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Sequentially continuous linear mappings in constructive analysis

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1 Introduction

A mapping $u: X \to Y$ between metric spaces is sequentially continuous if for each sequence (x_n) converging to $x \in X$, $(u(x_n))$ converges to u(x). It is well known in classical mathematics that a sequentially continuous mapping between metric spaces is continuous; but, as all proofs of this result involve the law of excluded middle, there appears to be a constructive distinction between sequential continuity and continuity. Although this distinction is worth exploring in its own right, there is another reason why sequential continuity is interesting to the constructive mathematician: Ishihara [8] has a version of Banach's inverse mapping theorem in functional analysis that involves the sequential continuity, rather than continuity, of the linear mappings; if this result could be upgraded by deleting the word "sequential", then we could prove constructively the standard versions of the inverse mapping theorem and the closed graph theorem.

Troelstra [9] showed that in Brouwer's intuitionistic mathematics (INT) a sequentially continuous mapping on a separable metric space is continuous. On the other hand, Ishihara [6], [7] proved constructively that the continuity of sequentially continuous mappings on a separable metric space is equivalent to a certain boundedness principle for subsets of \mathbb{N} ; in the same paper, he showed that the latter principle holds within the recursive constructive mathematics (RUSS) of the Markov School. Since it is not known whether that principle holds within Bishop's constructive mathematics (BISH), of which INT and RUSS are models and which can be regarded as the constructive core of mathematics, the exploration of sequential continuity within BISH holds some interest

In this paper we derive some results about sequentially continuous linear mappings within BISH. These results tend to reinforce our hope that such mappings may turn out to be bounded (continuous) after all. For background material on BISH, see [1], and for information about the relation between BISH, INT, and RUSS, see [2].

2 Sequential continuity preserves Cauchyness

The main result of this section and of the entire paper is Theorem 1 below, in which we show that a linear mapping is sequentially continuous if and only if it preserves the Cauchyness of sequences.¹ To do this we shall need the constructive *least-upper-bound principle*:

¹It is trivial to show that a bounded linear mapping preserves Cauchyness.

Let S be a nonempty set of real numbers that is bounded above. Then $\sup S$ exists if and only if for all real numbers a, b with a < b, either b is an upper bound of S or else there exists $s \in S$ such that s > a([1], Ch. 2, (4.3)).

LEMMA 1 Let $u: X \to Y$ be a sequentially continuous linear mapping between normed spaces, (x_n) a Cauchy sequence in X, and let 0 < a < b. Then either $||u(x_n)|| < b$ for all n or else there exists n such that $||u(x_n)|| > a$.

PROOF. In view of the linearity of u, we may assume that b - a > 1. Choosing a strictly increasing sequence $(N_k)_{k=1}^{\infty}$ of positive integers such that $||x_m - x_n|| < 2^{-3k}$ for all $m, n \ge N_k$, write

$$s_k = \max \{ \|u(x_n)\| : 1 \le n \le N_k \}.$$

Construct an increasing binary sequence $(\lambda_k)_{k=1}^{\infty}$ such that

$$\begin{array}{ll} \lambda_k = 0 & \Rightarrow & \forall j \le k \ (s_j < b - 2^{-2j}), \\ \lambda_k = 1 & \Rightarrow & \exists j \le k \ (s_j > b - 2^{-2j+1}) \end{array}$$

We may assume that $\lambda_1 = \lambda_2 = 0$. Now construct a sequence (z_k) in X as follows. If $\lambda_{k+1} = 0$, or if $\lambda_{k+1} = \lambda_k = 1$, set $z_k = 0$. If $\lambda_{k+1} = 1$ and $\lambda_k = 0$, then

$$||u(x_{N_k})|| \le s_k < b - 2^{-2k}$$

and $s_{k+1} > b - 2^{-2k-1}$, so we can choose j such that $N_k < j \le N_{k+1}$ and $||u(x_j)|| > b - 2^{-2k-1}$; setting $z_k = 2^{2k}(x_j - x_{N_k})$, we note that $||z_k|| < 2^{-k}$ and

$$\begin{aligned} \|u(z_k)\| &= 2^{2k} \|u(x_j) - u(x_{N_k})\| \\ &\geq 2^{2k} (\|u(x_j)\| - \|u(x_{N_k})\|) \\ &> 2^{2k} (b - 2^{-2k-1} - (b - 2^{-2k})) = \frac{1}{2} \end{aligned}$$

This completes the construction of a sequence (z_k) converging to 0 in X. So, by the sequential continuity of u, $\lim_{k\to\infty} u(z_k) = 0$. Now choose K such that $||u(z_k)|| < 1/2$ for all $k \ge K$; then $\lambda_k \ne 1 - \lambda_k$ for all $k \ge K$. If $\lambda_K = 1$, then there exists $n \le N_K$ such that

$$||u(x_n)|| > b - 2^{-2n+1} > a.$$

If $\lambda_K = 0$, then $\lambda_k = 0$ for all $k \ge K$ and therefore for all k, so $||u(x_k)|| < b$ for all k. \Box

