

# On the Existence of an Optimally Failure Tolerant 7R Manipulator Jacobian

Anthony A. Maciejewski

Purdue University  
1285 Electrical Engineering Building  
West Lafayette, Indiana 47907-1285  
maciejew@ecn.purdue.edu  
Tel (317) 494-9855  
Fax (317) 494-6951

Rodney G. Roberts

Department of Electrical Engineering  
FAMU/FSU College of Engineering  
2525 Pottsdamer St.  
Tallahassee, FL 32310

## Abstract

Kinematically redundant manipulators have been proposed for use in remote or hazardous environments due to their potential for tolerance to joint failures. This article defines a local measure of failure tolerance based on the worst-case dexterity following a locked joint failure. Optimal values of this measure are identified and related to the required structure of a Jacobian matrix. It is shown that it is trivial to derive manipulator configurations that correspond to the desired Jacobians with the single exception of a manipulator with seven rotational joints that is required to perform tasks which require fully general linear and rotational velocities of the end effector.

## I. INTRODUCTION

Kinematically redundant manipulators have long been proposed for complicated tasks due to their inherently higher degree of dexterity. Recently, there has been an increased interest in the use of such manipulators for use in remote or hazardous environments due to their potential for failure tolerance [4]. Researchers have analyzed the reliability of robotic systems [11], developed techniques for identifying failures [12], and even designed joint drives with multiple actuators [13]. This article is concerned with the most basic aspect of a robotic manipulator, i.e. its kinematics.

It has been previously shown that the kinematic structure of a redundant manipulator must be carefully designed to guarantee that the additional degrees of freedom support failure tolerance [9]. The classic example of a non-failure tolerant design is the common kinematic structure that mimics the human arm. Locking the elbow joint of such an arm constrains the workspace of the wrist to lie on a sphere centered at the shoulder despite the fact that there remain six degrees of freedom in the mechanism.

To design a kinematically failure tolerant redundant manipulator, one must first decide on a definition of failure tolerance. One possibility is to focus on local dexterity measures centered around the manipulator Jacobian [7],[9]. An alternative is to consider the global characteristics of the manipulator such as its workspace [8], [10]. Both of these measures are related since workspace boundaries result in manipulator singularities that are easily identified through the Jacobian. In this work, the local definition of failure tolerance centered on desirable properties of the manipulator Jacobian is used and is mathematically defined in the following section. Section III then presents several examples of optimally failure tolerant Jacobians as well as manipulator configurations derived from these Jacobians. Since most

manipulators consist of primarily rotational joints, the constraints imposed by such designs are considered in Section IV. Finally, the conclusions of this work are presented in the last section.

## II. A DEFINITION OF FAILURE TOLERANCE

The dexterity of manipulators is frequently quantified in terms of the properties of the manipulator Jacobian matrix which relates end-effector velocities to joint angle velocities. The Jacobian will be denoted by the  $m \times n$  matrix  $J$  where  $m$  is the dimension of the task space and  $n$  is the number of degrees of freedom of the manipulator. For redundant manipulators  $n > m$  and the quantity  $n - m$  is the degree of redundancy. The manipulator Jacobian can be written as a collection of columns

$$J_{m \times n} = [j_1 \quad j_2 \quad \cdots \quad j_n] \quad (1)$$

where  $j_i$  represents the end-effector velocity due to the velocity of joint  $i$ . For an arbitrary single joint failure at joint  $f$ , assuming that the failed joint can be locked, the resulting  $m$  by  $n - 1$  Jacobian will be missing the  $f$ th column, where  $f$  can range from 1 to  $n$ . This Jacobian will be denoted by a preceding superscript so that in general

$${}^f J_{m \times n-1} = [j_1 \quad j_2 \quad \cdots \quad j_{f-1} \quad j_{f+1} \cdots j_n]. \quad (2)$$

The properties of a manipulator Jacobian are perhaps best illustrated through the use

of the singular value decomposition (SVD) which can be defined as

$$J = UDV^T \quad (3)$$

where  $U$  is an  $m$  by  $m$  orthogonal matrix of the output singular vectors,  $V$  is an  $n$  by  $n$  orthogonal matrix of the input singular vectors, and  $D$  is a diagonal matrix of the form

$$D_{m \times n} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & \sigma_m & 0 & \cdots & 0 \end{bmatrix} \quad (4)$$

where the  $\sigma_i$  are the singular values which are typically ordered from largest to smallest. Most local dexterity measures can be defined in terms of simple combinations of these singular values such as their product (determinant) [14], sum (trace) [2], or ratio (condition number) [1],[5],[6]. Task specific dexterity measures also require knowledge of the output singular vectors [3]. The most significant of the singular values is  $\sigma_m$ , the minimum singular value, since it is by definition the measure of proximity to a singularity and tends to dominate the behavior of both the manipulability (determinant) and the condition number. The minimum singular value is also a measure of the worst-case dexterity over all possible end-effector motions.

