DIMACS Technical Report 97-59
September 1997

Chromatic Index Critical Graphs of Orders 11 and 12

by

Gunnar Brinkmann
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld, Germany

Eckhard Steffen[1]
Program in Applied and Computational Mathematics
Fine Hall, Washington Road
Princeton University
Princeton, New Jersey 08544

[1]supported by Deutsche Forschungsgemeinschaft

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs, Bellcore, and Bell Labs.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

A chromatic-index-critical graph G on n vertices is non-trivial if it has at most $\Delta\lfloor \frac{n}{2} \rfloor$ edges.

We prove that there is no chromatic-index-critical graph of order 12, and that there are precisely two non-trivial chromatic index critical graphs on 11 vertices.

Together with known results this implies that there are precisely three non-trivial chromatic-index-critical graphs of order ≤ 12.

1 Introduction

A famous theorem of Vizing [20] states that the chromatic index $\chi'(G)$ of a simple graph G is $\Delta(G)$ or $\Delta(G)+1$, where $\Delta(G)$ denotes the maximum vertex degree in G. A graph G is *class 1* if $\chi'(G) = \Delta(G)$ and it is *class 2* otherwise. A class 2 graph G is *(chromatic index) critical* if $\chi'(G-e) < \chi'(G)$ for each edge e of G. If we want to stress the maximum vertex degree of a critical graph G we say G is $\Delta(G)$*-critical.*

Critical graphs of odd order are easy to construct while not much is known about critical graphs of even order. One reason for this is that an overfull graph of odd order – that is a graph with more than $\Delta\lfloor\frac{|V|}{2}\rfloor$ edges – obviously is class 2, since it has too many edges. Thus it is not the specific structure of the graph which causes its colouring properties. The same holds true for many critical graphs with an odd number of vertices and $\Delta\lfloor\frac{|V|}{2}\rfloor + 1$ edges. We are interested in graphs which are class 2 for structural reasons, and define a critical graph with at most $\Delta\lfloor\frac{|V|}{2}\rfloor$ edges to be *non-trivial.* Clearly, each critical graph of even order is non-trivial.

Nevertheless the critical graph conjecture, independently formulated by Jakobsen [15] and Beineke, Wilson [1], claiming that every critical graph has odd order, is false.

Goldberg [12] constructed an infinite family of 3-critical graphs of even order. The smallest graph of this family has 22 vertices. Another counterexample – a 4-critical graph on 18 vertices – was independently found by Chetwynd and Fiol, cf. [14, 21]. Recently Grünewald and Steffen [13] constructed k-critical graphs of even order for each $k \geq 3$. It is still of interest which are the smallest k-critical graphs of even order, and Yap [21] posed the problem whether there are k-critical graphs of order 12, 14 or 16.

In [5] the authors showed that the graphs found by Goldberg and by Chetwynd, Fiol are the smallest 3- and 4-critical graphs of even order, respectively.

Based on results of [2] the first complete list of critical graphs of order $n \leq 8$ and of even order $n \leq 10$ was given in [10].

The gap for $n = 9$ was closed in [6] and hence for all $n \leq 10$ the critical graphs of order n are known.

It turned out that the Petersen graph minus a vertex is the only non-trivial critical graph on up to 10 vertices.

The aim of this paper is to determine all non-trivial critical graphs of order 11 or 12.

Theorem 1.1 *There are precisely two non-trivial critical graphs on 11 vertices.*

These two graphs are shown in Figure 1. Both can be obtained from the Petersen graph minus a vertex by replacing a vertex by a triangle.

Theorem 1.2 *There is no critical graph on 12 vertices.*

Together with the aforementioned results our theorems imply:

Corollary 1.3 *There are precisely three non-trivial critical graphs on up to 12 vertices.*

Corollary 1.4 *The smallest non-trivial critical graph is the Petersen graph minus a vertex, which is 3-critical.*

Corollary 1.4 motivates the following problem.

Problem 1.5 *For each $k \geq 4$, determine the smallest non-trivial k-critical graphs.*

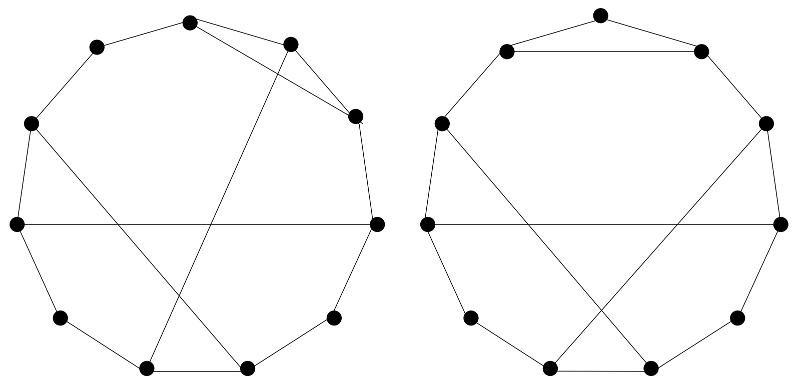

Figure 1

2 Proof of Theorem 1.1

For $k = 1, 2$ there are no non-trivial k-critical graphs on 11 vertices.

Let $3 \leq k \leq 10$. The following lower bounds for the number of edges in a k-critical graph are given by Corollary 5.4 and Theorem 5.7 in [21], and Theorems 13.2, 13.3 in [10].

