A Sobolov Space Analysis of Linear Regularization Methods for Ill-Posed Problems

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Abstract

In this paper we show that linear regularization methods can be decomposed as a smoothing of the generalized inverse or equivalently as the generalized inverse applied to smoothed data. We give conditions on the degree of the smoothing based on estimates in Sobolev norms. We also show that the regularization methods form a semi group. Conditions for the order optimality of the methods are provided. Finally we verify the results for the Radon transform as the model in computerized tomography.

1 Introduction

We study operator equations $Af = g$ for linear compact operators between Hilbert spaces X and Y. It is well known that in the case of infinite dimensional range of A the problem of solving $Af = g$ is ill-posed. In order to stabilize the solution regularization methods are applied. Prominent examples are Tikhonov-Phillips regularization, linear iterative methods, the truncated singular value decomposition which all can be written as filtered versions of the pseudoinverse, see [3, 6, 7, 12]. Mollifier methods as for example given by Andersson [1], Backus-Gilbert [2], Chavant [4], Eckhardt [5], Lions [9], Louis [10], Louis-Maass [17] or Murio [19] are also regularization methods. The recently introduced approximate inverse [14] can be viewed as a generalization of all these methods.

In this paper we discuss the above mentioned regularization methods in a general way. We show that they can be interpreted as a combination of the pseudoinverse and a smoothing operator in either order. To this end we start from a norm equivalence between $||Af||$ and a Sobolev norm of f and we repeat the definition of linear regularization methods. Then we study the version where first the pseudoinverse is applied and then the result is smoothed in suitable Sobolev norms. Recently Hegland-Andersson [8] called this approach domain mollification. Also we show that the regularization methods form a semigroup. After that we take a look on prewithening methods which means that we first smooth the data and then apply the pseudoinverse. In [8] this is called range mollification. One of the results of the paper is that the two methods produce the same results if the mollifiers are chosen accordingly. For a special combination this was shown in [24]. In the next section we give sufficient conditions for the smoothing operator to achieve order optimality of the regularization method. As an example we study Tikonov-Phillips regularizations and show that even norms with negative index lead to regularization as long as the index is at most half of the smoothing index of the operator. This means, if the smoothing index is α we can use as regularization term $||f||_{H^{-\beta}}$ with $\beta \leq \alpha/2$. This is validated for the Radon transform which serves as mathematical model for computerized tomography. After that we compare our results with those in [16] which were obtained by using another approach based on operator dependent norms.

2 Regularization Methods

Let A be a linear operator between the Hilbert spaces X and Y. For the sake of simplicity of the representation we assume A to be compact. Then $A : X \to Y$ has a complete singular system $\{v_n, u_n; \sigma_n\}_n$ with normalized v_n and u_n such that $Av_n = \sigma_n u_n$ and $A^* u_n = \sigma_n v_n$. The pseudoinverse $A^\dagger$ has the representation

$$A^\dagger g = \sum_n \sigma_n^{-1} \langle g, u_n \rangle v_n.$$

For nondegenerated A the singular values σ_n decay to zero, hence $A^\dagger$ is not defined on all of Y. So for finding an acceptable solution of $Af = g$ with noisy data $g \in Y$ we have to substitute this operator by a so called regularization.

Definition 2.1 *A* regularization *of $A^\dagger$ for finding the solution $f \in X$ of $Af = g$ is a family of operators*

$$\{T_\gamma\}_{\gamma > 0}, \ T_\gamma : Y \to X$$

with a mapping: $\gamma : \mathbb{R}^+ \times Y \to \mathbb{R}^+$, such that for all $g \in \mathcal{D}(A^\dagger)$ and for all $g^\epsilon \in Y$ with $||g - g^\epsilon|| \leq \epsilon$ the equality

$$\lim_{\epsilon \to 0, g^\epsilon \to g} T_{\gamma(\epsilon, g^\epsilon)} g^\epsilon = A^\dagger g$$

holds.

In the next section we want to give sufficient conditions so that a mapping T_γ is a regularization of $A^\dagger$. In order to do this we need some notion. In the following we concentrate on problems where X and Y are L_2 – spaces, possibly on different domains. For our discussion we use Sobolev spaces where the connection with the problem is achieved by an equivalence of the norm $||Af||$ and a Sobolev norm of f, see e.g. [22]. The Sobolev space $H^\alpha(\mathbb{R}^n)$ is defined as

$$H^\alpha = \{f \in \mathcal{S}'(\mathbb{R}^n) : ||f||^2_{H^\alpha} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi < \infty\}.$$

We make the assumption that there is a number $\alpha > 0$ and constants $c_1, c_2 > 0$, such that

$$c_1 ||f||_{H^{-\alpha}} \leq ||Af||_{L_2} \leq c_2 ||f||_{H^{-\alpha}} \text{ for all } f \in \mathcal{N}(A)^- \tag{1}$$

or

$$c_1 ||f||_{L_2} \leq ||Af||_{H^{\alpha}} \leq c_2 ||f||_{L_2} \text{ for all } f \in \mathcal{N}(A)^- \tag{2}$$

holds. If we define $A^\dagger g$ as the minimum norm solution of $Af = P_{\overline{\mathcal{R}(A)}} g$ with $f \in \mathcal{N}(A)^-$ and $\tilde{A}^\dagger$ the extension of $A^\dagger$ to all of Y then we are able to state the following theorem.

