On Ideal Closure Operators of $M$-sets

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Abstract

Closure operators have been used in Algebra and Topology. Well-known examples are the closure of a subspace of a topological space, or the normal closure of a subgroup of a group.

Category theory provides a variety of notions which expand on the lattice theoretic concept of closure operator which leads to a never ending stream of examples and applications in all areas of mathematics.

Actions of a monoid have always been a useful tool to study the mathematical structures, and recently have captured the interest of some computer scientists. For this reason and because of its closed relation to the category of sets, one can take the topos $\text{MSet}$ of $M$-sets, for a monoid $M$, as the "universe of discourse" to study mathematical notions in it.

Here, using the general definition of a closure operator on a category, we introduce and investigate some properties, such as idempotency, additivity, and hereditariness for ideal closure operators of the category of $M$-sets.

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1 Introduction

One of the very useful topoi in many branches of mathematics as well as in computer sciences is the category $\text{MSet}$, of sets with an action of a monoid $M$ on them. Here, we very briefly introduce this category. For more information see [8, 9].

1.1. Definition For a monoid $M$ with $e$ as its identity, a (left) $M$-set is a set $X$ together with a function $\lambda : M \times X \to X$, called the action of $M$ (or the $M$-action) on $X$, such that for $x \in X$ and $m, n \in M$ (denoting $\lambda(m, x)$ by $mx$)

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In fact, an $M$-set is an algebra $(X, (\lambda_m)_{m \in M})$ where each $\lambda_m : X \to X$ is a unary operation on $X$ such that $\lambda_e = id_X$, $\lambda_m \circ \lambda_n = \lambda_{mn}$, for each $m, n \in M$.

A morphism $f : X \to Y$ between $M$-sets $X, Y$ is an equivariant map; i.e. for $x \in X$, $m \in M$,

$$f(mx) = mf(x).$$

The category of all $M$-sets and equivariant maps between them is denoted by $\text{MSet}$.

1.2. Remark Limits in $\text{MSet}$ are calculated pointwise. In particular, the terminal object of $\text{MSet}$ is the singleton $\{0\}$, with the obvious $M$-action. Also, colimits in $\text{MSet}$ exist and are calculated as in $\text{Set}$ with a natural action of $M$ on them. In particular, $\emptyset$ with the empty action of $M$ on it is the initial object of $\text{MSet}$. Moreover, monomorphisms in $\text{MSet}$ are exactly one-one equivariant maps, and epimorphisms are precisely onto equivariant maps.

It is easily proved that, considering the monoid $M$ as a category $\text{M}$ with one object, the category $\text{MSet}$ is isomorphic to the functor category $\text{Set}^\text{M}$. Hence, using the well-known fact that every functor category $\text{Set}^\text{C}$, for any small category $\text{C}$, is a topos, we get that the category $\text{MSet}$ is a topos.

Denoting the lattice of all sub $M$-sets of an $M$-set $X$ by $\text{Sub}(X)$, following [5], for the general definition of closure operator on a category, we get

1.3. Definition A family $C = (C_X)_{X \in \text{MSet}}$, with $C_X : \text{Sub}(X) \to \text{Sub}(X)$, taking $Y \leq X$ to $C_X(Y)$, is called a Closure Operator on $\text{MSet}$ if it satisfies the following laws:

$(e_1)$ (Extension) $Y \leq C_X(Y),$

$(e_2)$ (Monotonicity) $Y_1 \leq Y_2$ implies $C_X(Y_1) \leq C_X(Y_2),$

$(e_3)$ (Continuity) $f(C_X(Y)) \leq C_Z(f(Y))$, for all morphisms $f : X \to Z$.

Now, one has the usual notions related to the notion of closure as follows.

1.4. Definition Let $Y \leq X$ be in $\text{MSet}$. We say that $Y$ is $C$-closed in $X$ if $C_X(Y) = Y$, and it is $C$-dense in $X$ if $C_X(Y) = X$. We take $\mathcal{M}_c$ to be the set of all $C$-closed, and $\mathcal{E}_c$ be the set of all $C$-dense monomorphisms.

Now we state some of the properties that a closure operator may have.

1.5. Definition A closure operator $C$ on $\text{MSet}$ is called idempotent (ID), if $C_X(C_X(Y)) = C_X(Y)$, for all $X$ and $Y \leq X$. And, $C$ is said to be weakly...
hereditary (WH), if for every $X$, every $Y \leq X$ is $C$-dense in $C_X(Y)$.

In general, one says that a category $C$ has $(\mathcal{E}, \mathcal{M})$ factorization if every morphism $f$ can be written as $f = me$ for some $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Now, following [7], we have

1.6. Theorem A closure operator $C$ on $\textbf{MSet}$ is (ID) and (WH) iff $\textbf{MSet}$ has $(\mathcal{E}^c, \mathcal{M}^c)$-factorization. □

1.7. Definition A closure operator $C$ on $\textbf{MSet}$ is called hereditary (HD), if for every $X$ and $Y_1 \leq Y_2 \leq X$ in $\textbf{MSet}$, $C_{Y_2}(Y_1) = C_X(Y_1) \cap Y_2$.