Lemma 2.1 *Let G be a non-trivial k-critical graph on 11 vertices.*

1. *If $k = 3$, then $|E(G)| = 15$.*
2. *If $k = 4$, then $19 \leq |E(G)| \leq 20$.*
3. *If $k = 5$, then $23 \leq |E(G)| \leq 25$.*
4. *If $k = 6$, then $25 \leq |E(G)| \leq 30$.*
5. *If $k = 7$, then $28 \leq |E(G)| \leq 35$.*
6. *If $k = 8, 9, 10$, then $\frac{1}{8}(3k^2 + 6k - 1) \leq |E(G)| \leq 5k$.*

Using the graph generator *makeg* ([16]), for each maximum degree, we generated all graphs on 11 vertices and edges within the range of possible numbers given by Lemma 2.1.

We filtered out the *candidates*, which are graphs with at least two vertices of maximum degree k, no vertex of degree 1, and which have $5k$ edges or adding an edge increases the maximum vertex degree.

We computed the class 2 graphs among the candidates and checked whether they had critical subgraphs of the same order.

Our results are given in the following table. They prove Theorem 1.1.

k	#graphs	# candidates	# class 2 graphs	# k-critical graphs
3	671	482	22	2
4	118600	30037	278	0
5	3521278	323325	572	0
6	66250465	695751	527	0
7	170091250	302921	107	0
8	221586717	28485	21	0
9	7251796	893	0	0
10	5700	25	0	0

3 Proof of Theorem 1.2

3.1 Basic Results

We will need the following results. If we do not prove them or cite a paper explicitely their proofs could be found e.g. in [10, 21].

Lemma 3.1 (Vizings Adjacency Lemma) *Let $k \geq 0$ and vw be an edge of a k-critical graph G with $d_G(v) = d$. Then w is adjacent to at least $k - d + 1$ vertices of degree k. Furthermore $d_G(v) + d_G(w) \geq k + 2$.*

Let v be a vertex of a graph G. We define $s(v) = \Delta(G) - d_G(v)$ to be the *deficiency of* v. The *deficiency of* G is $s(G) = \sum_{v \in V(G)} s(v)$. The minimum vertex degree in G is denoted by $\delta(G)$.

Lemma 3.2 *A critical graph $G = (V, E)$ of even order has deficiency at least $2(\Delta(G) - \delta(G) + 1)$.*

Lemma 3.3 *For all $k \geq 3$, a k-critical graph of even order contains at least 3 vertices of degree smaller than k.*

The following is a generalization of the well known Parity Lemma [3].

Lemma 3.4 (Parity Lemma) *Let G be a graph whose edges are coloured with colours $1, \ldots, c$, and let a_i be the number of vertices v in G such that no edge incident to v is coloured i. Then for all $i = 1, \ldots, c$: $a_i \equiv |V(G)| \pmod 2$.*

Proof. For $i = 1, \ldots c$ let E_i be the set of edges coloured i. Then $a_i = |V(G)| - 2|E_i|$, and hence $a_i \equiv |V(G)| \pmod 2$.

•

Lemma 3.5 *Let G be a k-critical graph ($k \geq 4$) of even order which has precisely three vertices v_1, v_2, v_3 of degree smaller than k and $d_G(v_1) = 2$. Then $d_G(v_2) + d_G(v_3) = k$.*

Proof. Both neighbours of v_1 have degree k, and $G - v_1$ is k-colourable. Let $\{1, \dots k\}$ be the set of colours. Since $|V(G - v_1)|$ is odd it follows from the Parity Lemma that each colour is missing in an odd number of vertices. Since G is critical it follows that at the neighbours of v_1 the same colour is missing. Thus each colour is missing either in v_2 or v_3 and therefore $d_G(v_2) + d_G(v_3) = k$.

•

The following theorem is due to Yap [21]:

Theorem 3.6 ([21]) *For all integers $k \geq 5$, $r \geq 0$, there is no k-critical graph with degree sequence $2^r k^{2r}$.*

Let $G = (V, E)$ be a graph. We define V_i to be the set of vertices having degree i in G, and $n_i = |V_i|$.

Theorem 3.7 *Let G be a k-critical graph. Then*

$$\sum_{\lfloor \frac{k+2}{2} \rfloor \leq i \leq k} n_i \geq \max\{i | 2 < i < \lfloor \frac{k+2}{2} \rfloor \quad \text{and} \quad n_i \neq 0\} + 2n_2.$$

Proof. Given a k-critical graph G. Since $d_G(v) + d_G(w) \geq k + 2$ if $vw \in E(G)$, it follows that $\bigcup_{2 \leq i < \lfloor \frac{k+2}{2} \rfloor} V_i$ is an independent set in G. Furthermore each neighbour of a vertex of degree two is not adjacent to a vertex of degree i, for $2 < i < k$. Hence there are $2n_2 + \sum_{2 \leq i < \lfloor \frac{k+2}{2} \rfloor} n_i$ vertices in G which are not neighbour of a vertex of degree i for $3 \leq i \leq \lfloor \frac{k+2}{2} \rfloor$. Therefore $\max\{i | 2 < i < \lfloor \frac{k+2}{2} \rfloor \quad \text{and} \quad n_i \neq 0\} \leq |G| - (2n_2 + \sum_{2 \leq i < \lfloor \frac{k+2}{2} \rfloor} n_i) = \sum_{\lfloor \frac{k+2}{2} \rfloor \leq i \leq k} n_i - 2n_2$, proving the Theorem.

•

Let G be a critical graph and G^* a graph obtained from G by adding a maximum number of edges such that $\Delta(G^*) = \Delta(G)$. With V^- we denote the set of vertices of G^* with degree smaller than $\Delta(G)$.