The definition of failure tolerance used in this work is based on the worst-case dexterity following an arbitrary locked joint failure. Since  ${}^f\sigma_m$  denotes the minimum singular value of  ${}^fJ$  then  ${}^f\sigma_m$  is a measure of the worst-case dexterity if joint  $f$  fails. If all joints are equally

likely to fail, then a measure of the worst-case failure tolerance is given by

$$\min_{f=1}^n({}^f\sigma_m). \tag{5}$$

To insure that manipulator performance is optimal prior to a failure, an optimally failure tolerant Jacobian is further defined as having all equal singular values due to the desirable properties of isotropic manipulator configurations [1],[5],[6]. Under these conditions, to guarantee that the minimum  ${}^f\sigma_m$  is as large as possible they should all be equal. Physically, this can be interpreted as attempting to balance the use of all joints so that they contribute equally to the velocity of the end effector. It is easy to show that the worst-case dexterity of an isotropic manipulator that experiences a single joint failure is governed by the inequality

$$\min_{f=1}^n({}^f\sigma_m) \leq \sigma \sqrt{\frac{n-m}{n}} \tag{6}$$

where  $\sigma$  denotes the norm of the original Jacobian. The best case of equality occurs if the manipulator is in an optimally failure tolerant configuration. The above inequality makes sense from a physical point of view since it represents the ratio of the degree of redundancy to the original number of degrees of freedom. This inequality can be used to determine the degree of redundancy required to maintain a minimum amount of dexterity in the event of any single joint failure.

Using the above definition of an optimally failure tolerant configuration one can identify the structure of the Jacobian required to obtain this property. In terms of the SVD of  $J$ , the matrix of output singular vectors,  $U$ , simply represents a rotation of the end-effector coordinate frame so that it does not affect the configuration of the manipulator and can be

arbitrarily set to identity without a loss of generality. In addition, since the manipulator is initially in an isotropic configuration only the matrix  $V^T$  needs to be considered. If  $V^T$  is partitioned between the  $m$  and  $m + 1$  rows in the following manner

$$V^T = \begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{n,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{n,2} \\ \vdots & \vdots & \cdots & \vdots \\ v_{1,m} & v_{2,m} & \cdots & v_{n,m} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ v_{1,m+1} & v_{2,m+1} & \cdots & v_{n,m+1} \\ \vdots & \vdots & \cdots & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{n,n} \end{bmatrix} \quad (7)$$

then the  $n - m$  rows  $v_{m+1}^T$  to  $v_n^T$  will span the null space and the upper  $m$  rows  $v_1^T$  to  $v_m^T$  will be equivalent to  $J$  so that the  $i$ th column of  $J$  can be taken to be

$$j_i = \begin{bmatrix} v_{i,1} \\ v_{i,2} \\ \vdots \\ v_{i,m} \end{bmatrix}. \quad (8)$$

The magnitude of the contribution of an individual joint  $i$  to the motion at the end effector is given by the norm of  $j_i$ . Since the definition of an optimally failure tolerant configuration requires that each joint contribute equally to the motion at the end effector, this translates into a constraint that all of the norms of the columns of the Jacobian be equal. In the following discussion, the scalar  $R_i$  will be used to denote the portion of joint  $i$ 's motion that

is transformed into the range space of  $J$ , so that

$$R_i = \|j_i\|^2 = \sum_{k=1}^m v_{i,k}^2. \quad (9)$$

The optimally failure tolerant criteria can be alternatively described as requiring each joint to contribute equally to the null space of the Jacobian transformation. Physically, this means that the redundancy of the robot is uniformly distributed among all the joints so that a failure at any joint can be compensated for by the remaining joints. An individual joint's contribution to the null space, denoted by  $N_i$ , is given by

$$N_i = \sum_{k=m+1}^n v_{i,k}^2. \quad (10)$$

It is easy to see that the condition that all of the  $R_i$  be equal is equivalent to all of the  $N_i$  being equal since

$$R_i + N_i = 1 \quad (11)$$

due to the fact that  $V$  is an orthogonal matrix. While these conditions are mathematically equivalent, one or the other may be computationally preferable depending on the degree of redundancy relative to the dimension of the end-effector space.

To summarize, an optimally failure tolerant Jacobian is defined as being isotropic, i.e.  $\sigma_i = \sigma$  for all  $i$ , and having a maximum worst-case dexterity following a failure, i.e. one for which  ${}^f\sigma_m = \sigma\sqrt{\frac{n-m}{n}}$  for all  $f$ . The second condition is equivalent to having the columns of the Jacobian have equal norms. The following section contains several illustrative examples of such manipulator Jacobians.

