Advances and improvements in the theory of standard bases and syzygies

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Introduction

The aim of this article is to describe recent advances and improvements on the tangent cone algorithm of T. Mora. This tangent cone algorithm is itself a variant of B. Buchberger's celebrated algorithm for constructing a Gröbner basis of an ideal in a polynomial ring over a field. In the same manner as the knowledge of a Gröbner basis allows the computation of numerous invariants of the coordinate ring of a projective algebraic variety, a standard basis (computed by the tangent cone algorithm) does so for invariants of the local ring of an algebraic variety at a given point. In this paper we describe a generalization which includes Buchberger's and Mora's algorithm as special cases. That is, we prove — with an appropriate definition of ecart — that Mora's algorithm terminates for any ordering on the monomials of $K[x_1, \ldots, x_n]$, which is compatible with the natural semigroup structure (a fact which was found independently by Gräbe [G]), in particular, the variables may have as well negative, positive or zero weights (cf. §1). More or less all algorithms using Gröbner bases (such as computation of syzygies, ideal theoretic operations, etc.) are now available in this general context. Our generalization provides also an easy manner to implement standard bases for modules over the Weyl algebra and for $\mathcal{D}$–modules. The general standard basis algorithm is described in §1.

In §2 we prove that Schreyer's method to compute syzygies generalizes to arbitrary semigroup orderings. It seems to be the first algorithmic proof of the fact that the length of a free resolution is equal to the number of variables which actually occur in the equations (and not on all variables of the ring) in the local and mixed (local–global) case. It follows basically Schreyer's original proof [S] but contains some new ideas, since Macaulay's lemma, which is usually applied, does not hold for orderings which are not well–orderings. As a consequence we obtain that the rings $\mathrm{Loc}_< K[x]$ (see below) are regular.

Chapter §3 contains a partial positive answer to Zariski's multiplicity conjecture. Although there are other partial positive answers known, e.g. by Zariski, Lê, Lipmann, Laufer, O'Shea, Yau and the first named author, it has basically resisted all attacks. Our result, which supports the conjecture, was prompted by computer experiments with an implementation of the above described algorithm in the computer algebra system SINGULAR. The proof (given in §3) does not use any computer computation but the computer experiments were essential in guessing the result. We include a proof that the module of leading terms, even in the case of general semigroup orderings, is a flat specialization of the original module. This is the basis of most applications, e.g. for computing Milnor numbers or multiplicities and Hilbert functions of singularities.

For a description of an implementation of the standard basis algorithm described in this paper, special strategies and many comparisons, also for syzygies, cf. [Gr et al].

1 A standard basis algorithm for any semigroup ordering

This algorithm is a generalization of Buchberger's algorithm (which works for wellorderings cf. [B1], [B2]) and Mora's tangent cone algorithm (which works for tangent cone orderings, cf. [M1], [MPT]) and which includes a mixture of both (which is useful for certain applications cf. [M2]). In fact, it is an easy extension of Mora's idea by introducing the "correct" definition of ecart. But we present it in a new way which, as we hope, makes the relation to the existing standard basis algorithms transparent.

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Let K be a field, $x = (x_1, \dots, x_n)$ and α, β, γ column vectors in $\mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $<$ be a **semigroup ordering** on the set of monomials $\{x^\alpha | \alpha \in \mathbb{N}^n\}$ of $K[x]$, that is, $<$ is a total ordering and $x^\alpha < x^\beta$ implies $x^\gamma x^\alpha < x^\gamma x^\beta$ for any $\gamma \in \mathbb{N}^n$. Robbiano proved that any semigroup ordering can be defined by a matrix $A \in GL(n, \mathbb{R})$ as follows:

Let $a_1, \dots, a_n$ be the rows of A, then $x^\alpha < x^\beta$ if and only if there is an i with $a_j \cdot \alpha = a_j \cdot \beta$ for $j < i$ and $a_i \cdot \alpha < a_i \cdot \beta$. Thus, $x^\alpha < x^\beta$ if and only if $A\alpha$ is smaller than $A\beta$ with respect to the lexicographical ordering of vectors in $\mathbb{R}^n$.

For $g \in K[x]$, $g \neq 0$, let $\mathbf{L(g)}$ be the **leading monomial** with respect to the ordering $<$ and $\mathbf{c(g)}$ the **leading coefficient** of g, that is $g = c(g)L(g) +$ smaller terms with respect to $<$.

Definition 1.1 *We define* $\mathbf{Loc_<K[x]} := S_<^{-1}K[x]$ *to be the localization of* $K[x]$ *with respect to the multiplicative closed set* $S_< := \{1 + g \mid g = 0$ *or* $g \in K[x]\backslash\{0\}$ *and* $1 > L(g)\}$.

Remark 1.2 1) $K[x] \subseteq \mathrm{Loc}_< K[x] \subseteq K[x]_{(x)}$, where $K[x]_{(x)}$ denotes the localization of $K[x]$ with respect to the maximal ideal $(x_1, \dots, x_n)$. In particular, $\mathrm{Loc}_< K[x]$ is noetherian and $K[x]$–flat and $K[x]_{(x)}$ is $\mathrm{Loc}_< K[x]$–flat.

2) If $x_1, \dots, x_r < 1$ and $x_{r+1}, \dots, x_n > 1$ then $1 + (x_1, \dots, x_r)K[x_1, \dots, x_r] \subseteq S_< \subseteq 1 + (x_1, \dots, x_r)K[x] =: S$, hence $K[x_1, \dots, x_r]_{(x_1, \dots, x_r)}[x_{r+1}, \dots, x_n] \subseteq \mathrm{Loc}_< K[x] \subseteq S^{-1}K[x]$.

