TATE COHOMOLOGY IN AXIOMATIC STABLE HOMOTOPY THEORY.

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Contents

1. Introduction

The purpose of the present note is to show how the axiomatic approach to Tate cohomology of [18, Appendix B] can be implemented in the axiomatic stable homotopy theory of Hovey-Palmieri-Strickland [32]. Much of the work consists of collecting known results in a single language and a single framework. The very effortlessness of the process is an effective advertisement for the language, and a call for further investigation of other instances. The main point is to recognize and compare incarnations of the same phenomenon in different contexts: the splitting and duality phenomena described in Sections 9 and 13 are particularly notable. More practically, Theorem 11.1 is new, and Theorem 12.1 extends results of [19].

A *stable homotopy category* [32, 1.1.4] is a triangulated category $\mathcal{C}$ with arbitrary coproducts, and so that all cohomology theories are representable. It is also required to have a compatible symmetric monoidal structure with unit S and a set $\mathcal{G}$ of strongly dualizable objects generating all of $\mathcal{C}$ using triangles, coproducts and retracts. If in addition the objects of $\mathcal{G}$ are small, the stable homotopy category is said to be *algebraic.*

We shall illustrate our constructions in several contexts specified in greater detail later. The following list gives the context followed by an associated stable homotopy category. Each of these admits a number of variations.

- Equivariant topology: the homotopy category of G-spectra (Section 4).
- Commutative algebra: the derived category of a commutative ring R (Section 5).
- Brave new commutative algebra: the homotopy category of modules over a highly structured commutative ring spectrum $\mathbf{R}$ (Section 6).
- Representation theory: the derived category of the group ring kG of a finite group G (Section 7).
- The bordism approach to stable homotopy theory: various chromatic categories (Section 8).

The paper is in three parts.

Part I: General formalities (Sections 2 and 3). In Section 2 we summarize necessary definitions and give the Tate construction in a stable homotopy category associated to a smashing localization. We establish the fundamental formal properties that make it reasonable to call this a Tate construction. In Section 3 we recall from [39] that finite localizations are smashing and hence give rise to Tate theories: the minor novelty is to emphasize the view that this is an Adams projective resolution in the sense of [1].

Part II: Examples (Sections 4 to 8). We describe the above contexts in more detail, and consider the construction in each one, identifying it in more familiar terms.

Part III: Special properties (Sections 9 to 13): The final sections give some more subtle results about the construction which require additional hypotheses. In Section 9, we discuss dichotomy results stating that the Tate construction is either periodic or split. We then turn to methods of calculation. The first is the familiar calculation using associative algebra, generalizing the use of group cohomology in descent spectral sequences (one uses homological algebra over the endomorphism ring of the basic building block). We describe this in Section 11, and give a new example in the case of equivariant topology with a compact Lie group of equivariance. This method applies fairly generally, provided the stable homotopy category arises from an underlying Quillen model category. Less familiar is the calculation in terms of commutative algebra. This arises when the (commutative) endomorphism ring of the unit object has a certain duality property (it is 'homotopically Gorenstein'). This is quite exceptional, but it applies in a surprisingly large number of familiar examples: in the cohomology of groups [19, 9, 8], in equivariant cohomology theories [17, 25], and in chromatic stable homotopy theory (Gross-Hopkins duality). Its occurrence in commutative algebra is investigated in [21], and shown to be very special.

2. Axiomatic Tate cohomology in a stable homotopy category.

In this section we describe the Tate construction. Since it depends on a suitable Bousfield localization, we briefly recall the terminology in a suitable form (see [32, Section 3] for more detail). We consider a functor $L : \mathcal{C} \longrightarrow \mathcal{C}$ on the stable homotopy category $\mathcal{C}$. The *acyclics* of L are the objects X so that $LX \simeq *$. The functor L is a Bousfield localization if it is exact, equipped with a natural tranformation $X \longrightarrow LX$, idempotent, and its class of acyclics is an ideal.

A Bousfield localization L is determined by its class $\mathcal{D}$ of acyclics as follows: Y is *L-local* if and only if $[D, Y]_* = 0$ for all D in $\mathcal{D}$, and a map $X \longrightarrow Y$ is the Bousfield localization if and only if Y is local and the fibre lies in $\mathcal{D}$. The usual notation for the localization triangle is $CX \longrightarrow X \longrightarrow LX$. Furthermore, any such class of acyclics is a *localizing ideal* (i.e. it is closed under completing triangles, sums and smashing with an arbitrary object).

A localization is said to be *smashing* if the natural map $X \wedge LS \longrightarrow LX$ is an equivalence for all X. It is equivalent to require either that L commutes with arbitrary sums, or that the class $L\mathcal{C}$ of L-local objects is a localizing ideal [32, 3.3.2].

We shall define a Tate construction associated to any smashing localization.

Notation 2.1. *(General context)*

- $\mathcal{C}$: a stable homotopy category
- $\mathcal{G}$: a set of generators for $\mathcal{C}$
- $\mathcal{D}$: the localizing ideal of acyclics for a smashing Bousfield localization $(\cdot)[\mathcal{D}^{-1}]$.
- $\mathcal{C}[\mathcal{D}^{-1}]$: the localizing ideal of $[\mathcal{D}^{-1}]$-local objects.

The notation $L_{\mathcal{D}}$ is often used for $(\cdot)[\mathcal{D}^{-1}]$; the present notation better reflects the character of a smashing localization, and corresponds to that in [24]. The idea is that we should think of $X[\mathcal{D}^{-1}]$ as a localization *away from* $\mathcal{D}$. More precisely the archetype is localization away from a closed subset in algebraic geometry. The notation comes from the case when the closed subset is defined by the vanishing of a single function f. In this very special case, the localization is realized by inverting the multiplicatively closed set $\{1, f, f^2, \dots\}$ in the sense of commutative algebra. We therefore use the corresponding 'sections with support' notation for the fibre of this localization:

$$\Gamma_{\mathcal{D}}(X) \longrightarrow X \longrightarrow X[\mathcal{D}^{-1}].$$

We can use this to define an associated completion.

Lemma 2.2. *The natural transformation $X \longrightarrow F(\Gamma_{\mathcal{D}}(S), X)$ is Bousfield completion whose class of acyclics is the class of $[\mathcal{D}^{-1}]$-local objects.*

Proof. First we must show that if E is $[\mathcal{D}^{-1}]$-local, then $[E, F(\Gamma_{\mathcal{D}}(S), X)]_* = 0$. By [32, 3.1.8], $S[\mathcal{D}^{-1}]$ is a ring object in $\mathcal{C}$ and $E = E[\mathcal{D}^{-1}]$ is a $S[\mathcal{D}^{-1}]$-module. Hence $E \wedge \Gamma_{\mathcal{D}}(S)$ is a retract of $E \wedge S[\mathcal{D}^{-1}] \wedge \Gamma_{\mathcal{D}}(S)$; since $[\mathcal{D}^{-1}]$ is idempotent and smashing, $S[\mathcal{D}^{-1}] \wedge \Gamma_{\mathcal{D}}(S) \simeq *$.

Secondly we must show that the fibre, $F(S[\mathcal{D}^{-1}], X)$ is $[\mathcal{D}^{-1}]$-local. However if D lies in $\mathcal{D}$ then $D \wedge S[\mathcal{D}^{-1}] \simeq D[\mathcal{D}^{-1}] \simeq *$. □

We write $X^{\wedge}_{\mathcal{D}} := F(\Gamma_{\mathcal{D}}(S), X)$ for this Bousfield completion, and also introduce the following notation for its fibre:

$$\Delta_{\mathcal{D}}(X) \longrightarrow X \longrightarrow X^{\wedge}_{\mathcal{D}}.$$

We now define the $\mathcal{D}$-Tate construction by

$$t_{\mathcal{D}}(X) = X^{\wedge}_{\mathcal{D}}[\mathcal{D}^{-1}].$$

This gives the diagram

$$\begin{array}{ccccc} \Gamma_{\mathcal{D}}(X) & \longrightarrow & X & \longrightarrow & X[\mathcal{D}^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_{\mathcal{D}}(X^{\wedge}_{\mathcal{D}}) & \longrightarrow & X^{\wedge}_{\mathcal{D}} & \longrightarrow & X^{\wedge}_{\mathcal{D}}[\mathcal{D}^{-1}] = t_{\mathcal{D}}(X) \end{array}$$

Lemma 2.3. *The map $\Gamma_{\mathcal{D}}(X) \longrightarrow \Gamma_{\mathcal{D}}(X^{\wedge}_{\mathcal{D}})$ is an equivalence.*

Proof. We need only remark that $\Gamma_{\mathcal{D}}(\Delta_{\mathcal{D}}(X)) \simeq *$; however by definition $\Delta_{\mathcal{D}}(X)$ lies in the class of $[\mathcal{D}^{-1}]$-local objects. □

Corollary 2.4. *(Hasse Principle) The diagram*

$$\begin{array}{ccc} X & \longrightarrow & X[\mathcal{D}^{-1}] \\ \downarrow & & \downarrow \\ X^\wedge_{\mathcal{D}} & \longrightarrow & t_{\mathcal{D}}(X) \end{array}$$

is a homotopy pullback square. □

Corollary 2.5. *(Warwick Duality* [18]*) There is an equivalence*

$$t_{\mathcal{D}}(X) = X^\wedge_{\mathcal{D}}[\mathcal{D}^{-1}] \simeq \Delta_{\mathcal{D}}(\Sigma\Gamma_{\mathcal{D}}(X)).$$

Proof. This is a composite of three equivalences.

$$X^\wedge_{\mathcal{D}}[\mathcal{D}^{-1}] \longleftarrow \Delta_{\mathcal{D}}(X^\wedge_{\mathcal{D}}[\mathcal{D}^{-1}]) \longrightarrow \Delta_{\mathcal{D}}(\Sigma\Gamma_{\mathcal{D}}(X^\wedge_{\mathcal{D}})) \longrightarrow \Delta_{\mathcal{D}}(\Sigma\Gamma_{\mathcal{D}}(X))$$

The first is an equivalence since $(\cdot)[\mathcal{D}^{-1}]^\wedge_{\mathcal{D}} \simeq *$ (the class $\mathcal{E}$ of acyclics for $(\cdot)^\wedge_{\mathcal{A}}$ consists of $[\mathcal{D}^{-1}]$-local objects) so that $X^\wedge_{\mathcal{D}}[\mathcal{D}^{-1}]^\wedge_{\mathcal{D}} \simeq *$. The second is an equivalence since $\Delta_{\mathcal{D}}(X^\wedge_{\mathcal{D}}) \simeq *$ (defining property of $\Delta_{\mathcal{D}}((\cdot))$ together with idempotence of $(\cdot)^\wedge_{\mathcal{D}}$). The third is an equivalence since $\Gamma_{\mathcal{D}}(\Delta_{\mathcal{D}}((\cdot))) \simeq *$ by 2.3 so that $\Delta_{\mathcal{D}}(\Sigma\Gamma_{\mathcal{D}}(\Delta_{\mathcal{D}}(X))) \simeq *$. □

This shows that the cohomology as well as the homology only depends on the localization away from $\mathcal{D}$. More precisely, the definition $t_{\mathcal{D}}(X) = F(\Gamma_{\mathcal{D}}(S), X)[\mathcal{D}^{-1}]$ shows that $T \wedge t_{\mathcal{D}}(X)$ only depends on the localization $T[\mathcal{D}^{-1}]$. The second avatar $t_{\mathcal{D}}(X) \simeq F(S[\mathcal{D}^{-1}], \Sigma\Gamma_{\mathcal{D}}(X))$ gives

$$[T, t_{\mathcal{D}}(X)]_* = [T \wedge S[\mathcal{D}^{-1}], \Sigma\Gamma_{\mathcal{D}}(X)]_* = [T[\mathcal{D}^{-1}], \Sigma\Gamma_{\mathcal{D}}(X)]_*,$$

which again only depends on $T[\mathcal{D}^{-1}]$.