### III. EXAMPLES

The simplest example of an optimally failure tolerant configuration is given by the following Jacobian

$$J = \begin{bmatrix} -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \\ 0 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}. \quad (12)$$

One planar 3R manipulator configuration which possesses this Jacobian is illustrated in Fig. 1. The null space at this configuration is given by

$$v_3 = \begin{bmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{bmatrix}. \quad (13)$$

which illustrates that each joint contributes equally to the null space motion thus distributing the redundancy proportionally to all degrees of freedom. Geometrically, it is easy to see that the three vectors  $j_1$ ,  $j_2$ , and  $j_3$  are all  $120^\circ$  apart which results in a balanced coverage of the planar workspace. If the three possible joint failures are considered, one can show that

$${}^f\sigma_2 = \sqrt{\frac{1}{3}} \quad (14)$$

for  $f = 1$  to 3, which satisfies the optimally failure tolerant criterion. The end-effector motion which suffers most from a failure at one of the joints is, intuitively, in the direction of the column of the Jacobian which is associated with that joint, so that

$${}^f u_2 = \frac{j_f}{\|j_f\|}. \quad (15)$$



Fig. 2 illustrates the two extremes of  $\min_{f=1}^n ({}^f\sigma_m)$  for a planar three degree-of-freedom manipulator.

It is instructive to consider the effects of adding another degree of freedom to this planar manipulator Jacobian and applying the conditions for optimal failure tolerance. For this case one can identify an entire family of optimally failure tolerant Jacobians which can be represented by the matrix

$$V^T = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \cos \alpha & -\sqrt{\frac{1}{2}} \sin \alpha \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \sin \alpha & \sqrt{\frac{1}{2}} \cos \alpha \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} \cos \alpha & \sqrt{\frac{1}{2}} \sin \alpha \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \sin \alpha & -\sqrt{\frac{1}{2}} \cos \alpha \end{bmatrix} \quad (16)$$

where  $\sin \alpha$  and  $\cos \alpha$  are the sine and cosine of an arbitrary parameter  $\alpha$  describing this family. This family of manipulator configurations is illustrated in Fig. 3. The worst-case dexterity for an arbitrary joint failure in this case is given by

$${}^f\sigma_2 = \sqrt{\frac{1}{2}} \quad (17)$$

for  $f = 1$  to 4. If one considers optimal failure tolerance to both single and double joint failures then the optimal Jacobian is given by  $\alpha = k\pi/4$  where  $k$  is any odd integer.

In order to specify an arbitrary end-effector velocity in three-dimensional space, the simplest redundant manipulator must possess four joints. An optimally failure tolerant

Jacobian for such a manipulator is given by

$$V^T = \begin{bmatrix} -\sqrt{\frac{3}{4}} & \sqrt{\frac{1}{12}} & \sqrt{\frac{1}{12}} & \sqrt{\frac{1}{12}} \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \\ 0 & 0 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \quad (18)$$

It is easy to see that the effects of the four joints of this manipulator symmetrically span the three-dimensional end-effector space. The worst-case dexterity for an arbitrary joint failure is given by

$$f \sigma_3 = \frac{1}{2} \quad (19)$$

for  $f = 1$  to 4.

The canonical structure of an optimally failure tolerant Jacobian is obvious so that one can specify such a matrix for a task space or manipulator of any dimension. Doing so for a seven degree-of-freedom manipulator required to perform the standard task of positioning and orienting its end effector results in

$$J = \begin{bmatrix} -\sqrt{\frac{6}{7}} & \sqrt{\frac{1}{42}} & \sqrt{\frac{1}{42}} & \sqrt{\frac{1}{42}} & \sqrt{\frac{1}{42}} & \sqrt{\frac{1}{42}} & \sqrt{\frac{1}{42}} \\ 0 & -\sqrt{\frac{5}{6}} & \sqrt{\frac{1}{30}} & \sqrt{\frac{1}{30}} & \sqrt{\frac{1}{30}} & \sqrt{\frac{1}{30}} & \sqrt{\frac{1}{30}} \\ 0 & 0 & -\sqrt{\frac{4}{5}} & \sqrt{\frac{1}{20}} & \sqrt{\frac{1}{20}} & \sqrt{\frac{1}{20}} & \sqrt{\frac{1}{20}} \\ 0 & 0 & 0 & -\sqrt{\frac{3}{4}} & \sqrt{\frac{1}{12}} & \sqrt{\frac{1}{12}} & \sqrt{\frac{1}{12}} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix} \quad (20)$$

Unfortunately, this canonical representation of failure tolerant Jacobians for task spaces

which include both linear and rotational velocities only correspond to manipulators whose joints are capable of an arbitrary screw motion. However, as noted earlier, premultiplication of any such matrix by an orthogonal matrix  $U$  does not change the Jacobian's failure tolerant properties. Thus, the question now becomes: Is it possible to identify an orthogonal matrix  $U$  such that premultiplying (20) by  $U$  results in a manipulator Jacobian that can be realized by a manipulator with seven rotational joints? We have not been able to identify such a matrix or prove that such a matrix does not exist. The following section outlines an alternative numerical approach for identifying failure tolerant Jacobians corresponding to configurations of manipulators that possess only rotational joints.