Note that $<$ is a wellordering if and only if $x^0 = 1$ is the smallest monomial and in this case $\mathrm{Loc}_< K[x] = K[x]$. We call such orderings also **global orderings**. If $1 > x_i$ for all i, then $\mathrm{Loc}_< K[x] = K[x]_{(x)}$. Such orderings are called **local orderings**. Orderings, where some of the variables are > 1 and others are < 1 are called **mixed orderings**. Important are **degree orderings** where each variable has an integer weight (positive or negative but not zero) and where the ordering refines the partial ordering induced by the weighted degree. Examples include the orderings ***w*–degrevlex** with $w = (w_1, \dots, w_n)$, $w_i \neq 0$, where $x^\alpha < x^\beta$ if $w \cdot \alpha < w \cdot \beta$ or $w \cdot \alpha = w \cdot \beta$ and the last non–zero entry of $\beta - \alpha$ is negative. We just write **degrevlex** (respectively **degrevlex$^-$**) if all $w_i = 1$ (respectively all $w_i = -1$).

Many applications require an **elimination ordering** for, say, $x_{r+1}, \dots, x_n$, which means that $L(g) \in K[x_1, \dots, x_r]$ implies $g \in K[x_1, \dots, x_r]$. Since $x^\alpha < 1$ implies $x^\alpha \in K[x_1, \dots, x_r]$ we see that this ordering is necessarily a wellordering on the set of monomials in $K[x_{r+1}, \dots, x_n]$. The usual lexicographical ordering **lex**, given by the matrix $A = \mathrm{id}$, is an elimination ordering for all $0 \leq r < n$, but the "local lexicographical" ordering **lex$^-$** given by $A = -\,\mathrm{id}$ is not an elimination ordering. If A_1 is an ordering for monomials in $x_1, \dots, x_r$ and A_2 for monomials in $x_{r+1}, \dots, x_n$, then the **product ordering** given by the direct sum $A_1 \oplus A_2$ of the matrices A_1 and A_2 is an elimination ordering for $x_1, \dots, x_r$.

We consider also **module orderings** $<_m$ on the set of monomials $\{x^\alpha e_i\}$ of $K[x]^r = \sum_{i=1,\dots,r} K[x]e_i$ which are compatible with the ordering $<$ on $K[x]$. That is for all monomials $f, f' \in K[x]^r$ and $p, q \in K[x]$ we have: $f <_m f'$ implies $pf <_m pf'$ and $p < q$ implies $pf <_m qf$.

We now fix an ordering $<_m$ on $K[x]^r$ compatible with $<$ and denote it also with $<$. Again we have the notion of coefficient $c(f)$ and leading monomial $L(f)$. $<$ has the important property:

$$\begin{array}{ll} L(qf) = L(q)L(f) & \text{for } q \in K[x] \text{ and } f \in K[x]^r, \\ L(f + g) \leq \max(L(f), L(g)) & \text{for } f, g \in K[x]^r. \end{array}$$

Definition 1.3 *Let* $I \subseteq K[x]^r$ *be a submodule.*

1) $\mathbf{L}(I)$ *denotes the submodule of* $K[x]^r$ *generated by* $\{L(f) | f \in I\backslash\{0\}\}$.

2) *A finite set* $G = \{f_1, \dots, f_s\} \subset I$ *is called a* **standard basis** *of* I *if* $\{L(f_1), \dots, L(f_s)\}$ *generates the* $K[x]$*–submodule* $L(I) \subset K[x]^r$.

3) *A standard basis* $\{f_1, \dots, f_s\}$ *is called* **reduced** *if, for any* i, $L(f_i)$ *does not divide any of the monomials of* $f_1, \dots, f_s$ *(except itself).*

4) *A finite set* $\{f_1, \dots, f_s\}$ *is called* **interreduced**, *if* $L(f_i) \nmid L(f_j)$ *for all* $i \neq j$.

Note that an interreduced standard basis does, while a reduced standard basis does not necessarily exist (cf. Remark 1.12).

Proposition 1.4 *If $\{f_1,\dots,f_s\}$ is a standard basis of I then $I Loc_< K[x] = (f_1,\dots,f_s) Loc_< K[x]$.*

The proof will be deduced from the normal form used in the standard basis algorithm (cf. Corollary 1.11). In general it is not true that $f_1,\dots,f_s$ generate I as $K[x]$–module (take $I = (x)K[x]$, $n = 1$, $f = x + x^2$ with lex$^-$). This is also not true if $I \subset K[x]$ is $(x_1,\dots,x_n)$–primary and if $\{f_1,\dots,f_s\}$ is a reduced standard basis (which answers a question of T. Mora): consider the ideal $I \subset K[x,y]$ generated by $x^{10} - y^2x^9$, $y^8 - x^2y^7$, $x^{10}y^7$ which is (x,y)–primary. The first two elements are a reduced standard basis of $I\,Loc_< K[x,y] = I\,K[x,y]_{(x,y)}$ where $<$ is degrevlex$^-$ und hence generate $I\,K[x,y]_{(x,y)}$ but they do not generate $I\,K[x,y]$. (Cf. also Remark 1.8.)

Notations: Let $f, g \in K[x]^r$, $L(f) = x^\alpha e_i$ and $L(g) = x^\beta e_j$. If $i = j$ and $x^\alpha | x^\beta$ then we write $L(f)|L(g)$.
If $i = j$ and $x^\gamma = lcm(x^\alpha, x^\beta)$, $\gamma = (\max(\alpha_1,\beta_1),\dots,\max(\alpha_n,\beta_n))$ then the **lowest common multiple** and the ***S*–polynomial** are defined as follows:

$$\mathbf{lcm}(\mathbf{L}(\mathbf{f}),\mathbf{L}(\mathbf{g})) := x^\gamma \text{ and}$$
$$\mathbf{spoly}(\mathbf{f},\mathbf{g}) := x^{\gamma-\alpha} f - \frac{c(f)}{c(g)} x^{\gamma-\beta} g.$$

If $i \neq j$ then, by definition, $L(f) \not| L(g)$, spoly$(f,g) := 0$ and $lcm(L(f),L(g)) := 0$.

Definition 1.5 *Let $\mathcal{F} = \{G \subseteq K[x]^r | G$ finite and ordered $\}$. A function $NF : K[x]^r \times \mathcal{F} \to K[x]^r, (p,G) \mapsto NF(p|G)$, is called a* **normal form** *if for any $p \in K[x]^r$ and any $G \in \mathcal{F}$ the following holds: if $NF(p|G) \neq 0$ then $L(g) \not| L(NF(p|G))$ for all $g \in G$. $NF(p|G)$ is called a* **normal form of p with respect to G**.