Theorem 2.2 *The operator $A^\dagger$ can be extended to $\tilde{A}^\dagger : L_2 \to H^{-\alpha}$ and $\tilde{A}^\dagger : \tilde{H}^\alpha \to L_2$ with $||\tilde{A}^\dagger|| \leq c_1^{-1}$. Here the space $\tilde{H}^\alpha$ is defined by*

$$\tilde{H}^\alpha = \{g = g_1 + g_2, g_1 \in \mathcal{N}(A^*)^-, g_2 \in \mathcal{N}(A^*) : ||g||_{\tilde{H}^\alpha} = ||g_1||_{H^\alpha} < \infty\}.$$

Proof: For $g \in L_2$ we have using relation (1) and $||A\tilde{A}^\dagger g|| = ||P_{\overline{\mathcal{R}(A)}} g|| \leq ||g||$ with the L_2-projection $P : L_2 \to \overline{\mathcal{R}(A)}$

$$||\tilde{A}^\dagger g||_{H^{-\alpha}} \leq c_1^{-1} ||g||_{L_2}.$$

Now we take $g = g_1 + g_2 \in \tilde{H}^\alpha$ with $g_1 \in \mathcal{N}(A^*)^-$, $g_2 \in \mathcal{N}(A^*)$ and get using relation (2)

$$||\tilde{A}^\dagger g||_{L_2} = ||\tilde{A}^\dagger g_1|| \leq c_1^{-1} ||g_1||_{H^\alpha} = c_1^{-1} ||g||_{\tilde{H}^\alpha}.$$

□

Looking at the above theorem we see that there are two theoretical ways to stabilize the problem of finding the solution f of the equation $Af = g$. The first one is based on some smoothness conditions for the data g, assuming $g \in H^\alpha$, which means that also the data errors have to be in H^α which is unrealistic in general. The second possibility is to change the concept of solution resulting in $f \in H^{-\alpha}$.

3 Smoothing the Pseudoinverse

In this section we produce regularization methods applying Theorem 2.2 with $\tilde{A}^\dagger : L_2 \to H^{-\alpha}$. In this setting the pseudoinverse $\tilde{A}^\dagger$ maps from L_2 to the space $H^{-\alpha}$. This space is too large, hence we need some smoothing operator to come back from this large space $H^{-\alpha}$ to L_2. In [8] this is called domain mollification because the regularization is achieved by smoothing in the domain of the operator A.

Theorem 3.1 *Let $M_\gamma : H^{-\alpha} \to L_2$ be a family of linear continuous operators such that*

$$\begin{array}{lll} i) & \|M_\gamma f\|_{L_2} \le c(\gamma)\|f\|_{H^{-\alpha}} & \forall f \in \mathcal{N}(A)^-, \\ ii) & \lim\limits_{\gamma\to 0}\|M_\gamma f - f\| = 0 & \forall f \in \mathcal{N}(A)^-, \\ iii) & c(\gamma)\epsilon \to 0 \text{ for } \gamma \to 0,\ \epsilon \to 0. & \end{array}$$

Then $T_\gamma = M_\gamma \tilde{A}^\dagger$ is a regularization of $A^\dagger$ for finding f.

Proof: Let $g \in \mathcal{D}(A^\dagger)$ and $g^\epsilon \in L_2$ such that $\|g^\epsilon - g\| \le \epsilon$, then we get with $A^\dagger g = \tilde{A}^\dagger g \ \forall\ g \in \mathcal{D}(A^\dagger)$:

$$\begin{aligned} \|T_\gamma g^\epsilon - A^\dagger g\| &\le \|T_\gamma(g^\epsilon - g)\| + \|T_\gamma g - A^\dagger g\| \\ &= \|M_\gamma \tilde{A}^\dagger(g^\epsilon - g)\| + \|T_\gamma g - \tilde{A}^\dagger g\| \\ &\le c(\gamma)\|\tilde{A}^\dagger(g^\epsilon - g)\|_{H^{-\alpha}} + \|M_\gamma \tilde{A}^\dagger g - \tilde{A}^\dagger g\| \\ &\le c(\gamma)c_1^{-1}\epsilon + \|M_\gamma \tilde{A}^\dagger g - \tilde{A}^\dagger g\| \\ &\stackrel{\epsilon\to 0}{\longrightarrow} 0 \text{ for } \epsilon \to 0 \text{ and } \gamma \text{ such that } c(\gamma)\epsilon \to 0. \end{aligned}$$

□

Our first example is the recently introduced approximative inverse (see [14]). The points, where the solution f is to be determined, are denoted by x. We choose a suitable smooth mollifier $e_\gamma(x, .)$ and determine for fixed x a reconstruction kernel $\psi_\gamma(x) \in L_2$ as minimum norm solution of

$$A^*\psi_\gamma(x) = e_\gamma(x, \cdot),$$

hence ψ_γ has a representation as

$$\psi_\gamma(x) = \sum_n \sigma_n^{-1}\langle e_\gamma(x, \cdot), v_n\rangle u_n.$$

We get

$$\langle f, e_\gamma\rangle = \langle f, A^*\psi_\gamma\rangle = \langle Af, \psi_\gamma\rangle = \langle g, \psi_\gamma\rangle =: S_\gamma g.$$

This statement holds with '=' if $e_\gamma \in \mathcal{R}(A^*)$ and with '$\simeq$' else. Now we define the operator M_γ as

$$M_\gamma v_n(x) = \langle e_\gamma(x, .), v_n\rangle$$

and get

$$S_\gamma = M_\gamma \tilde{A}^\dagger.$$

As a special case we consider a mollifier e_γ of convolution type. Then we can derive the following result using the Fourier transform $\hat{e}_\gamma$ of e_γ,

$$\hat{e}(\xi) = \mathcal{F}e(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e(x)\exp(-\imath\xi^T x)dx\ .$$

Theorem 3.2 *Let $e_\gamma(x,y)$ be of convolution type, i.e. $e_\gamma(x,y) = e_\gamma(x-y)$, and*

$$i) \quad (2\pi)^{n/2}\sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\} \le c(\gamma),$$

$$ii) \quad \sup_{\xi\in K}|(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1| \stackrel{\gamma\to 0}{\longrightarrow} 0 \qquad \text{for any } K \text{ compact},$$

$$iii) \quad \sup_\xi|\hat{e}_\gamma(\xi)| \le M.$$

Then $T_\gamma g(x) = \langle e_\gamma(x,.), \tilde{A}^\dagger g\rangle$ is a regularization of $A^\dagger$ for finding f.