#### IV. ROTATIONAL MANIPULATORS

Rotational joint variables result in columns of the manipulator Jacobian that are constrained to have the linear velocity component be orthogonal to the rotational velocity component. To restrict the columns of a manipulator Jacobian to be achievable by a rotational joint one can model the contribution of an individual joint by

$$\dot{j}_i = \begin{bmatrix} v_i \\ \text{---} \\ \omega_i \end{bmatrix} = \begin{bmatrix} \cos \gamma_i \begin{bmatrix} \cos \alpha_i \sin \beta_i \\ \sin \alpha_i \sin \beta_i \\ -\cos \beta_i \end{bmatrix} + \sin \gamma_i \begin{bmatrix} \sin \alpha_i \\ -\cos \alpha_i \\ 0 \end{bmatrix} \\ \text{---} \\ \begin{bmatrix} \cos \alpha_i \cos \beta_i \\ \sin \alpha_i \cos \beta_i \\ \sin \beta_i \end{bmatrix} \end{bmatrix} \quad (21)$$

where  $v_i$  and  $\omega_i$  are the linear and rotational velocity due to the  $i$ th joint velocity and  $\alpha_i, \beta_i$ , and  $\gamma_i$  are the variables defining the orientation of joint  $i$ . Note that this parameterization constrains the norm of the linear velocity component to be equal to the norm of the rotational

component. This is done in order to satisfy the requirements of an optimally failure tolerant Jacobian. The only other constraint is that the Jacobian composed of the  $j_i$ 's formed from (21) be an isotropic matrix. All isotropic matrices can be represented by scalar multiples of orthogonal matrices. Therefore, to obtain an optimally failure tolerant configuration one simply needs to determine the  $\alpha_i$ 's,  $\beta_i$ 's, and  $\gamma_i$ 's that result in a Jacobian with orthogonal rows of equal norm.

Consider  $J$  to consist of the six rows

$$J = \begin{bmatrix} v_x^T \\ v_y^T \\ v_z^T \\ \omega_x^T \\ \omega_y^T \\ \omega_z^T \end{bmatrix}. \quad (22)$$

The constraint that  $J$  have rows of equal norm is represented by the six equations

$$f_1 = 0 = v_x^T v_x - n/3 \quad (23)$$

$$f_2 = 0 = v_y^T v_y - n/3 \quad (24)$$

$$f_3 = 0 = v_z^T v_z - n/3 \quad (25)$$

$$f_4 = 0 = \omega_x^T \omega_x - n/3 \quad (26)$$

$$f_5 = 0 = \omega_y^T \omega_y - n/3 \quad (27)$$

$$f_6 = 0 = \omega_z^T \omega_z - n/3. \quad (28)$$

The row norm is equal to  $\sqrt{n/3}$  due to the form of (21) since both the linear and rotational components  $v_i$  and  $\omega_i$  were arbitrarily specified to be of unit norm. Note also that only four of these constraints are independent, also due to the manner in which (21) is specified. The constraints that  $J$  have orthogonal rows is represented by the 15 equations