Example 1.6 Let $<$ be a wellordering then the following procedure NFBuchberger is a normal form:

```
h := NFBuchberger (p|G)
    h := p
    WHILE exist f ∈ G such that L(f)|L(h) DO
          choose the first f ∈ G with this property
          h := spoly(h, f)
```

The principle for many standard basis algorithms depending on a chosen normal form is the following:

```
S := Standard (G, NF)
    S := G
    P := {(f, g)|f, g ∈ S}
    WHILE P ≠ ∅ DO
          choose (f, g) ∈ P; P := P\{(f, g)}
          h := NF(spoly (f, g) | S)
          IF h ≠ 0 THEN
                P := P ∪ {(h, f) | f ∈ S}
                S := S ∪ {h}
```

In this language **Buchberger's algorithm** is just

Buchberger(G) `= Standard (G, NFBuchberger).`

If $<$ is any ordering (not necessarily a wellordering) and A the corresponding matrix, then the matrix

$$\begin{pmatrix} 1 & 1\dots1 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$$

defines a wellordering on the monomials of $K[t,x]$ which we denote also by $<$. For $f \in K[x]$ let f^h be the **homogenization** of f with respect to t and for $G \subseteq K[x]$ let $G^h = \{f^h \mid f \in G\}$. If $f \in K[x]^r$, $f = \sum f_i e_i$, we define $f^h = \sum t^{\alpha_i} f_i^h e_i$ where $\deg f_i^h + \alpha_i = \deg f_j^h + \alpha_j$ for all i, j and the α_i minimal with this property. We set $G^h = \{f^h \mid f \in G\}$ for $G \subseteq K[x]^r$.

This ordering has the following property:

Lemma 1.7 *If there exists α and $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $t^\alpha > x^\gamma$ and $\alpha = \gamma_1 + \cdots + \gamma_n$ then $x^\gamma < 1$. Especially, $<$ is not a wellordering in this case on $K[x]$.*

The **Lazard method** (cf. [L]) to compute a standard basis is the following:

$S :=$ **Lazard (G)**
- $S := G^h$
- $S :=$ `Buchberger` (S)
- $S := S(t = 1)$

Remark 1.8 The result S of Lazard's method is, in general, much bigger than a standard basis computed by the algorithm "Standard basis" below. If we are only interested in a standard basis of $\langle G \rangle$ this algorithm computes usually too much and this might be the reason why it is often too slow. In Lazard's algorithm one may, and usually does, take an interreduced standard basis of $\langle G^h \rangle$ by deleting superfluous elements. The result still has the property that the $K[x]$–module $\langle G \rangle = GK[x]$ is generated by G (we need not pass to $\mathrm{Loc}_< K[x]$). This is not necessarily true if we take an interreduced standard basis of $\langle G \rangle$: let $G = \{x, x + x^2\}$, which is a standard basis of $\langle G \rangle = (x)K[x] \subset K[x]$ for lex$^-$. We may delete either x or $x + x^2$ to obtain an interreduced standard basis of $\langle G \rangle$ but if we delete x, then $x + x^2$ does not generate $\langle G \rangle$ (but, of course, $G\mathrm{Loc}_< K[x]$).

For tangent cone orderings and some mixed orderings (cf. [M1], [MPT]) Mora found an algorithm which computes a standard basis over $\mathrm{Loc}_< K[x]$. This algorithm can be generalized to any ordering and we can describe it as follows:

$S :=$ **Standard basis (G)**
- $S := G^h$
- $S :=$ `Standard` $(S,$ `NFMora`$)$
- $S := S(t = 1)$

Let $G \subseteq K[t,x]^r$ be a finite and ordered set of homogeneous elements and $p \in K[t,x]^r$ homogeneous. Note that an element of $K[t,x]^r$ is **homogeneous** if its components are homogeneous polynomials of the same degree. The generalization of Mora's normal form to any semigroup ordering is as follows:

```
h := NFMora (p|G)
    h := p
    T := G
    WHILE exist f ∈ T, such that L(f) | t^α L(h) for some α DO
        choose the first f ∈ T with L(f) | t^α L(h) and α minimal
        IF α > 0 THEN
            T := T ∪ {h}
        h := spoly (t^α h, f)
        IF t | h THEN
            choose α maximal such that t^α divides h
            h := h / t^α
```

Theorem 1.9 *1) NFMora terminates.*

2) If h is a normal form of p with respect to $G = \{f_1, \ldots, f_s\}$ computed by NFMora then there are homogeneous polynomials g, $\xi_1, \ldots, \xi_s \in K[t,x]$ such that

- $gp = \sum \xi_i f_i + h$
- $L(g) = t^\alpha$
- *$\deg p + \alpha = \deg \xi_i + \deg f_i = \deg(h)$ (if $\xi_i \neq 0$, $h \neq 0$)*

- $L(f_i) \not| t^\alpha L(h)$ *for all* i, α

If $<$ *is a wellordering on* $K[x]$ *then* $g = t^\alpha$.

Proof: 2) By induction suppose that after the ν–th step in NFMora we have

- $g_\nu p = \sum \xi_{i\nu} f_i + h_\nu$,
- $L(g_\nu) = t^{\alpha_\nu}$,
- $\deg p + \alpha_\nu = \deg \xi_{i,\nu} + \deg f_i = \deg h_\nu$ (if $\xi_{i,\nu} \neq 0,\ h_\nu \neq 0$)
- $t^{-\alpha_\mu} L(h_\mu) > t^{-\alpha_\nu} L(h_\nu)$ for $\mu < \nu$.

If $L(f_i) \not| t^\alpha L(h_\nu)$ for all i, α then we have finished.