Lemma 3.8 *Let G be a critical graph. Then the following holds true:*

1. *The subgraph $G^*[V^-]$ induced by V^- is a complete graph.*
2. $|V^-| \leq \Delta(G) - 2$
3. $\min\{d_{G^*}(v) | v \in V^-\} \geq |V^-| + 1$
4. *Let $xy \in E(G^*)$. If there are $v, w \in V^-$, such that $s(v) + s(w) \geq 2$ and $dist_{G^*}(x, v)$, $dist_{G^*}(y, w) \geq 2$ then $xy \in E(G)$.*

5. *Let $v \in V^-$ and let $xy \in E(G^*) \setminus E(G)$, $vy \notin E(G^*)$. Then G_1^* with $V(G_1^*) = V(G)$ and $E(G_1^*) = (E(G^*) - xy) \cup \{vy\}$ is another graph obtained from G by adding a maximum number of edges.*

Proof. Since G^* is obtained from G by adding maximum number of edges 1 is proved. Each vertex of G is adjacent to at least two vertices of maximum degree and hence 2 and 3 hold true.

Let $xy \in E(G^*)$, $v, w \in V^-$ and let $dist_{G^*}(x, v), dist_{G^*}(y, w) \geq 2$. Assume that $xy \notin E(G)$. Then G' with vertex set $V(G') = V(G)$ and edge set $E(G') = (E(G^*) - xy) \cup \{xv, yw\}$ has G as subgraph, it has maximum vertex degree k and $|E(G')| = |E(G^*)| + 1$. This contradicts the fact that G^* is obtained from G by adding a maximum number of edges. Thus $xy \in E(G)$.

Item 5 can be proven similarly.

•

Theorem 3.9 ([19]) *Let G be a graph on $n \geq 3$ vertices. If $\sum_{1 \leq i \leq k} n_i < k$ for each $1 \leq k < \frac{n-1}{2}$ (and $n_{\frac{n-1}{2}} \leq \frac{n-1}{2}$ if n is odd), then G is hamiltonian.*

It is well known that a regular graph of order $2n$ and vertex degree $r \in \{2n-1, 2n-2\}$ is class 1. Chetwynd and Hilton improved this result as follows.

Theorem 3.10 ([8]) *Let G be a regular graph of order $2n$ and vertex degree $r \in \{2n-3, 2n-4, 2n-5\}$. If $r \geq 2\lfloor \frac{1}{2}(n+1) \rfloor - 1$, then G is class 1.*

As a simple consequence we state

Corollary 3.11 *Let $k \geq 7$ and G be a k-critical graph on 12 vertices. Then G is not subgraph of a k-regular graph on 12 vertices.*

3.2 The Proof

Theorem 1.2 is true for $k = 2$. For $k = 3$ and 4 this is proved in [2] and [9, 11], respectively. For $k = 8$ this follows from more general results of [7], and for $k = 9, 10, 11$ it is a consequence of results of [17, 18].

Thus to prove Theorem 1.2 we have to solve the cases $k = 5, 6, 7, 8$. Our aim was to test as few graphs as possible. So we did not want to test every possibly critical graph, but tried to find a smaller set of supergraphs containing all these. This set was filtered for class 2 graphs and they were searched for critical subgraphs of the same order. So our aim was to have as few class 2 graphs as possible in the set of supergraphs – in particular we tried to avoid trivial class 2 graphs, that is: graphs containing an overfull subgraph of odd order. We proceeded as follows:

Assume there is a k-critical graph G of order 12. We add edges to G as long as we do not create a vertex of degree more than k, and we do not create a graph having a subgraph on 11 vertices with more than $5k$ edges. This allows to determine the set of possible degree sequences of such graphs. For $k = 5, 6$ we generate all possible graphs for these degree sequences and filter out the class 2 graphs. If there are class 2 graphs we look for k-critical subgraphs of order 12.

For $k = 7, 8$ in some cases the problem can be reduced: By theorem 3.9 these graphs have a hamiltonian cycle and therefore a 1-factor as well. Removing such subgraphs yields graphs having maximum vertex degree $k - 2$ or $k - 1$ that are class 2 if the original graphs are class 2. So some of the degree sequences can be reduced to sequences that also have to be checked for smaller k. If for these reduced sequences no class 2 graphs exist, the non-reduced sequences need not be tested.

In the following proofs we sometimes refer to the fact that the number of vertices of odd degree in a graph is even. We call this the *parity condition.*

Lemma 3.12 *For all $k \in \{5, 6, 7, 8\}$ we have: If there is a k-critical graph G on 12 vertices then there is a class-2-graph G' with degree sequence $\pi(G') \in \{4^2 5^{10}, 4^2 6^{10}, 4^2 7^{10}, 4^2 8^{10}, 45^2 6^9, 4567^9, 4578^9, 46^2 8^9, 5^4 6^8, 5^3 7^9, 5^2 6^{10}, 5^2 68^9, 56^2 78^8, 57^3 8^8, 6^4 8^8, 6^2 7^2 8^8, 67^4 8^7\}$, and G' is a supergraph of G with the maximal degree or can be obtained from a supergraph of G with the same maximal degree by deleting a perfect matching or a hamiltonian cycle.*

Proof of the Lemma. We check the cases succesively.

Claim 3.12.1 *Each 5-critical graph G of order 12 is subgraph of a graph G' with degree sequence $\pi(G') = 4^2 5^{10}$.*

Proof. If G^* is 5-regular, then there is an edge $e \in E(G^*) \setminus E(G)$ so that G is subgraph of $G' = G^* - e$, and $\pi(G') = 4^2 5^{10}$.

Let G^* be not 5-regular. It follows from Lemma 3.8 (2) and the parity condition that $|V^-| \in \{1, 2\}$.