$$f_7 = 0 = v_x^T v_y \quad (29)$$

$$f_8 = 0 = v_x^T v_z \quad (30)$$

$$f_9 = 0 = v_x^T \omega_x \quad (31)$$

$$f_{10} = 0 = v_x^T \omega_y \quad (32)$$

$$f_{11} = 0 = v_x^T \omega_z \quad (33)$$

$$f_{12} = 0 = v_y^T v_z \quad (34)$$

$$f_{13} = 0 = v_y^T \omega_x \quad (35)$$

$$f_{14} = 0 = v_y^T \omega_y \quad (36)$$

$$f_{15} = 0 = v_y^T \omega_z \quad (37)$$

$$f_{16} = 0 = v_z^T \omega_x \quad (38)$$

$$f_{17} = 0 = v_z^T \omega_y \quad (39)$$

$$f_{18} = 0 = v_z^T \omega_z \quad (40)$$

$$f_{19} = 0 = \omega_x^T \omega_y \quad (41)$$

$$f_{20} = 0 = \omega_x^T \omega_z \quad (42)$$

$$f_{21} = 0 = \omega_y^T \omega_z \quad (43)$$

of which only 14 are independent. The exact expressions for the 21 constraint equations represented by  $\mathbf{f} = 0$  in terms of the  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's that represent the joint variables are given in the Appendix. It is important to note that like the constraint equations, not all of the  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's are independent, however, these 21 simultaneous nonlinear equations can be minimized using an optimization scheme that uses derivative information since the equations are differentiable. This optimization was performed for several thousand randomly selected initial conditions. In all cases the optimization converged to a local minimum with a value of  $\mathbf{f}^T \mathbf{f} = 0.056827$ . A typical Jacobian corresponding to such a nearly optimal failure tolerant solution is given by

$$J = \begin{bmatrix} 1 & 0.4296 & 0.7495 & -0.5431 & 0.1401 & 0.3298 & -0.3783 \\ 0 & -0.6041 & 0.6479 & 0.4640 & -0.7889 & -0.1853 & -0.8047 \\ 0 & -0.6712 & -0.1357 & -0.6998 & 0.5983 & -0.9257 & -0.4575 \\ 0 & 0.7678 & 0.1449 & 0.8391 & 0.5831 & -0.6882 & -0.4296 \\ 1 & -0.1469 & -0.3607 & 0.3308 & -0.4226 & -0.7184 & 0.5904 \\ 0 & 0.6236 & -0.9214 & -0.4319 & -0.6938 & -0.1014 & -0.6832 \end{bmatrix}. \quad (44)$$

The singular values for this Jacobian, which are common to all solutions that have been identified, are given by

$$D = \text{diag}(1.5829 \ 1.5829 \ 1.5250 \ 1.5250 \ 1.4727 \ 1.4727) \quad (45)$$

which are close to the near optimal value of  $\sigma = \sqrt{n/3} = 1.5275$ . The worst-case failure for this Jacobian is when joint seven is locked, resulting in

$${}^7\sigma_6 = 0.5196 \quad (46)$$

which is approximately 10% poorer than the theoretically optimal worst-case scenario given by  ${}^f\sigma_6 = \sqrt{1/3} = 0.5774$ . It is possible to slightly improve this value if an optimization is explicitly performed on (5)

It is interesting to note that the 7R manipulator is the only case in which an optimally failure tolerant Jacobian has not been found. For example, if the numerical procedure is performed for an 8R manipulator, it immediately converges to a solution. In fact, there are entire families of solutions, an example of which is given by

$$J = \begin{bmatrix} 1 & 0.0223 & -0.9878 & 0.1851 & -0.6558 & 0.4505 & -0.1501 & -0.0246 \\ 0 & 0.7775 & -0.1554 & -0.9784 & 0.3045 & 0.1982 & -0.8740 & 0.4298 \\ 0 & -0.6284 & -0.0002 & -0.0917 & 0.6908 & 0.8705 & -0.4622 & -0.9026 \\ 0 & 0.4130 & -0.0978 & 0.5393 & 0.7384 & 0.8588 & 0.5993 & 0.7442 \\ 1 & -0.5796 & 0.6228 & 0.1791 & 0.4491 & -0.3626 & -0.4523 & 0.6108 \\ 0 & -0.7024 & -0.7762 & -0.8229 & 0.5030 & -0.3619 & 0.6606 & 0.2706 \end{bmatrix}. \quad (47)$$

As designed, this Jacobian is isotropic with equal singular values of

$$\sigma = \sqrt{n/3} = \sqrt{8/3} \quad (48)$$

and an optimal worst-case failure dexterity of

$${}^f\sigma_6 = \sigma\sqrt{(n-m)/n} = \sqrt{2/3} \quad (49)$$

for all joint failures  $f$ .

## V. CONCLUSIONS

This work has discussed a measure of failure tolerance for kinematically redundant manipulators that guarantees both desirable performance before a failure and optimal worst-case behavior following locked joint failures. Mathematically, these properties were defined as manipulator Jacobians that were both isotropic, i.e. all equal singular values, and that had columns of equal norm. It was shown that matrices with such properties could be easily created for an arbitrary dimension of both desired task space and manipulator joint variables. It was also shown that, in most cases, it was a trivial matter to map a matrix with such properties into a physical manipulator consisting of purely rotational joints that possessed a configuration with an optimally failure tolerant Jacobian. The one exception appears to be a manipulator consisting of seven rotational joints required to perform fully general linear and rotational end-effector velocities. However, it was shown that one can use a numerical optimization technique to identify manipulator configurations for such manipulators that possess Jacobians that are very close to being optimally failure tolerant.