Note that (HD) implies (WH).

1.8. Definition A closure operator $C$ on $\textbf{MSet}$ is said to be

(a) grounded (GR), if for every object $X$, $C_X(\emptyset) = \emptyset$.
(b) additive (AD), if for every $X$, $C_X(Y \cup Z) = C_X(Y) \cup C_X(Z)$.
(c) productive (PR), if for every family of subobjects $Y_i$ of $X_i$ in $\textbf{MSet}$, taking $Y = \prod_i Y_i$ and $X = \prod_i X_i$, $C_X(Y) = \prod_i C_X(Y_i)$.
(d) discrete, if $C_X(Y) = Y$, for every $X$ and every $Y \leq X$ in $\textbf{MSet}$.
(e) trivial, if $C_X(Y) = X$, for every $X$ and every $Y \leq X$ in $\textbf{MSet}$.

2 Ideal Closure Operators on $\textbf{MSet}$

In this section we study a particular set of closure operators in the category $\textbf{MSet}$.

Recall that a (right) ideal of a monoid $M$ is a subset $I$ of $M$ which is closed under (right) multiplication; that is for $a \in I$ and $s \in M$, $(as) \in I$ as, $sa \in I$. Denoting the set of all ideals of $M$ by $\text{Id}(M)$, it is clear that $\text{Id}(M)$ is a monoid with binary operation given by $IJ = \{ab : a \in I, b \in J\}$ and identity $M$. It is also a lattice with intersection and union as meet and join.

2.1. Definition For any (right) ideal $I$ of $M$, the $I$-closure $C^I = (C^I_X)_{X \in \text{MSet}}$ on $\textbf{MSet}$ is defined as

$$C^I_X(Y) = \{x \in X : Ix \subseteq Y\}$$

for any subalgebra $Y$ of an $M$-set $X$.

It is easily shown that $C^I$ is a closure operator in $\textbf{MSet}$. Because of the following theorem one may call $C^I$ the residual closure with respect to $I$ (see [3], p. 325).
2.2. Theorem  For each (right) ideal $I$ of $M$ and a subalgebra $Y$ of an $M$-set $X$, $C^I_X(Y)$ is the largest sub $M$-set of $X$ with the property $IC^I_X(Y) \subseteq Y$.

Proof: To show that $C^I_X(Y)$ is a sub $M$-set of $X$, let $x \in C^I_X(Y)$ and $m \in M$. Since $Ix \subseteq Y$ and $Im \subseteq I$, $I(mx) \subseteq Ix \subseteq Y$. Thus, $mx \in C^I_X(Y)$.

Now, it is easily shown that $C^I_X(Y)$ is the largest sub $M$-set of $X$ with the required property. □

The following theorem shows that the concept of residual closure operators relative to the right ideals of $M$ and that of relative to ideals of $M$ are the same.

First, note the following lemma.

2.3. Lemma  For a right ideal $I$ of $M$, $C^I = C^{MI}$.

Proof: This is because, for any sub $M$-set $Y$ of an $M$-set $X$ and $x \in X$, $Ix \subseteq Y$ iff $MIx \subseteq MY \subseteq Y$. □

2.4. Theorem  The set $C^r = \{C^I : I$ is a right ideal of $M\}$ and the set $C = \{C^I : I$ is an ideal of $M\}$ are equal.

Proof: Since, for any right ideal $I$ of $M$, $MI$ is an ideal of $M$, the result is immediate from the above lemma. □

The following theorem is easily proved using the definition of $C^I$.

2.5. Theorem  For each (right) ideal $I$ of $M$, $C^I$ is hereditary, weakly hereditary, productive, and also grounded, if $I \neq \emptyset$.

Recall that an element $a$ in a lattice $L$ is said to be join prime if whenever $a \leq b \lor c$ then $a \leq b$ or $a \leq c$. Now we have

2.6. Theorem  For an ideal $I$ of $M$, $C^I$ is additive iff for every $x$ in an $M$-set $X$, $Ix$ is join prime in the lattice $Sub(X)$.

Proof: Let $Y$ and $Z$ be sub $M$-sets of an $M$-set $X$ and $x \in C^I_X(Y \cup Z)$. Then, $Ix \subseteq Y \cup Z$ and hence, $Ix$ being $\lor$-prime, $Ix \subseteq Y$ or $Ix \subseteq Z$. Thus, $x \in C^I_X(Y) \cup C^I_X(Z)$. This shows that each $C^I_X$, and hence $C^I$, is additive.

Conversely, let $C^I$, and hence each $C^I_X$, be additive. Let $x \in X$ and $Ix \subseteq Y \cup Z$, where $Y$ and $Z$ are sub $M$-sets of $X$. Then, by monotonicity and additivity,

$$C^I_X(Ix) \subseteq C^I_X(Y \cup Z) = C^I_X(Y) \cup C^I_X(Z).$$

Now, since $x \in C^I_X(Ix)$, $x \in C^I_X(Y)$ or $x \in C^I_X(Z)$. Thus, $Ix \subseteq Y$ or $Ix \subseteq Z$, proving that $Ix$ is join prime in $Sub(X)$. □
Since $Ie = I$, we have the following.

