Pasting presentations for ω-categories*

Sjoerd E. Crans
Utrecht University†

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Abstract

The pasting theorem showed that pasting schemes are useful in studying free ω-categories. It was thought that their inflexibility with respect to composition and identities prohibited wider use. This is not the case: there is a way of dealing with identities which makes it possible to describe ω-categories in terms of generating pasting schemes and relations between generated pastings, i.e., with pasting presentations. In this chapter I develop the necessary machinery for this. The main result, that the ω-category generated by a pasting presentation is universal with respect to respectable families of realizations, is a generalization of the pasting theorem.

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*This paper is also Chapter 2 of the author's PhD thesis. If you fetch this paper, please acknowledge this by sending an email to the address below. The thesis is also available in printed form, requests for this can be sent to the same address.

†presently at: School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia crans_s@maths.su.oz.au

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1 Introduction

In [19], Street, following Roberts [15], suggests that n-cohomology should be developed using n-categories as coefficient objects. As already suggested there, weaker versions would eventually do a better job at this. Still, n-categories are worth studying, partly as an exercise for the weaker case, but also for homotopy theoretic reasons [4, 10], and in computer science [14].

Because of the combinatorics involved, it is quite difficult to say precisely what a *composable diagram* in an n-category should be. Closely related to this is the problem of describing free n-categories. Street defined the free n-category on an n-simplex: his n-th oriental [19]. Motivated by this, Johnson defined pasting schemes, and proved a pasting theorem for them [8]. Other suggested structures are Power's pasting schemes [12, 13], defined geometrically, Street's parity complexes [20], Steiner's directed complexes [17], and Street's inductively

defined n-computads [18, 21]. The major objection, as Power noted [13], to Johnson's pasting schemes is that they don't include all possible compositions of n-cells. But he also gives an example showing that his own definition doesn't solve this problem either. Other people seem to think that Johnson's theory is unnecessarily complicated [11].

In this chapter I show that one *can* model all possible composites of n-cells by using *equivalence classes* of *labeled* well-formed loop-free pasting schemes. These labelings include the possibility of specifying that a cell is to represent an identity, and because of the equivalences it is possible to insert identities to avoid composites not being loop-free. Moreover, the beauty of the resulting theory suggests that Johnson's pasting schemes are not *unnecessarily* complicated.

In chapter 3, I will use pasting presentations to give a detailed description of the tensor product on the collection of ω-categories, which is defined by Steiner [16], which extends Gray's tensor product on 2-categories [6] and which is closely related to Brown-Higgins's tensor product on ω-groupoids [2]. This description will lead to a better understanding of Gray-categories [5], so that hopefully methods of chapter 1 can be used to show the homotopical relevance of Gray-groupoids [9] and their higher dimensional generalizations, still to be defined. Another application of pasting presentations may be possible in connection with ideas of Tonks [22].

2 Pasting schemes according to Johnson

As a short introduction to the theory of pasting schemes, I will recall briefly some definitions and theorems from [8] and [7]. This will also fix the notation. For more details, see [7] which contains some applications as well.

2.1 Graded sets

A *graded set* A is a sequence of sets $(A_i)_{i\in\omega}$. If a particular member of the sequence, say A_i, is not specified then it is assumed to be empty. Elements of A_i are called *i-dimensional.* x is an *element of* A, denoted $x \in A$, if $x \in A_i$ for some i. Elements will also be called *cells.* The graded set A is called *n-dimensional* if $A_i = \varnothing$ for $i > n$ and $A_n \neq \varnothing$. The *n-dimensional skeleton* $|A|_n$ *of* A has $(|A|_n)_i = \varnothing$ for $i > n$ and $(|A|_n)_i = A_i$ for $i \leq n$. For graded sets A and B, A is said to be a *subgraded set of* B, denoted $A \subseteq B$, if for each i one has $A_i \subseteq B_i$. Graded sets inherit the usual operations on sets as union, intersection, difference by defining them dimensionwise. An n-dimensional element of A will

often be identified with the subgraded set X of A having $X_n = \{x\}$ and $X_i = \varnothing$ if $i \neq n$.

2.2 ω-categories

Street [19] allowed ω-categories to have infinite dimensional cells. Steiner [16] has the extra condition that every cell has to be finite dimensional, and calls them ∞-categories, following Brown and Higgins [1]. I will use Steiner's approach here since that's the one that reflects the notion of higher dimensional homotopies closest, but I will nonetheless call them ω-categories, and I agree with Verity's suggestion to call the other ones ω^+-categories.

Definition 2.1 An ω-category is a set $\mathbb{C}$ together with operations s_n, $t_n : \mathbb{C} \to \mathbb{C}$, also denoted d_n^- and d_n^+ and called n-source and n-target respectively, and an operation $\circ_n : \mathbb{C}\ {}_{s_n}\!\times_{t_n} \mathbb{C} \to \mathbb{C}$, called n-composition, for every n, such that

(i) $$\begin{aligned} d_m^\beta d_n^\alpha &= d_m^\beta \quad \text{if } m < n \\ &= d_n^\alpha \quad \text{if } m \geq n \end{aligned}$$
for all m, n and α, $\beta = \pm$,

(ii) $d_n^-(c) \circ_n c = c = c \circ_n d_n^+(c)$ for all $c \in \mathbb{C}$ and all n,

(iii) $$\begin{aligned} d_m^\alpha(c' \circ_n c) &= d_n^-(c) && \text{if } m = n \text{ and } \alpha = - \\ &= d_n^+(c') && \text{if } m = n \text{ and } \alpha = + \\ &= d_m^\alpha(c') \circ_n d_m^\alpha(c) && \text{otherwise} \end{aligned}$$
for all c, $c' \in \mathbb{C}$ and all m, n and $\alpha = \pm$,

(iv) (associativity) $(c'' \circ_n c') \circ_n c = c'' \circ_n (c' \circ_n c)$ for all c, c', $c'' \in \mathbb{C}$ and all n for which either side is defined,

(v) (interchange law) $(d' \circ_n d) \circ_m (c' \circ_n c) = (d' \circ_m c') \circ_n (d \circ_m c)$ for all c, c', d, $d' \in \mathbb{C}$ and all $m \neq n$ for which both sides are defined,

(vi) for all $c \in \mathbb{C}$ there exists n such that $c = d_n^-(c) = d_n^+(c)$. $\diamond$

Intuitively, an element of an ω-category $\mathbb{C}$, also called a *cell* of $\mathbb{C}$, can be pictured as running between lower dimensional cells, and composition can be pictured as a way of putting cells together. Condition (i) says sources and targets match up:

$$\begin{tikzcd} s_m(c) \arrow[r, bend left, "s_n(c)"] \arrow[r, bend right, "t_n(c)"'] & c \arrow[r] & t_m(c) \end{tikzcd} \quad .$$

Condition (ii) makes d_n^- and d_n^+ left and right identities respectively. Condition (iii) tells that the source and target of a composition look like

$$\begin{tikzcd} s_m(c)=s_m(c') \arrow[r, "c"] \arrow[rr, bend left, "s_n(c)"] \arrow[rr, bend right, "t_n(c')"'] \arrow[r, phantom, shift right=4ex, "c'"] & t_m(c)=t_m(c') \end{tikzcd} \quad .$$

Condition (iv) says n-composition is associative, and condition (v) states that the two possible ways of composing

$$\begin{tikzcd} \bullet \arrow[r, bend left, "c'"] \arrow[r, "c"'] & \bullet \arrow[r, bend left, "d'"] \arrow[r, "d"'] & \bullet \end{tikzcd}$$

are equal. Finally, condition (vi) ensures that the dimension of a cell can be defined as the least n for which (vi) holds, and that this dimension is always finite. An *n-cell* is a cell of dimension at most n, and the set of n-cells of $\mathbb{C}$ will be denoted by C_n.

For $n \in \omega$, an n-category is an ω-category for which all cells are n-cells. For an ω-category $\mathbb{C}$, the n-category consisting of its n-cells will be denoted by $|\mathbb{C}|_n$.

Definition 2.2 An ω-functor $f : \mathbb{C} \to \mathbb{D}$ is a function respecting all n-sources and n-targets, and all n-compositions. $\diamond$

It is also possible to define ω-categories multi-sorted instead of one-sorted. An ω-category then consists of a graded set $\mathbb{C}$ together with functions s_n, t_n : $C_m \to C_n$ for $m > n$, functions $\circ_n : C_m \; {}_{s_n}\!\times\!{}_{t_n} C_{m'} \to C_{\max(m,m')}$ for m, $m' > n$, and functions $\mathrm{id} : C_m \to C_n$ for $m < n$, satisfying conditions analogous to the ones above. The connection between the two definitions is that a non-identity n-cell of the graded set corresponds to an n-dimensional cell, and that identities are identified with the cell they are identity on. I will use both definitions, the one-sorted and the many-sorted, interchangeably, where many sorted is the default. Note however, that an element of C_n in the many-sorted definition doesn't need to be n-dimensional in the one-sorted definition.

2.3 Pasting schemes

The following is taken almost entirely from [8].

Well-formed loop-free pasting schemes were introduced in order to parametrize composable diagrams in an ω-category. The first ingredient to this is that the cells making up the diagram hang together, which is taken care of in the notion of *pasting scheme*.

A pasting scheme will be a graded set A, together with two collections of relations E^i_j and $\mathsf{B}^i_j \subseteq A_i \times A_j$ for $j \leq i$, which may be thought of as describing which j-cells are at the "end", respectively "beginning" of each of the i-cells. For such a graded set A with relations E^i_j and B^i_j, let R^i_j be the relation between A_i and A_j given by $x\mathsf{R}^i_j y$ when there exists a sequence $x = x_1, \ldots, x_k = y$ of cells of A satisfying $x_{k'}\mathsf{D}^{i_{k'}}_{i_{k'+1}} x_{k'+1}$ for all $1 \leq k' \leq k$ with $\mathsf{D}^i_j = \mathsf{E}^i_j$ or B^i_j.

If X is a subgraded set of A of dimension n then the graded set $\mathsf{E}^i(X)$ is defined by $\mathsf{E}^i(X)_j = \{y \in A_j |$ there exists $x \in X_i$ with $x\mathsf{E}^i_j y\}$. The graded set $\mathsf{E}^n(X)$ will be denoted $\mathsf{E}(X)$. Graded sets $\mathsf{B}(X)$ and $\mathsf{R}(X)$ are defined analogously. The grading will often be placed on the relation, thus denoting the set $\mathsf{E}(X)_j$ of j-dimensional elements of $\mathsf{E}(X)$ by $\mathsf{E}_j(X)$. The relation E^i_j is called *finitary* when, for any $x \in A_i$, the set $\mathsf{E}^i_j(x)$ is finite.

There is a duality between the "end" and "beginning" relations: for every proposition P and every i, the i-th dual of P is obtained from P by replacing all occurrences of E^i by B^i and vice versa.

Definition 2.3 A *pasting scheme* is a graded set A together with finitary relations E^i_j and B^i_j for $j \leq i$ such that

(i) E^i_j is a relation between A_i and A_j;

(ii) E^i_i is the identity relation on A_i;

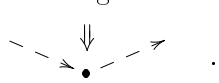
(iii) for $i > 0$ and any $x \in A_i$ there exists $y \in A_{i-1}$ with $x\mathsf{E}^i_{i-1}y$;

(iv) for $j < i$, $w\mathsf{E}^i_j x$ if and only if there exists u and v such that $w\mathsf{E}^i_{i-1}u\mathsf{E}^{i-1}_j x$ and $w\mathsf{E}^i_{i-1}v\mathsf{B}^{i-1}_j x$;

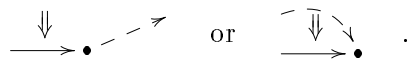
(v) if $w\mathsf{E}^i_{i-1}u\mathsf{E}^{i-1}_j x$, then either $w\mathsf{E}^i_j x$ or there exists v such that $w\mathsf{B}^i_{i-1}v\mathsf{E}^{i-1}_j x$

and dually (notice that there are four dual forms of condition (v)). $\diamond$

Informally, condition (iii) says that every i-cell ends at at least one $(i-1)$-cell, and dually begins at at least one $(i-1)$-cell. Condition (iv) ensures that low dimensional ends occur between higher dimensional ends:



Finally, condition (v) ensures that a cell's ends close up:



The second ingredient is that it is not allowed to encounter one specific cell twice.

In a pasting scheme A, define $\triangleleft_A$, written as $\triangleleft$ when there is no danger of confusion, as follows: for any i, and for any $a, b \in A_i$, say $a \triangleleft b$ if there is a sequence $a = a_0, \ldots a_k = b$, $k > 0$, of elements of A_i with, for all $k' < k$, $\mathsf{E}_{i-1}(a_{k'}) \cap \mathsf{B}_{i-1}(a_{k'+1}) \neq \varnothing$.

Definition 2.4 A pasting scheme A has *no direct loops* when, for any i and any $a, b \in A_i$, $\mathsf{B}(a) \cap \mathsf{E}(a) = \{a\}$ and $a \triangleleft b$ implies $\mathsf{B}(a) \cap \mathsf{E}(b) = \varnothing$. $\diamond$

If A is a pasting scheme and X is a finite subgraded set of A, define its domain $\mathrm{dom}(X)$ by $X - \mathsf{E}(X)$ and its codomain $\mathrm{cod}(X)$ by $X - \mathsf{B}(X)$.

Lemma 2.5 (Johnson) *If A is a finite, n-dimensional pasting scheme with no direct loops, then* $\mathrm{dom}(A)$ *is a $(n-1)$-dimensional graded set.* $\square$

Theorem 2.6 (Johnson) *If A is a finite pasting scheme with no direct loops then* $\mathrm{dom}\,\mathrm{dom}(A) = \mathrm{dom}\,\mathrm{cod}(A)$. $\square$

Thus, finite pasting schemes with no direct loops have sensible notions of domain and codomain. This can be extended to sensible notions of n-source and n-target, thereby incorporating the third ingredient that the highest dimensional cells must agree in orientation, by requiring pasting schemes to be *well formed*.

If A is a finite n-dimensional pasting scheme with no direct loops, write

$$\begin{array}{rcll} s_m(A) & = & A & \text{if } m \geq n \\ & = & \mathrm{dom}^{n-m}(A) & \text{if } m < n \end{array}$$

and

$$\begin{array}{rcll} t_m(A) & = & A & \text{if } m \geq n \\ & = & \mathrm{cod}^{n-m}(A) & \text{if } m < n. \end{array}$$

Notice that if $m < n$ then $s_m(A)$ and $t_m(A)$ are m-dimensional by lemma 2.5. $s_n(A)$ is called the n-source of A, and $t_n(A)$ the n-target of A.

Definition 2.7 A pasting scheme A of dimension $n > 0$ is *compatible* when for any x, $y \in A_n$, if $x \neq y$, then $\mathsf{B}_{n-1}(x) \cap \mathsf{B}_{n-1}(y) = \varnothing$ and dually. A zero-dimensional pasting scheme is compatible if it is a singleton.

A subgraded set X of a pasting scheme A is a *subpasting scheme of* A if $y \in \mathsf{R}(X)$ implies $y \in X$.

A finite pasting scheme A with no direct loops is *well formed* if

(i) A is compatible;

(ii) for all $m \geq 0$, both $s_m(A)$ and $t_m(A)$ are compatible subpasting schemes of A. $\diamond$

The fourth ingredient is quite technical, and serves to eliminate more subtle looping behaviour.