Proof: We check the conditions of Theorem 3.1.

$$\begin{aligned} ||M_\gamma f||^2 &= ||\mathcal{F}(M_\gamma f)||^2 \\ &= (2\pi)^n \int_{\mathbb{R}^n} |\hat{e}_\gamma(\xi)\hat{f}(\xi)|^2 d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\alpha}(1+|\xi|^2)^{\alpha}|\hat{e}_\gamma(\xi)|^2|\hat{f}(\xi)|^2 d\xi \\ &\le \underbrace{(2\pi)^n \sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\}^2}_{\le c(\gamma)^2} ||f||^2_{H^{-\alpha}}. \end{aligned}$$

$$||M_\gamma f - f||^2 = ||\mathcal{F}(M_\gamma f - f)||^2 = \int_{\mathbb{R}^n} |(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1|^2|\hat{f}(\xi)|^2 d\xi \stackrel{(*)}{\longrightarrow} 0.$$

The convergence at $(*)$ holds, because for every $\epsilon > 0$ we can find a compact set K and $\gamma_0 > 0$ such that

$$\int_{\mathbb{R}^n\setminus K} |\hat{f}(\xi)|^2 d\xi \le \epsilon \text{ and } \sup_{\xi\in K}|(2\pi)^{n/2}\hat{e}(\xi) - 1| \le \epsilon \text{ for all } \gamma \le \gamma_0.$$

It follows

$$\begin{aligned} \int_{\mathbb{R}^n} |(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1|^2|\hat{f}(\xi)|^2 d\xi &= \int_K |(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1|^2|\hat{f}(\xi)|^2 d\xi \\ &+ \int_{\mathbb{R}^n\setminus K} |(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1|^2|\hat{f}(\xi)|^2 d\xi \\ &\le ||f||^2_{L_2}\epsilon + ((2\pi)^{n/2}M + 1)^2\epsilon, \end{aligned}$$

because $|(2\pi)^{n/2}\hat{e}_\gamma(\xi) - 1|$ converges uniformly in K (see ii) and $|\hat{e}_\gamma| \le M$ (see iii).

□

Example 3.3 *Let $n = 1$ and*

$$e_\gamma(x-y) = \frac{1}{\gamma\pi}\text{sinc }(\frac{x-y}{\gamma})$$

with sinc $x = \sin x/x$. *Then we have*

$$\hat{e}_\gamma(\xi) = (2\pi)^{-1/2}\chi_{[-\frac{1}{\gamma},\frac{1}{\gamma}]}(\xi)$$

with $\chi_{[-\frac{1}{\gamma},\frac{1}{\gamma}]}$ *being the characteristic function of the intervall* $[-\frac{1}{\gamma},\frac{1}{\gamma}]$. *Checking the conditions of Theorem 3.2 we obtain:*

i) $$(2\pi)^{1/2}\sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\} = \left(1+\left|\frac{1}{\gamma}\right|^2\right)^{\alpha/2} = c(\gamma),$$

ii) $\sup_{\xi\in K}|(2\pi)^{1/2}\hat{e}_\gamma(\xi) - 1| \to 0$, *because for some* $\gamma > 0$ *we have* $K \subset [-\frac{1}{\gamma},\frac{1}{\gamma}]$,

iii) $$\sup_\xi|\hat{e}_\gamma(\xi)| \leq (2\pi)^{-1/2}.$$

Hence $M_\gamma\tilde{A}^\dagger$ *is a regularization of* $A^\dagger$.

Our second example are filter regularizations. Here the smoothing operator has the special structure

$$M_\gamma v_n = F_\gamma(\sigma_n)v_n.$$

These methods are discussed in detail in [12]. Examples are the truncated singular value decomposition with $F_\gamma(\sigma_n) = 1$ for $\sigma \geq \gamma$ and 0 otherwise, the Tikhonov-Phillips regularization with $F_\gamma(\sigma_n) = \sigma_n^2/(\sigma_n^2+\gamma^2\beta_n^2)$ (see [12]) or the Landweber method with $F_m(\sigma_n) = 1-(1-\beta\sigma_n^2)^m$ for $\gamma = 1/m$.
It is shown in [15] that the filter methods are a special case of the approximative inverse with

$$e_\gamma(x,y) = \sum_n F_\gamma(\sigma_n)v_n(x)v_n(y).$$

In a more general case we have the representation

$$e_\gamma(x,y) = \sum_{n,m} e_{\gamma,nm}v_n(x)v_m(y). \tag{3}$$

If we want to solve a problem $Af = g$, then we are often led to use more than only one smoothing operator M_γ. For example we first can discretize the problem with a suitable mesh size, see [20], and then apply the method of Tikhonov-Phillips [24]. That means we use two regularizations to solve our problem. Therefore we want to look whether the regularization effects of the smoothing operators are saved.
We notice that for $M_\gamma, M'_\gamma : H^{-\alpha} \to L_2$

$$M_\gamma \circ M'_\gamma : H^{-\alpha} \to L_2,$$

holds because $H^{-\alpha} \supset L_2$.

Theorem 3.4 *The set*

$$\mathcal{M} \quad = \{M_\gamma : H^{-\alpha} \to L_2 : ||M_\gamma f|| \le c(\gamma)||f||_{H^{-\alpha}} \ , \ ||M_\gamma||_{L_2 \to L_2} \le c \\ and \ ||M_\gamma f - f|| \stackrel{\gamma \to 0}{\longrightarrow} 0, \ \forall f \in \mathcal{N}(A)^-\}$$

(a set of regularizations which fulfill the conditions of Theorem 3.1) equipped with the obvious law of composition is a semi group.

Proof:
a) For $f \in H^{-\alpha}$ and $M_\gamma, M'_\gamma \in \mathcal{M}$ we have

$$||M_\gamma M'_\gamma f|| \le c||M'_\gamma f||_{L_2} \le \underbrace{c'(\gamma)c}_{c''(\gamma)}||f||_{H^{-\alpha}}.$$

b) For $f \in \mathcal{N}(A^-)$ and $M_\gamma, M'_\gamma \in \mathcal{M}$ we have

$$||M_\gamma M'_\gamma f - f|| \le \underbrace{||M_\gamma M'_\gamma f - M_\gamma f||}_{\to 0} + \underbrace{||M_\gamma f - f||}_{\to 0}.$$

c) The composition of continuous linear operators is associative and continuous.