Since T consists of elements $f_k \in G$ and of h_μ constructed in previous steps we have to consider two cases:

(a) If $L(f_k) \mid t^\alpha L(h_\nu)$ and α is minimal for all possible choices for $f_k \in G$ then

$$t^\alpha g_\nu p = \sum t^\alpha \xi_{i\nu} f_i + t^\alpha h_\nu - \eta f_k + \eta f_k$$

with $L(f_k)\eta = t^\alpha L(h_\nu)$. We obtain

$$\begin{aligned} h_{\nu+1} &= t^\alpha h_\nu - \eta f_k \\ g_{\nu+1} &= t^\alpha g_\nu \\ \xi_{i\nu+1} &= t^\alpha \xi_{i\nu} \quad \text{if } \nu \neq k \\ \xi_{k\nu+1} &= t^\alpha \xi_{k\nu} + \eta \end{aligned}$$

and the induction step follows with $\alpha_{\nu+1} = \alpha + \alpha_\nu$.

(b) If $L(h_\mu) \mid t^\alpha L(h_\nu)$ for some $\mu < \nu$ and α is minimal for all possible choices from T then

$$t^\alpha g_\nu p = \sum t^\alpha \xi_{i\nu} f_i + t^\alpha h_\nu - \eta h_\mu + \eta h_\mu$$

with $L(h_\mu)\eta = t^\alpha L(h_\nu)$. We have

$$\begin{aligned} h_{\nu+1} &= t^\alpha h_\nu - \eta h_\mu \\ g_{\nu+1} &= t^\alpha g_\nu - \eta g_\mu \\ \xi_{i\nu+1} &= t^\alpha \xi_{i\nu} - \eta \xi_{i\mu}. \end{aligned}$$

Now $t^{\alpha_\nu - \alpha_\mu} L(h_\mu) > L(h_\nu)$ implies $t^{\alpha+\alpha_\nu} > L(\eta) t^{\alpha_\mu}$, that is $t^{\alpha+\alpha_\nu} = L(g_{\nu+1})$. This proves 2).

To prove 1) let $I_\nu = \langle L(f) \mid f \in T_\nu \rangle$, T_ν be the set T after the ν–th reduction. Let N be an integer such that $I_N = I_{N+1} = \ldots$ (such N exists because $K[t,x]^r$ is noetherian). This implies $T_N = T_{N+1} = \ldots$. The algorithm continues with fixed T and terminates because $<$ is a wellordering on $K[t,x]^r$.

Remark 1.10 1) If the ordering $<$ on $K[x]$ is global, then the standard basis algorithm is equivalent to Buchberger's algorithm because then $x^\alpha \mid x^\beta$ implies $x^\alpha < x^\beta$. This shows that only elements from G are used for the reduction in NFMora. Moreover, if G is homogeneous but $<$ arbitrary, the standard basis algorithm even coincides with Buchberger's algorithm.

2) If $<$ is a tangent cone ordering then the algorithm is Mora's tangent cone algorithm. In his algorithm Mora uses the same normal form, just in another language. Instead of passing from $K[x]$ to $K[t,x]$ by homogenizing and extending the ordering, he uses the notion of **ecart**, where $\text{ecart}(p) = \deg_t(p^h)$. During the implementation of SINGULAR we discovered that the normal form with $\text{ecart}(p) := \deg_t(L(p^h))$ terminates for any ordering, not only for tangent cone orderings. This was found also by Gräbe (cf. [G]).

Corollary 1.11 *Let* $S = \{f_1, \ldots, f_s\}$ *be a finite subset of the submodule* $I \subseteq K[x]^r$.

1) If S *is a standard basis of* I *then:*

(i) For any $f \in K[x]^r$ there are $g, \xi_i \in K[x]$, $h \in K[x]^r$, such that

$$(1+g)f = \sum \xi_i f_i + h,$$

$L(g) < 1$ if $g \neq 0$, $L(\xi_i f_i) \leq L(f)$ if $\xi_i \neq 0$ and, for all i, $L(f_i) \not| L(h)$ if $h \neq 0$.

(ii) $f \in I$ if and only if NFMora $(f^h \mid S^h) = 0$.

(ii') $f \in I$ if and only if $(1+g)f = \sum \xi_i f_i$ for suitable $g, \xi_i \in K[x]$, $L(g) < 1$ if $g \neq 0$ and $L(\xi_i f_i) \leq L(f)$ if $\xi_i \neq 0$.

(iii) $I\, Loc_< K[x] = \langle S \rangle\ Loc_< K[x]$ that is S generates $I\, Loc_< K[x]$ as $Loc_< K[x]$–module.

2) The following are equivalent:

(i) S is a standard basis of I.

(ii) S^h = Standard (S^h, NFMora).

(iii) NFMora (spoly(f, g), S^h) = 0 for all $f, g \in S^h$.

(iv) One of the conditions (ii), (ii') of 1).

The corollary is an easy consequence of 1.9.

Remark 1.12 1) If one extends the ordering $<$ given by the matrix A on $K[x]$ to $K[t, x]$ by

$$\begin{pmatrix} 1 & w_1, \ldots, w_n \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}, \quad \text{all } w_i > 0$$

and uses homogenization with respect to the weights $w_1, \ldots, w_n$ then the standard basis algorithm works as well. Gräbe discovered (cf. [G]) that for a suitable choice of the weights adapted to the input (the polynomials should become as homogeneous as possible with respect to these weights) the algorithm can become faster. We call this the **(weighted) ecartMethod**. It is implemented in SINGULAR with an automatic choice of an "optimal" weight vector.

2) Given $<$ and G there exist, of course, many normal forms NF$(-|G)$ (choose, for instance, in the described NF–algorithms not the first element). But if $<$ is a global ordering, we can apply the normal form algorithm to each monomial of h and we can achieve that for any $f \in K[x]^r$,

$$(*) \qquad f = \sum \xi_i f_i + h,$$

for suitable $\xi_i \in K[x]$, $h \in K[x]^r$ such that $L(\xi_i f_i) \leq L(f)$ if $\xi_i \neq 0$ and, for all i, no monomial of h is divisible by $L(f_i)$; h is then unique. Hence, there exists a **distinguished normal form** NF$(-|G)$, characterized by the property that $L(g)$ does not divide any monial of NF(p, G) for any $g \in G$ and any $p \in K[x]^r$ with NF$(p|G) \neq 0$.