Definition 2.8 A pasting scheme A is *loop free* if

(i) A has no direct loops;

(ii) for any $x \in A$, $\mathsf{R}(x)$ is well formed;

(iv) for any well-formed j-dimensional subpasting scheme Y of A and any $x \in A$ with $s_j(\mathsf{R}(x)) \subseteq Y$, if u, $u' \in s_j(\mathsf{R}(x))$ and, for some $v \in Y_j$, $u \lhd_Y v \lhd_Y u'$, then $v \in s_j(\mathsf{R}(x))$

and dually. $\diamond$

Condition (iii) is omitted here since it is a consequence of the other three conditions and the pasting axioms, see [7].

2.4 The pasting theorem

If two well-formed subpasting schemes of a loop-free pasting scheme have n-source and n-target matching up, it is possible to paste them together to obtain another well-formed subpasting scheme:

Theorem 2.9 (Johnson) *Suppose that A is a loop-free pasting scheme and $\mathcal{P}(A)$ the collection of well-formed subpasting schemes of A, then $\mathcal{P}$ is an ω-category with s_n and t_n as n-source and n-target respectively, and with union as composition.* $\square$

If A is a loop-free pasting scheme then the ω-category $\mathcal{P}(A)$ is called the *ω-category of components of A*.

Theorem 2.10 (Johnson) *Suppose A is a loop-free pasting scheme, then the ω-category of components of A is the free ω-category generated by the $\mathsf{R}(x)$, $x \in A$.* $\square$

Well-formed loop-free pasting schemes are the appropriate parametrizing objects for composable diagrams in ω-categories. It remains to describe the parametrizations themselves and to establish that such a parametrization determines a unique cell called the *composite* of the diagram.

Definition 2.11 A *realization* (A, f_i) of a well-formed loop-free pasting scheme A in an ω-category $\mathbb{C}$ is a collection of functions $f_i : A_i \to C_i$. $\diamond$

The j-category $|\mathcal{P}(A)|_j$ is precisely the sub-ω-category of $\mathcal{P}(A)$ consisting of the well-formed subpasting schemes of A of dimension less than or equal j.

Definition 2.12 A realization (A, f_i) is *n-extendable* when there exists a unique ω-functor $f : |\mathcal{P}(A)|_n \to \mathbb{C}$ such that

$$\begin{tikzcd} |\mathcal{P}(A)|_i \arrow[r] & |\mathcal{P}(A)|_n \arrow[d, "f"] \\ A_i \arrow[u] \arrow[r, "f_i"'] & \mathbb{C} \end{tikzcd}$$

commutes for all $i \leq n$.

A realization (A, f_i) is extendable if it is n-extendable for all n. $\diamond$

Any realization will be called 0-*appropriate*, and is zero-extendable. Suppose that every n-appropriate realization is n-extendable and suppose given an n-appropriate realization (A, f_i) with n-extension f. (A, f_i) is called $(n+1)$-*appropriate* if $s_n\, f_{n+1} = f\, s_n\, \mathsf{R}$ and $t_n\, f_{n+1} = f\, t_n\, \mathsf{R}$. By freeness of $\mathcal{P}(A)$, theorem 2.10, (A, f_i) is then $(n+1)$-extendable. A realization is called *appropriate* if it is n-appropriate for all n.

Definition 2.13 A *composable diagram* in an ω-category $\mathbb{C}$ is a well-formed loop-free pasting scheme A together with an appropriate realization (A, f_i) of A in $\mathbb{C}$. $\diamond$

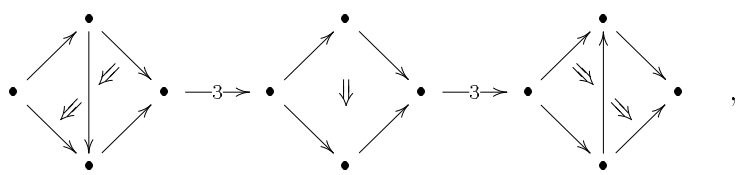
Theorem 2.14 (Johnson's pasting theorem) *If A is a well-formed loop-free pasting scheme of dimension n then $A \in |\mathcal{P}(A)|_n$. Furthermore, appropriate realizations are extendable. Thus a composable diagram (A, f_i) in an ω-category $\mathbb{C}$ determines uniquely, via $f(A)$, a cell of $\mathbb{C}$ called the* composite *of (A, f_i), where f is the extension of (A, f_i).* $\square$

Because of this theorem it is almost never necessary to write out in which order cells should be composed: it suffices that they make up a composable diagram, and then all sensible orders give the same result.

3 Strategy

The goal of this chapter is to make it possible to describe ω-categories with generators and relations. The objects parametrizing diagrams in ω-categories, i.e., well-formed loop-free pasting schemes, will be used to describe the generated elements. By theorem 2.10 the generating elements for the free ω-category on a loop-free pasting scheme are well-formed loop-free pasting schemes of the form $\mathsf{R}(x)$. In this more general setting I will use these as generators as well. The lower dimensional cells of a generating pasting scheme will be labeled to see that lower dimensional subpasting schemes are generated by the lower dimensional generators and relations, and to be able to compose generators.

In the free ω-category on a loop-free pasting scheme all compositions are given by pasting. Here, I need to be more careful since it is not always possible to paste generators: when two well-formed loop-free pasting schemes are given whose n-source and n-target match up, their union need not be a well-formed loop-free pasting scheme again. An example of this was given by Power [13]:



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as the pasting of both 3-cells results in a pasting scheme having a direct loop of dimension 1. On the other hand, given a well-formed loop-free pasting scheme, it can be obtained as the composite of $\mathsf{R}(x)$'s. I will show that situations as above can be avoided, and that pasting of well-formed loop-free pasting schemes can be used as composition. The key to this is, that I allow cells in a pasting scheme to designate that they are to be identities when realized in an ω-category. Thus, pasting schemes are not on their own, but there is a whole collection of well-formed loop-free pasting schemes denoting essentially the same diagram, the only difference between them lying in the presence and the position of identities. Two well-formed loop-free pasting schemes can now be composed by adding enough

identities on both sides in order to make sure the result of pasting is a well-formed loop-free pasting scheme as well. Or, to put it more boldly, diagrams can be composed by pasting, by choosing the right parametrizing objects for them.

Relations between generated pastings will be considered only when both sides are of equal dimension and have equal domain and codomain. Another restriction on the relations will be that both pasting schemes have to be *round*, a technical condition making it possible to express the composite of a realization as the composite of a realization of a pasting scheme which has the same domain and codomain, but which has only one top-dimensional cell. The reason for not considering more general relations is that defining relations are to induce an equivalence relation on the collection of all generated pastings, where two such are equivalent if they differ only in a subpasting scheme and there is a relation between these subpasting schemes. The conditions on the relations make that the difficulties in comparing two pasting schemes which differ only in a subpasting scheme, and in particular, in comparing their composites when realized, can be kept under control.

Given such a set of generating pasting schemes and relations between generated pastings, the ω-category on this data will thus consist of equivalence classes of generated pasting schemes, where the equivalence relation is generated by

- two are equivalent if they agree up to related subpasting schemes,
- two are equivalent if they agree up to an identity.

The described ω-category satisfies the required universal property, namely that a function assigning to generators cells of an ω-category $\mathbb{C}$ such that this assignment is compatible with the labels, and such that related generated pastings have equal image, induces a unique ω-functor from this ω-category to $\mathbb{C}$. I will prove this following a procedure analogous to Johnson's proof of the pasting theorem [8].

In sections 4 to 6 I describe a class of pasting schemes especially suitable to replacements of subpasting schemes, the replacements themselves, and the insertion of identities. In section 7 I give the precise definition of a pasting presentation. In section 8 I describe how to enlarge a pasting scheme in a specific way by adding identities, which I use in section 9 to describe the ω-category on a pasting presentation. In section 10 I prove universality of this ω-category, and in section 11 I show that every ω-category has a pasting presentation. More examples of pasting presentations will occur in chapter 3.

4 More on pasting schemes

First some notation that will be used throughout this chapter. If not stated otherwise, A will be a well-formed loop-free pasting scheme of dimension n, and X will be a well-formed subpasting scheme of A of dimension p.

Definition 4.1 For a p-dimensional well-formed loop-free pasting scheme X, the *interior* of X is the subgraded set $\widetilde{X} = X - s_{p-1}(X) - t_{p-1}(X) = \mathsf{E}(X) \cap \mathsf{B}(X)$ of X, consisting of the highest dimensional cells of X together with all cells in between. $\diamond$

Lower dimensional cells in the interior do not occur in the definition of the i-sources and i-targets. But they can be used as circumstantial evidence:

Lemma 4.2 *Let A be a finite pasting scheme with no direct loops. If $x \mathsf{E}^i_j y$ and $y \in s_i(A)$, then there exists $x' \in s_i(A)$ with $x' \mathsf{E}^i_j y$.*

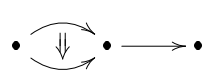
Proof. If $x \notin s_i(A)$, then there exists u with $u \mathsf{E}^{i+1}_i x$ by definition of s_i. It cannot be that $u \mathsf{E}^{i+1}_j y$ because that would contradict $y \in s_i(A)$, so applying pasting axiom (v) to this situation, there must be a v with $u \mathsf{B}^{i+1}_i v \mathsf{E}^i_j y$. Replace x by v and repeat. Because A is without direct loops every new v is different from the ones before, and because A is finite there must come a moment when the v under consideration is in $s_i(A)$. This v can then be taken for x'. $\square$

4.1 Roundness

As remarked before, it is necessary to compare a pasting scheme with the pasting scheme obtained from it by replacing the interior by one cell. The result of this repacing, however, is not always a well-formed loop-free pasting scheme. I will show that this is the case if and only if the pasting scheme is *round*.

Definition 4.3 A well-formed loop-free pasting scheme A is *round* if $s_{n-1}(A) \cap t_{n-1}(A) = s_{n-2}(A) \cup t_{n-2}(A)$. $\diamond$

The idea behind roundness is that the pasting scheme has no lower dimensional cells sticking out, as in



,

nor that there are "bumps" in the pasting scheme, as in

$$\bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \bullet \quad .$$

Replacing the interior of a well-formed loop-free pasting scheme A by one cell results in the graded set $A_t = (A - \tilde{A}) \coprod \{a_t\}$, where a_t is a cell of dimension n. Define relations E^i_j and B^i_j on A_t, by requiring that $(a, a') \in \mathsf{E}^i_j$ if and only if one of the following:

1. $a \in A - \tilde{A}$, $a' \in A - \tilde{A}$, $\quad a\mathsf{E}^i_j a'$ in A,

2. $a = a_t$, $a' \in \widetilde{t_{n-1}(A)}$.

$(a, a') \in \mathsf{B}^i_j$ is defined dually, namely by $(a, a') \in \mathsf{B}^i_j$ if and only if one of the following:

1. $a \in A - \tilde{A}$, $a' \in A - \tilde{A}$, $\quad a\mathsf{B}^i_j a'$ in A,

2. $a = a_t$, $a' \in \widetilde{s_{n-1}(A)}$.

Proposition 4.4 *A_t is a well-formed loop-free pasting scheme if and only if A is round.*

Proof. "$\Rightarrow$": I will first show that A_t is round. If x of dimension $n-1$ is in $s_{n-1}(A_t) \cap t_{n-1}(A_t) = \mathsf{R}(\mathsf{B}_{n-1}(A_t)) \cap \mathsf{R}(\mathsf{E}_{n-1}(A_t))$ ([8, Proposition 7]) then $a_t\mathsf{E}^n_{n-1}x$ and $a_t\mathsf{B}^n_{n-1}x$, contradicting A_t having no direct loops. So $s_{n-1}(A_t) \cap t_{n-1}(A_t)$ is at most $(n-2)$-dimensional. If x of lower dimension is inthere but not in $s_{n-2}(A_t) \cup t_{n-2}(A_t)$ then there are $(n-1)$-cells u and v in $s_{n-1}(A_t)$ with $u\mathsf{E}^{n-1}_j x$ and $v\mathsf{E}^{n-1}_j x$, and likewise $(n-1)$-cells u' and v' in $t_{n-1}(A_t)$ with $u'\mathsf{E}^{n-1}_j x$ and $v'\mathsf{E}^{n-1}_j x$. It follows, using pasting axiom (iv), that $a_t\mathsf{E}^n_j x$ and $a_t\mathsf{B}^n_j x$, again contradicting A_t having no direct loops. So A_t is round. But $s_{n-1}(A) = s_{n-1}(A_t)$ and $t_{n-1}(A) = t_{n-1}(A_t)$, so A is round as well.

"$\Leftarrow$": first observe that A being round implies that $\widetilde{t_{n-1}(A)} \cap \widetilde{s_{n-1}(A)} = \varnothing$. I will show that A_t is a pasting scheme, that it has no direct loops, that it is well formed, and that it is loop free.

Pasting axioms (i) and (ii) are trivial. Pasting axiom (iii) holds because $t_{n-1}(A)$ is $(n-1)$-dimensional, so a_t ends in at least one $(n-1)$-cell, and for other x there is a y as in A. Pasting axioms (iv) and (v) for dimensions below n are immediate from the corresponding facts in A. For pasting axiom (iv) "$\Rightarrow$"

with $w = a_t$, $x \in \widetilde{t_{n-1}(A)}$ implies $x \notin \widetilde{s_{n-1}(A)}$ as remarked above, so there is a $w' \in A$ with $w'\mathsf{E}^n_j x$. Apply pasting axiom (iv) "$\Rightarrow$" in A to this, obtaining u' and v'. Now lemma 4.2 gives u and v in $t_{n-1}(A)$, and because they are $(n-1)$-dimensional they are in $\widetilde{t_{n-1}(A)}$. Thus $w\mathsf{E}^n_{n-1}u\mathsf{E}^{n-1}_j x$ and $w\mathsf{E}^n_{n-1}v\mathsf{B}^{n-1}_j x$ in A_t, as required. For pasting axiom (iv) "$\Leftarrow$" with $w = a_t$, I have to show that if $u\mathsf{E}^{n-1}_j x$ and $v\mathsf{B}^{n-1}_j x$ in A with u and $v \in \widetilde{t_{n-1}(A)}$, then $x \in \widetilde{t_{n-1}(A)}$. Well, x must be in $t_{n-1}(A)$ because u is and $t_{n-1}(A)$ is a subpasting scheme of A, and it has incoming and outgoing $(n-1)$-cells in $t_{n-1}(A)$, so it is in $\widetilde{t_{n-1}(A)}$. For pasting axiom (v) with $w = a_t$, I have to examine the situation $u\mathsf{E}^{n-1}_j x$ for $u \in \widetilde{t_{n-1}(A)}$. As before, x must be in $t_{n-1}(A)$, if x is in $\widetilde{t_{n-1}(A)}$ then $a_t\mathsf{E}^n_j x$, otherwise $x \in s_{n-2}(A) \cup t_{n-2}(A)$. Using lemma 4.2, x cannot be in $s_{n-2}(A)$ because it has u incoming, so it has an incoming $(n-1)$-cell in $s_{n-1}(A)$, which can be taken for v. This proves that A_t is a pasting scheme.

From the observation above it follows that $\mathsf{B}(a_t) \cap \mathsf{E}(a_t) = \{a_t\}$. The rest of A_t having no direct loops is immediate from the corresponding facts in A.

A_t is well formed because it has only one top-dimensional cell and its i-sources and i-targets are equal to those of A.