□

It is shown in Seidman [25] that for a fixed regularization method the set $\{T_\gamma : \gamma > 0\}$ forms also a semi group.

4 Prewithening

Now we first smooth our data $g \in L_2$ and then apply the pseudoinverse. Therefore we use Theorem 2.2 with $\tilde{A}^\dagger : \tilde{H}^\alpha \to L_2$. In [8] this is called range mollification because the regularization is achieved by smoothing in the 'range' of the operator A.
By using the same techniques as in the proof of Theorem 3.1 we can give a result similar to Theorem 3.1.

Theorem 4.1 *Let $\tilde{M}_\gamma : L_2 \to \tilde{H}^\alpha$ be a family of linear continuous operators such that*

$$\begin{array}{lll} i) & ||\tilde{M}_\gamma g||_{\tilde{H}^\alpha} \le c(\gamma)||g||_{L_2} & \forall g \in L_2, \\ ii) & \lim_{\gamma \to 0} ||\tilde{M}_\gamma g - g||_{\tilde{H}^\alpha} = 0 & \forall g \in \mathcal{D}(A^\dagger), \\ iii) & c(\gamma)\epsilon \to 0 \ for \ \gamma \to 0, \ \epsilon \to 0. & \end{array}$$

Then $T_\gamma = A^\dagger \tilde{M}_\gamma$ is a regularization of $A^\dagger$ for finding f.

There is a very interesting connection between the methods described in section 3 and the prewithening.

Theorem 4.2 *i) Let $M_\gamma : H^{-\alpha} \to L_2 \cap \mathcal{N}(A)^-$ be given, define $\tilde{M}_\gamma : L_2 \to \tilde{H}^\alpha$ as*

$$\tilde{M}_\gamma = A M_\gamma \tilde{A}^\dagger,$$

then we have

$$M_\gamma \tilde{A}^\dagger = A^\dagger \tilde{M}_\gamma.$$

ii) Let $\tilde{M}_\gamma : L_2 \cap \mathcal{N}(A^*)^- \to \tilde{H}^\alpha$ be given, define $M_\gamma : H^{-\alpha} \to L_2$ as

$$M_\gamma = A^\dagger \tilde{M}_\gamma A,$$

then we also have

$$M_\gamma \tilde{A}^\dagger = A^\dagger \tilde{M}_\gamma.$$

Proof: i) Let $g = g_1 + g_0 \in L_2$ with $g_1 \in \overline{\mathcal{R}(A)}$ and $g_0 \in \mathcal{R}(A)^-$. Then $\tilde{A}^\dagger g = \tilde{A}^\dagger g_1 \in H^{-\alpha}$ and $M_\gamma \tilde{A}^\dagger g \in L_2$. Using (2) we see that $\tilde{M}_\gamma g = A(M_\gamma \tilde{A}^\dagger g) \in \tilde{H}^\alpha$ which proofes that $\tilde{M}_\gamma : L_2 \to \tilde{H}^\alpha$. Because $M_\gamma \tilde{A}^\dagger g \in \mathcal{N}(A)^-$ we can apply $A^\dagger$ and obtain $A^\dagger \tilde{M}_\gamma g = M_\gamma \tilde{A}^\dagger g$.
ii) follows analoguosly.

□

As a consequence of this result we can interpret any regularization method either as a smoothing of the pseudoinverse or as smoothing of the data followed by the application of the pseudoinverse.
Again we are able to show some semi group structure for the prewithening similar to Theorem 3.4.

5 Order-Optimality

In order to compare different regularization methods we consider the unavoidable error for a problem with erroneous data. Let

$$E_{\alpha,\beta}(\epsilon, \rho, T_\gamma) = \sup\{||T_\gamma g^\epsilon - A^\dagger g|| : ||g^\epsilon - g||_{L_2} \leq \epsilon, ||A^\dagger g||_{H_\beta} \leq \rho\}.$$

The unavoidable error of the best regularization method T_γ is

$$E_{\alpha,\beta}(\epsilon, \rho) = \inf_{T_\gamma} E_{\alpha,\beta}(\epsilon, \rho, T_\gamma).$$

This error is bounded, see e.g. [22, 26], by

$$E_{\alpha,\beta}(\epsilon, \rho) \leq \epsilon^{\beta/(\alpha+\beta)} \rho^{\alpha/(\alpha+\beta)}.$$

Definition 5.1 *A regularization method T_γ is said to be* order optimal *for β if for all $\epsilon \geq 0$ and $\rho \geq 0$ there exists a parameter $\gamma(\epsilon, \rho)$ such that*

$$E_{\alpha,\beta}(\epsilon, \rho, T_\gamma) \leq c\epsilon^{\beta/(\alpha+\beta)} \rho^{\alpha/(\alpha+\beta)}$$

holds for a constant $c > 0$.

Theorem 5.2 *Let $M_\gamma : H^{-\alpha} \to L_2$ be continuous with*

$$\begin{array}{ll} i) & \|M_\gamma f\|_{L_2} \le c\gamma^{-\theta}\|f\|_{H^{-\alpha}} \quad \forall f \in \mathcal{N}(A)^- \\ ii) & \|M_\gamma f - f\| \le c_\beta \gamma^{\theta\frac{\beta}{\alpha}}\|f\|_{H^\beta} \quad \forall f \in \mathcal{N}(A)^- \end{array}$$

for a number $\theta \in \mathbb{R}^+$. Then $T_\gamma = M_\gamma \tilde{A}^\dagger$ is order optimal for β with the parameter choice

$$\gamma = \eta \left(\frac{\epsilon}{\rho}\right)^{\frac{\alpha}{\theta(\beta+\alpha)}}.$$

The error bound becomes minimal for

$$\gamma = \left(\frac{c\alpha}{\beta c_1 c_\beta}\frac{\epsilon}{\rho}\right)^{\frac{\alpha}{\theta(\beta+\alpha)}}.$$

Proof: We use the same ideas as in Theorem 3.1 and get

$$\|T_\gamma g^\epsilon - A^\dagger g\| \le c c_1^{-1}\gamma^{-\theta}\epsilon + c_\beta \gamma^{\theta\frac{\beta}{\alpha}}\rho.$$

With the choice of γ it follows

$$\|T_\gamma g^\epsilon - A^\dagger g\| \le C\epsilon^{\beta/(\alpha+\beta)}\rho^{\alpha/(\alpha+\beta)},$$

with $C = c c_1^{-1}\eta^{-\theta} + c_\beta \eta^{\theta\frac{\beta}{\alpha}}$.
In order to minimize the error bound we consider C as a function of η to get the above result.