## APPENDIX

The equal norm row constraints specified by (23-28) when applied to a manipulator Jacobian whose columns have been constrained to be in the form of (21) are given by:

$$f_1 = 7/3 - \sum_{i=1}^n \cos^2 \alpha_i \sin^2 \beta_i \cos^2 \gamma_i + 2 \sin \alpha_i \cos \alpha_i \sin \beta_i \sin \gamma_i \cos \gamma_i + \sin^2 \alpha_i \sin^2 \gamma_i$$

$$f_2 = 7/3 - \sum_{i=1}^n \sin^2 \alpha_i \sin^2 \beta_i \cos^2 \gamma_i - 2 \sin \alpha_i \cos \alpha_i \sin \beta_i \sin \gamma_i \cos \gamma_i + \cos^2 \alpha_i \sin^2 \gamma_i$$

$$f_3 = 7/3 - \sum_{i=1}^n \cos^2 \beta_i \cos^2 \gamma_i$$

$$f_4 = 7/3 - \sum_{i=1}^n \cos^2 \alpha_i \cos^2 \beta_i$$

$$f_5 = 7/3 - \sum_{i=1}^n \sin^2 \alpha_i \cos^2 \beta_i$$

$$f_6 = 7/3 - \sum_{i=1}^n \sin^2 \beta_i$$

Note, once again, that due to this particular formulation only four of these constraints are independent.

The orthogonal row constraints specified by (29-43) when applied to a manipulator Jacobian whose columns have been constrained to be in the form of (21) are given by:

$$f_7 = \sum_{i=1}^n \sin \alpha_i \cos \alpha_i \sin^2 \beta_i \cos^2 \gamma_i - \cos^2 \alpha_i \sin \beta_i \sin \gamma_i \cos \gamma_i + \sin^2 \alpha_i \sin \beta_i \sin \gamma_i \cos \gamma_i - \sin \alpha_i \cos \alpha_i$$

$$f_8 = \sum_{i=1}^n -\cos \alpha_i \sin \beta_i \cos \beta_i \cos^2 \gamma_i - \sin \alpha_i \cos \beta_i \sin \gamma_i \cos \gamma_i$$

$$f_9 = \sum_{i=1}^n \cos^2 \alpha_i \sin \beta_i \cos \beta_i \cos \gamma_i + \sin \alpha_i \cos \alpha_i \cos \beta_i \sin \gamma_i$$

$$f_{10} = \sum_{i=1}^n \sin \alpha_i \cos \alpha_i \sin \beta_i \cos \beta_i \cos \gamma_i + \sin^2 \alpha_i \cos \beta_i \sin \gamma_i$$

$$f_{11} = \sum_{i=1}^n \cos \alpha_i \sin^2 \beta_i \cos \gamma_i + \sin \alpha_i \sin \beta_i \sin \gamma_i$$

$$f_{12} = \sum_{i=1}^n -\sin \alpha_i \sin \beta_i \cos \beta_i \cos^2 \gamma_i + \cos \alpha_i \cos \beta_i \sin \gamma_i \cos \gamma_i$$

$$f_{13} = \sum_{i=1}^n \sin \alpha_i \cos \alpha_i \sin \beta_i \cos \beta_i \cos \gamma_i - \cos^2 \alpha_i \cos \beta_i \sin \gamma_i$$

$$f_{14} = \sum_{i=1}^n \sin^2 \alpha_i \sin \beta_i \cos \beta_i \cos \gamma_i - \sin \alpha_i \cos \alpha_i \cos \beta_i \sin \gamma_i$$

$$f_{15} = \sum_{i=1}^n \sin \alpha_i \sin^2 \beta_i \cos \gamma_i - \cos \alpha_i \sin \beta_i \sin \gamma_i$$

$$f_{16} = \sum_{i=1}^n -\cos \alpha_i \cos^2 \beta_i \cos \gamma_i$$

$$f_{17} = \sum_{i=1}^n -\sin \alpha_i \cos^2 \beta_i \cos \gamma_i$$

$$f_{18} = \sum_{i=1}^n -\sin \beta_i \cos \beta_i \cos \gamma_i$$

$$f_{19} = \sum_{i=1}^n \sin \alpha_i \cos \alpha_i \cos^2 \beta_i$$

$$f_{20} = \sum_{i=1}^n \cos \alpha_i \sin \beta_i \cos \beta_i$$

$$f_{21} = \sum_{i=1}^n \sin \alpha_i \sin \beta_i \cos \beta_i$$

Note, once again, that due to this particular formulation only fourteen of these constraints are independent for a total of eighteen independent constraints.