If $V^- = \{v\}$ then $d_{G^*}(v) = 3$. Then Lemma 3.3 implies that there is an edge $xy \in E(G^*) \setminus E(G)$ with $dist_{G^*}(v, y) \geq 2$. Thus by Lemma 3.8 (5) there is a supergraph G' of G with $\pi(G') = 4^2 5^{10}$.

If $V^- = \{v, w\}$ then $d_{G^*}(v) = d_{G^*}(w) \in \{3, 4\}$. Assume $d_{G^*}(v) = 3$. Then $vw \notin E(G)$ and hence $d_G(v) = d_G(w) = 2$. Since all neighbours of v or w have degree 5 in G and they are pairwise different, vw is the only edge which is added to G to obtain G^*. But this contradicts Lemma 3.3.

Claim 3.12.2 *Each 6-critical graph G on 12 vertices is subgraph of a graph G' with degree sequence $\pi(G') \in \{4^2 6^{10}, 45^2 6^9, 5^2 6^{10}, 5^4 6^8\}$.*

Proof. If G^* is 6-regular then there is an edge $e \in E(G^*) \setminus E(G)$ such that G is subgraph of $G' = G^* - e$ and $\pi(G') = 5^2 6^{10}$.

If G^* is not 6-regular, then by Lemma 3.8 (2) it follows that $1 \leq |V^-| \leq 4$.

If $V^- = \{v\}$ then due to the maximality of G^* we have $d_{G^*}(v) = 4$. Lemmas 3.2 and 3.3 imply that there is an edge $xy \in E(G^*) \setminus E(G)$ such that $G' = G^* - xy$ has degree sequence $45^2 6^9$.

If $V^- = \{v, w\}$ it follows that $d_{G^*}(v), d_{G^*}(w) \geq 3$. The only possible cases which are not mentioned in the assertion are $d_{G^*}(v) = d_{G^*}(w) = 3$ and $d_{G^*}(v) = 3$ and $d_{G^*}(w) = 5$. Lemmas 3.2 and 3.3 imply that there is an edge $xy \in E(G^*) \setminus E(G)$ such that $x \neq v, w$.

In the first case $y \neq v, w$ and hence we obtain a contradiction to the maximality of G^*.

In the second case it follows that $dist_{G^*}(v, x) \geq 2$. Hence by Lemma 3.8 (5), G is subgraph of a graph G' with $\pi(G') \in \{4^2 6^{10}, 45^2 6^9\}$.

If $V^- = \{v_1, v_2, v_3\}$ then Lemma 3.8 (3) implies $d_{G^*}(v_i) \geq 4$, $i = 1, 2, 3$.

Assume $\pi(G^*) \neq 45^2 6^9$. Then $d_{G^*}(v_i) = 4$ for $i = 1, 2, 3$, and from Lemma 3.1 follows that V^- is an independent set in G. In fact, v_i and v_j, $1 \leq i < j \leq 3$ do not even have a neighbour in common. Thus identifying v_1, v_2 and v_3 yields a 6-regular class 2 graph on 10 vertices. But such graphs do not exist, cf. [4].

If $|V^-| = 4$ then follows from Lemma 3.8 (3) that $\pi(G^*) = 5^4 6^8$.

Claim 3.12.3 *If there is a 7-critical graph G on 12 vertices then there is a class 2 graph G' with degree sequence $\pi(G') \in \{4^2 6^{10}, 4^2 7^{10}, 45^2 6^9, 4567^9, 5^4 6^8, 5^3 7^9, 5^2 6^{10}\}$.*

Proof. We show that G is subgraph of a graph H with degree sequence $\pi(H) \in \{4^2 7^{10}, 4567^9, 5^3 7^9, 5^2 7^{10}, 56^2 7^9, 6^4 7^8, 6^2 7^{10}\}$. In some cases we remove a 1-factor from H to reduce the sequence to one already appearing in Claim 3.12.2.

Because of the parity condition, Lemma 3.8 (2,3) and Corollary 3.11 we have $1 \leq |V^-| \leq 4$.

If $V^- = \{v\}$ then it is easy to see that $d_{G^*}(v) = 5$. Lemmas 3.3 and 3.8 (4) imply that there is $xy \in E(G^*) \setminus E(G)$ with $x, y \neq v$. So G is subgraph of a graph with degree sequence $56^2 7^9$.

Let $V^- = \{v_1, v_2\}$ and assume $\delta(G^*) = d_{G^*}(v_1) = 3$. If $v_1 v_2 \in E(G)$ then $d_G(v_2) = 6$, a contradiction to the parity condition.

Thus $v_1 v_2 \notin E(G)$ and $d_G(v_1) = 2$. Due to the maximality of G^* there is an edge $v_2 x \in E(G^*) \setminus E(G)$, $x \neq v_1$ and $x v_1 \notin E(G^*)$. Thus $d_{G^*}(v_2) = 5$, and G is subgraph of $G' = (G^* - v_2 x) + v_1 x$ having degree sequence $4^2 7^{10}$.

If $\pi(G^*) = 467^{10}$ ($d_G(v_1) = 4$) then there is an edge $v_2 x \in E(G^*) \setminus E(G)$ such that $x \neq v_1$. Thus G is subgraph of $G^* - v_2 x$ having degree sequence 4567^9. So the case $|V^-| = 2$ is complete.

Let $V^- = \{v_1, v_2, v_3\}$, then $d_{G^*}(v_i) \geq 4$. Assume $d_{G^*}(v_1) = 4$, then precisely one vertex of V^- has degree 5, say v_2. We have to show that $4^2 57^9$ must not be considered, so assume $d_{G^*}(v_3) = 4$.