If we try the same for local or mixed orderings, this procedure will, in general, not terminate. We can only derive a presentation $(*)$ with $\xi_i \in K[[x]]$ and $h \in K[[x]]^r$ (formal power series) having the above properties. In particular, a distinguished normal form does only exist as a function with values in $K[[x]]^r$.

3) A reduced standard basis is uniquely determined by I and $<$. If $<$ is a wellordering or if $\dim_K Loc_< K[x]^r/I < \infty$ then there exists always a reduced standard basis in $K[x]^r$. In general, it exists only in $K[[x]]^r$.

2 On Schreyer's method to compute syzygies

In this chapter we shall prove that Schreyer's method to compute syzygies (cf. [S], [E]) extends to any semigroup ordering $<$ on $K[x]^r = \sum_{i=1}^r K[x]e_i$. For the treatment of syzygies in a different context, or for different algorithms see [Ba], [MM1], [MM2] and [MMT].
Let $S = \{g_1, \ldots, g_q\}$ be a standard basis of $I \subseteq K[x]^r$.

For $K[x]^q = \sum_{i=r+1}^{q+r} K[x]e_i$ we choose the following **Schreyer ordering** $<_1$ (depending on S): $x^\alpha e_{i+r} <_1 x^\beta e_{j+r}$ if and only if either $L(x^\alpha g_i) < L(x^\beta g_j)$ or $L(x^\alpha g_i) = L(x^\beta g_j)$ and $i > j$.

For g_i, g_j having the leading term in the same component, that is $L(g_i) = x^{\alpha_i} e_k, L(g_j) = x^{\alpha_j} e_k$ we consider $\text{spoly}(g_i, g_j) := m_{ji} g_i - m_{ij} g_j$ with $m_{ji} = c(g_j) \frac{lcm(L(g_i), L(g_j))}{x^{\alpha_i}}$.

Because S is a standard basis we obtain (Corollary 1.11)

$$(1 + h_{ij})(m_{ji} g_i - m_{ij} g_j) = \sum \xi_\nu^{ij} g_\nu$$

with $L(h_{ij}) < 1$ if $h_{ij} \neq 0$ and $L(\xi_\nu^{ij} g_\nu) < L(m_{ji} g_i)$.

For $j > i$ such that g_i, g_j have leading term in the same component, let

$$\tau_{ij} := (1 + h_{ij})(m_{ji} e_{i+r} - m_{ij} e_{j+r}) - \sum \xi_\nu^{ij} e_{\nu+r}.$$

Let $\ker(K[x]^q \to K[x]^r, \sum w_i e_{i+r} \mapsto \sum w_i g_i)$ denote the **module of syzygies, syz**(I), of $\{g_1, \ldots, g_q\}$. The following proposition is essentially due to Schreyer.

Proposition 2.1 *With respect to the ordering $<_1$ the following holds:*

1) $L(\tau_{ij}) = m_{ji} e_{i+r}$.

2) $\{\tau_{ij} \mid i < j$ s.t. $L(g_i), L(g_j)$ are in the same component $\}$ is a standard basis for syz(I).

Proof: 1) $L(\tau_{ij}) = L(m_{ji} e_{i+r} - m_{ij} e_{j+r}) = m_{ji} e_{i+r}$ holds by definition of $<_1$. To prove 2) it has to be shown that $L(\text{syz}(I)) = \langle\{m_{ji} e_{i+r}\}\rangle$.

Let $\sum w_i g_i = 0$, that is $\tau := \sum w_i e_{i+r} \in \text{syz}(I)$, and let $m e_{k+r} = L(\tau)$ with respect to $<_1$. Let

$$T := \{n e_{r+l} \mid n e_{r+l} \text{ be a monomial of } \tau, L(n g_l) = L(m g_k)\}.$$

Then, obviously, $\tau|_T := \sum_{n e_{r+l} \in T} n e_{r+l}$ is a syzygy of $L(g_1), \ldots, L(g_q)$. Especially, $\#T \geq 2$. Choose l such that $n e_{r+l} \in T$ for some n and $n e_{r+l} \neq m e_{k+r}$. Because $L(\tau) = m e_{k+r}$ and the definition of $<_1$ we have $k < l$. Since $m L(g_k) = n L(g_l)$ we have $m_{lk} \mid m$. But $L(\tau_{kl}) = m_{lk} e_{k+r}$ implies $L(\tau_{kl}) \mid \tau$, that is $L(\tau) \in L(\langle\{m_{ji} e_{i+r}\}\rangle)$, which proves the proposition.

The algorithm "Standard basis" of paragraph 1, together with repeated application of the algorithm "Syz", provides an effective way to construct finite $\text{Loc}_< K[x]$–free resolutions and gives a sharpened version of Hilbert's syzygy theorem which generalizes Schreyer's proof (cf. [E], [S]).

Lemma 2.2 *Let $\{g_1, \ldots, g_q\}$ be a standard basis of $I \subset K[x]^r = \sum_{i=1,\ldots,r} K[x] e_i$. We assume that the leading terms are a basis vector of $K[x]^r$, that is $L(g_i) = e_{\nu_i}$ for suitable ν_i. We set $J = \{\nu \mid \exists i$ s.t. $\nu = \nu_i\}$ and for $\nu \in J$ we choose exactly one g_{i_ν} such that $L(g_{i_\nu}) = e_\nu$. Then $I Loc_< K[x]$ is a free $Loc_< K[x]$–module with basis $\{g_{i_\nu} \mid \nu \in J\}$ and $(Loc_< K[x])^r / I Loc_< K[x]$ is $Loc_< K[x]$–free with basis represented by the $\{e_j \mid j \notin J\}$.*

Proof: Let us renumber the g_i such that $g_{i_\nu} = g_\nu$ for $\nu \in J$. First of all, the subset $\{g_\nu \mid \nu \in J\} \subset \{g_1, \ldots, g_q\}$ remains a standard basis of I since the set of leading terms is not changed. Hence, we may assume that all leading terms are different. By Proposition 1.4, $\{g_\nu \mid \nu \in J\}$ generates $I\text{Loc}_< K[x]$. Now consider a relation

$$\sum_{j \notin J} \xi_j e_j = \sum_{j \in J} \xi_j g_j, \quad \xi_j \in \text{Loc}_< K[x].$$