For any $a \in A_t$, $\mathsf{R}(a)$ is well formed: for a_t this is well-formedness of A_t, and for other, lower dimensional, a there is no difference with A. Finally, condition (iv) of loop-freeness, for which the only interesting case is $j < n$ and $x = a_t$. Then $s_j(\mathsf{R}(a_t)) = s_j(A) \subseteq Y$. If the next v in the sequence connecting u and u' is not in $s_j(A)$ then the cell in between u and v has an outgoing cell in $s_j(A)$ as well, by lemma 4.2, which contradicts well-formedness of Y. By induction on the length of the sequence, *all* v in between u and u' are in $s_j(\mathsf{R}(a_t))$. $\square$

5 Replacing a subpasting scheme

To compare two pasting schemes which differ only in a subpasting scheme, I will consider one of them as a deformation of the other one, where the deformation consists of the replacement of a subpasting scheme by another pasting scheme. In fact, the process of replacing a subpasting scheme is basic here, and only afterwards the result will be compared with another pasting scheme. An example of such a replacement already occurred in section 4. In this section I describe the general process of replacing, and I compare composites of related realizations of the original and the result.

5.1 The graded set $A[X'/X]$

Let X be a p-dimensional well-formed subpasting scheme of the well-formed loop-free pasting scheme A, and let X' be a p-dimensional well-formed loop-free pasting scheme with $s_{p-1}(X') = s_{p-1}(X)$ and $t_{p-1}(X') = t_{p-1}(X)$. Define $A[X'/X] = (A - \widetilde{X}) \coprod \widetilde{X'}$, which is a graded set. Define relations E and B on $A[X'/X]$, by requiring that $(a, a') \in \mathsf{E}^i_j$ if and only if one of the following:

1. $a \in A - \widetilde{X}$, $a' \in A - \widetilde{X}$, $\quad a\mathsf{E}^i_j a'$,
2. $a \in A - \widetilde{X}$, $a' \in \widetilde{X'}$, $\quad$ there exists $x \in \widetilde{X}$ such that $a\mathsf{E}^i_{j'}x$ for some j',
3. $a \in \widetilde{X'}$, $a' \in A - \widetilde{X}$, $\quad a\mathsf{E}^i_j a'$ in X',
4. $a \in \widetilde{X'}$, $a' \in \widetilde{X'}$, $\quad a\mathsf{E}^i_j a'$ in X'.

$(a, a') \in \mathsf{B}^i_j$ is defined dually (which in this case is the same as replacing every occurrence of E by B and conversely).

Thus, the interior of X gets replaced by the interior of X', and the cells around it end or begin in the cells of $\widetilde{X'}$ if they did end or begin in the cells of $\widetilde{X}$.

5.2 Elementary replaceability

For the result of a replacement to be sensible, some conditions are needed on the original pasting scheme and the subpasting scheme that is going to be replaced. For example, if there is a cell ending in only part of the interior of X, then replacement shouldn't be allowed since the result could fail to be well formed, or, more subtlely, induced realizations could fail to be appropriate. Also, it must not be allowed that a low-dimensional cell outside X ends in the interior of X since it may end in nothing after replacement.

Definition 5.1 Let X be a p-dimensional well-formed subpasting scheme of the well-formed loop-free pasting scheme A. *X is elementary replaceable in A*, or just elementary replaceable if A is clear from the context, if

1. for every $(p+1)$-dimensional cell $a \in A$ and every $x \in \widetilde{X}$, if $a\mathsf{E}^{p+1}_j x$ then for every $x' \in \widetilde{X}$, $a\mathsf{E}^{p+1}_{j'} x'$,
2. for every $(p+1)$-dimensional cell $a \in A$ and every $x \in X$, if $a\mathsf{E}^{p+1}_j x$ then for every $x' \in X$ there exists $a' \in A$ with $a'\mathsf{E}^{p+1}_{j'} x'$, and

3. for every i-dimensional cell $a \in A$ with $i \leq p$ and every $x \in \widetilde{X}$, if $a\mathsf{E}^i_j x$ then $a \in X$

and dually. $\diamond$

In the sequel I will also need condition 1 for higher dimensional cells.

Lemma 5.2 *If X is elementary replaceable in A then for every i-dimensional cell $a \in A$ with $i \geq p+1$, if $a\mathsf{E}^i_j x$ for some $x \in \widetilde{X}$ then $a\mathsf{E}^i_j x$ for* all *$x \in \widetilde{X}$, and dually.*

Proof. Induction on i. For $i = p+1$ this is condition 1. Now suppose the statement is proven for $i-1$, and let $a\mathsf{E}^i_j x$ for some $x \in \widetilde{X}$. Find u and v via pasting axiom (iv), apply the induction hypothesis and its dual to u and v, and apply the other direction of pasting axiom (iv). This gives $a\mathsf{E}^i_j x$ for all $x \in \widetilde{X}$. $\square$

A slight variation on the previous theme:

Lemma 5.3 *If X is elementary replaceable in A and there is an $x \in \widetilde{X}$ in $s_i(A)$ then $X \subseteq s_i(A)$, and dually.*

Proof. By definition of $\widetilde{X}$ it cannot be that $i \leq p$. Suppose the statement does not hold for $i = p+1$, i.e., $a\mathsf{E}^{p+1}_j x'$ for some $a \in A$ and some $x' \in X$. Then by condition 2, there exists $a' \in A$ satisfying $a'\mathsf{E}^{p+1}_j x$, contradicting the assumption that $x \in s_p(A)$. Proceed inductively, as in the proof of lemma 5.2, using this lemma in order to be able to use the same a' for both u and v. $\square$

This lemma makes clear why condition 2 is needed: although it is immediate from lemma 5.2 that the assumption implies $\widetilde{X} \subseteq s_i(A)$, cells in the boundary of X, especially cells totally irrelevant for the replacement, might spoil the conclusion of lemma 5.3. This conclusion eliminates anomalous behaviour of the i-source of $A[X'/X]$.

An elementary replacement can be undone:

Lemma 5.4 *Let X be elementary replaceable in A and let $A' = A[X'/X]$ be a well-formed loop-free pasting scheme. Then X' is elementary replaceable in A', and $A'[X/X'] = A$.*

Proof. Just check the definitions. $\square$

5.3 Pasting scheme

The result of an elementary replacement is a pasting scheme:

Proposition 5.5 *If X is elementary replaceable in A then for every p-dimensional well-formed loop-free pasting scheme X' with $s_{p-1}(X') = s_{p-1}(X)$ and $t_{p-1}(X') = t_{p-1}(X)$ the graded set $A[X'/X]$ is a pasting scheme.*

Proof. Pasting axiom (i) holds by definition, and pasting axiom (ii) is straightforward.

For pasting axiom (iii), take $x \in (A[X'/X])_i$. If x is in X' then there exists $y \in X'_{i-1}$ by pasting axiom (iii) in X', and case 3 or 4 applies. If $x \in A - \widetilde{X}$, find $y \in A_{i-1}$ by applying pasting axiom (iii) in A. If $y \notin \widetilde{X}$ then case 1 applies, if $y \in \widetilde{X}$ then y must be p-dimensional since otherwise x would be in $\widetilde{X}$ by condition 3, take any p-dimensional cell of $\widetilde{X'}$ and case 2 applies.

For the "$\Rightarrow$" part of pasting axiom (iv), suppose that $w\mathsf{E}^i_j x$ in $A[X'/X]$. First consider case 1. If $w \in X$ then u and v can be found via pasting axiom (iv) in X, and then they cannot be in $\widetilde{X}$ because this would conflict with $w \notin \widetilde{X}$, so case 1 applies twice. If $w \notin X$, find u and v in A. The dimension of w is greater than p by condition 3 of elementary replaceability. If $\dim w > p+1$ then $u, v \notin \widetilde{X}$ and case 1 applies twice. If $\dim w = p+1$ and $u \notin \widetilde{X}$ then case 1 applies twice as well, if $u \in \widetilde{X}$ then $x \in X$ but $\notin s_{p-1}(X) = s_{p-1}(X')$ since u prevents that, so there is a $u' \in X'$ with $u'\mathsf{E}^p_{j'}x$ in X', and cases 2 and 4 apply. This finishes case 1. In case 2, $\dim w > p$ by condition 3, so $w \notin X$. If $\dim w > p+1$, take some $x' \in \widetilde{X}$, find u and v by applying pasting axiom (iv) in A to $w\mathsf{E}^i_j x'$, and cases 1 and 2 apply. If $\dim w = p+1$ then $x \notin s_{p-1}(X')$, which gives a u to which cases 2 and 4 apply, and $x \notin t_{p-1}(X')$, which gives a v. This finishes case 2. The other cases can be done completely in X'.

For the "$\Leftarrow$" part of pasting axiom (iv), suppose that $w\mathsf{E}^n_{n-1}u\mathsf{E}^{n-1}_j x$ and $w\mathsf{E}^n_{n-1}v\mathsf{B}^{n-1}_j x$ in $A[X'/X]$. The only non-trivial, i.e., not immediately holding in A or in X', cases are those with $w \in A - \widetilde{X}$, and one or more of u, v and x in $\widetilde{X'}$. If u and x are in $\widetilde{X'}$, then $\dim w = p+1$, and $w\mathsf{E}^{p+1}_p u$ gives that there exists $x' \in \widetilde{X}$ such that $wE^{p+1}_{j'}x'$, and case 2 applies. This argument can also be used if v and x are in $\widetilde{X'}$. If neither of u and v are in $\widetilde{X'}$, but x is, then, by condition 3, $\dim w > p+1$, and lemma 5.2 can be applied to u and v, yielding that all $x' \in \widetilde{X}$ are in $\mathsf{E}(u)$ and in $\mathsf{E}(v)$. An application of pasting axiom (iv) for any such x' gives that case 2 applies. The last case is that $u \in \widetilde{X'}$, $x \notin \widetilde{X'}$, which makes $x \in X$ since X is a subpasting scheme of A. Again $\dim w = p+1$, and $w\mathsf{E}^{p+1}_p u$ gives that there exists $u' \in \widetilde{X}$ such that $wE^{p+1}_{j'}u'$. By condition 1

this holds for any $u' \in \widetilde{X}$, in particular, since u prevents that $x \in s_{p-1}(X)$, for some p-dimensional $u' \in \widetilde{X}$ which satisfies $u'\mathsf{E}^p_j x$. v cannot be in $\widetilde{X'}$ at the same time without x being in $\widetilde{X'}$ so pasting axiom (iv) can be applied to w, u', v and x, and case 1 applies.

For pasting axiom (v), suppose that $w\mathsf{E}^n_{n-1}u\mathsf{E}^{n-1}_j x$. The only interesting cases are when $w \in A - \widetilde{X}$, and u or x or both in $\widetilde{X'}$. If they are both in $\widetilde{X'}$ then the same argument as in one of the cases of (iv), "$\Leftarrow$" gives that case 2 applies. If only u is in $\widetilde{X'}$ then u prevents x from being in $s_{p-1}(X') = s_{p-1}(X)$, so there is a p-dimensional $u' \in X$ with $u'\mathsf{E}^p_j x$. Applying pasting axiom (v) in A gives $w\mathsf{E}^{p+1}_j x$ and case 1 applies, or $w\mathsf{B}^{p+1}_p v\mathsf{E}^p_k x$ for some v. If $v \notin \widetilde{X}$ this takes place outside $\widetilde{X}$ and case 1 applies twice, and $v \in \widetilde{X}$ cannot occur since then x would not be in $t_{p+1}(\mathsf{R}(w))$. If only $x \in \widetilde{X'}$ then replace x by any $x' \in \widetilde{X}$, apply pasting axiom (v) in A, replace back, and case 2 applies. $\square$

5.4 s_m and t_m

Before I can show that $A[X'/X]$ is well formed I need to determine $s_m(A[X'/X])$ and $t_m(A[X'/X])$. Some care is needed here since the definition of well-formedness assumes that A contains no direct loops. But the reason this needs to be assumed is that dom and cod then behave nicely, which is needed in the definition of s_m and t_m. But even if $A[X'/X]$ contains a direct loop, it is sufficiently well-behaved to make it possible to speak of s_m and t_m.

For $m \geq \dim(A)$, $s_m(A[X'/X])$ and $t_m(A[X'/X])$ are trivially equal to $A[X'/X]$. Now assume that $\dim(A) > \dim(X) = p$, and consider $m = \dim(A) - 1 = n-1$, i.e., I now want to know what $\mathrm{dom}(A[X'/X])$ looks like. I can restrict myself to s_m since t_m goes dually. If there is an $x \in \widetilde{X}$ for which there is an $a \in A$ with $a\mathsf{E}^{m+1}_{j'}x$ then by case 2 $a\mathsf{E}^{m+1}_j x'$ for all $x' \in \widetilde{X'}$. So there is no cell of $\widetilde{X'}$ in $\mathrm{dom}(A[X'/X])$, and because for all other cells of $A[X'/X]$ the E-relations are as in A, it follows that in this case $\mathrm{dom}(A[X'/X]) = \mathrm{dom}(A)$. If there is no $x \in \widetilde{X}$ for which there is an $a \in A$ with $a\mathsf{E}^{m+1}_{j'}x$ then $X \subseteq \mathrm{dom}(A)$ by lemma 5.3, and because the conditions have to hold for fewer cells X' is then elementary replaceable for X in $\mathrm{dom}(A)$. This makes it possible to speak about $\mathrm{dom}(A)[X'/X]$. But the cells of this are exactly the cells a of $A[X'/X]$ for which there is no $a' \in A$ with $a'\mathsf{E}^{m+1}_j a$, so in this case $\mathrm{dom}(A[X'/X]) = \mathrm{dom}(A)[X'/X]$. Continuing this for lower dimensions, it follows that for every $m \geq p$ either $s_m(A[X'/X]) = s_m(A)[X'/X]$ or $s_m(A[X'/X]) = s_m(A)$, depending on whether $X \subseteq s_m(A)$ or not.

If $\dim(A) = \dim(X) = p$ I want to show that $\mathrm{dom}(A[X'/X]) = \mathrm{dom}(A)$. This follows if for all $a \in A - \widetilde{X}$, there exists a p-cell $x \in X$ with $x\mathsf{E}^p_j a$ if and only if there exists a p-cell $x' \in X'$ with $x'\mathsf{E}^p_j a$, since cells in $\widetilde{X'}$ resp. in $\widetilde{X}$ are not in $\mathrm{dom}(A[X'/X])$, resp. $\mathrm{dom}(A)$ anyway. But this statement can be rephrased to the statement that for all $a \in A$, $a \notin s_{p-1}(X)$ if and only if $a \notin s_{p-1}(X')$, and this holds because $s_{p-1}(X) = s_{p-1}(X')$ by assumption. Consequently, for every $m < p$, $s_m(A[X'/X]) = s_m(A)$.

5.5 Well-formedness

Because of the no direct loops assumption in the definition of well-formedness, the proposition that the result of an elementary replacement is well formed needs to be stated with care.

Proposition 5.6 *If X is elementary replaceable in A then for every p-dimensional well-formed loop-free pasting scheme X' with $s_{p-1}(X') = s_{p-1}(X)$ and $t_{p-1}(X') = t_{p-1}(X)$ the pasting scheme $A[X'/X]$, if it contains no direct loops, is well formed. Moreover, if this is the case then for every $a \in A[X'/X]$, $\mathsf{R}_{A[X'/X]}(a)$ is well formed.*

Proof. The proof falls apart in two: that $s_m(A[X'/X])$ is a subpasting scheme of $A[X'/X]$, and that $A[X'/X]$ is compatible. The description of $s_m(A[X'/X])$ above then gives that for every $A[X'/X]$, every $s_m(A[X'/X])$ is a compatible subpasting scheme.