If $v_1 v_2$,$v_2 v_3$ or $v_1 v_3 \in E(G)$ then at least three neighbours of v_1 or v_3 in G have degree 7 in G. Thus $d_{G^*}(v_1) > 4$ or $d_{G^*}(v_3) > 4$, a contradiction.

Thus $v_1 v_2, v_1 v_3, v_2 v_3 \notin E(G)$, and hence $d_G(v_1) = d_G(v_3) = 2$ and $d_G(v_2) \leq 3$. By Lemma 3.5 and the maximality of G^* there is $x \in V^-$, $d_G(x) = 6$ with $x v_2 \in G^* \setminus G$. Thus $d_G(v_3) = 2$.

Since all second neighbours of v_1, v_2, v_3 in G have degree 7 in G, there are $3+6=9$ vertices in G which cannot be neighbour of x. Thus $d_G(x) \leq 2$, a contradiction.

All possible degree sequences with $d_{G^*}(v) > 4$ for all $v \in V^-$ are in the list.

Let $V^- = \{v_1, v_2, v_3, v_4\}$ then $d_{G^*}(v_i) \geq 5$. Assume $d_{G^*}(v_1) = 5$, then there is a second vertex $v_2 \in V^-$ with $d_{G^*}(v_2) = 5$.

Assume there is an edge v_1v_i, $2 \leq i \leq 4$. Since v_1 is adjacent to at most two vertices of degree 7 in G, it follows that $d_G(v_i) = 6$. Thus $v_1v_2 \notin E(G)$ and $v_iv_j \in E(G)$ for all $1 \leq i < j \leq 4, j > 2$. But then v_1, v_2 are adjacent to at most three vertices of degree 7 in G and hence $d_G(v_1) = 5$, and therefore $v_1v_2 \in E(G)$, a contradiction.

Thus $v_1v_i, v_2v_j \notin EG$ for $i = 2,3,4$ and $j = 3,4$. Hence $d_G(v_1) = d_G(v_2) = 2$, $d_G(v_3), d_G(v_4) \leq 4$. From Lemma 3.1 follows that V^- is an independent set in G and therefore $d_G(v_3), d_G(v_4) \leq 3$. In fact v_1, v_2 can not even share a neighbour with v_3, v_4.

Identifying v_1, v_3 and v_2, v_4 in $G' = G^* - \{v_iv_j | 1 \leq i < j \leq 4\}$ yields a class 2 graph H on 10 vertices having degree sequence 4^27^8 or 5^27^8. This graph contains a 7-critical subgraph of order at most 9.

By the results of [6] there are no 7-critical graphs on less than 9 vertices and those of order 9 have one of the following degree sequences: 27^8, 367^7, 457^7, 46^27^6, 5^267^6, 56^37^5 and 6^57^4. None of them is exendable to H.

Claim 3.12.4 *If there is an 8-critical graph G on 12 vertices then there is a class 2 graph G' with degree sequence $\pi(G') \in \{4^26^{10}, 4^28^{10}, 45^26^9, 4567^9, 4578^9, 46^28^9, 5^46^8, 5^37^9, 5^268^9, 56^278^8, 57^38^8, 6^48^8, 6^27^28^8, 67^48^7\}$.*

Proof. We show that G is subgraph of a graph H with degree sequence $\pi(H) \in \{4^28^{10}, 4578^9, 46^28^9, 5^268^9, 56^278^8, 5678^9, 57^38^8, 6^48^8, 6^38^9, 6^27^28^8, 6^28^{10}, 67^48^7, 67^28^9, 7^48^8\}$.

Applying Theorem 3.9 we sometimes remove a hamiltonian circuit or a 1-factor from H to obtain the desired result.

Because of Corollary 3.11 V^- is not empty. If $V_2(G) \neq \emptyset$, then – due to Lemma 3.1 – G contains at most four vertices of degree smaller than 8 and hence $|V^-| \leq 4$ in this case.

If $|V^-| = 6$ then $\pi(G^*) = 7^68^6$. Let $\partial(V^-)$ be the set of edges with precisely one end in V^-. Since each vertex of V^- is adjacent to precisely two vertices of degree 8 it follows $|\partial(V^-)| = 12$.

On the other hand just six vertices of G^* have degree 8, and hence each vertex of degree 8 is adjacent to at least three vertices of degree 7. Thus $|\partial(V^-)| = 18$, a contradiction, and hence $|V^-| \leq 5$.

Let $V^- = \{v_1, \ldots, v_5\}$. Then $d_{G^*}(v_i) \geq 6$ for all i. Because of the parity condition at least one vertex has degree 6, say v_1.

Since $d_G(v_1) \geq 3$ it follows that there is another vertex of V^-, say v_2, so that $v_1v_2 \in E(G)$. In addition v_1 is adjacent to precisely two vertices of degree 8 in G. Therefore $d_G(v_2) = 7$ and hence $v_2v_i \in E(G)$ for all $i \neq 2$ and v_2 is adjacent to at most three vertices of degree 8 in G. This implies $d_G(v_1) = 6$ and hence $v_1v_i \in E(G)$ for all $i \neq 1$. Therefore $d_G(v_i) = 7$ for $i \neq 1$ and hence $\pi(G^*) = \pi(G) = 67^48^7$ in this case.

Let $V^- = \{v_1, \ldots, v_4\}$. Then $d_{G^*}(v_i) \geq 5$ for all i.

All possible degree sequences with $d_{G^*}(v) > 5$ for all $v \in V^-$ are listed, so assume $d_{G^*}(v_1) = 5$.