After clearing denominators we may assume that $\xi_j \in K[x]$. Since the leading terms involve different e_i on each side, we obtain $\xi_1 = \cdots = \xi_n = 0$. This shows that the g_ν, $\nu \in J$ are linear independent and that the e_j, $j \notin J$, are independent modulo $I\text{Loc}_< K[x]$. Since $\{L(g_j) \mid j \in J\} \cup \{e_i \mid i \notin J\}$ generate $L(K[x]^r) = (e_1, ..., e_r) K[x]$, $\{g_j \mid j \in J\} \cup \{e_i \mid i \notin J\}$ is a standard basis of $K[x]^r$ and this set generates $(\text{Loc}_< K[x])^r$ by Corollary 1.11. Therefore, $\{e_j \mid j \notin J\}$ generates $(\text{Loc}_< K[x])^r / I\ \text{Loc}_< K[x]$ over $\text{Loc}_< K[x]$.

Theorem 2.3 *Let $S = \{g_1, \ldots, g_q\}$ be a standard basis of $I \subseteq K[x]^r$. Order S in such a way that whenever $L(g_i)$ and $L(g_j)$ involve the same component, say $L(g_i) = x^{\alpha_i} e_k$ and $L(g_j) = x^{\alpha_j} e_k$, then $\alpha_i \geq \alpha_j$ in the lexicographical ordering if $i < j$. If $L(g_1), \ldots, L(g_q)$ do not depend on the variables $x_1, \ldots, x_s$, then the $L(\tau_{ij})$ do not depend on the variables $x_1, \ldots, x_{s+1}$ and*

$$M := (Loc_< K[x])^r / I\ Loc_< K[x]$$

has a $Loc_< K[x]$–free resolution of length $\leq n - s$. In particular, M always has a free resolution of length $\leq n$ and, by Serre's theorem, $Loc_< K[x]$ is a regular ring.

Proof: For $i < j$ and $L(g_i) = x^{\alpha_i}e_k$, $L(g_j) = x^{\alpha_j}e_k$ we have $\alpha_i = (0,\ldots,0,\alpha_{i,s+1},\ldots), \alpha_j = (0,\ldots,0,\alpha_{j,s+1},\ldots)$ with $\alpha_{i,s+1} \geq \alpha_{j,s+1}$. Therefore, $L(\tau_{ij}) = m_{ji}e_{i+r}$ does not depend on $x_1,\ldots,x_{s+1}$.

Let $q_1 := q$ and $\varphi_1 : K[x]^{q_1} \to K[x]^r$ the morphism given by $\{g_i\}$, $\sum w_i e_{i+r} \mapsto \sum w_i g_i$, and $\varphi_2 : K[x]^{q_2} \to K[x]^{q_1}$ the analog morphism given by the standard basis $\{\tau_{ij}\}$, $q_2 = \#\{\tau_{ij}\}$. Applying the same construction as above to $\mathrm{syz}^1(I) := \mathrm{syz}(I) = \ker(\varphi_1)$ and $\{\tau_{ij}\}$ we obtain a standard basis $\{\tau^2_{kl}\}$ of $\mathrm{szy}^2(I) := \mathrm{syz}(\mathrm{syz}(I)) = \ker(\varphi_2)$ such that the leading terms of τ^2_{kl} do not depend on $x_1,\ldots,x_{s+2}$.

Continuing in the same way we obtain an exact sequence

$$0 \to K[x]^{q_{n-s}}/\ker(\varphi_{n-s}) \xrightarrow{\varphi_{n-s}} K[x]^{q_{n-s+1}} \to \ldots \xrightarrow{\varphi_2} K[x]^{q_1} \xrightarrow{\varphi_1} K[x]^r \to K[x]^r/I \to 0.$$

Moreover, $\ker(\varphi_{n-s}) = \mathrm{syz}^{n-s}(I)$ has a standard basis $\{\tau^{n-s}_{k,l}\}$ such that none of the variables appear in $L(\tau^{n-s}_{k,l})$. Hence, by the preceding lemma, $K[x]^{q_{n-s}}/\ker(\varphi_{n-s})$ becomes free after tensoring with $\mathrm{Loc}_< K[x]$. If we tensor the whole sequence with $\mathrm{Loc}_< K[x]$ it stays exact (since $\mathrm{Loc}_< K[x]$ is $K[x]$–flat) and is the desired free resolution of M.

Remark 2.4 The above algorithm almost never gives a minimal free resolution (in the local or in the homogeneous case), on the contrary, every syzygy module is generated by a standard basis. Nevertheless, it is often quite fast (cf. [Gr et al]).

3 Zariski's question, Milnor numbers and multiplicities

The generalization of Buchberger's algorithm presented in this paper has many applications, in particular to local algebra and local algebraic geometry. For instance, most of the algorithms described in [E], II.15 can be transferred from $k[X]$ to $\mathrm{Loc}_< K[X]$. Some use extra tag variables to be eliminated later, hence they require mixed orderings even for pure local computations. Here we shall only explain how the implementation in SINGULAR helped to find a partial answer to Zariski's multiplicity question and prove the theoretically relevant results (cf. Proposition 3.3 and Corollaries 3.4, 3.6) which justify such kinds of applications.

Zariski asked in 1971 (cf. [Z]) whether two complex hypersurface singularities f and g with the same embedded topological type have the same multiplicity, where for $f \in \mathbb{C}\{x_1,\ldots,x_n\} = \mathbb{C}\{x\}$, $f = \sum c_\alpha x^\alpha$, $f(0) = 0$, a not constant convergent powerseries, $\mathrm{mult}(f) = \min\{|\alpha| \mid c_\alpha \neq 0\}$ is the multiplicity of f. Zariski's question (usually called Zariski's conjecture) is, in general, unsettled but the answer is known to be yes in the case $n = 2$, that is for plane curve singularities (Zariski, Lê Dũng Trang), and if f is semiquasihomogeneous and g is a deformation of f (Greuel, O'Shea).