**2.7. Corollary** For an ideal $I$ of $M$, if $C^I$ is additive then $I$ is join prime in the lattice $Id(M)$, of ideals of $M$. □

**2.8. Corollary** Let $I$ be an ideal of $M$. If $I$ has a right identity element $e_0$, then $C^I$ is additive.

**Proof:** Note that, in this case, for any sub $M$-set $X$ of an $M$-set $Y$, $x \in C^I_X(Y)$ iff $e_0x \in Y$. This is because, $ix = (ie_0)x = i(e_0x)$, for each $i \in I$.

Now, if $Y$ and $Z$ are sub $M$-sets of $X$ and $Ix \subseteq Y \cup Z$, for $x \in X$, then one can easily see that $Ix \subseteq Y$ or $Ix \subseteq Z$, depending on $e_0x \in Y$ or $e_0x \in Z$, respectively. □

**2.9. Lemma** For any (right) ideal $I$ of $M$ and any sub $M$-set $K$ of the $M$-set $M$, $C^I_M(K) = M$ iff $Ie \subseteq K$.

**Proof:** Note that, $C^I_M(K) = M$ iff $e \in C^I_M(K)$, iff $I = Ie \subseteq Y$. □

**2.10. Theorem** For any (right) ideal $I$ of $M$, $C^I$ is idempotent iff $I^2 = I ((MI)^2 = MI)$.

**Proof:** Let $C^I$ be idempotent. Since $x \in C^I_M(C^I_M(I^2))$ iff $I^2x \subseteq I^2$, we get that $e \in C^I_M(C^I_M(I^2)) = C^I_M(I^2)$. Thus, by the above lemma, $I \subseteq I^2$, and hence $I^2 = I$.

Conversely, let $I^2 = I$ and $Y$ be a sub $M$-set of an $M$-set $X$. Since, for $x \in X$, $I^2x \subseteq Y$ iff $Ix \subseteq Y$, we get the result. □

## 3 Lattice of Ideal Closure Operators

Let $C = \{C^I : I$ is an ideal of $M\}$. Following [5], for closure operators $C^I$ and $C^J$ in $C$, we have the composition of closure operators $C^I \circ C^J$ given by $(C^I \circ C^J)_X(Y) = C^I_X(C^J_X(Y))$, for $M$-set $X$ and $Y \subseteq X$. It is easily shown that, the composition $C^I \circ C^J$ is equal to $C^{IJ}$, and $C$ is a monoid, under $\circ$, with $C^M$ as its identity element. Moreover, $C^\emptyset$ is the zero (absorbing) element of $C$.

**3.1. Lemma** For every ideals $I$ and $J$ of $M$, $C^I = C^J$ iff $I = J$.

**Proof:** Let $C^I = C^J$. Then, by Lemma 2.9, $C^I(I) = M = C^J(I)$. Using Lemma 2.9 again, we get that $J \subseteq I$. Similarly, $I \subseteq J$ and hence $I = J$. □
As a corollary of the above lemma and Lemma 2.3, we get the following result.

3.2. Corollary For any two right ideals $I$ and $J$ of $M$, $C^I = C^J$ iff $MI = MJ$. □

Now, once again, one gets the following theorem.

3.3. Theorem Let $I$ be an ideal of $M$. Then, $C^I$ is idempotent iff $I$ is idempotent, that is, $I^2 = I$.

Proof: Note that $C^I \circ C^I = C^{I^2}$, and by Theorem 2.10, $C^{I^2} = C^I$ iff $I^2 = I$. □

Following [5], $C^I \leq C^J$ if, for any sub $M$-set $Y$ of an $M$-set $X$, $C^I_X(Y) \subseteq C^J_X(Y)$.

Clearly, for ideals $I$ and $J$ of $M$, $C^I \leq C^J$ iff $J \subseteq I$. Thus, $(C, \leq)$ is a partially ordered monoid. This is because, for ideals $I, J, K$ of $M$, if $J \subseteq I$ then $JK \subseteq IK$ and $KJ \subseteq KI$, and hence $C^K \circ C^I \leq C^K \circ C^J$ and $C^I \circ C^K \leq C^J \circ C^K$. Moreover, $C^M$ and $C^\emptyset$ are the smallest and the largest elements of $C$, respectively.

Note also that $C$ is a lattice with $C^I \lor C^J = C^{I \lor J}$ and $C^I \land C^J = C^{I \land J}$. In fact, we have the following.

3.4. Theorem $(C, \lor, \land)$ is a (dual) $l$-monoid.

Proof: For any ideals $I, J, K$ of $M$, we have

$$C^I \circ (C^J \land C^K) = C^{I \circ (C^J \land C^K)} = C^{(C^I \lor C^K)J} = C^{(C^I \circ C^K)J} = C^{(C^J \land C^K)} \land (C^I \circ C^K)$$

Similarly, $(C^J \land C^K) \circ C^I = (C^J \circ C^K) \land (C^I \circ C^K)$. □

References


