundation for a *practice-based theory of mathematical knowledge for teaching* (Ball & Bass, 2003).

We approach the problem in two ways. First, we conduct extensive qualitative analyses of video of teaching practice. Second, we design measures of mathematical knowledge for teaching based on hypotheses formulated from our qualitative studies.

From our analyses, we have developed a working definition of "mathematical knowledge for teaching." By this phrase, we mean the *mathematical knowledge that teachers need to carry out their work as teachers of mathematics*. Obviously, teachers need to know the content they teach and that students are expected to master. Our question is whether they need to know more, and if so, what they need to know and in what ways they need to know this mathematics to use it in their teaching? The most prevalent hypothesis evident in the policy world and society at large is that teachers need to know whatever mathematics is in the curriculum plus some additional number of years of further study in college mathematics. A second hypothesis is that teachers need to know the curriculum plus pedagogical content knowledge. In both cases, it is unclear what exactly it is that makes up the *extra* knowledge of mathematics. We work on this problem by considering the range of mathematical demands that arise in the work teachers do, and in particular, those that require *a distinctive brand of mathematical knowledge rather unique to the work of teachers*.

This definition of mathematical knowledge for teaching, explicitly framed in terms of the work teachers do, may seem like a minor point, but it is perhaps more significant than it seems. For instance, it suggests that the way to decide whether teachers should be taught particular content, such as calculus, is by considering when and where such knowledge would bear on what teachers need to do. It also suggests that the connections between subject matter knowledge and teaching be made explicit. Defining mathematical knowledge for teaching in this way addresses two important problems; it provides a basis for setting priorities for what teachers are taught, and it increases the likelihood that teachers will be able to use what they are taught when they teach.

In our analyses, we also noticed that the nature of that mathematical knowledge and skill seemed itself to be of different types. We hypothesized that teachers' opportunities to learn mathematics for teaching could be better designed if we could identify those types more clearly. If mathematical knowledge required for teaching is indeed multidimensional then professional education could be organized to help teachers learn the range of knowledge and skill they need in focused ways. If, however, it is basically all the same as general mathematical ability, then discriminating professional learning opportunities would be unnecessary.

To represent our current hypotheses, we propose a refinement to Shulman's categories. Figure 2 shows the correspondence between our current map of the domain of content knowledge for teaching and Shulman's (1986) initial categories: subject matter knowledge and pedagogical content knowledge.

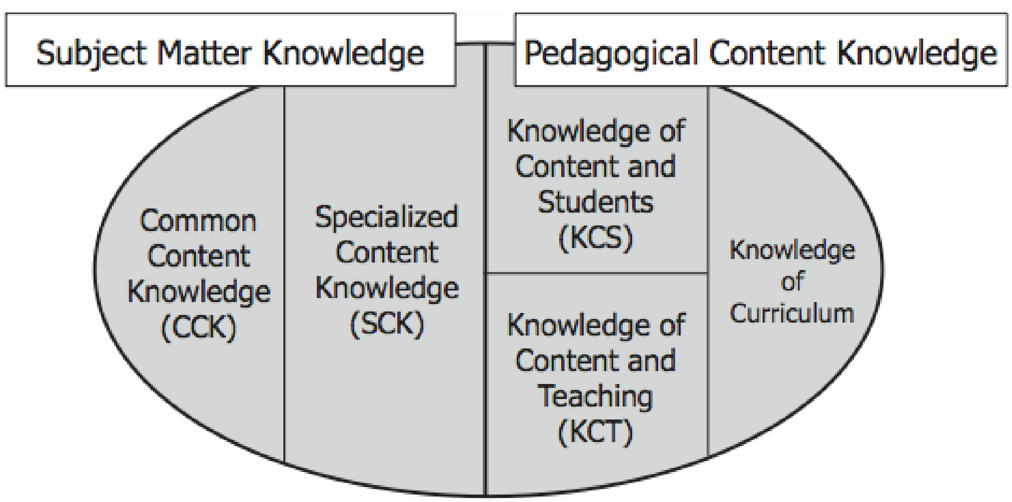

Figure 2. Shulman's Original Category Scheme (1985) Compared to Ours

Based on our analysis of the mathematical demands of teaching, we hypothesize that Shulman's categories of content knowledge and pedagogical content knowledge can be subdivided into *common content knowledge* and *specialized content knowledge*, on the one hand, and *knowledge of content and students* and *knowledge of content and teaching*, on the other. Perhaps of most interest to us is evidence of the second category — specialized content knowledge. Like pedagogical content knowledge it is closely related to practice, but unlike pedagogical content knowledge it does not require additional knowledge of students or teaching. It is distinctly mathematical knowledge, but is not necessarily mathematical knowledge familiar to mathematicians. We now turn to defining and illustrating these proposed categories.