Recall that f is called semiquasihomogeneous if there exists an analytic change of coordinates and positive weights for the new coordinates such that the sum of terms of smallest weighted degree has an isolated singularity.

The idea for the search for a counter example to Zariski's conjecture is as follows: let $f_t(x) = f(x)+tf_1(x)+t^2f_2(x)+\ldots$ be a deformation of $f(x)$ and $\mu(f_t) = \dim_{\mathbb{C}} \mathbb{C}\{x\}/(\partial f_t/\partial x_1,\ldots,\partial f_t/\partial x_n)$ the Milnor number of f_t which we assume to be finite for $t = 0$ (then it is finite for t close to 0). Then, if the topological type of f_t is independent of t, the Milnor number $\mu(f_t)$ is independent of t (for t sufficiently close to 0). The converse is also known to be true if $n \neq 3$. Hence, if $\mu(f_t)$ is constant but $\mathrm{mult}(f_t)$ is not, we get a counter example (at least if $n \neq 3$). Because of the above mentioned positive results, a candidate for a counter example must have a big Milnor number which cannot be computed by hand. The standard basis algorithm of §1, together with a good choice of strategies and special improvements for zero–dimensional ideals (cf. [Gr et al]), as implemented in SINGULAR, allowed these Milnor numbers to be computed for several series of candidates (all other systems failed). The failure to find a counter example led to the following positive result.

Let $f_t(x)$ be a (1–parameter) holomorphic **family of isolated hypersurface singularities**, that is $0 \in \mathbb{C}^n$ is an isolated critical point of f_t for each t close to $0 \in \mathbb{C}$. The **polar curve** of such a family is the curve singularity in $\mathbb{C}^n \times \mathbb{C}$ defined by the ideal $(\partial f_t/\partial x_1,\ldots,\partial f_t/\partial x_n) \subset \mathbb{C}\{x,t\}$.

Lemma 3.1 *Let f_t be a family of isolated hypersurface singularities. Let $H \cong \mathbb{C}^{n-1}$ be a hyperplane through 0 such that formation of the polar curve is compatible with restriction to H. That is: polar curve$(f_t \mid H) =$ polar curve$(f_t) \cap H$. Then*

$$\mu(f_t) = \text{ constant } \Rightarrow \mu(f_t|H) = \text{ constant.}$$

Proof: We may assume that $H = \{x_n = 0\}$ and then the polar curve$(f_t|H)$ is given by $(\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_{n-1}, x_n)$ while polar curve$(f_t) \cap H$ is given by $(\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_n, x_n)$. Hence, the assumption is equivalent to $\partial f_t/\partial x_n \in (\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_{n-1}, x_n)$.

We shall use the valuation test for μ–constant by Lê and Saito ([LS]):
$\mu(f_t) =$ constant $\Leftrightarrow$ for any holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$ we have $\text{val}(\partial f_t/\partial t(\gamma(s))) \geq \min\{\text{val}(\partial f_t/\partial x_i(\gamma(s))),\ i = 1, \ldots, n\}$. Moreover, this is equivalent to "$\geq$" replaced by "$>$". (val denotes the natural valuation with respect to s.)

Now let $\gamma(s)$ be any curve in $H = \{x_n = 0\}$. Then $\partial f_t/\partial x_n \in (\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_{n-1}, x_n)$ implies that $\text{val}(\partial f_t/\partial x_n(\gamma(s)) \geq \min\{\text{val}(\partial f_t/\partial x_i(\gamma(s))),\ i = 1, \ldots, n-1\}$.

Applying the valuation test to f_t and to $f_t \mid H$, the result follows.

Proposition 3.2 *Let $f_t(x_1, \ldots, x_n) = g_t(x_1, \ldots, x_{n-1}) + x_n^2 h_t(x_1, \ldots, x_n)$ be a family of isolated hypersurface singularities. Let g_0 be semiquasihomogeneous or let $n = 3$. If the topological type of f_t is constant then the multiplicity of g_t is constant (for t close to 0). In particular, if mult$(g_t) \leq$ mult$(x_n^2 h_t)$ then mult(f_t) is constant.*

Proof: Since f_t has an isolated singularity we may add terms of sufficiently high degree without changing the analytic type of f_t. If $n = 3$ we may replace g_t by $g_t(x_1, x_2) + x_1^N + x_2^N$, N sufficiently big, which has an isolated singularity and the same multiplicity as $g_t(x_1, x_2)$. Hence, in any case we may assume that g_t has an isolated singularity. Applying the preceding lemma to the hyperplane $\{x_n = 0\}$ we obtain $\mu(g_t)$ constant. But since Zariski's conjecture is true for plane curve singularities and for deformations of semiquasihomogeneous singularities ([Gr]), mult(g_t) is constant.

The Milnor number $\mu(f)$ of an isolated singularity can be computed as the number of monomials in $K[x_1, \ldots, x_n]/L(I)$ where I is the leading ideal of $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ with respect to any local ordering. This follows from Corollary 3.4, for which we need the following construction:

Let $g_1, \ldots, g_q$ be a standard basis of $I \subset K[x]^r = \sum_{i=1,\ldots,r} K[x]e_i$. Any monomial $x^\alpha e_k$ may be identified with the point $(\alpha_1, \ldots, \alpha_n, 0, \ldots, 1, \ldots 0) \in \mathbb{N}^{n+r}$. For a weight vector $w = (w_1, \ldots, w_{n+r}) \in \mathbb{Z}^{n+r}$ we define

$$\deg_w x^\alpha e_k = w_1\alpha_1 + \cdots + w_n\alpha_n + w_{n+k}$$

to be the weighted degree of $x^\alpha e_k$. Let $\text{in}_w(f)$ the initial term of $f \in K[x]^r$, that is the sum of terms (monomial times coefficient) of f with maximal weighted degree and $\text{in}_w(I)$ the submodule generated by all $\text{in}_w(f)$, $f \in I$.