Remark 2.6. The definition of the Tate construction we have given is at a natural level of generality. One might be tempted to consider $L_{\mathcal{D}}L_{\mathcal{E}}X$ for arbitrary $\mathcal{D}$ and $\mathcal{E}$. However, if one wants Warwick Duality, one requires (i) $L_{\mathcal{E}}L_{\mathcal{D}}X \simeq *$, so that $\mathcal{E} \supseteq L_{\mathcal{D}}\mathcal{C}$ and (ii) $C_{\mathcal{D}}C_{\mathcal{E}}X \simeq *$, so that $C_{\mathcal{E}}X$ is $L_{\mathcal{D}}$-local, and $\mathcal{E} \subseteq L_{\mathcal{D}}\mathcal{C}$. Thus we require $\mathcal{E} = L_{\mathcal{D}}\mathcal{C}$, and this must be a localizing ideal. Thus $L_{\mathcal{D}}$ must be smashing, and determines $\mathcal{E}$.

3. Finite localizations.

In this section we describe one very fruitful source of smashing localizations. This is explicit in Section 3.3 of [32], and especially Theorem 3.3.5. It generalizes the finite localization of Mahowald–Sadofsky and Miller [36, 39]. We recall the construction for future reference, and emphasize the connection with Adams projective resolutions.

Recall that a full subcategory is *thick* if it is closed under completing triangles and taking retracts. The piece of data we need is a $\mathcal{G}$-ideal $\mathcal{A}$ of small objects (ie a thick subcategory of small objects, closed under smashing with elements of $\mathcal{G}$). If $\mathcal{C}$ is not algebraic, we must suppose in addition that $\mathcal{A}$ is essentially small, consists of strongly dualizable objects and is closed under Spanier Whitehead duality; if $\mathcal{C}$ is algebraic these conditions are automatic. In practice we will specify $\mathcal{A}$ by giving a set $\mathcal{T}$ of small generators: $\mathcal{A} = \mathcal{G}\text{-ideal}(\mathcal{T})$. We then need to form the localizing ideal $\mathcal{D} = \text{locid}(\mathcal{A})$ generated by $\mathcal{A}$: this is the smallest thick subcategory containing $\mathcal{A}$ which is closed under arbitrary sums and smashing with arbitrary elements of $\mathcal{C}$.

Context 3.1. (for a finite localization)

- $\mathcal{C}$: a stable homotopy category
- $\mathcal{G}$: a set of generators for $\mathcal{C}$
- $\mathcal{T}$: a set of small objects of $\mathcal{C}$
- $\mathcal{A} = \mathcal{G}\text{-ideal}(\mathcal{T})$
- $\mathcal{D} = \mathrm{locid}(\mathcal{T}) = \mathrm{locid}(\mathcal{A})$

In these circumstances, we write write $\mathcal{A}$ or $\mathcal{T}$ in place of $\mathcal{D}$ in the notation, so that $t_{\mathcal{A}}(X) = t_{\mathcal{T}}(X) = t_{\mathcal{D}}(X)$ and so forth.

Miller has shown that there is a smashing localization functor $(\cdot)[\mathcal{A}^{-1}]$ whose acyclics are precisely $\mathcal{D}$, and whose small acyclics are precisely $\mathcal{A}$; this is known as a *finite localization* and the notation $L^f_{\mathcal{A}}$ is used in [32]. The construction is described in 3.3 below. The associated functor $(\cdot)^{\wedge}_{\mathcal{A}}$ whose acyclics are the objects $X[\mathcal{A}^{-1}]$ is denoted by $L_{\mathcal{A}}$ in [32].

There is a convenient lemma for showing a set of elements in a localizing subcategory is a generating set. It would be more traditional to view it as a convergence theorem for a projective resolution in the sense of Adams [1].

Proposition 3.2. *If $\mathcal{T} \subseteq \mathcal{D}$ is a set of objects then $\mathcal{D} = \mathrm{locid}(\mathcal{T})$ provided one of the two following conditions holds.*
(i) $\mathcal{T}$ is a set of small objects and detects triviality in $\mathcal{D}$, in the sense that if X is in $\mathcal{D}$ then $[T, X]_* = 0$ for all T in $\mathcal{T}$ implies $X \simeq *$.
(ii) The objects of $\mathcal{G}$ are small, and for any $X \in \mathcal{D}, S' \in \mathcal{G}$ and any $x \in [S', X]_*$ there is a map $t : \Sigma^n T \longrightarrow X$ with $x \in \mathrm{im}(t_* : [S', \Sigma^n T]_* \longrightarrow [S', X]_*)$ and T in $\mathcal{T}$.

Proof. We need to prove that if X is an arbitrary object of $\mathcal{D}$, we may form X from copies of elements of $\mathcal{T}$ using sums and completion of triangles. We give the proof assuming that Condition (ii) holds; the proof when Condition (i) holds is similar except that $[S', \cdot]_*$ for $S' \in \mathcal{G}$ is replaced by $[T, \cdot]_*$ for $T \in \mathcal{T}$.

By hypothesis we may form a projective resolution in the sense of Adams:

$$\begin{array}{ccccccccc} X & = & X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \ldots \\ & & \uparrow{\scriptstyle t_0} & & \uparrow{\scriptstyle t_1} & & \uparrow{\scriptstyle t_2} & & \\ & & T_0 & & T_1 & & T_2 & & \end{array}$$

Thus each T_i is a sum of suspensions of elements of $\mathcal{T}$, each t_i is surjective in $[S', \cdot]_*$ for all $S' \in \mathcal{G}$ and X_{i+1} is formed as the cofibre of $t_i : T_i \longrightarrow X_i$. Note that $X_\infty = \mathrm{tel}_n X_n$ has trivial $[S', \cdot]_*$ for all $S' \in \mathcal{G}$ since i_s is zero in $[S', \cdot]_*$ by construction; thus $X_\infty \simeq *$ since $\mathcal{G}$ gives a set of generators. Defining X^i as the fibre of $X \longrightarrow X_i$ we find that X^i is constructed from sums of suspensions of elements of $\mathcal{T}$ by a finite number of cofibre sequences. Passing to direct limits, we obtain a cofibre sequence $X^\infty \longrightarrow X \longrightarrow X_\infty$, so that $X^\infty \simeq X$. □

Remark 3.3. Note that the argument essentially gives the construction of a finite localization. Take a set $\mathcal{T}$ of small generators of the $\mathcal{G}$-ideal $\mathcal{A}$ and the localizing ideal $\mathcal{D}$, and ensure it is closed under duality. We now form a projective resolution as in the proof of 3.2, but without assuming that X lies in $\mathcal{D}$. Ensure t_i is surjective in $[T, \cdot]_*$ for each i. Then the triangle $X^\infty \longrightarrow X \longrightarrow X_\infty$ has X^∞ in $\mathcal{D}$ by construction, and $[T, X^\infty]_* = 0$ for all T in $\mathcal{T}$.

This completes the discussion of formalities. In the rest of the paper we want to discuss a number of examples from this point of view, and show how comparisons between the examples give rise to means of calculation.

4. The category of G-spectra

In this section we consider the category $\mathcal{C} = G$-**spectra** of G-spectra for a compact Lie group G, and localizations associated to a family $\mathcal{F}$ of subgroups. We recover the constructions of [23]; indeed these constructions motivated investigation of its other manifestations.

Thus we suppose $\mathcal{F}$ is closed under conjugation and passage to subgroups, and we let $\mathcal{T} = \{G/H_+ \mid H \in \mathcal{F}\}$. Thus $\mathcal{A}$ is the class of retracts of finite $\mathcal{F}$-spectra, and $\mathcal{D}$ is the class of all $\mathcal{F}$-spectra. We recall that in the homotopy category of G-spectra, the class of $\mathcal{F}$-spectra can be described in three ways, as is implicit in [34].

Lemma 4.1. *The following three classes of G-spectra are equal, and called $\mathcal{F}$-spectra.*
(i) G-spectra formed from spheres $G/H_+ \wedge S^n$ with $H \in \mathcal{F}$
(ii) G-spectra X so that the natural map $E\mathcal{F}_+ \wedge X \stackrel{\simeq}{\longrightarrow} X$ is an equivalence and
(iii) G-spectra X so that the geometric fixed point spectra $\Phi^H X$ are non-equivariantly contractible for $H \in \mathcal{F}$.

Proof: The equality of Classes (i) and (ii) is straightforward.