If V^- is an independent set in G, then $d_G(v_i) \leq 4$ $(i = 2, 3, 4)$ and $d_G(v_1) = 2$. Thus G contains precisely four vertices with degree smaller than 8. Theorems 3.6 and 3.7 imply $\pi(G) \in \{23^2 48^8, 2^2 3^2 8^8, 2^2 4^2 8^8\}$. In case $\pi(G) = 23^2 48^8$ we have $\pi(G^*) = 56^2 78^8$, which is in the list. Otherwise there are two pairs of vertices with deficiency so that identifying the two vertices of a pair yields a class 2 graph H on 10 vertices with degree sequence $5^2 8^8$ or $6^2 8^8$. This graph contains an 8-critical subgraph H'. By the results of [6] H' has one of the following degree sequences: $57^3 8^5, 6^3 8^6, 6^2 7^2 8^5, 67^4 8^4, 7^6 8^3$, and none of them is extendable to H. Thus V^- is not an independent set in G.

If $v_1 v_i \in E(G)$ $(i \in \{2, 3, 4\})$ then v_i is adjacent to precisely four vertices of degree 8 in G, and hence $d_G(v_1) = 5$ and $d_G(v_i) = 7$. Therefore $v_1 v_2, v_1 v_3, v_1 v_4 \in E(G)$, $d_G(v_2) = d_G(v_3) = d_G(v_4) = 7$, and $G[V^-] = K_4$. Since all neighbours of $v_1, \ldots, v_4$ in $V(G) \setminus V^-$ have degree 8 in G, by maximality we have $\pi(G) = 57^3 8^8$. But then $s(G) = 6$, a contradiction to Lemma 3.2, and hence $v_i v_j \notin E(G)$ for $v_i, v_j \in V^-$ with $d_{G^*}(v_i) = 5$.

Thus we have $d_G(v_1) = 2$, and G has precisely four vertices with degree smaller than 8. Since V^- is not independent in G there is an edge between two vertices of V^-, say $v_2 v_3 \in E(G)$. Thus $d_{G^*}(v_2), d_{G^*}(v_3) \geq 6$. If $d_{G^*}(v_2) = 6$ then — since $v_1 v_2 \notin E(G)$ and v_3 is adjacent to at most four vertices of degree 8 — $d_G(v_2) = 5$ and hence $v_2 v_4 \in E(G)$, too. Thus v_2 is adjacent to at most three vertices of degree 8 in G and since $v_1 v_3, v_1 v_4 \notin E(G)$, $d_G(v_3) = d_G(v_4) = 6$ in contradiction to the parity condition. Hence $d_{G^*}(v_i) = 7$ for any vertex v_i which is adjacent in G to another vertex of V^-.

Thus $\pi(G^*) = 5^2 7^2 8^8$ or $57^3 8^8$. In the first case we have $\pi(G) = 2^2 5^2 8^8$. Identifying each vertex of degree 2 with one of the degree 5 vertices yields a class 2 graph on ten vertices with degree sequence $7^2 8^8$. As above we obtain a contradiction by applying the results of [6]. Thus $\pi(G^*) = 57^3 8^8$ in this case.

Let $V^- = \{v_1, v_2, v_3\}$. Then $d_{G^*}(v_i) \geq 4$ for all i.

Assume $d_{G^*}(v_1) = 4$.

If $v_1 v_2 \in E(G)$ then – since v_1 is adjacent to at most two vertices of degree 8 in G – $d_G(v_2) = 7$, and $d_G(v_1) = 4$. Thus $v_1 v_3, v_2 v_3 \in E(G)$, too. All neighbours of v_1, v_2 or v_3 not in V^- have degree 8 in G. Hence $\pi(G) = 47^2 8^9$ and $s(G) = 7$, in contradiction to Lemma 3.2.

Thus $d_G(v_1) = 2$, G contains at most 4 vertices with deficiency and if $d_{G^*}(v) = 4$ then $d_G(v) = 2$.

If G contains precisely three vertices with deficiency then Lemma 3.5 implies $d_G(v_2) + d_G(v_3) = 8$. Thus $\pi(G) \in \{2358^9, 24^2 8^9\}$. Furthermore V^- is an independent set in G and hence $\pi(G^*) \in \{4578^9, 46^2 8^9\}$ in this case.

If G has a fourth vertex x with degree smaller than 8, then $d_G(x) \geq 6$, $v_2 x$ or $v_3 x \in E(G^*) \setminus E(G)$, say $v_2 x$, and therefore $d_{G^*}(v_2) \geq 5$. Since $d_G(x) \geq 6$ there is at most one vertex with degree 2 in G, namely v_1. The only two possible degree sequences of G^* which are not asserted are $45^2 8^9$ and $47^2 8^9$.

Let $\pi(G^*) = 45^2 8^9$, then $v_2 v_3 \notin E(G)$ and hence $d_G(v_2) = d_G(v_3) = 3$. Thus there is no edge $v_2 x \in E(G^*) \setminus E(G)$, a contradiction.

If $\pi(G^*) = 47^2 8^9$ then $G' = (G^* - v_2 x) + v_1 x$ is a maximum graph and $\pi(G') = 5678^9$.

All sequences with $d_{G^*}(v_i) > 4$ for $1 \leq i \leq 3$ are contained in the list of possible degree sequences.

Let $V^- = \{v_1, v_2\}$.

If $d_{G^*}(v_1) = 3$ then $v_1 v_2 \notin E(G)$. Thus $d_G(v_1) = 2$ and G has at most four vertices with degree smaller than 8. Since G must have deficiency 14, we get $\pi(G) = 2^2 7^2 8^8$. So there are 6 vertices in G having only neighbours with degree 2 or 8. Thus G cannot contain two vertices of degree 7. Hence $d_{G^*}(v_1), d_{G^*}(v_2) \geq 4$.