□

In the special case of e_γ being of convolution type we can state:

Theorem 5.3 *Let $e_\gamma(x,y)$ be of convolution type and a regularization method in the sense of Theorem 3.2. If e_γ satisfies*

$$\begin{array}{ll} i) & (2\pi)^{n/2}\sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\} \le c\gamma^{-\theta}, \\ ii) & \sup_\xi\{(1+|\xi|^2)^{-\beta^*/2}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|\} \le c_\beta \gamma^{\theta\frac{\beta^*}{\alpha}} \end{array}$$

for numbers $\theta > 0$ and $\beta^* > 0$, then $T_\gamma = M_\gamma \tilde{A}^\dagger$ is order optimal for all β, $0 < \beta \le \beta^*$ with the choice

$$\gamma = \eta \left(\frac{\epsilon}{\rho}\right)^{\frac{\alpha}{\theta(\beta+\alpha)}}.$$

The error bound becomes minimal for

$$\gamma = \left(\frac{c\alpha}{\beta c_1 c_\beta}\frac{\epsilon}{\rho}\right)^{\frac{\alpha}{\theta(\beta+\alpha)}}.$$

Proof: First we show that condition ii) holds for any β with $0 < \beta \leq \beta^*$.

$$
\begin{aligned}
\sup_\xi\{(1+|\xi|^2)^{-\beta/2}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|\} &\leq \sup_\xi\Big\{\left((1+|\xi|^2)^{-\beta^*/2}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|\right)^{\frac{\beta}{\beta^*}} \\
&\quad |(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|^{1-\frac{\beta}{\beta^*}}\Big\} \\
&\leq \left(c_{\beta^*}\gamma^{\theta\frac{\beta^*}{\alpha}}\right)^{\frac{\beta}{\beta^*}}(M+1)^{1-\frac{\beta}{\beta^*}} \\
&= c_\beta\gamma^{\theta\frac{\beta}{\alpha}}.
\end{aligned}
$$

Now we check the conditions of Theorem 5.2.

$$
\begin{aligned}
||M_\gamma f||^2 &= ||\mathcal{F}(M_\gamma f)||^2 \\
&= (2\pi)^n\int_{\mathbb{R}^n}|\hat{e}_\gamma(\xi)\hat{f}(\xi)|^2 d\xi \\
&= (2\pi)^n\int_{\mathbb{R}^n}(1+|\xi|^2)^{-\alpha}(1+|\xi|^2)^{\alpha}|\hat{e}_\gamma(\xi)|^2|\hat{f}(\xi)|^2 d\xi \\
&\leq \underbrace{(2\pi)^n\sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\}^2}_{c^2\gamma^{-2\theta}}||f||^2_{H^{-\alpha}}. \\
||M_\gamma f - f||^2 &= ||\mathcal{F}(M_\gamma f - f)||^2 \\
&= \int_{\mathbb{R}^n}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|^2|\hat{f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^n}(1+|\xi|^2)^{\beta}(1+|\xi|^2)^{-\beta}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|^2|\hat{f}(\xi)|^2 d\xi \\
&\leq \underbrace{\sup_\xi\{(1+|\xi|^2)^{-\beta/2}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|\}^2}_{c_\beta^2\gamma^{2\theta\frac{\beta}{\alpha}}}||f||^2_{H^{\beta}}.
\end{aligned}
$$

$\square$

Example 5.4 *Let*

$$e_\gamma(x-y) = \frac{1}{\gamma\pi}\text{sinc}\,(\frac{x-y}{\gamma}).$$

Then we have

$$\hat{e}_\gamma(\xi) = (2\pi)^{-1/2}\chi_{[-\frac{1}{\gamma},\frac{1}{\gamma}]}(\xi).$$

The conditions of Theorem 5.3 are fullfilled:

$$i)\quad (2\pi)^{1/2}\sup_\xi\{(1+|\xi|^2)^{\alpha/2}|\hat{e}_\gamma(\xi)|\} = \left(1+\left|\frac{1}{\gamma}\right|^2\right)^{\alpha/2} \overset{\gamma<1}{\leq} \left(\frac{2}{\gamma}\right)^{\alpha} \quad \Rightarrow \theta = \alpha,$$

$$ii)\quad \sup_\xi\{(1+|\xi|^2)^{-\beta/2}|(2\pi)^{n/2}\hat{e}_\gamma(\xi)-1|\} \leq (1+|\frac{1}{\gamma}|^2)^{-\beta/2} \leq \gamma^\beta.$$

This means for $\gamma < 1$ the operator $M_\gamma\tilde{A}^\dagger$ is order optimal for all $\beta > 0$.

In an analogous way sufficient conditions can be given for the order optimality of the prewithening.