Since Φ^H commutes with smash products [34, II.9.12], and it agrees with H-fixed point spaces on suspension spectra [34, II.9.9], it follows that $E\mathcal{F}_+ \wedge X$ lies in the third class, so Class (ii) is contained in Class (iii). Suppose then that X is in Class (iii); we must show it is also in Class (ii). By hypothesis, the map $E\mathcal{F}_+ \wedge X \longrightarrow X$ has the property that applying Φ^H gives a non-equivariant equivalence for all H. It remains to observe that geometric fixed points detect weak equivalences. This is well known, but I do not know a good reference: it follows from the fact that Lewis-May fixed points tautologically detect weak equivalences, by an induction on isotropy groups. The basis is the relation between geometric and Lewis-May fixed points [34, II.9.8]: for any H-spectrum X, $\Phi^H X \simeq (\tilde{E}\mathcal{P} \wedge X)^H$ where $\mathcal{P}$ is the family of proper subgroups of H. □

From the equality of Classes (i) and (ii) $E\mathcal{F}_+ \wedge X$ lies in $\mathcal{D}$, and from the fact that $\tilde{E}\mathcal{F}$ is $\mathcal{F}$-contractible we see that $X \longrightarrow \tilde{E}\mathcal{F} \wedge X$ is localization away from $\mathcal{D}$. Hence $\Gamma_{\mathcal{F}} S = E\mathcal{F}_+$ and $S[\mathcal{F}^{-1}] = \tilde{E}\mathcal{F}$. Now the equality of Classes (i) and (ii) can be recognized as the statement that localization away from the class of $\mathcal{F}$-spectra is smashing. It follows that in this case Diagram 2 is the diagram

$$\begin{array}{ccccc} E\mathcal{F}_+ \wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}\mathcal{F} \wedge X \\ \simeq\downarrow & & \downarrow & & \downarrow \\ F(E\mathcal{F}_+, E\mathcal{F}_+ \wedge X) & \longrightarrow & F(E\mathcal{F}_+, X) & \longrightarrow & t_{\mathcal{F}}(X), \end{array}$$

which is Diagram C of [23].

The skeletal filtration gives rise to spectral sequences for calculating the homotopy groups of these spectra based on group cohomology [23], and we discuss this in more abstract terms in Section 11. More interesting is that for well behaved cohomology theories (such as those which are Noetherian, complex orientable and highly structured), one may prove a local cohomology theorem in which case the homotopy groups may be calculated by commutative algebra [18]. We discuss this in more abstract terms in Section 12. The formal framework for such spectral sequences is described in Section 10.

5. The derived category of a commutative ring.

In this section we consider the category $\mathcal{C} = D(R)$ for a commutative ring R, and localizations associated to an ideal I of R. In particular, we obtain a new approach to the results of [18] and an improved perspective on the role of finiteness conditions.

We wish to consider the class of acyclics for a localization, and there are several candidates for this. The most natural is the class

$$\mathcal{D} = \{M \mid \mathrm{supp}(H_*(M)) \subseteq V(I)\},$$

but we should also consider

$$\mathcal{D}' = \{M \mid \text{every element of } H_*(M) \text{ is } I\text{-power torsion}\}.$$

It is straightforward to check they are both candidates.

Lemma 5.1. *The classes $\mathcal{D}$ and $\mathcal{D}'$ are localizing ideals.* □

It is also easy to see that $\mathcal{D}' \subseteq \mathcal{D}$.

Lemma 5.2. *If I is finitely generated then $\mathcal{D}' = \mathcal{D}$, but this is not true in general.*

Proof. Suppose M is a module with support in $V(I)$, and $x \in M$ has annihilator J. Since R/J has support $V(J)$, we see that $V(J) \subseteq V(I)$ so that $\sqrt{J} \supseteq \sqrt{I} \supseteq I$. If I is finitely generated, some power of I lies in J.

To give an example where equality fails we need only display an ideal J so that no power of $\sqrt{J}$ lies in J, since then we may take $I = \sqrt{J}$ and $M = R/J$. For instance if R is polynomial on a countably infinite number of generators, $x_1, x_2, x_3, \ldots$ over a field and $J = (x_1, x_2^2, x_3^3, \ldots)$ we find that $\sqrt{J}$ is the maximal ideal $(x_1, x_2, x_3, \ldots)$ no power of which lies in J. □

It is useful to have a specific generator for $\mathcal{D}$ as a localizing ideal. Perhaps the most natural candidate for a generator of $\mathcal{D}$ is R/I, but this can only generate $\mathcal{D}'$. For the rest of the section we assume that I is finitely generated, say $I = (x_1, \ldots, x_n)$, and thus $\mathcal{D} = \mathcal{D}'$. We show that R/I does give a generator, but there are other candidates which are usually convenient.

Warning 5.3. If R/I does not have a finite resolution by finitely generated projectives, it need not be small.

We may define the unstable Koszul complex for the sequence $x_1^d, x_2^d, \ldots, x_n^d$ by

$$UK_d^\bullet(\mathbf{x}) = (R \xrightarrow{x_1^d} R) \otimes \cdots \otimes (R \xrightarrow{x_n^d} R).$$

We also write $UK^\bullet(\mathbf{x}) := UK_1^\bullet(\mathbf{x})$. The unstable Koszul complexes have the advantage of being small, and explicitly constructed from free modules.

We may also define the stable Koszul complex

$$K^\bullet(I) = (R \longrightarrow R[1/x_1]) \otimes \cdots \otimes (R \longrightarrow R[1/x_n])$$

and define $\check{C}^\bullet(I)$ by the existence of a fibre sequence $K^\bullet(I) \longrightarrow R \longrightarrow \check{C}^\bullet(I)$. It is not hard to check [17] that both $K^\bullet(I)$ and $\check{C}^\bullet(I)$ are independent of the generators of the ideal, up to quasi-isomorphism. Since $K^\bullet(I)$ and $\check{C}^\bullet(I)$ are only complexes of flat modules and not projective modules, it is necessary to replace them by complexes $PK^\bullet(I)$ and $P\check{C}^\bullet(I)$ of projectives when calculating maps out of them in the derived category.

We start by showing what can be constructed from $UK^\bullet(\mathbf{x})$.

Lemma 5.4. *(i) Provided $d_1, d_2, \ldots, d_n \geq 1$, the unstable Koszul complex $UK^\bullet(x_1^{d_1}, x_2^{d_2}, \ldots x_n^{d_n})$ lies in the thick subcategory generated by $UK^\bullet(\mathbf{x})$.*
(ii) The stable Koszul complex $K^\bullet(I)$ lies in the localizing subcategory generated by the unstable Koszul complex $UK^\bullet(\mathbf{x})$.

Proof. (i) First we deal with the case $n = 1$. We proceed by induction on d using the square

$$\begin{array}{ccc} R & \xrightarrow{x^{d-1}} & R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle x} \\ R & \xrightarrow{x^d} & R \end{array}$$

to construct a cofibre sequence $UK^\bullet_{d-1}(x) \longrightarrow UK^\bullet_d(x) \longrightarrow UK^\bullet(x)$. The general case follows since the argument remains valid after tensoring with any free object.

(ii) The map $R \longrightarrow R[1/x]$ is the direct limit of the maps $R \xrightarrow{x^d} R$, and hence $K^\bullet(x)$ is equivalent to the homotopy direct limit of the terms $UK^\bullet(x^d)$. Tensoring these together and using the fact that $\underset{\rightarrow}{\mathrm{holim}}_d$ commutes with tensor products, we find

$$K^\bullet(I) \simeq \underset{\rightarrow}{\mathrm{holim}}_d\, UK^\bullet_d(\mathbf{x}).$$

□

We also need a related result in the other direction.

Lemma 5.5. *The unstable Koszul complex $UK^\bullet(\mathbf{x})$ lies in the thick subcategory generated by the stable Koszul complex $K^\bullet(I)$.*

Proof. Consider the self-map of the cofibre sequence $K^\bullet(x) \longrightarrow R \longrightarrow R[1/x]$ given by multiplication by x. Since x is an equivalence of $R[1/x]$, the octahedral axiom shows there is a fibre sequence $UK^\bullet(x) \longrightarrow K^\bullet(x) \xrightarrow{x} K^\bullet(x)$. We may tensor this argument with any object X, so that we find a fibre sequence

$$K^\bullet(x_1, \ldots, x_{n-1}) \otimes UK^\bullet(x_n) \longrightarrow K^\bullet(I) \xrightarrow{x_n} K^\bullet(I).$$

Repeating this, we see that $UK^\bullet(\mathbf{x})$ lies in the thick subcategory generated by $K^\bullet(I)$. □

Proposition 5.6. *If $I = (x_1, x_2, \ldots, x_n)$ is finitely generated, the class $\mathcal{D}$ is generated as a localizing ideal by R/I, by $K^\bullet(I)$ and by $UK^\bullet(\mathbf{x})$.*

Proof. We start by showing that $\mathcal{D}$ is generated by $UK^\bullet(\mathbf{x})$. Since $UK^\bullet(\mathbf{x})$ is small, we may apply Proposition 3.2 (i). It suffices to check that $UK^\bullet(\mathbf{x})$ detects triviality of objects D of $\mathcal{D}$. Suppose then that $H_*(X)$ is I-power torsion and $t \in H_*(X)$. It suffices by 5.4 to show that the corresponding map $t : R \longrightarrow X$ extends over $R \longrightarrow UK^\bullet(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n})$ for some $d_1, d_2, \ldots, d_n \geq 1$.

Suppose by induction on m that t has been extended to $t' : UK^\bullet(x_1^{d_1}, x_2^{d_2}, \ldots, x_m^{d_m}) \longrightarrow X$. This is clear for $m = 0$, so the induction starts, and we suppose $0 < m < n$. Now note that t' is I-power torsion, since $[T, X]$ is I-power torsion for any finite complex T of free modules. Choose d_{m+1} so that $x_{m+1}^{d_{m+1}} t' = 0$. Construct a cofibre sequence by tensoring

$$R \stackrel{x_{m+1}^{d_{m+1}}}{\longrightarrow} R \longrightarrow UK^\bullet(x_{m+1}^{d_{m+1}})$$

with $UK^\bullet(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_m})$. Exactness of $[\cdot, X]$ shows that t' extends along

$$UK^\bullet(x_1^{d_1}, x_2^{d_2}, \cdots, x_m^{d_m}) \longrightarrow UK^\bullet(x_1^{d_1}, x_2^{d_2}, \cdots, x_m^{d_m}, x_{m+1}^{d_{m+1}}),$$

completing the inductive step. This completes the proof that $UK^\bullet(\mathbf{x})$ generates $\mathcal{D}$.

By 5.5 it follows that $K^\bullet(I)$ also generates $\mathcal{D}$, and the fact that R/I generates $\mathcal{D}$ follows if we can show $UK^\bullet(\mathbf{x})$ lies in the localizing ideal generated by R/I.

Lemma 5.7. *The localizing ideal containing R/I contains any complex X so that $H_*(X)$ is bounded in both directions and I-power torsion.*

Proof. First, we prove by induction on k that a module M (regarded as an object of the derived category in degree 0 with zero differential) lies in locid(R/I) provided $I^k M = 0$. If $k = 0$ this means $M = 0$, so we suppose $k \geq 1$. First, the short exact sequence $I^k/I^{k+1} \longrightarrow R/I^{k+1} \longrightarrow R/I^k$ gives a triangle, with the first and third term already known to be in the ideal, so R/I^{k+1} lies in the ideal. Now suppose $I^{k+1}M = 0$. There is a surjective map $T_0 \longrightarrow M_0 = M$ of modules where T_0 is a sum of copies of R/I^{k+1}, and the kernel K_0 also satisfies $I^{k+1}K_0 = 0$. We may thus iterate the construction and apply 3.2 (ii) to deduce M lies in the localizing ideal generated by R/I^{k+1}.