For the first part, note that the induced E and B relations on the subgraded set $s_m(A)$ or $s_m(A)[X'/X]$ of $A[X'/X]$ are the same as these relations on this graded set as a pasting scheme. Thus, it remains to be shown that for $X \subseteq s_m(A)$, if $a\mathsf{E}^i_j a'$ in $A[X'/X]$ with $a \in s_m(A)[X'/X]$ then $a' \in s_m(A)[X'/X]$, and that for $X \not\subseteq s_m(A)$, if $a\mathsf{E}^i_j a'$ in $A[X'/X]$ with $a \in s_m(A)$ then $a' \in s_m(A)$. In the case $X \not\subseteq s_m(A)$, case 2 of the definition of $A[X'/X]$ cannot occur since if there exists an $x \in \widetilde{X}$ such that $a\mathsf{E}^i_k x$ then $x \in s_m(A)$ because $s_m(A)$ is a subpasting scheme of A, contradicting $x \in \widetilde{X}$. In the case $X \subseteq s_m(A)$, cases 2, 3 and 4 follow from $X' \subseteq s_m(A)[X'/X]$, while in case 1 $a' \in s_m(A)$, again because $s_m(A)$ is a subpasting scheme of A, and $a' \notin \widetilde{X}$, so $a' \in s_m(A)[X'/X]$, as required.

This leaves compatibility. If A is compatible then also $A[X'/X]$ if its dimension is at least $p+1$ since they don't differ in dimensions above $p+1$, and if $a\mathsf{E}^{p+1}_p x$ for some $x \in \widetilde{X}$ then by lemma 5.2 $a\mathsf{E}^{p+1}_j x$ for *all* $x \in \widetilde{X}$. If the dimension of A is equal to p (it cannot be lower since $X \subseteq A$), the only case to

check is that $x' \in \widetilde{X'}$ and $a \in A$ have $\mathsf{B}_{p-1}(x') \cap \mathsf{B}_{p-1}(a) \neq \varnothing$, say it contains x. Now $x' \mathsf{E}^p_{p-1} x$ implies $x \notin t_{p-1}(X') = t_{p-1}(X)$, which means there exists $x'' \in X$ such that $x'' \mathsf{E}^p_{p-1} x$. By compatibility of A, it must be that $x'' = a$, contradicting $a \notin \widetilde{X}$.

For the moreover part, $\mathsf{R}_{A[X'/X]}(a)$ is either $\mathsf{R}_A(a)[X'/X]$ or $\mathsf{R}_A(a)$, and the above can be used, or $\mathsf{R}_{X'}(a)$. □

5.6 Full replaceability

In comparing composites of realizations of A and $A[X'/X]$, I will calculate both composites by first composing (the induced realizations of) X and X' respectively, and I want to conclude from these composites being equal that the composites of the realizations of A and $A[X'/X]$ are equal as well. To make this work, I will use $A[X_t/X]$, where X_t is the pasting scheme obtained from X by replacing its interior by one cell as defined in section 4, as an intermediate. For this intermediate to be sensible, X has to satisfy a stronger condition than elementary replaceability.

Definition 5.7 *X is fully replaceable in A* if X is elementary replaceable in A, X is round, and $A[X_t/X]$ is loop free. ◇

Full replaceability works in both directions:

Lemma 5.8 *If X is fully replaceable in A and $A[X'/X]$ is loop free then X' is fully replaceable in $A[X'/X]$, and $(A[X'/X])[X/X'] = A$.*

Proof. Mostly immediate from lemma 5.4, except that X' is round, and that $(A[X'/X])[X_t/X']$ is loop free. But the former of these is immediate from roundness of X since X and X' have equal $(n-1)$-source and $(n-1)$-target, and the latter one is trivial after the remark that $(A[X'/X])[X_t/X'] = A[X_t/X]$, which is easily checked from the definitions. □

In case that the interior of the whole pasting scheme needs to be replaced, for example in defining a relation, only roundness needs to be checked:

Lemma 5.9 *A is fully replaceable in A if and only if A is round.*

Proof. Necessity is trivial. Sufficiency comes down to elementary replaceability, for which cases 1 and 2 are void, and case 3 is immediate from $s_{n-1}(A)$ and $t_{n-1}(A)$ being subpasting schemes. □

5.7 Subpasting schemes of $A[X_t/X]$

Before considering realizations and their composites, I need to look at well-formed subpasting schemes of $A[X_t/X]$. I want to relate these to subpasting schemes of A.

First consider a well-formed subpasting scheme Y' of $A[X_t/X]$ not containing x_t. In this case, Y' can be seen as a subgraded set of A not containing any element of $\widetilde{X}$, the interior of X. Because the E and B relations on A and $A[X_t/X]$ for cells outside $\widetilde{X}$ and $\widetilde{X'}$ respectively coincide, Y' is a well-formed subpasting scheme of A if it is one of $A[X_t/X]$.

Next consider a well-formed subpasting scheme Y' of $A[X_t/X]$ which does contain x_t, in which case $X_t \subseteq Y'$ because $X_t = \mathsf{R}_{A[X_t/X]}(x_t)$ and Y' is a subpasting scheme of $A[X_t/X]$. Take $Y = (Y' - \{x_t\}) \coprod \widetilde{X}$, which can be seen as a subgraded set of A.

Lemma 5.10 *In this case, Y is a well-formed subpasting scheme of A, and $Y[X_t/X] = Y'$.*

Proof. Essentially the same as for $x_t \notin Y'$, but a little bit more involved.

Subpasting scheme: suppose $y\mathsf{E}z$ in A for some $y \in Y$, to show that $z \in Y$. If $y \in Y' - \{x_t\}$ then because the E and B relations on A and $A[X_t/X]$ for cells outside $\widetilde{X}$ and $\widetilde{X'}$ respectively coincide, $y\mathsf{E}z$ in $A[X_t/X]$, which implies $z \in Y'$, hence $z \in Y$ because z cannot be x_t. If $y \in \widetilde{X}$ then $z \in X \subseteq Y$ because X is a subpasting scheme of A and because Y' is a subpasting scheme of $A[X_t/X]$.

Well formed: in fact, Y is defined as $Y'[X/X_t]$, thus depending on whether $x_t \in s_m(Y')$ or not, $s_m(Y) = s_m(Y')[X/X_t]$ or $s_m(Y) = s_m(Y')$. So I only have to show that for all Y', Y is compatible. The only interesting case in this is when one of the cells, y say, is in $Y' - \{x_t\}$ and the other one is in $\widetilde{X}$. In this case $\dim(Y) \geq p$ because $\widetilde{X} \subseteq Y$ and $\dim(Y) \leq p$ because a cell in X is highest dimensional. Hence Y is p-dimensional, and the cell in $\widetilde{X}$ is p-dimensional as well. Chasing the definitions now gives that in Y' the cells y and x_t end in the same cell, contradicting compatibility of Y'.

Now Y is a well-formed loop-free pasting scheme, and X is elementary replaceable by X_t in Y because it is in A. Inspecting the definition of $Y[X_t/X]$ shows it is exactly Y'. □

So a well-formed subpasting scheme Y' of $A[X_t/X]$ induces a well-formed subpasting scheme Y of A such that if $x_t \notin Y'$ then $Y' = Y$ and if $x_t \in Y'$ then $Y' = Y[X_t/X]$. In the latter case, $Y = Y'[X/X_t]$, and it will be convenient to

extend the same notation to the former case, so that its meaning becomes: in Y', replace X_t by X if X_t is a subpasting scheme of Y', otherwise do nothing.

Next, composition in the ω-category of components of $A[X_t/X]$ is related to composition in the ω-category of components of A:

Lemma 5.11 *Let Y' and Z' be subpasting schemes of $A[X_t/X]$ such that $t_m(Y') = s_m(Z')$. Then $t_m(Y'[X/X_t]) = s_m(Z'[X/X_t])$, and $(Y' \cup Z')[X/X_t] = Y'[X/X_t] \cup Z'[X/X_t]$.*

Proof. There are four cases to consider, depending on whether x_t is in Y' or in Z'. If x_t is in neither of both then the statement becomes trivial. If x_t is only in Y' (only in Z' goes dual) then apparently $x_t \notin t_m(Y')$, so $t_m(Y'[X/X_t]) = t_m(Y') = s_m(Z') = s_m(Z'[X/X_t])$. If x_t is in both then $x_t \in t_m(Y')$, so $t_m(Y'[X/X_t]) = t_m(Y')[X/X_t] = s_m(Z')[X/X_t] = s_m(Z'[X/X_t])$. In all cases, the last statement is straightforward from the definitions. □

5.8 Realizing and composing $A[X'/X]$

To compare composites of related realizations of A and $A[X'/X]$, the composite of X, i.e., the value of the extension of the appropriate realization of A at X, needs to be considered. This composite will be substituted in the realization of A, yielding a realization of $A[X_t/X]$, which explains the necessity to consider this intermediate pasting scheme. This realization will be used to show that the realizations of A and $A[X'/X]$ have equal composite. As before, $A[X'/X]$ is considered to be obtained from A, and so realizations of $A[X'/X]$ will be obtained from realizations of A and X'.

Suppose an appropriate realization (A, f_i) of A in some ω-category $\mathbb{C}$ together with an appropriate realization (X', g_i) of X' in $\mathbb{C}$ is given, such that for every $x \in X - \widetilde{X}$, $f_i(x) = g_i(x)$, and such that $f(X) = g(X')$. This information induces a realization $(A[X'/X], f_i')$ of $A[X'/X]$ in $\mathbb{C}$, given by

$$\begin{array}{rcll} f_i'(a) & = & f_i(a) & \text{if } a \in A - \widetilde{X} \\ & = & g_i(a) & \text{if } a \in \widetilde{X'}. \end{array}$$

Proposition 5.12 *The induced realization $(A[X'/X], f_i')$ is appropriate, and $f'(A[X'/X]) = f(A)$.*

Proof. By induction. The information above also induces a realization $(A[X_t/X], f_i^t)$ of $A[X_t/X]$ in $\mathbb{C}$, given by

$$\begin{array}{rcll} f_i^t(a) & = & f_i(a) & \text{if } a \neq x_t \\ & = & f(X) & \text{if } a = x_t. \end{array}$$

I will prove the following induction hypothesis: $(A[X_t/X], f_i^t)$ is n-appropriate with n-extension $f^t : |\mathcal{P}(A[X_t/X])|_n \to \mathbb{C}$ given by

$$f^t(Y') = f(Y'[X/X_t]),$$

and $(A[X'/X], f_i')$ is n-appropriate.

Realizations are always 0-appropriate, so if $n = 0$ I only need to show the formula for the 0-extension f^t of $(A[X_t/X], f_i^t)$. But well-formed 0-dimensional pasting schemes are singletons, so this is immediate, also if X is 0-dimensional.

Now suppose the induction hypothesis holds up to n, I will first show that $(A[X_t/X], f_i^t)$ is $(n+1)$-appropriate. For $a = x_t$,

$$\begin{array}{lll} f^t\left(s_n(\mathsf{R}_{A[X_t/X]}(a))\right) = & & \\ \quad = & f\left(s_n(\mathsf{R}_{A[X_t/X]}(a))[X/X_t]\right) & \\ \quad = & f(s_n(X)) & \text{because } \mathsf{R}_{A[X_t/X]}(a) = X_t \\ \quad = & s_n(f(X)) & \text{because } f \text{ is an } \omega\text{-functor} \\ \quad = & s_n f_{n+1}^t(a) & \text{by definition of } f_i^t. \end{array}$$

For $a \neq x_t$,

$$\begin{array}{lll} f^t\left(s_n(\mathsf{R}_{A[X_t/X]}(a))\right) = & & \\ \quad = & f\left(s_n(\mathsf{R}_A(a))\right) & \text{because } (\mathsf{R}_{A[X_t/X]}(a))[X/X_t] = \mathsf{R}_A(a) \\ \quad = & s_n(f_{n+1}(a)) & \text{because } (A, f_i) \text{ is } (n+1)\text{-appropriate} \\ \quad = & s_n(f_{n+1}^t(a)) & \text{by definition of } f_i^t. \end{array}$$

Thus indeed $(A[X_t/X], f_i^t)$ is $(n+1)$-appropriate.

Next, I will prove the formula for the $(n+1)$-extension f^t of $(A[X_t/X], f_i^t)$. This amounts to showing that f^t is an ω-functor, that it extends $(A[X_t/X], f_i^t)$, and that it is the unique function $|\mathcal{P}(A[X_t/X])|_{n+1} \to \mathbb{C}$ having these properties. It is an ω-functor because for Y' a well-formed subpasting scheme of $A[X_t/X]$,

$$\begin{array}{lll} s_m(f^t(Y')) = & & \\ \quad = & s_m(f(Y'[X/X_t])) & \\ \quad = & f(s_m(Y'[X/X_t])) & \text{because } f \text{ is an } \omega\text{-functor} \\ \quad = & f(s_m(Y')[X/X_t]) & \text{using in case } x_t \notin Y' \text{ the notational convention} \\ \quad = & f^t(s_m(Y')), & \end{array}$$

because if $t_m(Y') = s_m(Z')$ then

$$\begin{array}{rcll} f^t(Y' \cup Z') & = & f((Y' \cup Z')[X/X_t]) & \\ & = & f(Y'[X/X_t] \cup Z'[X/X_t]) & \text{by lemma 5.11} \\ & = & f(Z'[X/X_t]) \circ_m f(Y'[X/X_t]) & \text{because } f \text{ is an } \omega\text{-functor} \\ & = & f^t(Z') \circ_m f^t(Y'), & \end{array}$$

and because there is nothing to do for identities since $|\mathcal{P}(A[X_t/X])|_{n+1}$ and $\mathbb{C}$ are considered one-sorted. It extends $(A[X_t/X], f_i^t)$ because for $a = x_t \in A[X_t/X]_k$ with $k \leq n+1$,

$$f^t(\mathsf{R}_{A[X_t/X]}(a)) = f(X) = f_k^t(a),$$

and for $a \in A[X_t/X]_k$ with $a \neq x_t$,

$$f^t(\mathsf{R}_{A[X_t/X]}(a)) = f(\mathsf{R}_A(a)) = f_k(a) = f_k^t(a).$$

Uniqueness is immediate from the $(n+1)$-appropriateness of $(A[X_t/X], f_i^t)$ since that implies $(n+1)$-extendability. For $(n+1)$-appropriateness of f_i', note that $(A[X'/X])[X_t/X'] = A[X_t/X]$, and that the n-extension of f_i' satisfies

$$f^t(Y') = f'(Y'[X'/X_t]).$$

And because $s_m(\mathsf{R}(a))$ is always either of the form $Y'[X'/X_t]$ or a subpasting scheme of X' $(n+1)$-appropriateness of f_i^t and of g_i give that f_i' is $(n+1)$-appropriate.