Mathematical Knowledge for Teaching: An Example

Our analyses of teachers' practice reveal that the mathematical demands of teaching are substantial. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning. To better understand what we mean by this, we offer an example based on a simple subtraction computation: 307 – 168. Most readers will know an algorithm to produce the answer 139, such as:

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  2 9
  3̸ ¹0̸ ¹7
- 1  6  8
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  1  3  9
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Teachers must be able to themselves perform this calculation. This is mathematical knowledge we would expect a well-educated adult to know, and we refer to it as *common content knowledge* (CCK). It is closely related to the content of the curriculum, but not to a particular curriculum. It includes knowing when students have answers wrong, recognizing when the textbook gives an inaccurate definition, and being able to use terms and notation correctly when speaking and writing at the board. In short, it is the knowledge teachers need in order to be able to *do the work* that they are assigning their students.

In analyzing video of teaching, it became obvious, especially when teachers lacked common content knowledge, that such knowledge is essential. When a teacher mispronounced terms, made calculation errors, or got stuck trying to solve a problem, instruction suffered and valuable time was lost. In mapping out the mathematical knowledge needed by teachers, it is important not to lose sight of the critical role that a basic understanding of the mathematics in the student curriculum plays in planning and carrying out instruction.

Returning to our subtraction problem, however, we see that being able to carry out the procedure is necessary, but not sufficient, for teaching it. Many third graders struggle with this algorithm, often making errors. One common error is:

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  307
- 168
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  261
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A teacher needs to be able to spot that 261 is incorrect. However, a teacher who can see only that this is not the correct answer is not well equipped to help a student learn to get it right. Skillful teaching requires being able to size up the source of a mathematical error. Further, this is work that teachers often must do very quickly, since, in a classroom, students cannot wait as a teacher puzzles over the mathematics. Here, for example, a student has, in each column, calculated the difference between the two digits, or subtracted the smaller digit from the larger one. A teacher who is mystified about what could have produced 261 as an answer will arguably move more slowly and with less precision to help correct the student's problem.

Consider another error that teachers may see with this subtraction problem.

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

What line of thinking would produce this error? In this case, the student has "borrowed" one from the hundreds column, "carried the one" to the ones place, and subtracted 8 from 17, yielding 9. The thinking might continue by "bringing down" the 6 and subtracting 2 – 1 = 1. Teachers need to be able to perform this kind of mathematical error analysis efficiently and fluently.

These two errors stem from different difficulties with the algorithm for subtracting multi-digit numbers. In the first, the student considered the difference between digits with no thought to the relationships among columns. In the second, the student attempts to regroup the number, but without careful consideration of the value of the places and the conservation of the value of the number. Seeing both answers as simply "wrong" does not equip a teacher with the detailed mathematical understanding required for a skillful treatment of the problems these students face.

Analysis such as this are characteristic of the distinctive work teachers do and they require a kind of mathematical reasoning that most adults do not need to do on a regular basis. And although mathematicians engage in analyses of error, often of failed proofs, the analysis used to uncover a student error appears to be related to, but not the same as, other error analysis in the discipline. Further, there is no demand on mathematicians to conduct their work quickly as students wait for guidance.

It is also common in instruction for students to come up with non-standard approaches that are unfamiliar to the teacher. For instance, what mathematical issues confront a teacher if a student asserts that she would "take 8 away from both the top and the bottom," yielding the easier problem:

$$\begin{array}{r} 299 \\ -\ 160 \\ \hline \end{array}$$

Is it okay to do this? Why? Would it work in general? Is it easier for some numbers and harder for others? How might you describe the method the student is using and how would you justify it mathematically? Being able to engage in this sort of mathematical inner dialogue, and to provide mathematically sound answers to these questions, is a crucial foundation for determining what to *do* in teaching this mathematics.

Teachers confront all kinds of student solutions. They have to figure out what students have done, whether the thinking is mathematically correct for the problem, and whether the approach would work in general. Consider the following three executions of our original subtraction problem. What is going on mathematically in each case?

307	11	307
- 168	307	- 168
-1	- 168	2
-60	11	30
200	139	107
139		139

In fact, these examples are all correct and could be generalized in plausible ways, but figuring this out is not a straightforward task for those who only know how to do the subtraction as they themselves learned it in third grade.