LEMMA 2 Let $u: X \to Y$ be a sequentially continuous linear mapping between normed spaces, and (x_n) a Cauchy sequence in X. Then $\sup_{n\geq 1} ||u(x_n)||$ exists. PROOF. We first show that the sequence $(||u(x_n)||)$ is bounded. To do so, choose R > 0 such that $||x_n|| \leq R$ for all n. Taking a = 1 and b = 2 in Lemma 1, we may assume that there exists n_1 such that $||u(x_{n_1})|| > 1$. Set $\lambda_1 = 0$. Using Lemma 1 repeatedly, we now construct an increasing binary sequence (λ_n) , and an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, such that

$$egin{array}{lll} \lambda_k = 0 & \Rightarrow & \|u(x_{n_k})\| > k ext{ and } n_k > n_{k-1}, \ \lambda_k = 1 & \Rightarrow & (u(x_n)) ext{ is a bounded sequence and } n_{k+1} = n_k. \end{array}$$

Assume that we have found λ_k and n_k . If $\lambda_k = 1$, we set $\lambda_{k+1} = \lambda_k$ and $n_{k+1} = n_k$. If $\lambda_k = 0$, then $||u(x_{n_j})|| > j$ for all $j \leq k$. We then apply Lemma 1 to the Cauchy sequence $(x_j)_{j>n_k}$. Either we obtain $n_{k+1} > n_k$ such that $||u(x_{n_{k+1}})|| > k + 1$, or else $||u(x_j)|| < k + 2$ for all $j > n_k$. In the first case we set $\lambda_{k+1} = 0$, and in the second, noting that $(u(x_n))_{n=1}^{\infty}$ is bounded, we set $\lambda_{k+1} = 1$ and $n_{k+1} = n_k$.

If $\lambda_k = 0$, set $z_k = k^{-1}x_{n_k}$; if $\lambda_k = 1$, set $z_k = 0$. Then $||z_k|| \leq Rk^{-1}$ for each k, so $z_k \to 0$ and therefore, by the sequential continuity of u, $u(z_k) \to 0$. Choose K such that $||u(z_k)|| < 1$ for all $k \geq K$. If $\lambda_K = 0$, then $||u(z_k)|| = k^{-1} ||u(x_{n_k})|| > 1$, a contradiction. Hence $\lambda_K = 1$ and so $(||u(x_n)||)$ is bounded.

It follows immediately from this, Lemma 1, and the constructive least-upperbound principle that $\sup_{n>1} ||u(x_n)||$ exists. \Box

THEOREM 1 A linear mapping $u: X \to Y$ between normed spaces is sequentially continuous if and only if it maps Cauchy sequences to Cauchy sequences. PROOF. Assume first that u is sequentially continuous. Given a Cauchy sequence (x_n) in X, choose a strictly increasing sequence $(N_k)_{k=1}^{\infty}$ of positive integers such that $||x_m - x_n|| < 2^{-k}$ for all $m, n \ge N_k$. For each k let

$$s_k = \sup_{n \geq N_k} \left\| u(x_n) - u(x_{N_k})
ight\|_{2}$$

which exists in view of Lemma 2. Given $\varepsilon > 0$, we show that $s_k < \varepsilon$ for some k. To this end, construct an increasing binary sequence (λ_k) such that

$$\lambda_k = 0 \Rightarrow s_k > \varepsilon/4$$

 $\lambda_k = 1 \Rightarrow s_k < \varepsilon/2.$

We may assume that $\lambda_1 = 0$. If $\lambda_k = 0$, choose $j \ge N_k$ such that $||u(x_j) - u(x_{N_k})|| > \varepsilon/4$ and set $z_k = x_j - x_{N_k}$. If $\lambda_k = 1$, set $z_k = 0$. Then $||z_k|| < 2^{-k}$ for each k, so $z_k \to 0$. Since u is sequentially continuous, $u(z_k) \to 0$ and we can choose K such that $||u(z_k)|| < \varepsilon/4$ for all $k \ge K$. If $\lambda_K = 0$, then $||u(z_K)|| > \varepsilon/4$, which is absurd; so $\lambda_K = 1$ and therefore $s_k < \varepsilon/2$. It follows that for all $j, k \ge N_K$,

$$\begin{aligned} \|u(x_j) - u(x_k)\| &\leq \|u(x_j) - u(x_{N_k})\| + \|u(x_k) - u(x_{N_k})\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since ε is arbitrary, $(u(x_n))$ is a Cauchy sequence in Y.

Now assume, conversely, that u maps Cauchy sequences to Cauchy sequences. If (x_n) is a sequence converging to 0 in X, then $(u(x_n))$ is a Cauchy sequence in Y; so in order to prove that $(u(x_n))$ converges to 0, it will suffice to find a subsequence of it that converges to 0. Accordingly, choose a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) such that $||x_{n_k}|| < 1/k^2$ for each k. Then $(kx_{n_k})_{k=1}^{\infty}$ converges to 0 in X, so $(u(kx_{n_k}))_{k=1}^{\infty}$ is a Cauchy sequence in Y. Hence there exists M > 0 such that, for each k, $||ku(x_{n_k})|| \leq M$ and therefore $||u(x_{n_k})|| \leq M/k$. Thus $\lim_{k\to\infty} u(x_{n_k}) = 0$ and our proof is complete. \Box

Theorem 1 enables us to extend a sequentially continuous linear map to the completion of its domain.