This proves the induction hypothesis for $(n+1)$. Finally, the formulae in this proof give that

$$\begin{array}{rcl} f'(A[X'/X]) & = & f'((A[X_t/X])[X'/X_t]) \\ & = & f^t(A[X_t/X]) \\ & = & f((A[X_t/X])[X/X_t]) \\ & = & f(A). \end{array}$$

□

6 Inserting an identity

To compare two pasting schemes differing by an identity, I will consider the larger one as a deformation of the smaller one, where the deformation here

consists of the insertion of an extra cell that is to be an identity in all realizations. As in the case of replacing a subpasting scheme, the process of inserting is basic. In this section I describe the general process of insertion, and I will compare composites of related realizations of the original and the result. Because of the many possibilities for inserting an identity on a particular subpasting scheme, there will be a condition that the result is a well-formed loop-free pasting scheme.

This insertion of identities is a generalization of the "plastering" of pasting schemes considered by Buckland [3].

6.1 Where

It is not enough to specify that an identity on the subpasting scheme X is to be inserted in A, since this can be done in many different ways. To indicate *where* id_X has to go in A more information needs to be given. This witnessing information consists of functions W_{id} and W_{ba}, as follows.

W_{id} tells what position id_X will have with respect to cells $a \in A$ of dimension greater than $p+1$ that satisfy $X \subseteq \mathsf{R}(a)$ but not $a\mathsf{D}x$ for any $x \in X$. To each such cell, W_{id} assigns one of the four dual forms of the specification "ends in id_X and X_e", or the specification "neither of these". For example, in the pasting scheme

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & x \end{tikzcd} ,$$

if a is specified to end in $\mathrm{id}_{\mathsf{R}(x)}$ and in $\mathsf{R}(x)_b$ then the result will be

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Downarrow a"] & x_e \\ & x_b \arrow[ur, "\mathrm{id}_x"'] & \end{tikzcd} ,$$

if a is specified to begin in $\mathrm{id}_{\mathsf{R}(x)}$ and in $\mathsf{R}(x)_b$ then it will be

$$\begin{tikzcd} & x_b \arrow[dr, "\mathrm{id}_x"] \arrow[d, phantom, "\Downarrow a"] & \\ \bullet \arrow[ur] \arrow[rr] & & x_e \end{tikzcd} ,$$

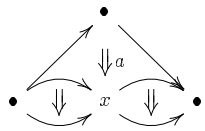
and if a is neither of these, then

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow a"] & x_b \arrow[r, "\mathrm{id}_x"] & x_e \end{tikzcd} .$$

Note that depending on the position of X not all dual forms necessarily make sense.

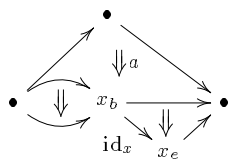
W_{ba} tells what position cells $a \in A - \widetilde{X}$ which satisfy $a\mathsf{D}x$ for some $x \in X$ will have with respect to id_X. To each such cell, W_{ba} assigns one of the specifications

"is before X", "is after X" and "is both". Only cells of dimension greater than $p+1$ can be specified to be both. For example, in the pasting scheme



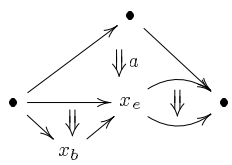
,

if a is specified to be before x then the result will be



,

if a is specified to be after x then it will be



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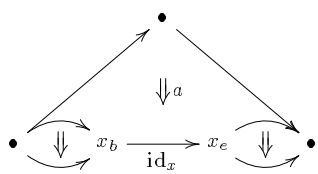
and if a is both then



.

An example of a situation where a is of dimension less than $p+1$ is left to the reader.

The collection of witnessing specifications W_{id} and W_{ba} for the position of id_X will usually be denoted by W. Of course merely giving W doesn't ensure that the result of inserting id_X just there will be a pasting scheme, let alone that it will be well formed and loop free. The examples above already indicate that the witnessing specifications need to be related in a very specific way for the result to be sensible.

6.2 The graded set $A[\mathrm{id}_X : W]$

Let X be a p-dimensional well-formed subpasting scheme of the well-formed loop-free pasting scheme A, and let W be a collection of witnessing specifications.

Define $A[\mathrm{id}_X{:}W] = (A - \widetilde{X}) \coprod \{\mathrm{id}_X\} \coprod (\widetilde{X} \coprod \widetilde{X})$, which is a graded set, where id_X has degree $p+1$. The two copies of $\widetilde{X}$ in $A[\mathrm{id}_X{:}W]$ will be distinguished by writing $\widetilde{X}_b$ and $\widetilde{X}_e$ respectively. Define relations E and B on $A[\mathrm{id}_X{:}W]$, by requiring that $(a, a') \in \mathsf{E}^i_j$ if and only if one of the following:

1. $a \in A - \widetilde{X}$, $a' \in A - \widetilde{X}$, $\quad a\mathsf{E}^i_j a'$,
2. $a \in A - \widetilde{X}$, $a' = \mathrm{id}_X$, $\quad$ either a ends in id_X or a is both,
3. $a \in A - \widetilde{X}$, $a' \in \widetilde{X}_e$, $\quad$ either a ends in X_e or $a\mathsf{E}^i_j a'$ and a is after X,
4. $a \in A - \widetilde{X}$, $a' \in \widetilde{X}_b$, $\quad$ either a ends in X_b or $a\mathsf{E}^i_j a'$ and a is before X,
5. $a = \mathrm{id}_X$, $a' = \mathrm{id}_X$,
6. $a = \mathrm{id}_X$, $a' \in \widetilde{X}_e$,
7. $a \in \widetilde{X}_e$, $a' \in A - \widetilde{X}$, $\quad a\mathsf{E}^i_j a'$,
8. $a \in \widetilde{X}_b$, $a' \in A - \widetilde{X}$, $\quad a\mathsf{E}^i_j a'$,
9. $a \in \widetilde{X}_e$, $a' \in \widetilde{X}_e$, $\quad a\mathsf{E}^i_j a'$,
10. $a \in \widetilde{X}_b$, $a' \in \widetilde{X}_b$, $\quad a\mathsf{E}^i_j a'$.

$(a, a') \in \mathsf{B}^i_j$ is defined in a dual way. This duality is *not* the same as replacing every occurrence of E by B and conversely, but also takes into account the duality that exists between $_e$ and $_b$, and between "is after" and "is before".

These relations make that $A[\mathrm{id}_X{:}W]$ is equal to the pasting scheme A with a new $(p+1)$-dimensional cell id_X inserted, whose place is determined by the witnessing specifications W.

6.3 Elementary insertability

Because of the intricacy of witnessing specifications, I will not give a large set of conditions for the result of an insertion to be sensible. Instead, I will use only conditions to eliminate anomalous behaviour of the i-source of $A[\mathrm{id}_X{:}W]$, and I will eventually require that the result is a well-formed loop-free subpasting scheme.

Definition 6.1 Let X be a p-dimensional well-formed subpasting scheme of the well-formed loop-free pasting scheme A, and let W be a collection of witnessing specifications. id_X *is elementary insertable in A at W*, or just elementary insertable if A and W are clear form the context, if

- for every $(p+2)$-dimensional cell $a \in A$ and every $x \in \widetilde{X}$, if $a \mathsf{E}_j^{p+2} x$ and a is both then for every $x' \in \widetilde{X}$, $a \mathsf{E}_{j'}^{p+1} x'$,
- for every $(p+2)$-dimensional cell $a \in A$, if a is after X then there exists $a' \in A$ which ends in id_X and in X_b

and dually. $\diamond$

6.4 s_m and t_m

Under the assumption that $A[\mathrm{id}_X : W]$ is a well-formed pasting scheme, and under the condition of elementary replaceability, I will now calculate its i-sources and i-targets.

As before I can restrict myself to s_m since t_m goes dually. For m at least $\dim(A[\mathrm{id}_X : W])$ it is trivially equal to $A[\mathrm{id}_X : W]$. Now assume $\dim(A) > p+1$, and consider $m = \dim(A) - 1 = n - 1$, i.e., look at $\mathrm{dom}(A[\mathrm{id}_X : W])$. If there is an $a \in A$ such that $a \mathsf{E}_{p+1}^{m+1} \mathrm{id}_X$ in $A[\mathrm{id}_X : W]$, then either a ends in id_X or a is both. In the former case a ends in all cells of $\widetilde{X}_e$ or in all cells of $\widetilde{X}_b$, so $\mathrm{dom}(A[\mathrm{id}_X : W]) \cong \mathrm{dom}(A)$, and in the latter case a ends in all cells of $\widetilde{X}_e$ *and* $\widetilde{X}_b$, so $\mathrm{dom}(A[\mathrm{id}_X : W]) = \mathrm{dom}(A)$. If there is no $a \in A$ such that $a \mathsf{E}_{p+1}^{m+1} \mathrm{id}_X$ in $A[\mathrm{id}_X : W]$ then $\mathrm{id}_X \in \mathrm{dom}(A[\mathrm{id}_X : W])$, and if $A[\mathrm{id}_X : W]$ is well formed X_b and $X_e \subseteq \mathrm{dom}(A[\mathrm{id}_X : W])$ as well because of the second condition of elementary insertability. Furthermore, elementary insertability of id_X in A implies that id_X is elementary insertable in $\mathrm{dom}(A)$, and comparing the E and B relations it follows that $\mathrm{dom}(A[\mathrm{id}_X : W]) = \mathrm{dom}(A)[\mathrm{id}_X : W|_{\mathrm{dom}(A)}]$, where the witnessing specifications $W|_{\mathrm{dom}(A)}$ consist of the part of the witnessing specifications W that live in $\mathrm{dom}(A)$. Continuing this for lower dimensions, it follows that for every $m \geq p+1$ either $s_m(A[\mathrm{id}_X : W]) \cong s_m(A)$ or $s_m(A[\mathrm{id}_X : W]) = s_m(A)[\mathrm{id}_X : W|_{s_m(A)}]$, depending on whether there exists an $a \in A$ such that $a \mathsf{E}_{p+1}^{m+1} \mathrm{id}_X$ in $A[\mathrm{id}_X : W]$ or not.

If $\dim(A) = p+1$ then id_X and $\widetilde{X}_e$ are not in $\mathrm{dom}(A[\mathrm{id}_X : W])$, so this last one will be isomorphic to $\mathrm{dom}(A)$ if A contains $(p+1)$-dimensional cells, or isomorphic to A if A is p-dimensional. Consequently, for every $m < p+1$, $s_m(A[\mathrm{id}_X : W]) \cong s_m(A)$.

6.5 A after $s_p(A)$

An important situation of an insertion is where $X = s_p(A)$ and the witnessing specifications are all "neither of these" or "after $s_p(A)$", and the dual situation of this. Insertions of this kind will be used in composing pasting schemes, see

sections 8 and 9. I will show that, provided that X is round, which will be needed later on anyway, the result of such an insertion is a well-formed loop-free pasting scheme.

Let A be a well-formed loop-free pasting scheme such that $s_p(A)$ is round. Define a collection of witnessing specifications W^{af} by specifying that all cells $a \in A$ of dimension greater than $p+1$ that satisfy $s_p(A) \subseteq \mathsf{R}(a)$ but not $a\mathsf{D}x$ for any $x \in s_p(A)$ are neither of these, and by specifying that all cells $a \in A - \widetilde{s_p(A)}$ which satisfy $a\mathsf{D}x$ for some $x \in s_p(A)$ are after $s_p(A)$. Dually, for round $t_p(A)$ the collection of witnessing specifications W^{be} is defined, where all cells are either "neither of these" or "before $t_p(A)$".

Lemma 6.2 $\mathrm{id}_{s_p(A)}$ *is elementary insertable in A at W^{af}.*

Proof. For $x \in s_p(A)$ there is no $a \in A$ with $a\mathsf{D}_j^{p+2}x$ since $s_p(A) \subseteq s_{p+1}(A)$ and $\subseteq t_{p+1}(A)$, so there is nothing to check. □

Proposition 6.3 $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$ *is a pasting scheme.*

Proof. Pasting axiom (i) holds by definition, and pasting axiom (ii) is straightforward.

For pasting axiom (iii), take $x \in (A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}])_i$. If $x = \mathrm{id}_{s_p(A)}$ then take y_e for any p-dimensional $y \in s_p(A)$, and case 6 applies. If $x \in \widetilde{s_p(A)_e}$ then find y in $s_p(A)$ and case 7 applies or case 9 applies to y_e. If $x \in A - \widetilde{s_p(A)}$, find y in A, if $y \notin \widetilde{s_p(A)}$ then case 1 applies, otherwise take y_e and case 3 applies.

For the "$\Rightarrow$" part of pasting axiom (iv), suppose that $w\mathsf{E}_j^i x$ in $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$. First consider case 1. If $w \in s_p(A)$ then $x \in s_p(A)$. Find u and v in $s_p(A)$, they are not in $\widetilde{s_p(A)}$ so case 1 applies twice. If $w \notin s_p(A)$ then find u and v in A. If $\dim w > p+1$ then u and $v \notin \widetilde{s_p(A)}$ so case 1 applies twice. If $\dim w \leq p+1$ then if $u \notin \widetilde{s_p(A)}$ case 1 applies twice, otherwise take u_e and cases 3 and 7 apply because w is after $s_p(A)$. This finishes case 1. In case 3 $w \notin s_p(A)$, find u and v in A, then they are not in $s_p(A)$ as well, so cases 1 and 3 apply because u and v are after $s_p(A)$. This finishes case 3. In case 6 $x \in \widetilde{s_p(A)}$ implies the existence of u and v, and cases 6 and 9 apply to u_e and v_e. Cases 2 and 4 don't occur, while the other cases are trivial or can be done in $s_p(A)$.

For the "$\Leftarrow$" part of pasting axiom (iv), suppose that $w\mathsf{E}_{i-1}^i u\mathsf{E}_j^{i-1}x$ and $w\mathsf{E}_{i-1}^i v\mathsf{B}_j^{i-1}x$ in $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$. u, v and x cannot be $\mathrm{id}_{s_p(A)}$ because no cell of A ends in $\mathrm{id}_{s_p(A)}$ or is both, so the only non-trivial case is $w = \mathrm{id}_{s_p(A)}$.

Then u and v are in $\widetilde{s_p(A)}_e$, and x must be in $s_p(A)$. But u and v make that $x \notin s_{p-1}(s_p(A)) \cup t_{p-1}(s_p(A))$, so $x \in \widetilde{s_p(A)}_e$ and cases 6 and 9 apply.

For pasting axiom (v), suppose that $w\mathsf{E}^i_{i-1}u\mathsf{E}^{i-1}_j x$. The only interesting case is when $w = \mathrm{id}_{s_p(A)}$. If $x \in \widetilde{s_p(A)}_e$ then case 6 applies, if not then case 8 applies to u_b. □

It follows from lemma 6.2 and from the previous calculations on s_m and t_m that for every $m \geq p+1$, $s_m(A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]) = s_m(A)[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$, and for every $m < p+1$, $s_m(A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]) \cong s_m(A)$.

Lemma 6.4 *The pasting scheme $A[id_{s_p(A)}\colon W^{\mathrm{af}}]$ has no direct loops.*

Proof. There are two interesting cases, namely walking through $\mathrm{id}_{s_p(A)}$ and through cells in $\widetilde{s_p(A)}_e$ or $\widetilde{s_p(A)}_b$. There are no $(p+1)$-cells ending in an element of $\widetilde{s_p(A)}_b$, so if $a \triangleleft_{A[id_{s_p(A)}:W^{\mathrm{af}}]} b$ via $\mathrm{id}_{s_p(A)}$ then $a = \mathrm{id}_{s_p(A)}$. For the same reason, $\mathsf{E}(b) \cap \widetilde{s_p(A)}_b = \varnothing$, as is then required. Also, $\mathrm{id}_{s_p(A)}$ cannot occur in a sequence of dimension $p+2$ since there are no $(p+2)$-cells beginning or ending in $\mathrm{id}_{s_p(A)}$.