6 Application to Tikhonov – Phillips Regularization

In the follwoing we consider the Tikhonov–Phillips regularization with a penalty term based on a Sobolev norm, see also Natterer [21, 23]:

$$||Af - g||^2 + \gamma ||f||^2_{H^s} \ .$$

We use the Bessel potential

$$\widehat{J^s f}(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$$

such that

$$||f||_{H^s} = ||J^s f||_{L_2}$$

to rewrite the above functional as

$$||Af - g||^2 + \gamma ||J^s f||^2$$

leading to the normal equation

$$(A^* A + \gamma J^{2s}) f = A^* g \tag{4}$$

where we have used that $(J^s)^* J^s = J^{2s}$. Then we can rewrite the regularization scheme as

$$T_\gamma g = M_\gamma A^\dagger g$$

with the smoothing operator

$$M_\gamma = \left(A^* A + \gamma J^{2s}\right)^{-1} A^* A \ . \tag{5}$$

The equivalence (1) can be expressed as $A^* A \simeq J^{-2\alpha}$ leading to

$$M_\gamma \simeq (J^{-2\alpha} + \gamma J^{2s})^{-1} J^{-2\alpha} \simeq \gamma^{-1} J^{-2(\alpha + s)} \ .$$

In order to have a regularization method the operator M_γ has to smooth α steps in the Sobolev scale, which we interpret as $M_\gamma \simeq J^{-t}$ with $t \geq \alpha$ leading to

$$2(\alpha + s) \geq \alpha$$

or

$$s \geq -\alpha/2 \ .$$

Hence for $s \geq -\alpha/2$ we have the necessary smoothing property for the smoothing operator M_γ which means that we can use in the above Tikhonov–Phillips functional a norm with a negative index, as long as $s \geq -\alpha/2$. A similar result based on the singular value decomposition is given in [12]. A regularization is achieved with a penalty term based on a compact operator which is at most 'half as ill-posed' as the operator A.

7 Application to the Radon transform

The mathematical model describing x-ray computerized tomography is in two dimensions the Radon transform, see [13, 15, 22]. In n dimensions the Radon transform is defined as

$$Rf(\omega, s) = \int_{\mathbb{R}^n} f(x)\delta(s - x^\top \omega)dx$$

where $\omega \in S^{n-1}$, the n - dimensional sphere, and $s \in \mathbb{R}$. The operator R is a compact mapping

$$R : L_2(\Omega) \to L_2(Z)$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $Z = S^{n-1} \times \mathbb{R}$.
It is well known that the Radon transform smoothes the function $(n-1)/2$ steps in the Sobolev scale. If the domain Ω is bounded then we also have the inequality relation, see [22]

$$c_1(t,n)||f||_{H^t} \leq ||Rf||_{H^{t+(n-1)/2}} \leq c_2(t,n)||f||_{H^t} \text{ for all } t \in \mathbb{R}. \tag{6}$$

So in our setting of section 2 the number α has the value $\alpha = (n-1)/2$.
Now we consider a Tikhonov-Phillips regularisation with the functional

$$\Gamma_\gamma(f) = ||Rf - g||^2_{L_2(Z)} + \gamma ||f||^2_{H^s}.$$

The regularized solution f_γ is then defined as

$$\Gamma_\gamma(f_\gamma) \leq \Gamma_\gamma(f) \text{ for all } f \in H^s.$$

In the following we denote with $\mathcal{F}$ the n-dimensional Fourier transform and with $\mathcal{F}_1$ the one dimensional Fourier transform.
For the residuum we have, using the projection theorem,

$$\begin{aligned}
||Rf - g||^2_{L_2(Z)} &= \int_{S^{n-1}} ||Rf(\omega,.) - g(\omega,.)||^2_{L_2(\mathbb{R})} d\omega \\
&= \int_{S^{n-1}} ||\mathcal{F}_1 Rf(\omega,.) - \mathcal{F}_1 g(\omega,.)||^2_{L_2(\mathbb{R})} d\omega \\
&= \int_{S^{n-1}} \int_{\mathbb{R}} |(2\pi)^{(n-1)/2} \mathcal{F}f(\sigma\omega) - \mathcal{F}_1 g(\omega,\sigma)|^2 d\sigma d\omega.
\end{aligned}$$

And for the penalty term we get

$$||f||^2_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathcal{F}f(\xi)|^2 d\xi = \int_{S^{n-1}} \int_{\mathbb{R}+} \sigma^{n-1}(1+\sigma^2)^s |\mathcal{F}f(\sigma\omega)|^2 d\sigma d\omega.$$

Alltogether we obtain

$$\begin{aligned}
\Gamma(f) &= ||Rf - g||^2_{L_2(Z)} + \gamma ||f||^2_{H^s} \\
&= \int_{S^{n-1}} \int_{\mathbb{R}} |(2\pi)^{(n-1)/2} \mathcal{F}f(\sigma\omega) - \mathcal{F}_1 g(\omega,\sigma)|^2 \\
&\qquad + \gamma \frac{|\sigma|^{n-1}}{2} (1+\sigma^2)^s |\mathcal{F}f(\sigma\omega)|^2 d\sigma d\omega.
\end{aligned}$$

Hence $\Gamma_\gamma(f)$ is minimized by

$$\mathcal{F} f_\gamma(\sigma\omega) = \frac{(2\pi)^{(n-1)/2}\mathcal{F}_1 g(\omega,\sigma)}{(2\pi)^{n-1} + \gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}.$$

We define the operator S_γ as

$$\mathcal{F}(S_\gamma g)(\sigma\omega) = \frac{(2\pi)^{(n-1)/2}\mathcal{F}_1 g(\omega,\sigma)}{(2\pi)^{n-1} + \gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}. \tag{7}$$

Theorem 7.1 *If $s \geq (1-n)/4$ then the operator S_γ defined in (7) is a regularization for $R^\dagger$ for finding f.*