Assume $d_{G^*}(v_1) = 4$. The only sequence not contained in the list is if $d_{G^*}(v_2) = 6$. Since $s(G) \geq 10$, there must be a vertex $x \notin V^-$, $v_2 x \in E(G^*) \setminus E(G)$. So G is subgraph of $G' = G^* - v_2 x$ and $\pi(G') = 4578^9$.

Thus we may assume $\delta(G^*) \geq 5$.

Let $\pi(G^*) = 5^2 8^{10}$. If $v_1 v_2 \notin E(G)$ then G is subgraph of $G^* - v_1 v_2$ having degree sequence $4^2 8^{10}$. If $v_1 v_2 \in E(G)$ then $d_G(v_1) = d_G(v_2) = 5$ and hence all neighbours of v_1, v_2 not contained in V^- have degree 8 in G in contradiction to the maximality of G^*.

Let $\pi(G^*) = 578^{10}$. Let $d_{G^*}(v_2) = 7$. Then there is an edge $v_2 x \in E(G^*) \setminus E(G)$ with $x \neq v_1$. Thus G is subgraph of $G' = G^* - v_2 x$ having degree sequence 5678^9.

If $\pi(G^*) = 7^2 8^{10}$ then G is subgraph of a graph having degree sequence $7^4 8^8$ or $67^2 8^9$.

Let $V^- = \{v\}$, then $d_{G^*}(v)$ is even and $d_{G^*}(v) \geq 4$.

If $d_{G^*}(v) = 6$. Then by Lemma 3.3 there is an edge $xy \in E(G^*) \setminus E(G)$ such that $G^* - xy$ has degree sequence $67^2 8^9$.

Thus let us assume that $d_{G^*}(v) = 4$.

If $d_G(v) = 2$ then $s(G) \geq 14$ and Lemma 3.5 implies that G has four vertices with degree smaller than 8. Since $d_{G^*}(v)$ is even, $vw \notin E(G^*)$ for exactly one vertex w with $d_G(w) < 8$. So $\pi(G) = 25^2 68^8$ and the set of vertices of degree smaller than 8 is independent. Hence three edges can be added to G to obtain a graph with degree sequence 4578^9.

If $d_G(v) = 3$ then $s(G) \geq 12$. In this case there are vertices w, x, y such that $vw, wx, wy \in E(G^*) \setminus E(G)$ and $vx, vy \notin E(G^*)$. Thus G is subgraph of $G' = (G^* - \{wx, wy\}) + \{vx, vy\}$ and $\pi(G') = 6^2 8^{10}$.

If $d_G(v) = 4$ then $s(G) \geq 10$. But G^* is maximum and hence only one edge was added to obtain G^* from G. Thus $s(G) \leq 6$, a contradiction.

•

Lemma 3.13 *There is no class 2 graph G with degree sequence $\pi(G) \in \{4^2 6^{10}, 4^2 7^{10}, 4^2 8^{10}, 45^2 6^9, 4567^9, 4578^9, 46^2 8^9, 5^4 6^8, 5^3 7^9, 5^2 6^{10}, 5^2 68^9, 56^2 78^8, 57^3 8^8, 6^4 8^8, 6^2 7^2 8^8, 67^4 8^7\}$,*

and there is no 5-critical graph on 12 vertices which is subgraph of a graph with degree sequence $4^2 5^{10}$.

Proof. With a computer aided check we proved that there are no class 2 graphs with degree sequence in the given set.

There are six class 2 graphs with degree sequence $4^2 5^{10}$, but none of them contains a 5-critical subgraph on 12 vertices.

•

Lemmas 3.12 and 3.13 imply Theorem 1.2.

4 An independent proof

For **all** kinds of results that are obtained with the help of a computer, an independent check is a very useful thing to do. We want to emphasize that we do not think that an error in a computer assisted proof is more likely than in a long proof done by hand and that all the programs used here have been carefully programmed and checked against all data available to us. But since computer programs are very hard to check and even hardware or compiler errors might occur, an independent implementation – or even better: an implementation of a completely independent method – reduces the probability of a wrong result caused by a program error.

We checked Theorem 1.2 using the following method:

A graph where every additional edge that can be inserted cannot be contained in a critical graph due to Vizings Adjacency Lemma is called a VAL-maximal graph. Obviously every critical graph is contained in at least one VAL-maximal graph of the same order and with the same maximum degree.

Some informal reasoning lead us to the expectation that there are less VAL-maximal graphs than graphs where no edges at all can be inserted without changing the maximum degree. And in fact in all the cases observed this was the case. Since for 9, 10 and 11 the result follows theoretically and since for 3 and 4 the result is well known [11][9][21][5], we had to generate all graphs on 12 vertices with maximum degree between 5 and 8 and filter them for VAL-maximal graphs. We used the graph generator *makeg* [16] for this. Since makeg only allows to give an upper bound for the maximum degree, we restricted the generation to graphs with maximum degree at most 8 (there are 112 458 045 313 graphs) and deleted those with maximum degree 3 or 4 (6 800 637 graphs).

The remaining graphs were filtered for VAL-maximal graphs, which is a fast test (in the worst case quadratic in the number of vertices). In all, 74 064 621 graphs fulfilled the Vizing criterion, 691 920 of them being VAL-maximal. They had to be tested by the colouring routine, which determined 203 177 graphs to be class 2 graphs. They were tested for critical subgraphs of the same order – without finding any. In fact it turned out that 203 168 of them were class 2 because of an overfull subgraph on 11 vertices, 3 of them because of an overfull subgraph on 9 vertices (maximal valency 8) and 6 of them because of an overfull subgraph on 7 vertices (maximal valency 5).