If there is a direct loop in $A[id_{s_p(A)}\colon W^{\mathrm{af}}]$ containing elements of $\widetilde{s_p(A)}_e$ or $\widetilde{s_p(A)}_b$, then this gives a direct loop in A by replacing every x_e or x_b by x. □

Proposition 6.5 *The pasting scheme $A[id_{s_p(A)}\colon W^{\mathrm{af}}]$ is well formed.*

Proof. The proof falls apart in two: that $s_m(A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}])$ is a subpasting scheme of $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$, and that $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$ is compatible. The description of $s_m(A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}])$ above then gives that for every $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$, every $s_m(A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}])$ is a compatible subpasting scheme.

For the first part, note that the E and B relations on $s_m(A)[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$ and on $s_m(A)$ are the same as on $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$. In all cases that occur it can be used that $s_m(A)$ is a subpasting scheme of A, except in case 6 where one uses that $s_p(A) \subseteq s_m(A)$ for $m \geq p$.

Compatibility is only non-trivial in dimension $p+1$ because $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$ has at least dimension $p+1$ and there are no $(p+2)$-cells $a \in A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$ with $a\mathsf{D}\,\mathrm{id}_{s_p(A)}$. The only interesting case in dimension $p+1$ is the combination of 3 and 6, but since there are no $(p+1)$-cells $a \in A$ with $a\mathsf{E}x$ for some $x \in s_p(A)$, this case is void. □

Lemma 6.6 *For every $a \in A[id_{s_p(A)}\colon W^{\mathrm{af}}]$, $\mathsf{R}(a)$ is well formed.*

Proof. For $a \neq \mathrm{id}_{s_p(A)}$, $\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}(a) \cong \mathsf{R}_A(a)$, which is well formed.

For $a = \mathrm{id}_{s_p(A)}$, $\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}(a)$ has only one top-dimensional cell, and $\mathrm{dom}\,\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}(a) \cong s_p(A) \cong \mathrm{cod}\,\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}$, using roundness, which is well formed as well. □

Proposition 6.7 *The pasting scheme $A[id_{s_p(A)}\colon W^{\mathrm{af}}]$ is loop free.*

Proof. Conditions (i) and (ii) are lemmas 6.4 and 6.6, so this leaves condition (iv).

If $x \in \widetilde{s_p(A)}_e$ then $s_j(\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}(x)) = s_j(\mathsf{R}_A(x))_e$. For u and u' inthere, condition (iv) in A gives that any $v \in A$ with $u \triangleleft_A v \triangleleft_A u'$ must be in $s_j(\mathsf{R}_A(x))$. In this dimension $\triangleleft_{A[id_{s_p(A)}:W^{\mathrm{af}}]}$ implies $\triangleleft_A$, which gives that any $v \in A[id_{s_p(A)}\colon W^{\mathrm{af}}]$ with $u \triangleleft_{A[id_{s_p(A)}:W^{\mathrm{af}}]} v \triangleleft_{A[id_{s_p(A)}:W^{\mathrm{af}}]} u'$ must be in $s_j(\mathsf{R}_{A[id_{s_p(A)}:W^{\mathrm{af}}]}(x))$, unless $v \in \widetilde{s_p(A)}_b$. But this v_b cannot be immediately $\triangleleft$-after u since in that case apparently v_b and v_e are in the well-formed j-dimensional subpasting scheme Y of $A[id_{s_p(A)}\colon W^{\mathrm{af}}]$, both beginning in some $z \in s_{p-1}(s_p(A))$ or $t_{p-1}(s_p(A))$, contradicting well-formedness of Y. Induction on the length of $u \triangleleft v_b$ now finishes the case $x \in \widetilde{s_p(A)}_e$.

If $x = \mathrm{id}_{s_p(A)}$ and $j = p$ then $s_j(\mathsf{R}(x)) = s_p(A)_b$. Consider again v immediately $\triangleleft$-after u via z, say. u must be in $\widetilde{s_p(A)}_b$, if $z \in \widetilde{s_p(A)}_b$ then v must be inthere as well because $s_p(A)_b \subseteq Y$, and if $z \notin \widetilde{s_p(A)}_b$ then $z \in s_{p-1}(s_p(A)) = s_{p-1}(A)$ so u' will not be reached. If $x = \mathrm{id}_{s_p(A)}$ and $j < p$ then $s_j(\mathsf{R}(x)) = s_j(A)$ as subpasting scheme of $A[\mathrm{id}_{s_p(A)}\colon W^{\mathrm{af}}]$. Then for v immediately $\triangleleft$-after u via z, say, lemma 4.2 gives a $v' \in s_j(A)$ beginning in z. By compatibility, this must be v itself, which is thus in $s_j(A)$.

If x is outside $\mathsf{R}(\mathrm{id}_{s_p(A)})$ then $\mathrm{id}_{s_p(A)}$ cannot play a rôle because there are no cells beginning or ending in it, and there are no $(p+1)$-cells ending in cells in $s_p(A)_b$. If cells of $s_p(A)_b$ or $s_p(A)_e$ play a rôle, do the argument in A, and eliminate the possibility $v \in s_p(A)_b$ as in the first case. □

6.6 Full insertability

To be able to compare composites of realizations of A and $A[\mathrm{id}_X\colon W]$, I will need that all cells of X indeed occur around the cell id_X in $A[\mathrm{id}_X\colon W]$. This will be

the case if X is required to be round.

Definition 6.8 id_X *is fully insertable at* W *in* A if id_X is elementary insertable in A, X is round and $A[\mathrm{id}_X\colon W]$ is a well-formed loop-free pasting scheme. $\diamond$

Lemma 6.9 $\mathrm{id}_{s_p(A)}$ *is fully insertable at* W^{af} *in* A.

Proof. Elementary replaceability is lemma 6.2, roundness is assumed, and well-formed loop-free pasting scheme is propositions 6.3 to 6.7. $\square$

6.7 Subpasting schemes of $A[\mathrm{id}_X\colon W]$

Before considering realizations and their composites, I need to look at well-formed subpasting schemes of $A[\mathrm{id}_X\colon W]$. As in the case of replacements, I want to relate these to subpasting schemes of A.

Let Y' be a well-formed subpasting scheme of $A[\mathrm{id}_X\colon W]$. Take $Y = ((A - \widetilde{X}) \cap Y') \cup \{x \in \widetilde{X} | x_e \in Y'\} \cup \{x \in \widetilde{X} | x_b \in Y'\}$, which is a subgraded set of A. It will be convenient to denote it by $Y'[-\mathrm{id}_X]$.

Lemma 6.10 *For every* Y', $s_m(Y'[-\mathrm{id}_X]) = s_m(Y')[-\mathrm{id}_X]$.

Proof. Straightforward, but lengthy. $\square$

Lemma 6.11 Y *is a well-formed subpasting scheme of* A. *Moreover, if* $\mathrm{id}_X \in Y'$ *then* $Y[\mathrm{id}_X\colon W|_Y] = Y'$.

Proof. Subpasting scheme: suppose $y\mathsf{E}z$ in A for some $y \in Y$, to show that $z \in Y$. If z, being in A, is not in $\widetilde{X}$, then $y\mathsf{E}z$ in $A[\mathrm{id}_X\colon W]$, so $z \in (A - \widetilde{X}) \cap Y'$ because Y' is a subpasting scheme of $A[\mathrm{id}_X\colon W]$. If $z \in \widetilde{X}$ then, depending on whether y is after X, before X or both, $y\mathsf{E}z_e$, $y\mathsf{E}z_b$ in $A[\mathrm{id}_X\colon W]$, or both. In any case z will be in Y.

Well formed: because of the formula for s_m in lemma 6.10, I only need to show compatibility of Y for every Y'. So suppose a and a' are top dimensional in Y and end in the same cell z. Then they are either in $A - \widetilde{X}$ or in $\widetilde{X}$, and these possibilities are there as well for z. If $z \in A - \widetilde{X}$ then no matter where a and a' come from in Y', they both give rise to cells ending in z, contradicting compatibility of Y'. If $z \in \widetilde{X}$ then it can come from X_e or from X_b, but in both cases, and no matter where a and a' are, the condition that Y' is a subpasting scheme of $A[\mathrm{id}_X\colon W]$ implies that such a situation occurs in Y' as well, and

whether it is with a, a_e or a_b is not important because in all cases there is a contradiction with compatibility of Y'.

Now Y is a well-formed loop-free pasting scheme, and if $\mathrm{id}_X \in Y'$ then id_X is fully insertable in Y at $W|_Y$ because it is fully insertable in A at W. Inspecting the definition of $Y[\mathrm{id}_X : W|_Y]$ shows it is exactly Y'. □

As in the case of replacements, composition in the ω-category of components of $A[\mathrm{id}_X : W]$ is related to composition in the ω-category of components of A:

Lemma 6.12 *Let Y' and Z' be subpasting schemes of $A[\mathrm{id}_X : W]$ such that $t_m(Y') = s_m(Z')$. Then $t_m(Y'[-\mathrm{id}_X]) = s_m(Z'[-\mathrm{id}_X])$, and $(Y' \cup Z')[-\mathrm{id}_X] = Y'[-\mathrm{id}_X] \cup Z'[-\mathrm{id}_X]$.*

Proof. Easy, using lemma 6.10 and the definition of $Y'[-\mathrm{id}_X]$. □

The following lemma describes $\mathsf{R}(a)$ for $a \in A[\mathrm{id}_X : W]$, which will be needed shortly.

Lemma 6.13 *If $a \in A - \widetilde{X}$ then $(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X] = \mathsf{R}_A(a)$. If $a = x_e$ then $(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X] = \mathsf{R}_A(x)$. If $a = \mathrm{id}_X$ then $(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X] = X$.*

Proof. Just write it out. Roundness of X ensures that indeed all cells of X appear around the cell id_X. □

6.8 Realizing and composing $A[\mathrm{id}_X : W]$

To compare composites of related realizations of A and $A[\mathrm{id}_X : W]$, realizations of the latter one will be obtained from the former one. The realizations will have equal composite.

Suppose an appropriate realization (A, f_i) of A in some ω-category $\mathbb{C}$ given. This information induces a realization $(A[\mathrm{id}_X : W], f'_i)$ of $A[\mathrm{id}_X : W]$ in $\mathbb{C}$, given by

$$\begin{array}{rcll} f'_i(a) & = & f_i(a) & \text{if } a \in A - \widetilde{X} \\ & = & \mathrm{id}_{f(X)} & \text{if } a = \mathrm{id}_X \\ & = & f_i(x) & \text{if } a = x_e \text{ or } a = x_b \text{ for some } x \in \widetilde{X} \end{array}$$

Proposition 6.14 *The realization $(A[\mathrm{id}_X : W], f'_i)$ defined above is appropriate, and $f'(A[\mathrm{id}_X : W]) = f(A)$.*

Proof. By induction. I will prove the following induction hypothesis: the realization $(A[\mathrm{id}_X : W], f'_i)$ is n-appropriate with n-extension $f' : |\mathcal{P}(A[\mathrm{id}_X : W])|_n \to \mathbb{C}$ given by

$$f'(Y') = f(Y'[-\mathrm{id}_X]).$$

Realizations are always 0-appropriate, so if $n = 0$ I only need to show the formula for the 0-extension f' of $(A[\mathrm{id}_X : W], f'_i)$. But well-formed 0-dimensional pasting schemes are singletons, and identities are at least one-dimensional, so this is immediate.

Now suppose the induction hypothesis holds up to n, I will first show that $(A[\mathrm{id}_X : W], f'_i)$ is $(n+1)$-appropriate. For $a = \mathrm{id}_X$,

$$\begin{aligned}
& f'\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))\right) = \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X]\right) \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X])\right) && \text{because of lemma 6.10} \\
&= f(X) && \text{because of lemma 6.13} \\
&= s_n(\mathrm{id}_{f(X)}) && \text{because } f \text{ is an } \omega\text{-functor} \\
&= s_n(f'_{n+1}(a)) && \text{by definition of } f'_i.
\end{aligned}$$

For $a = x_e$ or $a = x_b$,

$$\begin{aligned}
& f'\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))\right) = \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X]\right) \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X])\right) && \text{because of lemma 6.10} \\
&= f\left(s_n(\mathsf{R}_A(x))\right) && \text{because of lemma 6.13} \\
&= s_n(f_{n+1}(x)) && \text{because } f \text{ extends } (A, f_i) \\
&= s_n(f'_{n+1}(a)).
\end{aligned}$$

For $a \in A - \widetilde{X}$,

$$\begin{aligned}
& f'\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))\right) = \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a))[-\mathrm{id}_X]\right) \\
&= f\left(s_n(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X])\right) && \text{because of lemma 6.10} \\
&= f\left(s_n(\mathsf{R}_A(a))\right) && \text{because of lemma 6.13} \\
&= s_n(f_{n+1}(a)) && \text{because } f \text{ extends } (A, f_i) \\
&= s_n(f'_{n+1}(a)).
\end{aligned}$$

This proves $(n+1)$-appropriateness of $(A[\mathrm{id}_X : W], f'_i)$.

Next, the formula for its $(n+1)$-extension f'. Thus, I will show that it is the unique ω-functor $|\mathcal{P}(A[\mathrm{id}_X\colon W])|_{n+1} \to \mathbb{C}$ extending $(A[\mathrm{id}_X\colon W], f')$. It is an ω-functor because for Y' a well-formed subpasting scheme of $A[\mathrm{id}_X\colon W]$,

$$\begin{array}{rcll} s_m(f'(Y')) & = & s_m(f(Y'[-\mathrm{id}_X])) & \\ & = & f(s_m(Y'[-\mathrm{id}_X])) & \text{because } f \text{ is an } \omega\text{-functor} \\ & = & f(s_m(Y')[-\mathrm{id}_X]) & \text{because of lemma 6.10} \\ & = & f'(s_m(Y')) & \end{array}$$

and because if $t_m(Y') = s_m(Z')$ then

$$\begin{array}{rcll} f'(Y' \cup Z') & = & f((Y' \cup Z')[-\mathrm{id}_X]) & \\ & = & f(Y'[-\mathrm{id}_X] \cup Z'[-\mathrm{id}_X]) & \text{by lemma 6.12} \\ & = & f(Z'[-\mathrm{id}_X]) \circ_m f(Y'[-\mathrm{id}_X]) & \text{because } f \text{ is an } \omega\text{-functor} \\ & = & f'(Z') \circ_m f'(Y'), & \end{array}$$

and because there is nothing to do for identities since $|\mathcal{P}(A[\mathrm{id}_X\colon W])|_{n+1}$ and $\mathbb{C}$ are considered one-sorted. It extends $(A[\mathrm{id}_X\colon W], f')$ because for $a = \mathrm{id}_X$,

$$\begin{array}{rcll} f'(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)) & = & f(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X]) & \\ & = & f(X) & \text{by lemma 6.13} \\ & = & \mathrm{id}_{f(X)} & \text{considering } \mathbb{C} \text{ one-sorted} \\ & = & f'_{n+1}(a), & \end{array}$$

for $a = x_e$ or $a = x_b$,

$$f'(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)) = f(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X]) = f(\mathsf{R}_A(x)) = f_{n+1}(x) = f'_{n+1}(a),$$

and for $a \in A - \widetilde{X}$,

$$f'(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)) = f(\mathsf{R}_{A[\mathrm{id}_X:W]}(a)[-\mathrm{id}_X]) = f(\mathsf{R}_A(a)) = f_{n+1}(a) = f'_{n+1}(a).$$

Uniqueness is immediate from the $(n+1)$-appropriateness of $(A[\mathrm{id}_X\colon W], f'_i)$ since that implies $(n+1)$-extendability. $\square$

7 Generators and relations

After the technical preparations of the previous sections, I can now define what data constitute a pasting presentation.