The modules M are Eilenberg-MacLane objects, and we show that if X is bounded, it has a finite Postnikov tower. After suspension we may suppose $H_i(X) = 0$ for $i < 0$. Since X is equivalent to the subcomplex X' zero in negative degrees, with X'_0 the 0-cycles, and agreeing with X in positive degrees, we may suppose X is zero in negative degrees. There is then a canonical map $X = X^0 \longrightarrow M_0$ which is an isomorphism in degree 0 where $M_0 = H_0(X)$. The fibre X^1 then has $H_i(X^1) = 0$ for $i < 1$, and $H_i(X^1) \cong H_i(X)$ for $i \geq 1$, and we may iterate the construction. Defining X_k by the triangle $X^k \longrightarrow X \longrightarrow X_k$ we see that $X_0 \simeq 0$, and by the octahedral axiom there is a cofibre seqence

$$\Sigma^k M_k \longrightarrow X_{k+1} \longrightarrow X_k.$$

Since M_k lies in the localizing ideal generated by R/I, so does X_k for all k. By the boundedness hypothesis, $X^N \simeq 0$ for N sufficiently large, and so $X_N \simeq X$. □

Since $UK^\bullet(\mathbf{x})$ satisfies the conditions of the lemma, this completes the proof of 5.6. □

It is not hard to construct the relevant localizations and completions.

Lemma 5.8. *If I is finitely generated,*
(i) $M[\mathcal{D}^{-1}] = M \otimes \check{C}^\bullet(I)$
(ii) $\Gamma_{\mathcal{D}}(M) = M \otimes K^\bullet(I))$
(iii) $M_{\mathcal{D}}^\wedge = \mathrm{Hom}(PK^\bullet(I), M)$

Proof. (i) To see that $M \otimes \check{C}^\bullet(I)$ is local, we need only check it admits no morphism from $UK^\bullet(\mathbf{x})$ except zero. However $\check{C}^\bullet(I)$ admits a finite filtration with subquotients $R[1/x]$ for $x \in I$ so it suffices to show $[UK^\bullet(\mathbf{x}), R[1/x]]_* = 0$. This follows since x is nilpotent on $UK^\bullet(\mathbf{x})$ and an isomorphism on $R[1/x]$. To see that $M \longrightarrow M \otimes \check{C}^\bullet(I)$ is a $\mathcal{D}$-equivalence we need only verify that the $M \otimes K^\bullet(I)$ can be built from $UK^\bullet(\mathbf{x})$. However M can be built from R, and we saw in 5.4 that $K^\bullet(I)$ can be built from $UK^\bullet(\mathbf{x})$.

Part (ii) follows from the defining fibre sequence of $\check{C}^\bullet(I)$, and Part (iii) follows from 2.2. □

We write
$$H_I^*(M) := H^*(K^\bullet(I) \otimes M) = H_*(\Gamma_{\mathcal{D}}(M)) :$$
this is the local cohomology of M, and if R is Noetherian it calculates the right derived functors of
$$\Gamma_I(M) = \{x \in M \mid I^n x = 0 \text{ for } n >> 0\}$$
for modules M [29]. We write
$$H_*^I(M) := H_*(\mathrm{Hom}(PK^\bullet(I), M)) = H_*(M_{\mathcal{D}}^\wedge) :$$
this is the local homology of M [22]. If, in addition, R is Noetherian or good in the sense of [22], then this local homology gives the left derived functors of completion. In particular, if M is of finite type, $M_{\mathcal{D}}^\wedge = M_I^\wedge$. Furthermore, the Tate cohomology coincides with that of [18]. As pointed out in [18], Warwick duality is a generalization of the isomorphism $\mathbb{Z}_p^\wedge[1/p] = \lim\limits_{\leftarrow}(\mathbb{Z}/p^\infty, p)$.

Remark 5.9. If I is finitely generated, we have described both a construction and a method of calculation for useful localizations. It would be interesting to have analogues when I is not finitely generated.

6. The category of modules over a highly structured ring

In this section we suppose that $\mathbf{R}$ is a commutative S-algebra in the sense of [14], and we allow the equivariant case. Such objects are essentially equivalent to E_∞ ring spectra, so there is a good supply: in particular, any commutative ring R gives rise to the Eilenberg-MacLane S-algebra HR. We then let $\mathcal{C}$ denote the homotopy category of highly structured module spectra over R and consider localizations and completions associated to a finitely generated ideal I of the coefficient ring $\mathbf{R}_*$

Much of the discussion of the previous section applies in the present case, and was presented in [24], so we shall be brief. Thus we may form the stable and unstable Koszul modules by using cofibre sequences and smash products. Thus for example, $UK^\bullet(x)$ is the fibre of $\mathbf{R} \stackrel{x}{\longrightarrow} \mathbf{R}$; we avoid the common notation $\Sigma^{-1}\mathbf{R}/x$ for fear of confusion. Now $UK^\bullet(\mathbf{x}) = UK^\bullet(x_1) \wedge_{\mathbf{R}} UK^\bullet(x_2) \wedge_{\mathbf{R}} \ldots \wedge_{\mathbf{R}} UK^\bullet(x_n)$; similarly $K^\bullet(x)$ is the fibre of $\mathbf{R} \longrightarrow \mathbf{R}[1/x]$, and $K^\bullet(I) = K^\bullet(x_1) \wedge_{\mathbf{R}} K^\bullet(x_2) \wedge_{\mathbf{R}} \ldots \wedge_{\mathbf{R}} K^\bullet(x_n)$ where $I = (x_1, x_2, \ldots, x_n)$. We take $\mathcal{A}$ to be the class of retracts of finite $\mathbf{R}$-modules $\mathbf{M}$ so that $\mathbf{M}_*$ is I-power torsion This is generated by $\mathcal{T} = \{UK^\bullet(\mathbf{x})\}$, and generates the localizing ideal of all $\mathbf{M}$ so that each element of $\mathbf{M}_*$ is I-power torsion (i.e. $\mathbf{M}_*$ is in the class $\mathcal{D}(\mathbf{R}_*, I)$ in the sense of Section 5). We write $\Gamma_I(\mathbf{M}) := \Gamma_{\mathcal{D}}(\mathbf{M})$, $\mathbf{M}[I^{-1}] := \mathbf{M}[\mathcal{D}^{-1}]$ and $t_I(\mathbf{M}) := t_{\mathcal{D}}(\mathbf{M})$.

The statement and proof of Lemma 5.8 apply without change. Because the construction comes with an evident filtration we may obtain spectral sequences by taking homotopy, and the E_1 term is a chain complex representing the corresponding constructions of Section 5. This gives spectral sequences

$$\check{H}^*_I(\mathbf{R}_*; \mathbf{M}_*) \Longrightarrow \mathbf{M}[I^{-1}]_*$$

$$\hat{H}^*_I(\mathbf{R}_*; \mathbf{M}_*) \Longrightarrow t_I(\mathbf{M})_*$$

$$H^*_I(\mathbf{R}_*; \mathbf{M}_*) \Longrightarrow \Gamma_I(\mathbf{M})_*$$

$$H^I_*(\mathbf{R}_*; \mathbf{M}_*) \Longrightarrow (\mathbf{M}^\wedge_I)_*$$

for calculating their homotopy.

7. The derived category of kG

For a finite group G and a field k we consider the derived category $\mathcal{C} = D(kG)$, and take $\mathcal{A}$ to be the category of finite complexes of projectives. This is generated by $\mathcal{T} = \{kG\}$, and the generation is so systematic algebraically that it leads to the usual method for calculating Tate cohomology using projective resolutions and their duals. The relationship of the derived category $D(kG)$ to the category of G-spectra is analogous to the relationship of $D(\mathbf{R}_*)$ to the category of highly structured modules over $\mathbf{R}$.

It is proved in [32, 9.6] that the localization $M \longrightarrow M[\mathcal{A}^{-1}]$ is obtained by tensoring with a Tate resolution. Since any Tate resolution admits a finite filtration with subquotients $R[1/x]$ as in [19], it follows that every object of $\mathcal{C}$ with bounded cohomology is already complete. Thus we find that if M has bounded cohomology, $t_{\mathcal{A}}(M) = M[\mathcal{A}^{-1}] = M \otimes t_{\mathcal{A}}(k)$ and so the Tate construction defined by localization agrees with Tate homology in the classical sense.

There are at least three other examples to consider here, but some work is needed to give them substance. Recall that an indecomposable module M has vertex H if it is a summand in a module induced from H but not from any proper subgroup of H.

Variation 7.1. Consider a family $\mathcal{F}$ of subgroups, and the category $\mathcal{A}_{\mathcal{F}}$ of finite complexes of modules with vertex in $\mathcal{F}$. The case $\mathcal{F} = \{1\}$ is that given above. The $\mathcal{G}$-ideal $\mathcal{A}_{\mathcal{F}}$ is generated by $\mathcal{T}_{\mathcal{F}} = \{k[G/H] \mid H \in \mathcal{F}\}$. It is then appropriate to use Amitsur-Dress $\mathcal{F}$-cohomology [13]. Perhaps there is again a local cohomology theorem in the sense of Section 12 below, using the ideal of positive degree elements, but the appropriate theory of varieties has not been developed. It would also be interesting to know the relationship to ordinary group cohomology and the ideal $I_{\mathcal{F}}$ of cohomology elements restricting to zero in the cohomology of H for all $H \in \mathcal{F}$.

Variation 7.2. We choose a block β of kG and take $\mathcal{A}_\beta$ to be the category of finite complexes of projectives in β.

Variation 7.3. We may consider the stable module category $\mathcal{C} = StMod(kG)$, which is proved in [32, 9.6.4] to be a localization of $D(kG)$. It would then be interesting to investigate complexity quotients in the sense of Carlson-Donovan-Wheeler [10, 11, 5, 6] from the present point of view.

8. Chromatic categories.

Another important class of examples arises in the approach to stable homotopy theory through bordism. For background and further information see [40]. Thus we work in the stable homotopy category of spectra in the sense of algebraic topology, and we choose a prime $p > 0$. For $0 \leq n \leq \infty$ we shall need the spectrum $E(n)$ representing Johnson-Wilson cohomology theory and the Morava K-theory spectrum $K(n)$. For $0 < n < \infty$ these have coefficient rings $E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n, v_n^{-1}]$, and $K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}]$. The cases $n = 0, \infty$ are somewhat exceptional: by convention, for $n = 0$ we have $E(0) = K(0) = H\mathbb{Q}$ and for $n = \infty$ we have $E(\infty) = BP$ and $K(\infty) = H\mathbb{Z}/p$. Recall that a spectrum is of type n if $K(i)_*(X) = 0$ for $i < n$ and $K(n)_*(X) \neq 0$.