## REFERENCES

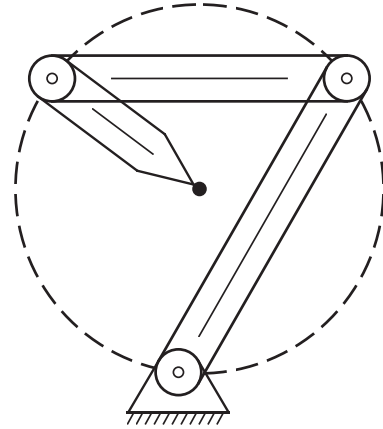
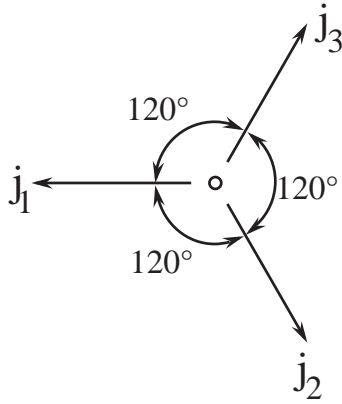
- [1] J. Angeles, "The design of isotropic manipulator architectures in the presence of redundancies," *International Journal of Robotics Research*, vol. 11, no. 3, pp. 196–201, June 1992.
- [2] J. Baillieul, "A constraint oriented approach to inverse problems for kinematically redundant manipulators," in *Proc. 1987 International Conference on Robotics and Automation*, pp. 1827–1833, (Raleigh, NC), March 31-April 3 1987.
- [3] S. L. Chiu, "Task compatibility of manipulator postures," *International Journal of Robotics Research*, vol. 7, no. 5, pp. 13–21, October 1988.
- [4] R. Colbaugh and M. Jamshidi, "Robot manipulator control for hazardous waste-handling applications," *Journal of Robotic Systems*, vol. 9, no. 2, pp. 215–250, 1992.
- [5] C. A. Klein and B. E. Blaho, "Dexterity measures for the design and control of kinematically redundant manipulators," *International Journal of Robotics Research*, vol. 6, no. 2, pp. 72–83, Summer 1987.
- [6] C. A. Klein and T. A. Miklos, "Spatial robotic isotropy," *International Journal of Robotics Research*, vol. 10, no. 4, pp. 426–437, August 1991.
- [7] C. L. Lewis and A. A. Maciejewski, "Dexterity optimization of kinematically redundant manipulators in the presence of failures," *Computers and Electrical Engineering: An International Journal*, vol. 20, no. 3, pp. 273–288, May 1994.
- [8] C. L. Lewis and A. A. Maciejewski, "An example of failure tolerant operation of a kinematically redundant manipulator," in *Proc. 1994 International Conference on Robotics and Automation*, pp. 1380–1387, (San Diego, CA), May 8-13 1994.

- [9] A. A. Maciejewski, "Fault tolerant properties of kinematically redundant manipulators," in *Proc. 1990 International Conference on Robotics and Automation*, pp. 638–642, (Cincinnati, OH), May 13-18 1990.
- [10] C. J. J. Paredis, W. K. F. Au, and P. K. Khosla, "Kinematic design of fault tolerant manipulators," *Computers and Electrical Engineering: An International Journal*, vol. 20, no. 3, pp. 211–220, May 1994.
- [11] D. L. Schneider, D. Tesar, and J. W. Barnes, "Development & testing of a reliability performance index for modular robotic systems," in *Proc. 1994 Proceedings of the Annual Reliability and Maintainability Symposium*, pp. 263–270, (Anaheim, CA), January 24-27 1994.
- [12] M. L. Visinsky, I. D. Walker, and J. R. Cavallaro, "Layered dynamic fault detection and tolerance for robots," in *Proc. 1993 International Conference on Robotics and Automation*, pp. 180–187, (Atlanta, Georgia), May 2-6 1993.
- [13] E. Wu, M. Diftler, J. Hwang, and J. Chladek, "A fault tolerant joint drive system for the space shuttle remote manipulator system," in *Proc. 1991 International Conference on Robotics and Automation*, pp. 2504–2509, (Sacramento, CA), April 9-11 1991.
- [14] T. Yoshikawa, "Manipulability of robotic mechanisms," *International Journal of Robotics Research*, vol. 4, no. 2, pp. 3–9, Summer 1985.

## Three Link Planar

$$\mathbf{J} = \begin{bmatrix} \sqrt{2/3} & \sqrt{1/6} & \sqrt{1/6} \\ 0 & -\sqrt{1/2} & \sqrt{1/2} \end{bmatrix}$$

$$\mathbf{n}_J^T = \mathbf{v}_3^T = \begin{bmatrix} \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \end{bmatrix}$$



$$f_{\sigma_2} = \sqrt{1/3}$$

$$f_{\mathbf{u}_2} = \frac{\mathbf{j}_f}{\|\mathbf{j}_f\|}$$

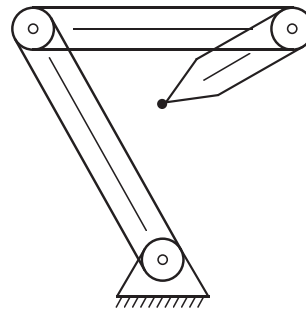
Fig. 1. Optimally failure tolerant configuration for a planar 3R manipulator generated from an optimally failure tolerant Jacobian.