Interpreting student error and evaluating alternative algorithms is not all that teachers do, however. Teaching also involves knowing rationales for procedures, meanings for terms, and explanations for concepts. For example, teachers need effective ways of representing the *meaning* of the subtraction algorithm — not just to confirm the answer, but to show what the steps of the procedure mean, and why they make sense. Our point here is not about what teachers need to teach, but about what they themselves need to know and be able to do in order to carry out any responsible form of teaching.

Our study of the mathematical demands of teaching has yielded a wealth of tasks that require mathematical knowledge and skill. What caught us by surprise, however, was how much purely *mathematical* knowledge was required, even in many everyday tasks of teaching — assigning student work, listening to student talk, grading or commenting on student work. We were also surprised to see that many of the tasks teachers do require mathematical knowledge independent of knowledge of students or teaching. For instance, deciding whether a method or procedure would work in general often requires mathematical knowledge and skill independent of knowing anything about students or teaching per se.

Of course, mathematical consideration of this kind is worthwhile only if a teacher knows enough about students and teaching to make use of it, but the point we want to make here is that the work teachers do constitutes a form of mathematical problem solving that lives inside the work of teaching. Likewise, determining the validity of a mathematical argument, or selecting appropriate mathematical representations, requires mathematical knowledge and skill important for teaching yet not necessarily about students or teaching. In our research we began to notice how rarely these mathematical demands were ones that could be addressed with mathematical knowledge learned in university mathematics courses. We began to hypothesize that there were aspects of subject matter knowledge — not pedagogical content knowledge — that need to be uncovered, mapped, organized, and included in mathematics courses for teachers.

Looking back across our subtraction example, many of the tasks, such as determining whether an alternative method will work in general, explaining the meaning of a procedure, or recognizing mathematical properties offered by different materials or models, involve deep and explicit knowledge of the subtraction algorithm — more than simply knowing how to perform the calculation. Yet, these tasks of teaching may or may not draw on knowledge of *students* or knowledge of *teaching*.

Consider, for instance, the first subtraction error.

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

This is a very common error and many teachers are familiar with it as something students do. In a real sense, knowledge of this error typically derives from experience with students and knowledge of their thinking. Because of this, we categorize it as *knowledge of content and students (KCS)*, a type of pedagogical content knowledge that combines knowing about students and knowing about mathematics.

When choosing an example, teachers need to predict what students will find interesting and motivating. When assigning a task, they need to anticipate what students are likely to do with it and whether they will find it easy or hard. They must also be able to hear and interpret students' emerging and incomplete thinking. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

A related domain, *knowledge of content and teaching (KCT)*, is knowledge that combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require mathematical knowledge that interacts with the design of instruction. Teachers need to sequence particular content for instruction, deciding which example to start with and which examples to use to take students deeper into the content. They need to evaluate the instructional advantages and disadvantages of representations used to teach a specific idea. During a classroom discussion, they have to decide when to ask for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning. Each of these requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning.

The fourth category, *specialized content knowledge (SCK)*, is mathematical knowledge *beyond* that expected of any well-educated adult but not yet requiring knowledge of students or knowledge of teaching. Many of the common tasks of teaching require significant mathematical resources, but do not yet necessarily require knowing about students or teaching (Fig. 3).

Presenting mathematical ideas

Responding to students' "why" questions

Finding an example to make a specific mathematical point

Recognizing what is involved in using a particular representation

Linking representations to underlying ideas and to other representations

Connecting a topic being taught to topics from prior or future years

Explaining mathematical goals and purposes to parents

Appraising and adapting the mathematical content of textbooks

Modifying tasks to be either easier or harder

Evaluating the plausibility of students' claims (often quickly)

Giving or evaluating mathematical explanations

Choosing and developing useable definitions

Using mathematical notation and language and critiquing its use

Asking productive mathematical questions

Selecting representations for particular purposes

Inspecting equivalencies

Figure 3. Mathematical Tasks of Teaching

Each of these is something teachers routinely do. Taken together, they make for rather unique mathematical requirements. In part, such tasks imply that teachers need to know a body of mathematics not typically taught to students. Teachers need to understand different interpretations of the operations in ways that students do not. They need to know the difference between "take away" and "comparison" models of subtraction, and between "measurement" and "partitive" models of division. They also need to know features of mathematics that they may never teach to students, such as a range of non-standard methods or the mathematical structure of student errors. These knowledge demands are distinct from those described by Shulman under the label of pedagogical content knowledge.