Proof: We use Theorem 3.1 and therefore we define the smoothing operator

$$M_\gamma = S_\gamma R.$$

Then we obtain with the help of inequality (6)
i)

$$\begin{aligned}
\|M_\gamma f\|^2 &= \|f_\gamma\|^2_{L_2(\Omega)} = \|\mathcal{F} f_\gamma\|^2_{L_2(\mathbb{R}^n)} \\
&= \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}} |\sigma|^{n-1}|\mathcal{F} f_\gamma(\sigma\omega)|^2 d\sigma d\omega \\
&= \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}} |\sigma|^{n-1}\left|\frac{(2\pi)^{(n-1)/2}\mathcal{F}_1 Rf(\omega,\sigma)}{(2\pi)^{n-1}+\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}\right|^2 d\sigma d\omega \\
&= \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}} |\mathcal{F}_1 Rf(\omega,\sigma)|^2 \cdot \\
&\qquad \left|\frac{\sigma^{(n-1)/2}}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}\right|^2 d\sigma d\omega \\
&\leq \frac{1}{2}\sup_\sigma \left|\frac{\sigma^{(n-1)/2}}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}\right|^2 \cdot \\
&\qquad \int_{S^{n-1}} \|\mathcal{F}_1 Rf(\omega,.)\|^2_{\mathbb{R}} d\omega \\
&\leq \frac{1}{2}\sup_\sigma \left|\frac{\sigma^{(n-1)/2}}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}\right|^2 \cdot \\
&\qquad c_2((1-n)/2,n)^2\|f\|^2_{H^{-(n-1)/2}(\Omega)}.
\end{aligned}$$

Because $s \geq (1-n)/4$ the supremum

$$\sup_\sigma \left|\frac{\sigma^{(n-1)/2}}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}\right|$$

is bounded by some constant $c(\gamma)$.
ii)

$$\|M_\gamma f - f\|^2 = \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}} |\sigma|^{n-1}|\mathcal{F} f_\gamma(\sigma\omega) - \mathcal{F} f(\sigma\omega)|^2 d\sigma d\omega$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{S^{n-1}}\int_{\mathbb{R}}\left|\frac{1}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}-1\right|^2 \cdot \\
&\qquad |\sigma|^{n-1}|\mathcal{F}f(\sigma\omega)|^2 d\sigma d\omega \\
&\le \frac{1}{2}\sup_{\sigma}\left|\frac{1}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}-1\right|^2 \cdot \\
&\qquad \int_{S^{n-1}}\int_{\mathbb{R}}|\sigma|^{n-1}|\mathcal{F}f(\sigma\omega)|^2 d\sigma d\omega \\
&= \sup_{\sigma}\left|\frac{1}{1+(2\pi)^{1-n}\gamma|\sigma|^{n-1}(1+\sigma^2)^s/2}-1\right|^2 ||f||^2_{L_2(\Omega)} \\
&\to 0 \text{ for } \gamma\to 0.
\end{aligned}
$$

iii) Because $c(\gamma)$ is a continuous function with respect to γ, $\gamma > 0$, we can always choose a γ such that $c(\gamma)\epsilon \to 0$ for $\epsilon \to 0$, $\gamma \to 0$.

□

With $\alpha = \frac{n-1}{2}$ we see that this is exactly the result we should expect concerning section 6. A similar result for the Tikhonov Phillips regularisation is given in [11].

8 Operator Dependent Norms

In this section we give a summary of some results of [16]. The approach which is used in [16] is based on spaces and norms constructed with the operator A under consideration.
Here we introduce the pre Hilbert scale

$$X_\nu = \mathcal{R}((A^*A)^{\nu/2})$$

where

$$||f||^2_{X_\nu} = \sum_n \sigma_n^{-2\nu}|\langle f, v_n\rangle|^2.$$

Obviously we have $X_{\nu+\mu} \subset X_\nu$ for $\mu > 0$. In an analogous way we define a pre Hilbert scale of spaces Y_ν as

$$Y_\nu = \mathcal{R}((AA^*)^{\nu/2})$$

where

$$||g||^2_{Y_\nu} = \sum_n \sigma_n^{-2\nu}|\langle g, u_n\rangle|^2.$$

We then obtain the result:

Theorem 8.1 *$\bar{A}^\dagger : Y_\nu \oplus \mathcal{N}(A^*) \to X_{\nu-1}$ with $||\bar{A}^\dagger|| = 1$. For $\nu \ge 1$ we have $\bar{A}^\dagger = A^\dagger$.*

The next theorem corresponds to Theorem 3.1.

Theorem 8.2 *Let $M_\gamma : X_{-1} \to X$ be a family of continuous operators such that*

$$\begin{array}{ll} i) & \|M_\gamma f\| \le c(\gamma)\|f\|_{X_{-1}}, \quad \forall\ f \in \mathcal{N}(A)^-, \\ ii) & \lim\limits_{\gamma\to 0}\|M_\gamma f - f\| = 0, \quad \forall\ f \in \mathcal{N}(A)^-. \end{array}$$

Then $T_\gamma = M_\gamma \bar{A}^\dagger$ is a regularization of $A^\dagger$ for finding f.

The above theorem immediately yields to the theorem for filter methods.

Theorem 8.3 *Let F_γ be such that*

$$\begin{array}{ll} i) & \sup\limits_n |F_\gamma(\sigma_n)\sigma_n^{-1}| = c(\gamma) < \infty, \\ ii) & \lim\limits_{\gamma\to 0} F_\gamma(\sigma_n) = 1 \text{ pointwise in } \sigma_n, \\ iii) & |F_\gamma(\sigma_n)| \le M \text{ for all } \gamma \text{ and } \sigma_n. \end{array}$$

Then $T_\gamma = M_\gamma \bar{A}^\dagger$ with $M_\gamma v_n = F_\gamma(\sigma_n)v_n$ is a linear regularization of $A^\dagger$ for finding f.

In the case of prewithening we obtain:

Theorem 8.4 *Let $\tilde{M}_\gamma : Y \to Y_1$ be a family of continuous operators such that*

$$\begin{array}{ll} i) & \|\tilde{M}_\gamma g\|_{Y_1} \le \tilde{c}(\gamma)\|g\|_Y \quad \forall g \in Y, \\ ii) & \lim\limits_{\gamma\to 0}\|\tilde{M}_\gamma g - g\|_{Y_1} = 0 \quad \forall g \in \mathcal{D}(A^\dagger). \end{array}$$

Then $T_\gamma = A^\dagger \tilde{M}_\gamma$ is a linear regularization of $A^\dagger$ for finding f.

Finally we give a result for order optimality.