Bousfield localization L_n with respect to $E(n)$ is the localization whose acyclics are the spectra X with $E(n) \wedge X \simeq *$. A well known theorem of Hopkins-Ravenel states that L_n is smashing. The usual notation is $C_nX \longrightarrow X \longrightarrow L_nX$. The completion $X^\wedge_{\mathcal{D}} = F(C_nS, X)$ is more mysterious, but when $n = 0$ it is profinite completion $F(S^{-1}\mathbb{Q}/\mathbb{Z}, X)$.

n	$E(n)$	$K(n)$	$F(n)$	L_{n-1}	$L_{K(n)}$
0	$H\mathbb{Q}$	$H\mathbb{Q}$	$S^0_{(p)}$	$*$	rationalization
1	$K_{(p)}$	K/p	S^0/p^k	invert p	$L_{K(1)}$
2	Ell	$Ell/(p, v_1)$	$S^0/(p^k, v_1^l)$	$L_1 =$ “$(\cdot)[(p, v_1)^{-1}]$”	$L_{K(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	BP	$H\mathbb{F}_p$	$*$	p-localization	p-adic completion

Following [33], let us consider a slightly simpler example. Let $\mathcal{C}$ be the $E(n)$-local category, and $\mathcal{A}$ the thick subcategory generated by $L_nF(n)$ for a finite type n spectrum $F(n)$. In this case $X[\mathcal{A}^{-1}] = L_{n-1}X$ [33, 6.10] and $X^\wedge_{\mathcal{A}} = L_{K(n)}X$ [33, 7.10]. The fibre of $X \longrightarrow L_{n-1}X$ isusually known as the nth monochromatic piece when X is $E(n)$-local, so we have $\Gamma_{\mathcal{A}}(X) = M_nX$. The fibre of $K(n)$ completion is sometimes known as $C_{K(n)}$, but we simply write $\Delta_{\mathcal{A}}(X) = \Delta_{K(n)}(X)$.

Corollary 8.1. *(Warwick Duality) If X is $E(n)$-local then*

$$L_{n-1}L_{K(n)}X \simeq \Sigma\Delta_{K(n)}(M_nX). \quad \square$$

We note that if $n = 0$ this states M_0X is rational, and if $n = 1$ it states that the cofibre of $M_1X \longrightarrow L_{K(1)}M_1X$ is the rationalization of $L_{K(1)}X$.

If we take $\mathcal{C}$ to be the entire category of p-local spectra there are two related examples. Indeed, we may still consider the smashing localization $L_{n-1} = L_{E(n-1)}$, but it does not seem so easy to describe the associated completion. In particular it is not equal to $L_{K(n)}$ (indeed, although $L_{n-1}S^0$ is $K(n)$-acyclic, there are many spectra, such as $F(n+1)$, which are $K(n)$-acyclic but not $E(n-1)$-local). We may also consider spectra $Tel(n) = F(n)[1/v_n]$, and the smashing localization L^f_{n-1} which is Bousfield localization with respect to $Tel(0) \vee Tel(1) \vee \ldots \vee Tel(n-1)$; this is finite localization with respect to $F(n)$ [39] and it is therefore smashing, and we may again consider the associated completion, which is again different from $L_{K(n)}$ for similar reasons. There is a natural transformation $L^f_n \longrightarrow L_n$, which is believed not to be an isomorphism for $n \geq 2$ [35].

9. Splittings of the Tate construction.

We describe two different classes splittings of the Tate construction. Each requires special properties of the localization.

First, continuing with the notation of the previous section, note that $t_{\mathcal{A}}(X) = L_{n-1}L_{K(n)}X$ is the subject of Hopkins's chromatic splitting conjecture [31, 4.2]. When $X = S^0$ (albeit not in the $E(n)$-local category $\mathcal{C}$) this is conjectured to split into 2^n pieces. More precisely there is a cofibre sequence

$$L_{n-1}S^0_p \longrightarrow L_{n-1}L_{K(n)}S^0 \longrightarrow \Sigma F(L_{n-1}S^0, L_nS^0_p),$$

which is conjectured to split, and furthermore, $F(L_{n-1}S^0, L_nS^0_p)$ is also conjectured to split as a wedge of $2^n - 1$ suitable localizations of spheres. To obtain the cofibre sequence, apply $F(\cdot, X)[\mathcal{D}^{-1}]$ to the cofibre sequence $\Sigma^{-1}S[\mathcal{D}^{-1}] \longrightarrow \Gamma_{\mathcal{D}}(S) \longrightarrow S$ to obtain

$$X[\mathcal{D}^{-1}] \longrightarrow t_{\mathcal{D}}(X) \longrightarrow \Sigma F(S[\mathcal{D}^{-1}], X)$$

since $F(S[\mathcal{D}^{-1}], X)$ is already $[\mathcal{D}^{-1}]$-local.

Secondly, there is a dichotomy between the periodic and split behaviour of the Tate construction, typified by the cohomology of finite groups. Although Tate cohomology is often associated with periodic behaviour, it is the split case that is generic. On the one hand, when G has periodic cohomology there is a 'periodicity element' x in $H^*(G)$ and the Tate cohomology $\hat{H}^*(G) = H^*(G)[1/x]$ is periodic under multiplication by x. By contrast, when group cohomology $H^*(G)$ has a regular sequence of length 2, Benson-Carlson [4] and Benson-Greenlees [7] have shown that the mod p Tate cohomology $\hat{H}^*(G)$ of a finite group splits

$$\hat{H}^*(G) = H^*(G) \oplus \Sigma^1 H_*(G)$$

(where the suspension is homological) both as a module over $H^*(G)$ and as a module over the Steenrod algebra. Even this context does not provide a true dichotomy, since there are groups with depth 1 which are not periodic, but this mixed behaviour is exceptional.

The analogous statement concerns the standard cofibre sequence

$$X^\wedge_{\mathcal{D}} \longrightarrow t_{\mathcal{D}}(X) \longrightarrow \Sigma\Gamma_{\mathcal{D}}(X)$$

when $X = S$. The dichotomy principle would suggest that in most cases, either $t_{\mathcal{D}}(S)$ is obtained from $S^\wedge_{\mathcal{D}}$ by inverting some multiplicatively closed subset of $\pi_*(S^\wedge_{\mathcal{D}})$, or else the cofibre sequence splits, and that the split case is generic. The hypotheses for a splitting must include the requirement that the norm map $\Sigma^{-1}\Gamma_{\mathcal{D}}(X) \longrightarrow X^\wedge_{\mathcal{D}}$ is zero in homotopy, and probably also that $\pi_*(X^\wedge_{\mathcal{D}})$ is of depth at least 2. However the proofs from the case of group cohomology do not extend in any simple way since they use the fact that homology and cohomology are identified in the Tate cohomology by their occurrence in positive and negative degrees.

A second case where the dichotomy holds is in commutative algebra [18]. When the ring is Noetherian and of Krull dimension 1, the rationality theorem [18, 7.1] holds: $\hat{H}^*_I(R) = S^{-1}(R^\wedge_I)$ where S is the set of regular elements of R. This is the periodic case. It is immediate that if the ring is of I-depth two or more the Tate cohomology splits since the local homology is in degree 0 whilst the local cohomology is only non-zero at or above the depth.

10. Calculation by comparison.

We discuss two quite different methods of calculation. To introduce the discussion, we explain the two methods as they apply to calculating the homology $H_*(G;M)$ of a finite group with coefficients in a chain complex M of kG-modules. The first is quite familiar, and states there is a spectral sequence

$$H^*(G;H_*(M)) \Longrightarrow H_*(G;M).$$

The second method is the local cohomology theorem, stating that there is a spectral sequence

$$H^*_I(H^*(G;M)) \Longrightarrow H_*(G;M)$$

where I is the ideal of positive codegree elements of $H^*(G)$ [19], and $H^*_I(\cdot)$ denotes local cohomology in the sense of Grothendieck [29] (the definition was recalled in Section 5).

The generalization we have in mind concerns finite localizations in the case that $\mathcal{A}$ is generated by a single object A. We require that A is a commutative comonoid in the sense that it has a commutative and associative coproduct $A \longrightarrow A \wedge A$ and a counit $A \longrightarrow S$. We require in addition that A is strongly dualizable and self-dual up to an invertible element, in the sense that $DA \simeq A \wedge S^{-\tau}$ for some object $S^{-\tau}$ admitting a smash inverse $S^{-\tau} \wedge S^{\tau} \simeq S$.

Example 10.1. (i) The motivating example has $\mathcal{C} = \mathcal{D}(kG)$ for a finite group G and $A = kG$. Note that we have an augmentation $kG \longrightarrow k$, and a diagonal map $kG \longrightarrow kG \otimes kG$. Furthermore kG is self-dual.
(ii) Alternatively, for a compact Lie group G, we may take $\mathcal{C}$ to be a category of G-spectra (or of module G-spectra over a ring G-spectrum $\mathbf{R}$) and $A = G_+$ (or $\mathbf{R} \wedge G_+$). Again we have an augmentation $G_+ \longrightarrow S^0$, and a diagonal map $G_+ \longrightarrow G_+ \wedge G_+$. We also have the duality statement $DG_+ \simeq \Sigma^{-d} G_+$ where $d = dim(G)$. This helps explain the notation $S^{-\tau}$, which is chosen since, in the geometric context, Atiyah duality shows τ corresponds to the tangent bundle.
(iii) Rather differently, we may take $\mathcal{C}$ to be the category of p-local spectra, (or of p-local $\mathbf{R}$-module spectra over a ring spectrum $\mathbf{R}$) and $A = \Sigma^{-d} F(n)$ (or $A = \mathbf{R} \wedge \Sigma^{-d} F(n)$) where $F(n) = S^0/(p^{i_0}, v_1^{i_1}, v_2^{i_2}, \ldots, v_{n-1}^{i_{n-1}})$ for suitable $i_0, i_1, \ldots, i_{n-1}$ and $d = dim(F(n))$. Collapse onto the top cell gives an augmentation $\Sigma^{-d} F(n) \longrightarrow S^0$. In favourable cases we have the duality statement $DF(n) \simeq \Sigma^{-d} F(n)$, and $F(n)$ may be taken to be a commutative ring spectrum [12], and the dual to the product gives a coproduct map $\Sigma^{-d} F(n) \longrightarrow \Sigma^{-d} F(n) \wedge \Sigma^{-d} F(n)$.