Definition 7.1 A labeled pasting scheme $(A, \mathfrak{L})$ is a well-formed loop-free pasting scheme A with labels attached via a function $\mathfrak{L} : A \to L$ for some set L such that each two cells a and a' having the same label are of equal dimension. $\diamond$

Usually the set L in the above definition will not be specified. It is *not* required that $\mathsf{R}(a) = \mathsf{R}(a')$ for cells a and a' having equal labels.

The notion of pasting presentation will be defined inductively. A pasting presentation will consist of generators, which give rise to generated pastings, and of relations between generated pastings, which give rise to an equivalence relation on the collection of generated pastings. The conditions are to ensure that the data are compatible with the data in lower dimensions.

Definition 7.2

- A *generator* of dimension n is an n-dimensional labeled pasting scheme $(A, \mathfrak{L})$ with one top-dimensional cell g such that $A = \mathsf{R}(g)$, and such that $\mathrm{dom}(A)$ and $\mathrm{cod}(A)$ are generated pastings of dimension $n-1$. The label of the top-dimensional cell must determine the generator uniquely.

- Given a set of generators of dimension $\leq n$, and a set of relations between generators of dimension $< n$, a *generated pasting* of dimension n is an n-dimensional labeled pasting scheme such that each i-dimensional cell a is labeled with the name of an i-dimensional generator g where $\mathrm{dom}(\mathsf{R}(a))$ and $\mathrm{dom}(\mathsf{R}(g))$ (resp. $\mathrm{cod}(\mathsf{R}(a))$ and $\mathrm{cod}(\mathsf{R}(g))$) with induced labeling are equivalent generated pastings of dimension $i-1$, *or* with a *new* label $\mathrm{id}_{\overline{X}}$ where X is a generated pasting of dimension $i-1$ which is equivalent to both $\mathrm{dom}(\mathsf{R}(a))$ and $\mathrm{cod}(\mathsf{R}(a))$ (it is understood that if $i = 0$ then the condition on domain and codomain of a is void).

- A *relation* between two generated pastings $(A, \mathfrak{L})$ and $(A', \mathfrak{M})$ of dimension n can exist if A is fully replaceable by A', and the labelings on $\mathrm{dom}(A)$ and $\mathrm{dom}(A')$ resp. on $\mathrm{cod}(A)$ and $\mathrm{cod}(A')$ coincide.

- Two generated pastings of dimension n are *equivalent* if they are so in the equivalence relation generated by $\sim$, which is defined by $(A, \mathfrak{L}) \sim (A', \mathfrak{M})$ if

  - X is a well-formed subpasting scheme of the generated pasting A, X is related to X' via one of the given relations, X is fully replaceable by X' in A, and $A' \cong A[X'/X]$, or

– X is a well-formed subpasting scheme of the generated pasting A, id_X is fully insertable in A, $A' \cong A[\mathrm{id}_X : W]$ and id_X, as an element of A', is labeled by $\mathrm{id}_{\overline{X}}$.

The equivalence class of the generated pasting $(A, \mathfrak{L})$ will be denoted by $\overline{(A, \mathfrak{L})}$ or $\overline{A, \mathfrak{L}}$ or $\overline{A}$.

A *pasting presentation* $(\underline{G}, \underline{R})$ is a set $\underline{G}$ of generators and a set $\underline{R}$ of relations.
$\diamond$

Two trivial properties of generated pastings for later use:

Lemma 7.3 *Every generator, with the top-dimensional cell labeled by the name of the generator, is a generated pasting.*

Proof. Every lower dimensional cell is labeled correctly because $\mathrm{dom}(\mathsf{R}(g))$ and $\mathrm{cod}(\mathsf{R}(g))$ are generated pastings, and the top-dimensional cell is labeled correctly because the domain and the codomain of the cell are indeed equivalent, even equal, to the domain and the codomain of the label. $\square$

Lemma 7.4 *A well-formed subpasting scheme of a generated pasting, with its induced labeling, is a generated pasting.*

Proof. If all cells are labeled correctly, then certainly some subset of them.
$\square$

8 Globularizations

In this section I will describe a way of obtaining a representative of the equivalence class of a generated pasting which is suitable for composing generated pastings by pasting. The idea is to insert lots of identities around the pasting scheme, in such a way that the copies of the subpasting schemes on which the identities are inserted can be treated independently, thereby making it possible to replace these by other, equivalent, pasting schemes. This will appear to be extremely useful, not only in defining composition, but also in proving its associativity and the interchange law, in section 9.

There will be three types of globularizations: a left one, a right one, and a middle one. It would be possible to use only the middle one, but in order to build it one comes across the other ones as well.

8.1 Inserting lots of identities

To define the globularizations of a well-formed loop-free pasting scheme A, I will start with inserting lots of identities in A. They will be inserted around A, that is, before k-sources and after k-targets, as follows. Define well-formed loop-free pasting schemes $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k for all $k \leq n$ by

$$\begin{array}{lcl} A^0 & = & A, \\ A^{k,\mathfrak{l}} & = & A^k[\mathrm{id}_{s_k(A^k)}\colon W^{\mathrm{af}}], \\ A^{k,\mathfrak{r}} & = & A^k[\mathrm{id}_{t_k(A^k)}\colon W^{\mathrm{be}}], \\ A^{k+1} & = & A^{k,\mathfrak{r}}[\mathrm{id}_{s_k(A^{k,\mathfrak{r}})}\colon W^{\mathrm{af}}], \\ & \cong & A^{k,\mathfrak{l}}[\mathrm{id}_{t_k(A^{k,\mathfrak{l}})}\colon W^{\mathrm{be}}]. \end{array}$$

These are all well-formed loop-free pasting schemes because previous steps have ensured that the subpasting schemes on which identities are inserted are round, so that they are fully insertable, and propositions 6.3 to 6.7 can be used. They will be labeled by giving a cell which is a copy of a cell in A the same label as in A, and by giving identities on k-sources and on k-targets the labels $\mathrm{id}_{s_k(A)}$ and $\mathrm{id}_{t_k(A)}$ respectively.

Lemma 8.1 *The labeled pasting schemes $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k are all generated pastings.*

Proof. I have to show that the labeling of the identities is justified. Thus, I have to show that $s_k(A^{k,\mathfrak{r}})$ and $s_k(A^k)$ are equivalent to $s_k(A)$, and that $t_k(A^{k,\mathfrak{l}})$ and $t_k(A^k)$ are equivalent to $t_k(A)$. This follows, using induction on k, from the statement that for every $m \leq k$, $s_m(A^{k,\mathfrak{r}})$ is equivalent to $s_m(A^k)$, $s_m(A^{k+1})$ is equivalent to $s_m(A^{k,\mathfrak{r}})$ and to $s_m(A^{k,\mathfrak{l}})$, $t_m(A^{k,\mathfrak{r}})$ is equivalent to $t_m(A^k)$, and $t_m(A^{k+1})$ is equivalent to $t_m(A^{k,\mathfrak{r}})$ and to $t_m(A^{k,\mathfrak{l}})$. But this follows directly from the description of $s_m(A[\mathrm{id}_X\colon W])$ and $t_m(A[\mathrm{id}_X\colon W])$ in section 6. □

Lemma 8.2 *A is equivalent to all the $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k.*

Proof. Trivial. □

8.2 Pasting schemes from sets of pasting schemes

In the pasting schemes $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k the copies of the j-sources and j-targets of A are independent of each other, precisely because there are lots of identities

separating them. To describe this situation in precise terms, consider a set $\mathcal{A}$ of $2k+2$ well-formed loop-free pasting schemes, two of them, say A_{j-} and A_{j+}, having dimension j for every $j < k$, one, say A_{k-}, having dimension k and one, say A, having dimension at least k. For example, every well-formed loop-free pasting scheme A gives rise to such an $\mathcal{A}$ by taking $A_{j-} = s_j(A)$, $A_{j+} = t_j(A)$, and $A = A$. Now define a graded set $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ consisting of:

- the $A_{j\alpha}$'s and A,
- for every $q \leq k$ a $(q+1)$-dimensional cell, denoted $\mathrm{id}_{A_{q-}}$,
- for every $q < k$ a $(q+1)$-dimensional cell, denoted $\mathrm{id}_{A_{q+}}$,
- for every $q < q' \leq k$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}^{q'-}_{A_{q-}}$ and $\mathrm{id}^{q'-}_{A_{q+}}$,
- for every $q < q' < k$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}^{q'+}_{A_{q-}}$ and $\mathrm{id}^{q'+}_{A_{q+}}$.

Define relations E and B on $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ by requiring that for $j < i$, $(c, c') \in \mathsf{E}^i_j$ if and only if one of the following:

1. $c \in A$, $c' \in A$, $\quad c\mathsf{E}^i_j c'$ in A,
2. $c \in A_{q\alpha}$, $c' \in A_{q\alpha}$ for $\alpha = \pm$, $\quad c\mathsf{E}^i_j c'$ in $A_{q\alpha}$,
3. $c = \mathrm{id}_{A_{(i-1)-}}$, $c' \in s_{i-1}(A)$ or $c' = \mathrm{id}_{A_{(j-1)\alpha}}$ for $\alpha = \pm$,
4. $c = \mathrm{id}^{q'\beta}_{A_{(i-1)-}}$, $c' \in s_{i-1}(A_{q'\beta})$ or $c' = \mathrm{id}^{q'\beta}_{A_{(j-1)\alpha}}$ for α, $\beta = \pm$,
5. $c = \mathrm{id}_{A_{(i-1)+}}$ or $c = \mathrm{id}^{q'\beta}_{A_{(i-1)+}}$, $c' \in A_{(i-1)+}$ or $c' = \mathrm{id}^{(i-1)+}_{A_{(j-1)\alpha}}$ for α, $\beta = \pm$.

From almost the same data one can also define $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and $\mathrm{Gl}_{k+1}(\mathcal{A})$. The first one is dual to $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, and the second one differs from $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ only in also having A_{k+}, $\mathrm{id}_{A_{k+}}$ and all $\mathrm{id}^{k+}_{A_{q\alpha}}$'s. See figure 1 for a picture of $\mathrm{Gl}_4(\mathcal{A})$.

In the above example, $\mathrm{Gl}_k(\mathcal{A})$ is exactly A^k and $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ is exactly $A^{k,\mathfrak{l}}$.

In the next few propositions I will show that all $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and $\mathrm{Gl}_k(\mathcal{A})$ are well-formed loop-free pasting schemes.

Proposition 8.3 *All* $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ *and* $\mathrm{Gl}_k(\mathcal{A})$ *are pasting schemes.*

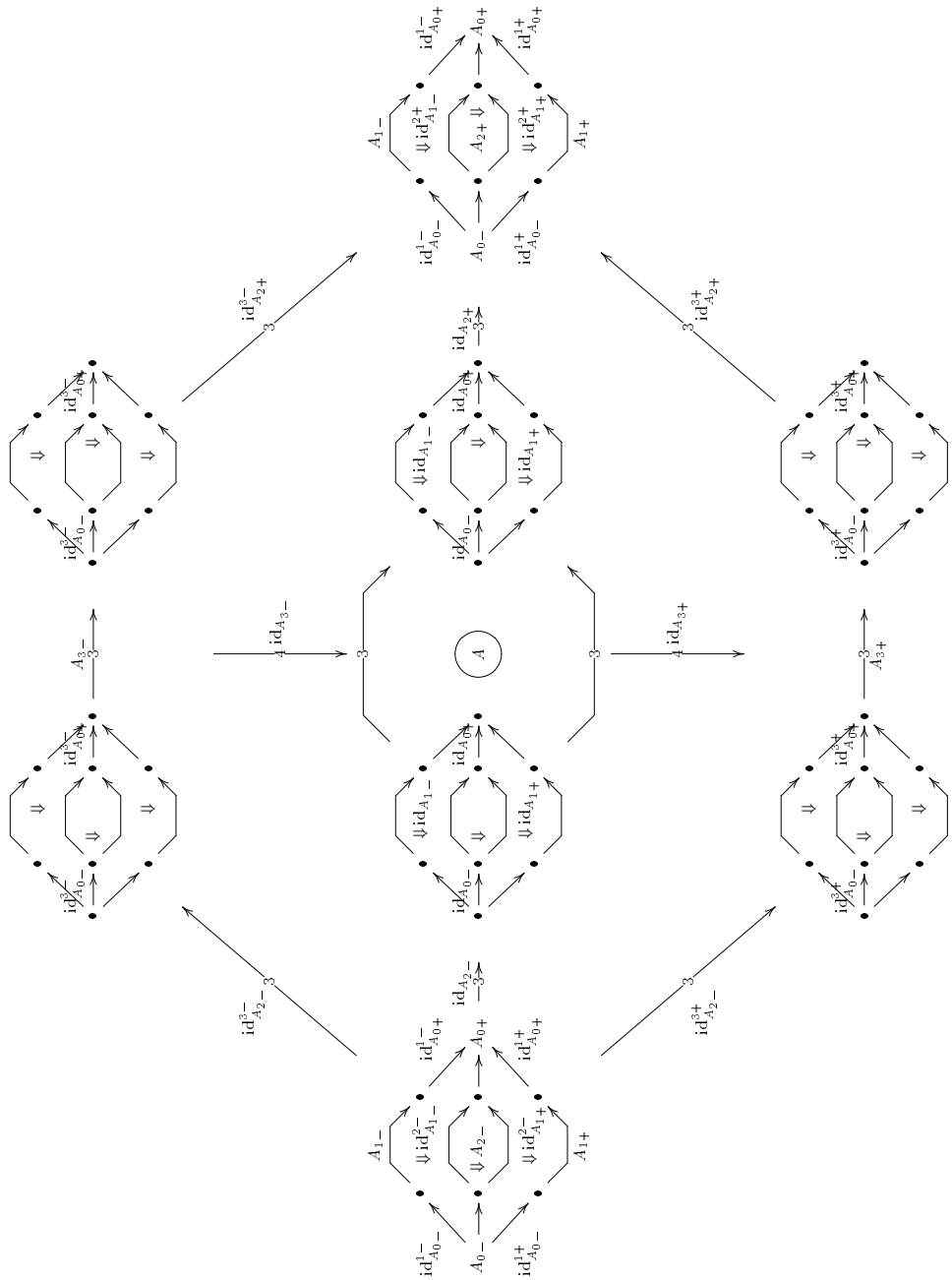


Figure 1: $\mathrm{Gl}_k(\mathcal{A})$ for $k = 4$

Proof. First note that when w is in A or in $A_{q\alpha}$ everything follows from these being pasting schemes. So I only have to consider the cases where w is an identity.

Pasting axiom (iii): the definition of E gives that other identities can be taken for y.

Pasting axiom (iv), "$\Rightarrow$": if w is an identity and x is an identity then x is in between two identities, if x is a cell of A or of $A_{q\alpha}$ then x is in between two cells there if they exist, otherwise x must be in s_{q-1} or t_{q-1} and then if $x \notin s_{q-2}$ and $\notin t_{q-2}$ then x is in between one identity and one cell of A or $A_{q\alpha}$, otherwise x is in between two identities.