Theorem 8.5 *Let $M_\gamma : X_{-1} \to X$ be continuous with*

$$\begin{array}{ll} i) & \|M_\gamma f\| \le c\gamma^{-\theta}\|f\|_{X_{-1}} \quad \forall f \in \mathcal{N}(A)^- \\ ii) & \|M_\gamma f - f\| \le c_\nu \gamma^{\theta\nu}\|f\|_{X_\nu} \quad \forall f \in \mathcal{N}(A)^- \end{array}$$

for a number $\theta \in \mathbb{R}^+$. Then $T_\gamma = M_\gamma \bar{A}^\dagger$ is order optimal for ν with the parameter choice

$$\gamma = \eta \left(\frac{\epsilon}{\rho}\right)^{1/\theta(\nu+1)}.$$

The error bound becomes minimal for

$$\gamma = \left(\frac{c}{\nu c_\nu}\frac{\epsilon}{\rho}\right)^{1/\theta(\nu+1)},$$

and we have

$$\|T_\gamma g^\epsilon - A^\dagger g\| \le (c\epsilon)^{\nu/(\nu+1)}(c_\nu \rho)^{1/(\nu+1)}(\nu+1)\nu^{-\nu/(\nu+1)}.$$

9 Discussion

There are some correspondences between results obtained with the Sobolev space analysis and the singular value decomposition, see for example Theorem 2.2 and Theorem 8.1 or Theorem 3.1 and Theorem 8.2. This seems astonishing on the first sight. But if we consider the norm equivalence (2) we obtain

$$||Av_n||_{H^\alpha} = \sigma_n ||u_n||_{H_\alpha} \simeq ||v_n||_{L_2} = 1.$$

and therefore

$$c_1 \sigma_n^{-1} \leq ||u_n||_{H^\alpha} \leq c_2 \sigma_n^{-1}.$$

This shows that there is a direct connection between the Sobolev norm and the singular values. But the spaces X_ν are based on the singular values and therefore we have a connection between the two approaches. This helps us to understand the similarities.
The parallels between Theorem 3.2 and Theorem 8.3 are of independent interest. Let us take two methods exemplarily, first the truncated singular value decomposition and second the mollifier $e_\gamma(x, y) = \text{sinc } (x - y)$. The truncated singular value decomposition cuts off all small singular values. Knowing that small singular values mean high frequencies the truncated singular value decomposition is a low-pass filter. This, on the other hand, is also achieved by a convolution with the sinc-funciton.
If we have a problem with pairwise different singular values, then we can state:

Theorem 9.1 *If the operator $A : X \to X$ has the singular value decomposition $\{v_n, u_n; \sigma_n\}$ and*

$$\begin{array}{ll} i) & \sigma_n \neq \sigma_m \text{ for } m \neq n \\ ii) & \mathcal{F}(A^\dagger g)(\xi) = k(\xi)\hat{g}(\xi) \quad \forall g \in \mathcal{R}(A), \end{array}$$

then every approximative inverse with mollifier $e_\gamma(x, y)$ of convolution type, such that

$$M_\gamma = \langle ., e_\gamma \rangle : X_{-1} \to X \text{ is continuous},$$

is a filter method, which means that $M_\gamma v_n = F_\gamma(\sigma_n) v_n$.

Proof: a) Let $g \in \mathcal{R}(A)$:

$$\begin{aligned} T_\gamma g &= M_\gamma A^\dagger g \\ &= \mathcal{F}^{-1}\mathcal{F}(M_\gamma A^\dagger g) \\ &= (2\pi)^{n/2}\mathcal{F}^{-1}\hat{e}_\gamma(\xi)k(\xi)\hat{g}(\xi) \\ &= (2\pi)^{n/2}\mathcal{F}^{-1}k(\xi)\hat{e}_\gamma(\xi)\hat{g}(\xi) \\ &= A^\dagger M_\gamma g. \end{aligned}$$

$A^\dagger M_\gamma g$ is well defined, because $X = Y$.

b) Let $g \in \mathcal{R}(A)^-$:

$$T_\gamma g = 0 = A^\dagger M_\gamma g.$$

c) $M_\gamma : X_{-1} \to X$ is continuous and therefore $M_\gamma : X \to X_1$ is continuous, too. So $M_\gamma A^\dagger$ and $A^\dagger M_\gamma$ are continuous.
Let $g \in \partial\mathcal{R}(A)$, then we have for any sequence $\{g_n\} \subset \mathcal{R}(A)$ with

$$\lim_{n\to\infty} g_n = g :$$

$$\begin{aligned} T_\gamma g = M_\gamma A^\dagger g \quad & \overset{M_\gamma A^\dagger \ cont}{=} \quad \lim_{n\to\infty} M_\gamma A^\dagger g_n \\ & \overset{g_n \in \mathcal{R}(A)}{=} \quad \lim_{n\to\infty} A^\dagger M_\gamma g_n \\ & \overset{A^\dagger M_\gamma \ cont}{=} \quad \lim_{n\to\infty} A^\dagger M_\gamma g_n \\ & = \quad A^\dagger M_\gamma g. \end{aligned}$$

Applying Theorem 4.2 and using formula (3) we get

$$e_{\gamma,nm} = \frac{\sigma_m}{\sigma_n} e_{\gamma,nm}.$$

Because of condition i) we have

$$e_{\gamma,nm} = F_\gamma(\sigma_n)\delta_{nm},$$

and this completes the proof.

□

Example 9.2 *Let*

$$Af(x) = \int_0^x f(y)dy = g(x)$$

and $X = Y = L_2(0,1)$. Then the solution f is given by

$$f = g' \text{ for } g \in \mathcal{R}(A).$$

The conditions of Theorem 9.1 are fulfilled, because $\sigma_n \neq \sigma_m$ for $m \neq n$ (see [12]) and

$$\mathcal{F}(A^\dagger g)(\xi) = \mathcal{F}g'(\xi) = i\xi\hat{g}(\xi).$$

This means, every mollifier of convolution type that is continuous from X_{-1} to X is a filter method.

Acknowledgement The research of the authors has been supported by the Deutsche Forschungsgemeinschaft under grant Lo310/4-2.

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