PROPOSITION 1 Let u be a sequentially continuous linear mapping of a normed space X into a Banach space Y. Then u extends to a sequentially continuous linear mapping of X^{\sharp} into Y, where X^{\sharp} is the completion of X.

PROOF. Let $(x_n), (x'_n)$ be sequences in X that converge to the same limit $x \in X^{\sharp}$. The foregoing theorem shows that

$$(u(x_1), u(x'_1), u(x_2), u(x'_2), \ldots)$$

is a Cauchy sequence in Y. Since Y is complete, this Cauchy sequence converges to a limit $y \in Y$. Hence each of the sequences $(u(x_n))$ and $(u(x'_n))$ converges to y; so

$$u^{\sharp}(x) \equiv \lim_{n \to \infty} u(x_n)$$

does not depend on the sequence (x_n) of elements of X converging to x. It is straightforward to show that u^{\sharp} is linear and coincides with u on X. Now let (x_n) be any sequence in X^{\sharp} converging to 0. By the definition of u^{\sharp} , for each n there exists $x'_n \in X$ such that $||x'_n - x_n|| \leq 1/n$ and $||u(x'_n) - u(x_n)|| < 1/n$. Then $\lim_{n\to 0} x'_n = 0$, so

$$0 = u^{\sharp}(0) = \lim_{n \to \infty} u(x'_n) = \lim_{n \to \infty} u(x_n).$$

Hence u^{\sharp} is sequentially continuous. \Box

3 Additional results on sequential continuity

To end the paper, we gather together some results connecting sequential continuity, boundedness, and normability for linear mappings.

PROPOSITION 1 Let $u: X \to Y$ be a sequentially continuous linear map between normed spaces, such that ker(u) is located. If $x_0 \in X$ and $u(x_0) \neq 0$, then $\rho(x_0, \text{ker}(u)) > 0$.

PROOF. Construct an increasing binary sequence (λ_n) such that

$$\begin{array}{ll} \lambda_n = 0 & \Rightarrow & \rho(x_0, \ker(u)) < 1/n^2, \\ \lambda_n = 1 & \Rightarrow & \rho(x_0, \ker(u)) > 1/(n+1)^2. \end{array}$$

If $\lambda_n = 0$, choose $y_n \in \ker(u)$ such that $||x_0 - y_n|| < 1/n^2$ and set $z_n = n(x_0 - y_n)$. If $\lambda_n = 1$, set $z_n = 0$. Then $||z_n|| < 1/n$ for each n, so $z_n \to 0$. By the sequential continuity of u, $(u(z_n))$ converges to 0 and therefore there exists N such that $||u(z_n)|| < 1$ for all $n \ge N$. If $\lambda_N = 0$, then

$$u(z_N) = N(u(x_0) - u(y_N)) = N(1 - 0) = N > 1,$$

a contradiction. Hence $\lambda_N = 1$ and therefore $\rho(x_0, \ker(u)) > 0$. \Box

COROLLARY Let $u: X \to Y$ be a nonzero sequentially continuous linear mapping with finite-dimensional range. Then u is compact if and only if ker(u) is located.

PROOF. Inspection of the proof of Theorem 1 in [3] shows that we need only prove that if $x_0 \in X$ and $u(x_0) = 1$, then $\rho(x_0, \ker(u)) > 0$; but this follows immediately from Proposition 2. \Box

COROLLARY Let $u: X \to \mathbb{C}$ be a nonzero sequentially continuous linear functional. Then u is normable if and only if ker(u) is located.

PROOF. This is a special case of the preceding corollary. \Box

Finally, we have a criterion for the boundedness of a sequentially continuous linear mapping. Note that if S is a subset of a normed space X, then

 $\sim S = \{x \in X : ||x - s|| > 0 \text{ for all } s \in S\}.$

PROPOSITION 3 A sequentially continuous linear mapping $u: X \to Y$ between normed spaces is bounded if and only if for each $\varepsilon > 0$ either there exists $x \in$ $\sim u^{-1}(B)$ or else $\sim u^{-1}(B)$ is bounded away from 0, where B is the closed unit ball of Y.

PROOF. Use a simple modification of the proof of Proposition 2 of [4]; we omit the details. \Box

All the foregoing results reinforce the hypothesis that sequential continuity and boundedness are equivalent properties of linear mappings within BISH. However, we should note that there is a sheaf model in which a certain linear mapping is sequentially continuous but not bounded (see page 293 of [5]); but the principle of countable choice does not hold in that model, and the foregoing proofs strongly suggest that we would have to use that principle in order to prove, within BISH, that sequential continuity entails boundedness for a linear map.

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