It is not difficult to see that there exists a weight vector $w \in \mathbb{Z}^{n+r}$ (indeed almost all w will do) such that $\text{in}_w(g_i) = c(g_i)L(g_i)$, $i = 1, \ldots, q$, and, moreover, $\text{in}_w(I) = L(I)$.

We choose such a w and shall now construct a deformation from $L(I)$ to I:
For $f \in K[x]^r$ we can write $f = f_p + f_{p-1} + f_{p-2} + \cdots$ such that the weighted degree of each monomial of f_ν is ν. Let t be one extra variable and put

$$\tilde{f}(x,t) = f_p(x) + tf_{p-1}(x) + t^2 f_{p-2}(x) + \cdots \in K[x,t]^r.$$

Let $\tilde{I} \subset K[x,t]^r$ be the submodule generated by all $\tilde{f}$, $f \in I$. On $K[x,t]^r$ we choose the following ordering: $x^\alpha t^p e_k < x^\beta t^q e_l$ if $p + \deg_w x^\alpha e_k < q + \deg_w x^\beta e_l$ or, if these terms are equal and $x^\alpha e_k < x^\beta e_l$.

With respect to this ordering we have $L(\tilde{f}) = L(f)$ and, moreover, $\tilde{g}_1, \ldots, \tilde{g}_q$ is a standard basis of $\tilde{I}$. (If $h \in \tilde{I}$ then $h = t^m \tilde{f}$, $f \in I$, hence, $L(h) = t^m L(f) \in \langle L(\tilde{g}_1), \ldots, L(\tilde{g}_q)\rangle$). In addition, setting $t = 0$ or 1, we obtain sections of the inclusion of multiplicative sets $S_<(K[x]) \subset S_<(K[x,t])$.

Let $R := \text{Loc}_< K[x]$, $S := \text{Loc}_< K[x,t]$ and $K(t)$ the quotient field of $K[t]$.

Proposition 3.3 *If $I \neq R^r$ then $S^r/\tilde{I}S$ is a faithfully flat $K[t]$–module with special fibre*

$$(S^r/\tilde{I}S) \otimes_{K[t]} K \cong R^r/L(I)R$$

and generic fibre

$$(S^r/\tilde{I}S) \otimes_{K[t]} K(t) \cong R^r/IR \otimes_K K(t).$$

Proof: The statements regarding the special and the generic fibres are easy. Note that for $\lambda \neq 0$, $(S^r/\tilde{I}S) \otimes_{K[t]} K[t]/(t-\lambda) \cong R^r/IR$. Therefore, if $I \neq R^r$ then the support of $S^r/\tilde{I}S$ is surjective over Spec $K[t]$ and hence it remains to show that t is a non–zero divisor of $S^r/\tilde{I}S$. Let $f \in S^r$ and $tf \in \tilde{I}S$. By Corollary 1.11 we have (after clearing denominators)

$$\text{NFMora}\,(tf^h \mid \{\tilde{g}_1^h, \ldots, \tilde{g}_q^h\}) = t\ \text{NFMora}\,(f^h \mid \{\tilde{g}_1^h, \ldots, \tilde{g}_q^h\}) = 0,$$

hence, $f \in \tilde{I}S$.

Corollary 3.4 *Let either $<$ be a wellordering or R^r/IR a finite dimensional K–vector space. Then the monomials in $K[x]^r \backslash L(I)$ represent a K–basis of R^r/IR.*

Proof: If $<$ is a wellordering, the monomials not in $L(I)$ are a basis of the free module $S^r/\tilde{I}S$ (Theorem of Macaulay, cf. [E]), hence the result. In general, it is easy to see that these monomial are linear independent modulo IR. (Use a standard basis of I and Corollary 1.11.) If R^r/IR is finite dimensional, there are only finitely many monomials in $K[x]^r \backslash L(I)$. The proposition implies that $S^r/\tilde{I}S$ is $K[t]$–free with these monomials as basis, hence they also generate R^r/IR.

Remark 3.5 In general, the monomials not in $L(I)$ are not a basis of $\text{Loc}_< K[x]/I$. Take, for example, $K[x]$ with lex$^-$ and $I = (0)$. Then $\text{Loc}_< K[x] = K[x]_{(x)}$ is not K–generated by monomials. If $<$ is a wellordering, then $S^r/\tilde{I}S$ is even free over $K[t]$ (cf. [E]).

Corollary 3.6 *For any module ordering* $\dim R^r/IR = \dim K[x]^r/L(I)$ *where dim denotes the Krull dimension.*

Proof: $I = R^r$ implies $L(I) = K[x]^r$, hence we may assume $I \neq R^r$. Faithful flatness implies that $\dim R^r/IR = \dim R^r/L(I)R$, hence the result.

Let us finish with a final remark about multiplicites in the local case:

Consider the local ring $R = K[x]_{(x)}$ with maximal ideal $(x) = (x_1, \ldots, x_n)$ and $M = R^r/IR$ a finitely generated R–module, where I is given as a submodule of $K[x]^r$ by finitely many generators. Let mult(M) denote the (Samuel–)multiplicity of M with respect to (x). Consider

$$gr\,M = \sum_{i \geq 0} (x)^i M/(x)^{i+1} M,$$

which is a graded module over $gr\,R = K[x]$. For any graded module N let h_N denote the Hilbert function of N and degree(h_N) the degree of the corresponding Hilbert polynomial.

The following proposition now follows easily.

Proposition 3.7 *Let $<$ be a degree ordering (cf. Chapter 1) on the monomials of $K[x]$ such that $w_i =$ degree $(x_i) = -1$ for $i = 1, \ldots, n$ which is extended to a module ordering on $K[x]^r$ arbitrarily. Let $M = R^r/IR$ be as above and $L(I)$ be the leading ideal of I. Then the Hilbert function $h_{gr\,M}$ coincides with the Hilbert function $h_{K[x]^r/L(I)}$ of the graded module $K[x]^r/L(I)$. In particular,* $\dim M = \dim K[x]^r/L(I)$ *and mult(M) = degree$(h_{K[x]^r/L(I)})$.*

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