We need to consider the graded commutative ring $k_* = [S,S]_*$, where S is the unit in $\mathcal{C}$, and two k_*-algebras. Firstly, since A is a commutative comonoid, $l_* = [A,S]_*$ is a commutative k_*-algebra, and $[A,Z]_*$ is a module over l_* for any Z. Secondly, we consider the k_*-algebra $\mathcal{E}_* = \mathrm{End}(A)_*$, which need not be commutative.

Context 10.2. (for calculation)

- A a commutative comonoid object
- $\mathcal{A}$ generated by A
- $DA \simeq S^{-\tau} \wedge A$
- $k_* = [S,S]_*$
- $l_* = [A,S]_*$

- $I = \ker(k_* \longrightarrow l_*)$
- $\mathcal{E}_* = [A, A]_*$

In our examples these are as follows.

Example 10.3. (i) When $A = kG$ we have $k_* = l_* = k$ and $\mathrm{End}(kG)_* = kG$.
(ii) When $A = \mathbf{R} \wedge G_+$ we have $k_* = \mathbf{R}^G_*$, $l_* = \mathbf{R}_*$ and $\mathrm{End}(\mathbf{R} \wedge G_+)_* = \mathbf{R}_*(G_+)$.
(iii) When $A = \Sigma^{-d} F(n)$ we have $k_* = \mathbf{R}_*$, $l_* = \mathbf{R}_*(F(n))$ and $\mathrm{End}(\mathbf{R} \wedge \Sigma^{-d} F(n))_* = \mathbf{R}_*(F(F(n), F(n)))$.

Given these data, there are two functors we can apply:

$$[A, \cdot]_* : \mathcal{C} \longrightarrow \mathrm{End}(A)_*\text{-mod}$$

(corresponding to non-equivariant homotopy in Example (ii)), and

$$[S, \cdot]_* : \mathcal{C} \longrightarrow k_*\text{-mod}$$

(corresponding to equivariant homotopy in Example (ii)).

It is then natural to seek spectral sequences reversing these two functors.

In the first case we may hope they take the form

10.4.

$$H_*(\mathrm{End}(A)_*; [A, S^\tau \wedge X]_*) \Longrightarrow (\Gamma_{\mathcal{A}}(X))_*$$

$$H^*(\mathrm{End}(A)_*; [A, X]_*) \Longrightarrow (X^\wedge_{\mathcal{A}})_*$$

and

$$\hat{H}^*(\mathrm{End}(A)_*; [A, X]_*) \Longrightarrow t_{\mathcal{A}}(X)_*.$$

A construction in some cases is given in Section 11, and the twisting S^τ in the first spectral sequence will be explained.

In the second case we let

$$I = \ker(k_* = [S, S]_* \longrightarrow [A, S]_*)$$

be the augmentation ideal, and apply local cohomology, local homology and local Tate cohomology as appropriate and hope the spectral sequences take the form

10.5.

$$H^*_I(X_*) \Longrightarrow (\Gamma_{\mathcal{A}}(X))_*$$

$$H^I_*(X_*) \Longrightarrow (X^\wedge_{\mathcal{A}})_*$$

and

$$\hat{H}^I_*(X_*) \Longrightarrow t_{\mathcal{A}}(X)_*.$$

A construction in some cases is given in Section 12. When the first spectral sequence exists we say that *the local cohomology theorem holds*. Provided this happens for good enough reasons, the other two spectral sequences exist as a consequence.

The content should be clearer when we give some examples. It is not surprising that to prove the existence of either set of spectral sequences we have to assume the existence of additional structure beyond that present in the stable homotopy category.

11. Calculations by associative algebra.

The point of this section is to generalize the Atiyah-Hirzebruch Tate spectral sequence of [16]:

$$\hat{H}^*(G; E^*(X)) \Longrightarrow t(E)^*_G(X)$$

for finite groups G, or in other words to prove the expectations suggested in 10.4 hold under suitable circumstances. The construction does not work entirely in a stable homotopy category, but rather relies on the existence of a suitable Quillen model category from which the stable homotopy category is formed by inverting weak equivalences.

We thus suppose given a stable homotopy category, and consider the $\mathcal{G}$-ideal $\mathcal{A}$ generated by a single object A. The aim is to find ways to calculate $\Gamma_A(X)_*$, $(X^\wedge_A)_*$ and $t_A(X)_*$ in terms of the $[A, A]_*$-module $[A, X]_*$. In view of the notational conflict we remind the reader that in the context of G-spectra, where $A = G_+$ the group $[S, \cdot]_*$ is *equivariant* homotopy and $[A, \cdot]_*$ is *non-equivariant* homotopy. The present discussion covers a number of new examples: the generalization is cruder than that of [23], but more general. The discussion of convergence in [23, Appendix B] applies without change.

To avoid the appearance of empty generalization, we state an unequivocal theorem in the equivariant homotopy context of Section 4 (with $\mathcal{A}$ generated by $\mathbf{R} \wedge G_+$).

Theorem 11.1. *Suppose G is a compact Lie group of dimension d, $\mathbf{R}$ is an equivariant S-algebra, and $\mathbf{M}$ an $\mathbf{R}$-module. Provided we have the Künneth theorem*
(KT1)

$$\mathbf{M}_*(G_+ \wedge T \wedge Y) = \mathbf{R}_*(G_+) \otimes_{\mathbf{R}_*} \mathbf{M}_*(T \wedge Y)$$

and the universal coefficient theorem
(UCT)

$$[G_+ \wedge T, \mathbf{M} \wedge Y]^G_* = [T, \mathbf{M} \wedge Y]_* = \mathrm{Hom}_{\mathbf{R}_*}(\mathbf{R}_*(T), \mathbf{M}_*(Y))$$

when $T = G_+^{\wedge s}$ for $s \geq 0$, there are spectral sequences

$$H_*(\mathbf{R}_*(G_+); \mathbf{M}_*(S^d \wedge Y)) \Longrightarrow \mathbf{M}^G_*(EG_+ \wedge Y),$$

$$H_*(\mathbf{R}_*(G_+); \mathbf{M}^*(Y)) \Longrightarrow \mathbf{M}^*_G(EG_+ \wedge Y),$$

and

$$\hat{H}_*(\mathbf{R}_*(G_+), \mathbf{M}_*(Y)) \Longrightarrow t(\mathbf{M})^G_*(Y),$$

where the homology and cohomology on the left is that of the Frobenius algebra $\mathbf{R}_*(G_+)$.

We return to this particular case at the end of the section. The rest of the discussion is conducted in general terms.

We want to view the construction of $\Gamma_A S$ as a "resolution" for $X = S$ using sums of objects of $\mathcal{A}$. More precisely, we use the method of 3.2 (i) without assuming X is in $\mathcal{D}$. The dual resolution is thus

$$\begin{array}{ccccccc} * =\!=\!= (S)^0 & \xrightarrow{j^0} & (S)^1 & \xrightarrow{j^1} & (S)^2 & \xrightarrow{j^2} & \cdots \\ \downarrow & & \downarrow q_0 & & \downarrow q_1 & & \\ * & & T_0 & & T_1 & & \end{array}$$

where each T_i is a sum of suspensions of objects of $\mathcal{A}$. This is associated with the sequence

$$T_0 \xleftarrow{\delta_1} \Sigma^{-1}T_1 \xleftarrow{\delta_2} \Sigma^{-2}T_2 \xleftarrow{\delta_3} \Sigma^{-3}T_3 \longleftarrow .$$

We want to apply simplicial methods, so we suppose there is an underlying model category, from which the stable homotopy category is formed by passage to homotopy. Furthermore, we require a compatible symmetric monoidal structure and that A is a strict comonoid object.

Example 11.2. The example relevant to the theorem is the homotopy category of modules over the equivariant S-algebra $\mathbf{R}$ for a compact Lie group G, and $A = \mathbf{R} \wedge G_+$. This is the homotopy category of the model category of equivariant $\mathbf{R}$-modules [14]. However, in this case it is more elementary to make the construction described below at the space level, apply the suspension spectrum functor and take the extended $\mathbf{R}$-module: this strategy will give parts of the theorem under weaker hypotheses.

We form the homogeneous bar construction [38] as a simplicial object, and take its geometric realization

$$\Gamma_A S = S^\infty = B(A, A, S).$$

This ensures $T_i = \Sigma^i A^{\wedge(i+1)}$.

By smashing with X we obtain a resolution for arbitrary X. Thus we may define

$$t_A(X) = F(B(A, A, S), X) \wedge \tilde{B}(A, A, S),$$

where $\tilde{B}(A, A, S)$ is the mapping cone of $B(A, A, S) \longrightarrow B(S, S, S) = S$.

To relate the resolution to an algebraic one, we apply a homology theory to obtain

$$[A, T_0]_* \stackrel{(\delta_1)_*}{\longleftarrow} [A, \Sigma^{-1}T_1]_* \stackrel{(\delta_2)_*}{\longleftarrow} [A, \Sigma^{-2}T_2]_* \stackrel{(\delta_3)_*}{\longleftarrow} [A, \Sigma^{-3}T_3]_* \longleftarrow .$$

In the equivariant context we have $[A, \Sigma^{-i}T_i]_* = \mathbf{R}_*(G_+^{\wedge i+1})$. To ensure it is a resolution, we assume there is a Künneth theorem

(KT1)

$$[A, A \wedge Z] = [A, A]_* \otimes_{[A,S]_*} [A, Z]_*$$

for relevant Z (namely $Z = A^{\wedge i}$). In the equivariant context this is a Künneth theorem for the (non-equivariant) homology theory $\mathbf{R}_*(\cdot)$. This ensures that the simplicial contraction in geometry is converted to one in algebra and the bar filtration spectral sequence for calculating $[A, B(A, A, S)]_*$ has its E^1 term given by the algebraic bar construction $B([A, A]_*, [A, A]_*, [A, X]_*)$. To calculate $\Gamma_A(X)_*$, we need the second Künneth theorem

(KT2)

$$[S, Z]_* = [A, S]_* \otimes_{[A,A]_*} [A, S^\tau \wedge Z]_*$$

for relevant Z (namely $Z = A^{\wedge(i+1)} \wedge X$). In the equivariant context, this states that the change of groups isomorphisms $[S, G_+ \wedge T]^G_* = [S, S^d \wedge T]_* = [G_+, T]^G_*$ are reflected in algebra.