The generator used was independent of the one used in the previous part and of course the filtering for VAL-maximal graphs also is. In order to keep also the colour testing part independent, in addition we tested the results in the first part using an independent program for vertex colouring and checking the chromatic number of the edge dual graphs. Since this program was very slow for large vertex degrees, we had to use reduced valency sequences whenever possible, even if the number of graphs for the reduced sequence was much larger. This test was much slower than the one with the special routine for edge colouring, which was astonishingly efficient. In both approaches only a small ratio of the CPU was used for checking colourability. It works as follows:

Suppose a graph with maximal valence k shall be checked for being k-colourable. If it has odd order, we first check whether it is overfull. If it has even order we check whether deletion of the vertex of minimum degree gives an overfull subgraph. In both cases the graph can of course not be coloured. If the graph has passed these tests, we proceed as follows:

We are looking for a matching that is not contained in a larger one and contains a fixed edge (we choose it as one containing vertices with smallest possible degree) and all vertices with maximum degree. In some tests we made, choosing the fixed edge in a different way decreased the performance of the program. The graph is k-colourable if and only if such a matching exists so that the graph obtained when removing this matching is $k-1$ colourable. This recursive routine turned out to be surprisingly fast and was also used in the critical subgraph determining program for the tests run on these graphs.

So the only program parts not checked independently are some subroutines of the critical subgraph determining program (in the first approach it was only used for the sequence $4^2 5^{10}$ and $k = 5$). We tested some cases for both approaches on various operating systems with different compilers, but did not do two complete independent runs on different machines and operating systems.

Outlook and CPU requirement

The first approach needed less than 13 hours (accumulated CPU) on a cluster of Alphas, DECs, Suns and 133MHZ Linux Pentium PCs. In this approach only 7 926 900 graphs were generated. So if it would be possible to determine all possible sequences for $n = 14$, it might also be possible to check the existence of a critical graph of order 14. Nevertheless doing this by hand would be a very hard thing to do and errors can easily occur. So an automatic routine would be needed for this.

The second approach needed an accumulated CPU of 160 days on the same cluster. In this approach 112 458 045 313 graphs were generated, but only 691 920 were tested for colourability. So in spite of the fact that this approach can not be applied for $n = 14$, a slight variation might be successful: Almost all of the time was used for generating graphs and filtering them for VAL-maximal ones. If this part would be replaced by a graph generation program generating only maximal or VAL-maximal graphs, this approach might also succeed for 14 vertices. This would be another important step on the way to determining the smallest critical graph of even order.

References

[1] L. W. Beineke, R. J. Wilson, *On the edge chromatic number of a graph*, Disc. Math. **5**

(1973) 15-20

[2] L. W. Beineke, S. Fiorini, *On small graphs critical with respect to edge colourings*, Disc. Math. **16** (1976) 109-121

[3] D. Blanuša, *Problem ceteriju boja (The Problem of four Colors)*, Hrvatsko Prirodoslovno Društvo Glasnik Math.-Fiz. Astr. Ser. II 1 (1946) 31-42

[4] G. Brinkmann, *Generating cubic graphs faster than isomorphism checking* SFB-Preprint 92-047, Universität Bielefeld, (1992)

[5] G. Brinkmann, E. Steffen, *3- and 4-critical graphs of small even order*, Disc. Math. **169** (1997) 193-197

[6] A. G. Chetwynd, H. P. Yap, *Chromatic index critical graphs of order 9*, Disc. Math. **47** (1983) 23-33

[7] A. G. Chetwynd, A. J. W. Hilton, *The chromatic index of graphs of even order with many edges* J. of Graph Th. **8** (1984) 463-470

[8] A. G. Chetwynd, A. J. W. Hilton, *Regular graphs of high degree are 1-factorizable* Pro. London Math. Soc. **3**, 50 (1985) 193-206

[9] S. A. Choudum, K. Kayathri, *There are no edge-chromatic 4-critical graphs of order 12*, Disc. Math. 141 (1995) 67-73

[10] S. Fiorini, R. J. Wilson, *Edge-colourings of graphs*, Research notes in math. no. 17, Pitman, London, San Francisco, Melbourne (1977)

[11] G. P. Gavrilov, *On the nonexistence of 4-critical graphs of twelfth order*, Diskretnaya Mathematika 4 (1992) 99-114

[12] M. K. Goldberg, *Construction of class 2 graphs with maximum vertex degree 3*, J. of Comb. Theory, Ser. B 31 (1981), 282-291

[13] S. Grünewald, E. Steffen, *Critical graphs of even order*, DIMACS Technical Report 97-39 (1997)

[14] A. J. W. Hilton, R. J. Wilson, *Edge-Colorings of Graphs: A Progress Report* in: M. F. Cabobianco et al. (Eds.) *Graph Theory and its Applications: East and West*, New York (1989) 241-249

[15] I. T. Jakobsen, *On critical graphs with chromatic index 4*, Disc. Math. **9** (1974) 265-276

[16] B. D. McKay, *Isomorph-free exhaustive generation* to appear in Journal of Algorithms

[17] M. Plantholt, *The chromatic index of graphs with a spanning star*, J. of Graph Th. **5** (1981) 45-53

[18] M. Plantholt, *The chromatic index of graphs with large maximum degree*, Disc. Math. **47** (1983) 91-96

[19] L. Pósa, *A theorem concerning hamilton lines*, A Mat. Kut. Int. Közl. VII.A/1-2 (1962) 225-226

[20] V. G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Anal. 3 (1964) 25-30

[21] H. P. Yap, *Some topics in graph theory*, London Math. Soc. LNS 108, Cambridge University Press (1986)