## Bound on Remaining Dexterity

$$0 \leq \min_{f=1}^n ({}^f \sigma_m) \leq \sigma \sqrt{(n-m)/n}$$

Best

$${}^1 \sigma_2 = {}^2 \sigma_2 = {}^3 \sigma_2 = \sqrt{1/3}$$



Worst

$${}^2 \sigma_2 = {}^3 \sigma_2 = 0$$

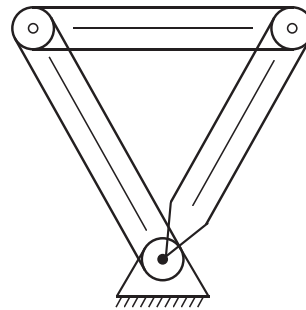
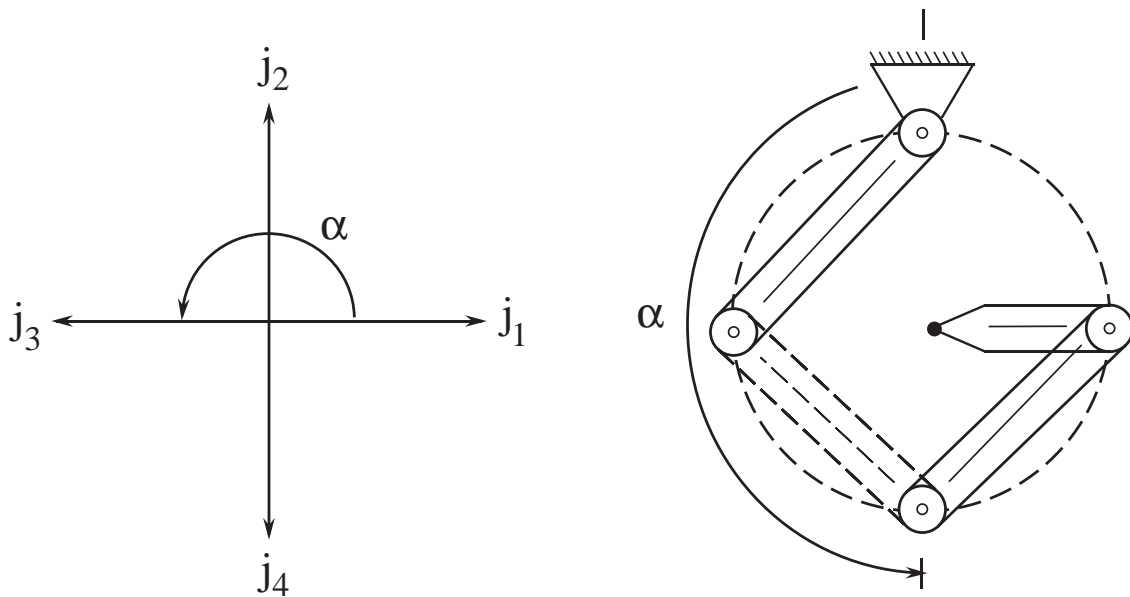


Fig. 2. An illustration of the extremes in the failure tolerance index. The optimal configuration is defined as  $\max(\min_{f=1}^n ({}^f \sigma_m))$  whereas an example of the worst possible failure tolerance measure is given by the configuration with  ${}^2 \sigma_2 = {}^3 \sigma_2 = 0$ .

## Four Link Planar

$$\mathbf{V}^T = \begin{bmatrix} \sqrt{1/2} & 0 & \sqrt{1/2} \cos \alpha & -\sqrt{1/2} \sin \alpha \\ 0 & \sqrt{1/2} & \sqrt{1/2} \sin \alpha & \sqrt{1/2} \cos \alpha \\ \sqrt{1/2} & 0 & -\sqrt{1/2} \cos \alpha & \sqrt{1/2} \sin \alpha \\ 0 & \sqrt{1/2} & -\sqrt{1/2} \sin \alpha & -\sqrt{1/2} \cos \alpha \end{bmatrix}$$



$$f_{\sigma_2} = \sqrt{1/2}$$

Fig. 3. A family of optimally failure tolerant configurations for a planar 4R manipulator corresponding to a family of optimally failure tolerant Jacobians.