Lemma 11.3. *The Künneth theorem (KT2) for $Z = A \wedge T$ follows from the Künneth theorem (KT1) for $Z = S^\tau \wedge T$.*

Proof: Assuming (KT1) for $Z = S^\tau \wedge T$, we calculate

$$\begin{aligned}
[A, S]_* \otimes_{[A,A]_*} [A, A \wedge S^\tau \wedge T]_* &= [A, S]_* \otimes_{[A,A]_*} [A, A]_* \otimes_{[A,S]_*} [A, S^\tau \wedge T]_* \\
&= [A, S]_* \otimes_{[A,S]_*} [A, S^\tau \wedge T]_* \\
&= [A, S^\tau \wedge T]_* \\
&= [S, S^\tau \wedge DA \wedge T]_* \\
&= [S, A \wedge T]_*
\end{aligned}$$

as required. □

This is enough to give a spectral sequence with

$$E^1 = [A,S]_* \otimes_{[A,A]_*} B([A,A]_*, [A,A]_*, [A,S^\tau \wedge X]_*);$$

it therefore takes the form

$$E^2_{*,*} = H_*(\mathrm{End}(A)_*; [A, S^\tau \wedge X]_*) \Longrightarrow \Gamma_A(X)_*.$$

It is easy to see this spectral sequence is conditionally convergent in the sense of Boardman [2]. The homology in the E_2-term is defined to be the homology of the bar construction, but in favourable cases it can be calculated in various other ways. For example in the case of G-spectra this spectral sequence takes the form

$$H_*(\mathbf{R}_*(G_+); (S^d \wedge X)_*) \Longrightarrow X^G_*(EG_+).$$

Note that we have two possible definitions of the $\mathbf{R}_*(G_+)$ module structure on X_* a diagram chase verifies they agree.

Lemma 11.4. *The action of $\mathbf{R}_*(G_+)$ on $X_* = [G_+, X]^G_* = [\mathbf{R} \wedge G_+, X]^{\mathbf{R},G}_*$ implied by the Künneth theorem and the action of G on X agrees with the action of $[\mathbf{R} \wedge G_+, \mathbf{R} \wedge G_+]^{\mathbf{R},G}_*$ by composition.* □

For cohomology we want to have universal coefficient theorem
(UCT)

$$[A \wedge Z, X]_* = \mathrm{Hom}_{[A,A]_*}([A, A \wedge Z]_*, [A,X]_*) = \mathrm{Hom}_{[A,S]_*}([A,Z]_*, [A,X]_*),$$

where the second equality is (KT1) and a change of rings isomorphism. This is enough to get a spectral sequence with

$$E_1 = \mathrm{Hom}_{[A,A]_*}(B([A,A]_*, [A,A]_*, [A,S]_*), [A,X]_*);$$

it therefore takes the form

$$E_2^{*,*} = H^*(\mathrm{End}(A)_*; [A,X]_*) \Longrightarrow [\Gamma_A(S), X]_* = (X^\wedge_A)_*.$$

Convergence is again conditional in the sense of Boardman. In the equivariant case this spectral sequence becomes

$$H^*(\mathbf{R}_*(G_+); X_*) \Longrightarrow X^*_G(EG_+).$$

When it comes to Tate cohomology we need to ask about splicing, both in topology and algebra. In topology we have

$$\longleftarrow DA^2 \longleftarrow DA \longleftarrow A \longleftarrow A^2 \longleftarrow \ldots$$

where the splicing is via

$$DA \xleftarrow{Dt_0} DS = S \xleftarrow{t_0} A.$$

To obtain a spectral sequence we may either apply $[A, \cdot \wedge X]_*$ and use the first avatar $t_A(X) = F(B(A,A,S), X) \wedge \tilde{B}(A,A,S)$, or apply $[A \wedge \cdot, X]_*$ and use the second avatar $t_A(X) = F(\tilde{B}(A,A,S), \Sigma X \wedge B(A,A,S))$. The first will make the relation to homology clearer and the second will make the relation to cohomology clearer, but since the resolution is self-dual, the two are essentially equivalent, and we only discuss the first. Convergence is again covered by the relevant argument (10.8) from [23].

In view of the equality $[A \wedge A, S]_* = [A, DA]_*$, we conclude that the E^2-term agrees with the homological one in positive filtration degrees, and with the cohomological one (shifted by one degree) in filtration degrees ≤ -2. More precisely, if $\mathcal{E}_* = \mathrm{End}(A)_* = [A, A]_*$, and $\tilde{\mathcal{E}}_* = [A \wedge A, S]^* = [A, DA]_*$, we have the algebraic resolution

$$\longleftarrow \tilde{\mathcal{E}}_*^{\otimes 2} \longleftarrow \tilde{\mathcal{E}}_* \longleftarrow \mathcal{E}_* \longleftarrow \mathcal{E}_*^{\otimes 2} \longleftarrow \ldots .$$

Using this particular resolution to define the E_2-term we have a spectral sequence

$$\hat{H}_*([A, A]_*, [A, X]_*) \Longrightarrow t_A(X)_*.$$

This is again conditionally convergent in the sense of Boardman. In the equivariant case this spectral sequence becomes

$$\hat{H}_*(\mathbf{R}_*(G_+), X_*) \Longrightarrow t^{\mathbf{R}}(X)_*^G.$$

For a more satisfactory account of the algebra, we assume $\mathcal{E}_*$ is projective as an l_*-module. Next, we express this in terms of a single type of resolution. Thus, by (KT1),

$$\tilde{\mathcal{E}}_* = [A, DA]_* = [A, A \wedge S^{-\tau}]_* = [A, A]_* \otimes_{[A,S]_*} [A, S^{-\tau}]_* = \mathcal{E}_* \otimes_{l_*} \lambda_*$$

where $\lambda_* = [A, S^{-\tau}]_*$. On the other hand, by (UCT),

$$\tilde{\mathcal{E}}_* = [A \wedge A, S]_* = \mathrm{Hom}([A, A]_*, [A, S]_*) = \mathrm{Hom}(\mathcal{E}_*, l_*),$$

so we conclude

$$\mathrm{Hom}(\mathcal{E}_*, l_*) = \mathcal{E}_* \otimes \lambda_*.$$

Next, we assume that the first Künneth theorem (KT1) applies also to $S^\tau \wedge S^{-\tau}$, so that λ_* is invertible and hence projective projective. Then we can specify a projective complete resolution by taking a resolution of l_*, dualizing and splicing. This is essentially the Tate cohomology of a Frobenius algebra, but with the twisting module inserted.

Proof of 11.1 We work in the category of $\mathbf{R}$-modules and take $X = \mathbf{M} \wedge Y$ in the first and third case, and $X = F(Y, \mathbf{M})$ in the second. □

12. Calculations by commutative algebra.

In this section we discuss the more subtle question of when the local cohomology theorem holds for $\mathcal{A}$ so that there is a calculation by commutative algebra in the sense of 10.5. This requires better behaviour of the cohomology theory concerned, and considerably more substance to the proof. We discuss two somewhat different methods for proving a local cohomology theorem. In a sense, the second method is a partial unravelling of the first: cellular constructions are replaced by multiple complexes. Both methods apply to the local cohomology theorem for finite groups, but beyond this they have different domains of relevance.

We discuss the more sophisticated example first [17, 24], because the formal machinery highlights the structure of the proof whilst hiding the technical difficulties.

Indeed if $\mathbf{R}$ is a highly structured commutative ring G-spectrum we have seen in Section 6 that, by construction, for any finitely generated ideal I in $\mathbf{R}_*^G$ we have spectral sequences

$$\hat{H}_I^*(\mathbf{R}_*^G; \mathbf{M}_*^G) \Longrightarrow t_I(\mathbf{M})_*^G$$

$$H_I^*(\mathbf{R}_*^G; \mathbf{M}_*^G) \Longrightarrow \Gamma_I(\mathbf{M})_*^G$$

$$H^I_*(\mathbf{R}^G_*;\mathbf{M}^G_*) \Longrightarrow (\mathbf{M}^\wedge_I)^G_*.$$

What we really want is to obtain similar spectral sequences for calculating $t_{\mathcal{A}}(\mathbf{M})^G_*, \Gamma_{\mathcal{A}}(\mathbf{M})^G_*$ and $(\mathbf{M}^\wedge_{\mathcal{A}})^G_*$ for the class $\mathcal{A}$ generated by G_+ using the ideal $I = \ker(\mathbf{R}^*_G \longrightarrow \mathbf{R}^*)$. We assume here that $\mathbf{R}^*_G$ is Noetherian, so that I is finitely generated, but see [25] for an example where this is not true. To obtain the desired spectral sequences we need to check that each of the constructions with $\mathcal{A}$ is equivalent to the corresponding construction on $\mathbf{R}$-modules for the ideal I. In fact, we need only check that

$$\Gamma_I(\mathbf{R}) \simeq \Gamma_I(\mathbf{R} \wedge EG_+) \simeq \Gamma_{\{G_+\}}\mathbf{R} = \mathbf{R} \wedge EG_+.$$

The second equivalence is a formal consequence of the fact that I restricts to zero non-equivariantly. The first equivalence contains the real work: it is equivalent to the statement that $\Gamma_I(\mathbf{R} \wedge \tilde{E}G) \simeq *$, where $\tilde{E}G$ is the unreduced suspension of EG. If G acts freely on a product of spheres (for example if it is a p-group) this follows from the existence of Euler classes (obviously elements of I) and the construction of $\tilde{E}G$ in terms of representation spheres [17]. To extend this to other groups some sort of transfer argument is necessary (see [20, 27] for examples).

This construction will give means of calculation whenever we have two suitably related smashing localizations. For example we may consider the localization $(\cdot)[\mathcal{D}^{-1}]$ with acyclics $\mathcal{D}$ and the localization $(\cdot)[I^{-1}]$ for an ideal I in the coefficient ring S_*. The requirements are then

- $\Gamma_I(S) \wedge S[\mathcal{D}^{-1}] \simeq *$ and
- $S[I^{-1}] \wedge \Gamma_{\mathcal{D}}(S) \simeq *$

Together, these give the equivalence

$$\Gamma_I(S) \simeq \Gamma_{\mathcal{D}}(S),$$

and hence the corresponding equivalences of other localization and colocalization functors. If we suppose $\mathcal{D}$ is generated by the single augmented object A as before, and define $I = \ker([S,S]_* \longrightarrow [A,S]_*)$, then the second requirement is again a formal consequence of the fact that elements of I restrict to zero. One expects the first requirement to use special properties of the context, as it did in the equivariant case.