For instance, teachers need "decompressed" mathematical knowledge that might be taught to students but with the eventual curricular goal of developing, in students, compressed forms of that knowledge as it is understood by competent adults for various uses. To help students learn, teachers need to not only be able to do mathematics but they need to unpack the elements of that mathematics to make its features apparent to students. They need to understand the place-value system in a self-conscious way that goes beyond the kind of tacit understanding of place value expected of others.

Moreover, teachers need extended expertise with certain mathematical practices. They need to be able to talk explicitly about how mathematical language is used (e.g., how the mathematical meaning of "edge" is different from the everyday reference to the edge of a table), how to choose, make, and use mathematical representations effectively (recognizing mathematical advantages and disadvantages for different options), and how to explain and justify one's mathematical ideas (e.g., why you invert and multiply to divide fractions). All of these are examples of ways in which teachers engage in particular mathematical practices and work with mathematics in its decompressed or unpacked form.

The mathematical demands of teaching require specialized mathematical knowledge, needed by teachers, but not needed by others. Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group needs to explain why, when you multiply by ten, you "add a zero." In developing survey questions to measure such knowledge, we ask, for example, whether an unusual method proposed by a student would work in general, which statement best explains why we find common denominators when adding fractions, and which of a set of given drawings could be used to represent 2 divided by $^{2}/_{3}$. These and questions like them are the daily fare of teaching. The demands of the work of teaching mathematics create the need for a body of mathematical knowledge that is specialized to teaching.

The lines between our four types of knowledge can be subtle. For instance, recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error may be either specialized content knowledge (SCK) or knowledge of content and students (KCS) depending on whether a teacher draws predominantly from her knowledge of mathematics and her ability to carry out a kind of mathematical analysis or instead draws from experience with students and familiarity with common student errors. Deciding how best to remediate the error may require knowledge of content and teaching (KCT).

Building a Map of Usable Professional Knowledge of Subject Matter

Several issues about our proposed categories are worth addressing — their relationship to pedagogical content knowledge, the "special" nature of specialized content knowledge, our use of teaching as a basis for defining the domains, and an appraisal of what they gain us.

From our definitions and examples it should be evident that this work may be understood as elaborating, not repudiating, the construct of pedagogical content knowledge. For instance, the two amalgam domains — knowledge of content and students and knowledge of content and teaching — coincide with the two central dimensions of pedagogical content knowledge identified by Shulman (1986):

- "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons," and;

- “the ways of representing and formulating the subject that make it comprehensible to others.”

However, we also see our work as developing in more detail the fundamentals of *subject matter knowledge for teaching* — defining, refining, and measuring Shulman’s categories.

We have been most struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is *not* intertwined with knowledge of pedagogy, students, curriculum, or other domains relevant to teaching. What distinguishes this sort of mathematical knowledge from other knowledge of mathematics is that it is subject matter knowledge needed by teachers for specific tasks of teaching, such as those in Figure 3. Still, it is clearly subject matter knowledge. These are tasks of teaching that depend on mathematical knowledge and that, significantly, have a life that is relatively independent of knowledge of students or of teaching. These are tasks that require knowing how knowledge is generated and structured in the discipline and much more. They are also knowledge and skills not typically taught to teachers in the course of their formal mathematical preparation.

Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime, or how and why the long division algorithm works? Teachers must know these sorts of things, and engage in these mathematical practices themselves in order to teach and they must also learn to teach them *to* students. Explicit knowledge and skill in these areas is vital for teaching.

Our current empirical results, based on factor analyses of data from instruments we have developed to measure mathematical knowledge for teaching, suggest that it is likely that content knowledge for teaching is multidimensional (Hill, Ball, & Schilling, 2004) and it positively affects student learning (Hill, Rowan, & Ball, 2005). Whether these categories, as we propose them here, are the “right” ones is not most important. Likely they are not — they likely need further testing and refinement.

Given all of this, a question that others regularly ask is: “Why are any new categories needed? Why not just refer to all of this as “pedagogical content knowledge”?” Three reasons capture our current thinking about the usefulness of refining the conceptual map of the content knowledge needed by teachers. First, in studying the relationships between teachers’ content knowledge and their students’ achievement, it would be useful to be able to ascertain whether there are aspects of teachers’ content knowledge that may predict students’ achievement more than others. If, for instance, teachers’ common content knowledge is the greatest predictor of students’ achievement, this might direct our efforts in ways different than if knowledge of content and students has the largest effect. Second, if this were the case, it could be useful to study whether and how different approaches to teacher development make different impacts on particular aspects of teachers’ content knowledge. Third, and closely related, a clearer sense of the categories of content knowledge for teaching might inform the design of support materials for teachers, as well as teacher education and professional development.

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