We now turn to the second method for proving a local cohomology theorem, and work with the group cohomology of a finite group in the derived category $D(kG)$ as in Section 7. We are considering the relationship with the derived category of the graded ring $R = H^*(G;k)$ and the ideal I of positive dimensional elements as in [19]. We may view these results as relating various completions and Tate cohomologies in the two categories by spectral sequences. We take this opportunity to extend the results of [19] to unbounded complexes. Since $H_*(G;M)$ is already I-complete if M is bounded below, the second spectral sequence is only of interest in the unbounded case.

Theorem 12.1. *Suppose G is a finite group, and M is a complex of kG-modules, and let I denote the ideal of positive codegree elements of the graded ring $H^*(G)$. There are spectral sequences*

$$H^*_I(H^*(G;M)) \Longrightarrow H_*(G;M),$$
$$H^I_*(H^*(G);H_*(G;M)) \Longrightarrow H^*(G;M^\wedge_{\{kG\}})$$

and

$$\hat{H}^I_*(H^*(G);H_*(G;M)) \Longrightarrow H^*(G;t_{\{kG\}}(M)).$$

We explain the changes that need to be made to the arguments of [19] to cover the unbounded case. The idea is to use the algebraic spheres of Benson-Carlson [3] to construct algebraic analogues of tori B on which G acts freely. Thus if k is of characteristic $p > 0$ and G is of p-rank r, then B is a complex graded over $\mathbb{Z}^r$ concentrated in a box with the lowest corner at the origin. From B we may construct a multigraded projective resolution T of k by stacking boxes in the region with all coordinates ≥ 0. More generally, if $\sigma \subseteq \{1, 2, \ldots, r\}$ we may form $T[\sigma]$ by stacking boxes to fill the region defined by requiring $n_i \geq 0$ if $i \neq \sigma$. Thus $T[\emptyset] = T$, and $T\{1, 2, \ldots, r\}$ fills all of $\mathbb{Z}^r$. We then form a dual Koszul complex $L_\bullet$ of multigraded chain complexes:

$$L_\bullet = \left(\bigoplus_{|\sigma|=r} T[\sigma] \longrightarrow \bigoplus_{|\sigma|=r-1} T[\sigma] \longrightarrow \cdots \longrightarrow \bigoplus_{|\sigma|=0} T[\sigma] \right).$$

The idea of the proof is to consider the double complex

$$\mathrm{Hom}(L_\bullet, M)^G.$$

If one takes homology in the Koszul direction first one obtains $\mathrm{Hom}(T^!, M)^G$, where $T^!$ is the complex concentrated in negative multidegrees; provided M is bounded below this is isomorphic to the rth suspension of $T \otimes_G M$, and this has homology $H_*(G; M)$ by definition. If M is not bounded below, the first complex has infinite products where the second has infinite sums.

Now

$$\mathrm{Hom}(T[\sigma], M) = \mathrm{Hom}(\lim_{\substack{\leftarrow \\ k}} \Sigma^{-k|\sigma|} T, M)$$

and, provided M is bounded below, this is equal to $\lim_{\substack{\rightarrow \\ k}} \mathrm{Hom}(\Sigma^{-k|\sigma|} T, M)$ because the limit is achieved in each total degree. Thus, if one takes homology in the kG-resolution direction first, one obtains the stable Koszul complex of $H^*(G; M)$. To avoid the requirement of boundedness we simply use the double complex

$$\lim_{\substack{\rightarrow \\ s}} \mathrm{Hom}(L[\geq s], M)$$

from the start, where $L[\geq s]$ is the quotient of L by the subcomplex of boxes which are at least s boxes below zero in some coordinate.

As is familiar from the case of commutative algebra, to construct the second spectral sequence we should consider the double complex

$$\operatorname*{holim}_{\substack{\leftarrow \\ s}} L[\geq s] \otimes_G M$$

If we take homology in the Koszul direction first we obtain

$$\begin{aligned} \operatorname*{holim}_{\substack{\leftarrow \\ s}} T^![\geq s] \otimes_G M &= \operatorname*{holim}_{\substack{\leftarrow \\ s}} \mathrm{Hom}((T^![\geq s])^*, M)^G \\ &= \mathrm{Hom}(\operatorname*{holim}_{\substack{\rightarrow \\ s}} (T^![\geq s])^*, M)^G \\ &\simeq \mathrm{Hom}(\Sigma^r T, M)^G \end{aligned}$$

On the other hand, if we take homology in the kG-resolution degree we obtain a homotopy inverse limit of complexes, each term of which is a suspension of $H_*(G; M)$, and so that the differentials are products of the chosen generators of I. By definition this is $\operatorname*{holim}_{\substack{\leftarrow \\ s}} UK_s(\mathbf{x}) \otimes_{H^*(G)} H_*(G; M)$, and by definition, its homology is the local homology in the statement.

For the Tate spectral sequence we combine these methods to form the double complex

$$\underset{\rightarrow}{\text{holim}}_t \, \text{Hom}(L[\leq t], \underset{\leftarrow}{\text{holim}}_s \, T^![\geq s] \otimes_G M).$$

13. Gorenstein localizations.

In this final section we point out that a local cohomology theorem in the sense of Section 12 implies a strong duality theorem in certain cases. The idea is that the local cohomology theorem gives a covariant equivalence of two objects that are quite generally contravariantly equivalent using a universal coefficient theorem. The composite contravariant self-equivalence is the duality.

To motivate the name, we recall that under mild hypotheses, a commutative complete local k-algebra (R, I, k) of dimension d is Gorenstein if $H^*_I(R) = H^d_I(R)$ (i.e. R is Cohen-Macaulay) and in addition

$$R = \text{Hom}_R(H^d_I(R), R^\vee) = \text{Hom}_{R/I}(H^d_I(R), R/I)$$

where $M^\vee = \text{Hom}_{R/I}(M, R/I)$. We want to consider a homotopy level version of the Gorenstein condition on the unit object S in a stable homotopy category $\mathcal{C}$. To make sense of this we need (i) a second stable homotopy category $\overline{\mathcal{C}}$ with unit object $\overline{S}$, (ii) a 'restriction' functor $r : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$, thought of as a forgetful map, and required to be lax monoidal, and (iii) an 'inflation' functor $i : \overline{\mathcal{C}} \longrightarrow \mathcal{C}$, splitting the forgetful map, and also required to be lax monoidal. This gives sense to the statement that S is an $\overline{S}$-algebra. Now take $I = \ker(S_* \longrightarrow \overline{S}_*)$, and say that S is *homotopically I-Gorenstein* if it is complete and there is an equivalence

$$S \simeq F(\Gamma_I(S), S^\vee) = F_{\overline{S}}(\Gamma_I(S), \overline{S})$$

where $X^\vee = F_{\overline{S}}(X, \overline{S})$, and where the $\overline{S}$-function object is an additional piece of structure.

To see that the homotopical Gorenstein statement has force, suppose $\overline{S}_*$ is a field. We then remark that if S is homotopically Gorenstein and S_* is Cohen-Macaulay then S_* is Gorenstein. Indeed, if S_* is Cohen-Macaulay of dimension d, then $\pi_*(\Gamma_I(S)) = H^d_I(S_*)$ from the spectral sequence of Section 6, and in the presence of a universal coefficient theorem we find a spectral sequence

$$\text{Ext}^{*,*}_{\overline{S}_*}(H^d_I(S_*), \overline{S}_*) \Rightarrow S_*.$$

If in addition $\overline{S}_*$ is a field, this states that S_* is the dual of $H^d_I(S_*)$ and so S_* is Gorenstein. See [21] for further investigation.

The principal example of the present formal setup is when $\mathcal{C}$ is the category of equivariant $\mathbf{R}$-modules for a highly structured split ring spectrum $\mathbf{R}$ and $\overline{\mathcal{C}}$ is the category of non-equivariant $\mathbf{R}$-modules. The relevant functors have been constructed by Elmendorf and May [15, 37].

In this case the augmentation is right adjoint to product with $A = G_+$, and there is additional structure since the completion $X^\wedge_{\mathcal{A}} = F(EG_+, X)$ and the torsion $\Gamma_{\mathcal{A}}(X) = EG_+ \wedge X$ both have homotopy described in nonequivariant terms. It is pointed out in the appendix to [21] that when there is a local cohomology theorem, $\mathbf{R}^\wedge_{\mathcal{A}} = F(EG_+, \mathbf{R})$ is homotopically Gorenstein. Recalling that $S = \mathbf{R}$ in the equivariant category and $\overline{S} = \mathbf{R}$ in the non-equivariant category, we may summarize the proof as follows

$$F_{\overline{S}}(\Gamma_I(S^\wedge_{\mathcal{A}}), \overline{S}) \simeq F_{\overline{S}}(\Gamma_{\mathcal{A}}(S^\wedge_{\mathcal{A}}), \overline{S}) \simeq F_{\overline{S}}(\Gamma_{\mathcal{A}}(S), \overline{S}) \simeq F_S(\Gamma_{\mathcal{A}}(S), S) = S^\wedge_{\mathcal{A}}.$$

The first equivalence is the local cohomology theorem, the second is 2.3 and the third is the split condition.

We remark that one expects a twisting in the application of the universal coefficient theorem when G is not a finite group. For example with a compact Lie group G, the twisting is given by the adjoint bundle in the Adams isomorphism. Similarly the twisting is given by the dualizing module for a virtual Poincaré duality group as in [8]. The twisting is built from the invertible object S^τ in the sense that it is essentially S^τ on each copy of A used to build $\Gamma_A(S)$. Thus, when G is a compact Lie group of dimension d, the adjoint bundle is a trivial d-dimensional bundle over any cell $S^n \wedge G_+$.

The existence and implications of the homotopy Gorentstein duality statement has been investigated for the cohomology of groups [19, 9, 8, 21], and for coefficients of equivariant cohomology theories in [17, 25, 26, 27]. We remark here that there is a precise formal similarity with Gross-Hopkins duality [28, 30, 41], which states that the Brown-Comenetz dual IM_nX of the monochromatic section M_nX is a twisted suspension of $L_{K(n)}DX$ for suitable finite spectra X, where M_nX is the fibre of $L_nX \longrightarrow L_{n-1}X$. Hopkins and Ravenel have proved there are spectral sequences for calculating the homotopy of M_nX and $L_{K(n)}X$ whose E_2-terms are the cohomology of the profinite group $\Gamma = S_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ with suitable coefficients, where S_n is the Morava stabilizer group. Furthermore, Γ is a p-adic Lie group; if it is p-torsion free it is a Poincaré duality group, and in general its cohomology has a local cohomology theorem as in [8] (the proof in the discrete case carries over to the profinite case in the category of Symonds-Weigel [42]). The local cohomology theorem at the E_2 level is the precise counterpart of the Gross-Hopkins duality between the spectra.

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