Pasting axiom (iv), "$\Leftarrow$": identities are only in between identities, and for x in A or in $A_{q\alpha}$, if u or v are inthere as well then okay, and if u and v are both identities then okay as well.

Pasting axiom (v): if x in A or in $A_{q\alpha}$ then if $q \leq \dim(w) - 1$ the second case holds, otherwise the first case holds. The same is the case for identities: the same division is made, this time looking at the upper index of the identity. $\square$

I will now analyse $a \triangleleft b$ for a and b in $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ or $\mathrm{Gl}_k(\mathcal{A})$. This can be completely in one of the $A_{q\alpha}$, or a or b or both are an identity, and then a must be $\mathrm{id}_{A_{q-}}$ for some q, possibly with an upper index, and similarly b must be $\mathrm{id}_{A_{q+}}$. In all cases, leaving out the identities leaves a sequence in only one $A_{q\alpha}$, or no sequence at all.

Lemma 8.4 *All* $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ *and* $\mathrm{Gl}_k(\mathcal{A})$ *have no direct loops.*

Proof. Obvious, since the $A_{q\alpha}$'s don't have them, and identities begin and end in different $A_{q\alpha}$'s and in different identities. $\square$

Before well-formedness, s_m must be considered. For $m > k$ these are $\mathrm{Gl}_k^{\mathfrak{l}}(s_m(A))$, $\mathrm{Gl}_k^{\mathfrak{r}}(s_m(A))$ and $\mathrm{Gl}_k(s_m(A))$ respectively. $s_k(\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})) = \mathrm{Gl}_k(A_{k-})$ and $s_k(\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})) = s_k(\mathrm{Gl}_k(\mathcal{A})) = \mathrm{Gl}_k(s_k(A))$. If $m < k$ then $s_m(\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})) = s_m(\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})) = s_m(\mathrm{Gl}_k(\mathcal{A})) = \mathrm{Gl}_m(A_{m-})$.

For $b \in A_{q\alpha}$, $\mathsf{R}_{\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})}(b) = \mathsf{R}_{A_{q\alpha}}(b)$, so $s_m(\mathsf{R}(b))$ is as in $A_{q\alpha}$. For x an identity I only need to know $\mathrm{dom}(\mathsf{R}(x))$ and $\mathrm{cod}(\mathsf{R}(x))$. $\mathrm{dom}(\mathsf{R}(\mathrm{id}_{A_{q\beta}}^{q'\alpha}))$ is either of the form $\mathrm{Gl}_q(A_{q\beta})$ or of the form $\mathrm{Gl}_q(d_q^\beta(A_{q'\alpha}))$, and $\mathrm{dom}(\mathsf{R}(\mathrm{id}_{A_{q\beta}}))$ is either of the form $\mathrm{Gl}_q(A_{q\beta})$ or of the form $\mathrm{Gl}_q(d_q^\beta(A))$.

Proposition 8.5 *All* $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ *and* $\mathrm{Gl}_k(\mathcal{A})$ *are well formed. Moreover for every* x *in one of these,* $\mathsf{R}(x)$ *is well formed.*

Proof. The m-sources and m-targets are clearly subpasting schemes because if c' is left out then also c because it is of or mentions an even higher dimension.

For compatibility, it suffices to show that for any $\mathcal{A}$, the pasting schemes $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and $\mathrm{Gl}_k(\mathcal{A})$ are compatible. I will do $\mathrm{Gl}_k(\mathcal{A})$, the other ones go likewise. If $\dim(A) > k$ then compatibility is immediate from compatibility of A. Otherwise, i.e., if $\dim(A) = k$, the only k-dimensional cells are k-cells of A and $\mathrm{id}_{A_{(k-1)\alpha}}$. The identities end only in elements of $s_{k-1}(A)$ or of $A_{(k-1)+}$, and there are no other cells ending in these. Compatibility of A does the rest.

The moreover part of the lemma follows from this and from well-formedness of the $A_{q\alpha}$'s, noting that $\mathsf{R}(x)$ is compatible because it has only one top-dimensional cell. □

Proposition 8.6 *All* $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ *and* $\mathrm{Gl}_k(\mathcal{A})$ *are loop free.*

Proof. Conditions (i) and (ii) are lemma 8.4 and the moreover part of proposition 8.5, so this leaves condition (iv).

For $x \in A_{q\alpha}$ the analysis of $a \triangleleft b$ above gives that everything $\triangleleft$-between u and u' must be in $A_{q\alpha}$ as well. Then apply condition (iv) to $A_{q\alpha}$ seen as a well-formed subpasting scheme of itself.

For x an identity $u \triangleleft v \triangleleft u'$ cannot escape from $s_j(\mathsf{R}(x))$ because there are no j-cells outside it beginning there. □

8.3 Independency of subpasting schemes

The $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and $\mathrm{Gl}_k(\mathcal{A})$ give the general form that the $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k take by taking other pasting schemes instead of the copies of $s_j(A)$ and $t_j(A)$. The interest now lies in the case that A is a generated pasting and $A_{q\alpha}$ is a generated pasting *equivalent* to $d_q^\alpha(A)$ for every q. Then $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and $\mathrm{Gl}_k(\mathcal{A})$ can be made into labeled pasting schemes by labeling $b \in A$ and $b \in A_{q\alpha}$ with their labels in A and $A_{q\alpha}$ respectively, by labeling $\mathrm{id}_{A_{q\alpha}}$ with $\mathrm{id}_{d_q^\alpha(A)}$ and labeling $\mathrm{id}^{q'\alpha}_{A_{q''\beta}}$ with $\mathrm{id}_{d^\beta_{q''}(A)}$.

Definition 8.7 Let A be a generated pasting and let $A_{q\alpha}$ be a generated pasting equivalent to $d_q^\alpha(A)$ for every q. Then $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ is a *left k-th globularization of* A, $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ is a *right k-th globularization of* A and $\mathrm{Gl}_k(\mathcal{A})$ is a k-th globularization of A. ◇

Thus, the m-source of any left k-th globularization of A is a left k-th globularization of $s_m(A)$ if $m > k$ and a m-th globularization of $s_m(A)$ if $m \leq k$.

Proposition 8.8 *All globularizations of a generated pasting A are themselves generated pastings, which are equivalent to A.*

Proof. Induction on k. $\mathrm{dom}(\mathsf{R}(\mathrm{id}_{A_{q\alpha}}))$ is a $(q-1)$-globularization of $A_{q\alpha}$ so by the induction hypothesis is equivalent to $A_{q\alpha}$ which was assumed to be equivalent to $d_q^\beta(A)$. Thus globularizations are generated pastings.

To show globularizations to be equivalent to A, it is enough to show that if X is replaceable by X' in $A_{q\alpha}$, then also in $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, in $\mathrm{Gl}_k^{\mathfrak{r}}(\mathcal{A})$ and in $\mathrm{Gl}_k(\mathcal{A})$, and if id_X is insertable in $A_{q\alpha}$ then also in globularizations, and that the result of this replacement respectively insertion is the same as $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$, $\mathrm{Gl}_k^{\mathfrak{l}}(\mathcal{A})$ and $\mathrm{Gl}_k(\mathcal{A})$ with $A_{q\alpha}[X'/X]$ respectively $A_{q\alpha}[\mathrm{id}_X{:}W]$ instead of $A_{q\alpha}$. This is indeed sufficient since then A is equivalent to $A^{k,\mathfrak{l}}$, $A^{k,\mathfrak{r}}$ and A^k, and inthere replacements and insertions can be carried out, in any order, and the results in between are always generated pastings since they are always globularizations of A. Now the fact that if $x \in \widetilde{X}$ is in $s_m(A_{q\alpha})$ then $X \subseteq s_m(A_{q\alpha})$ makes that full replaceability and full insertability are okay, and that the result is as needed is immediate for replacements and can be arranged by choosing the right collection of witnessing specifications for insertions. □

9 The ω-category $\boldsymbol{\omega}(\underline{G}, \underline{R})$

If $(\underline{G}, \underline{R})$ is a pasting presentation, the graded set of equivalence classes of generated pastings will be denoted by $\boldsymbol{\omega}(\underline{G}, \underline{R})$. I will define operations s_m, t_m, composition and identity on $\boldsymbol{\omega}(\underline{G}, \underline{R})$, using the globularizations of the previous section. These operations will make $\boldsymbol{\omega}(\underline{G}, \underline{R})$ into an ω-category.

9.1 s_m and t_m

m-sources and m-targets of equivalence classes of generated pastings will simply be equivalence classes of m-sources and m-targets of the pasting schemes.

Lemma 9.1 *For a generated pasting $(A, \mathfrak{L})$, $(s_m(A), \mathfrak{L}|_{s_m(A)})$ is a generated pasting for any m.*

Proof. Trivial from lemma 7.4. □

For a generated pasting $(A, \mathfrak{L})$, define

$$s_m(\overline{A, \mathfrak{L}}) = \overline{s_m(A), \mathfrak{L}|_{s_m(A)}}.$$

I have to show that s_m is well-defined on $\boldsymbol{\omega}(\underline{G}, \underline{R})$. For this, I only have to check that s_m behaves well with respect to the basic equivalences $A \sim A'$.

Suppose X is a well-formed subpasting scheme of A, X is related to X' via one of the given relations, X is fully replaceable in A, and $A' = A[X'/X]$. Then if $X \subseteq s_m(A)$, X is fully replaceable in $s_m(A)$, as is easily checked, and $s_m(A') = s_m(A)[X'/X]$, and if not then $s_m(A') = s_m(A)$. In both of these cases $s_m(A') \sim s_m(A)$.

Suppose X is a well-formed subpasting scheme of A, id_X is fully insertable in A at W and $A' = A[\mathrm{id}_X : W]$ for some collection of witnessing specifications W. Then depending on whether there is an $a \in A$ such that $a \mathsf{E}_{p+1}^{m+1} \mathrm{id}_X$ in $A[\mathrm{id}_X : W]$ or not, $s_m(A[\mathrm{id}_X : W]) \cong s_m(A)$ or $s_m(A[\mathrm{id}_X : W]) = s_m(A)[\mathrm{id}_X : W|_{s_m(A)}]$. In both of these cases $s_m(A') \sim s_m(A)$.

Thus s_m is well defined. For t_m there is a dual story.

9.2 Composition

To define composition in $\boldsymbol{\omega}(\underline{G}, \underline{R})$ it is not possible to take the union of the underlying pasting schemes since if for generated pastings $(A, \mathfrak{L})$ and $(B, \mathfrak{M})$ one has $\overline{t_m(A), \mathfrak{L}|_{t_m(A)}} = \overline{s_m(B), \mathfrak{M}|_{s_m(B)}}$ then not necessarily $t_m(A) = s_m(B)$, and even if this is the case, then the union need not be a well-formed loop-free pasting scheme, as the example in section 3 shows. This problem can be avoided by choosing appropriate representing generated pastings in the equivalence classes, i.e., I will show that if $\overline{t_m(A), \mathfrak{L}|_{t_m(A)}} = \overline{s_m(B), \mathfrak{M}|_{s_m(B)}}$ then there are generated pastings A' and B' with $\overline{A'} = \overline{(A)}$, $\overline{B'} = \overline{(B)}$, $t_m(A) = s_m(B)$ and $A \cup B$ a generated pasting. Of course, A' and B' will be suitable globularizations of A and B, chosen to make the m-th target of the first match up with the m-th source of the second. I will show that the result is independent of the choices made, i.e., that the equivalence class of the result is independent of the particular representing generated pastings of $\overline{A, \mathfrak{L}}$ and $\overline{B, \mathfrak{M}}$, and independent of the particular choice for A' and B'.

Given generated pastings $(A, \mathfrak{L})$ and $(B, \mathfrak{M})$ such that $\overline{t_m(A), \mathfrak{L}|_{t_m(A)}} = \overline{s_m(B), \mathfrak{M}|_{s_m(B)}}$, define two sets of $2m+2$ well-formed loop-free pasting schemes $\mathcal{A}$ and $\mathcal{B}$ by taking $A_{q\alpha} = B_{q\alpha} = d_q^\alpha(A)$ for $q < m$, $A_{m+} = B_{m-} = t_m(A)$, $A = A$

and $B = B$. These give rise to globularizations $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ of A and B respectively, because of the well-definedness of $d_{q\alpha}\left(\overline{s_m(B)}\right)$. By proposition 8.8 A is equivalent to $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$, and B is equivalent to $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$, and almost by definition $t_m(\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}))$ is *equal* to $s_m(\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B}))$.

And now $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ can be glued together: as a graded set, $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ consists of:

- a copy of A and a copy of B,
- for every $q \leq m$ a copy of $t_q(A)$, whose elements will be written a_{q+} for $a \in t_q(A)$,
- for every $q < m$ a copy of $s_q(A)$, whose elements will be written a_{q-} for $a \in s_q(A)$,
- for every $q \leq m$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}_{t_q(A)}$ and $\mathrm{id}_{s_q(B)}$,
- for every $q < m$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}_{s_q(A)}$ and $\mathrm{id}_{t_q(B)}$,
- for every $q < q' < m$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}^{q'-}_{d^\alpha_q(A)}$ for $\alpha = \pm$,
- for every $q < q' \leq m$ two $(q+1)$-dimensional cells, denoted $\mathrm{id}^{q'+}_{d^\alpha_q(A)}$ for $\alpha = \pm$.

The relations E and B are such that for $j < i$, $(c, c') \in \mathsf{E}^i_j$ if and only if one of the following:

1. $c \in A$, $c' \in A$, $\quad c\mathsf{E}^i_j c'$ in A,
2. $c = a_{q\alpha}$, $c' = a'_{q\alpha}$ for $\alpha = \pm$, $\quad a\mathsf{E}^i_j a'$ in A,
3. $c \in B$, $c' \in B$, $\quad c\mathsf{E}^i_j c'$ in B,
4. $c = \mathrm{id}_{s_{i-1}(A)}$, $c' \in s_{i-1}(A)$ or $c' = \mathrm{id}_{d^\alpha_{j-1}(A)}$ for $\alpha = \pm$,
5. $c = \mathrm{id}^{q'\beta}_{s_{i-1}(A)}$, $c' = a'_{q'\beta}$ with $a' \in s_{j-1}(A)$ or $c' = \mathrm{id}^{q'\beta}_{d^\alpha_{j-1}(A)}$ for α, $\beta = \pm$,
6. $c = \mathrm{id}_{s_{i-1}(B)}$, $c' \in s_{i-1}(B)$ or $c' = \mathrm{id}_{d^\alpha_{j-1}(B)}$ for $\alpha = \pm$,
7. $c = \mathrm{id}_{t_{i-1}(A)}$ or $c = \mathrm{id}^{q'\beta}_{t_{i-1}(A)}$ or $c = \mathrm{id}_{t_{i-1}(B)}$, $c' = a'_{(i-1)+}$ or $c' = \mathrm{id}^{(i-1)+}_{d^\alpha_{j-1}(A)}$ for α, $\beta = \pm$.

The labeling on $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ will be the one induced from $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ separately, which is possible since it doesn't give conflicts on the common part. The picture in figure 2 shows what $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ looks like in the case $m = 3$.

Lemma 9.2 *$\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ is a generated pasting. More generally, this holds for every $\mathcal{A}$ and $\mathcal{B}$ with $B_{q\alpha} = A_{q\alpha}$ and both equivalent to both $d^{\alpha}_q(A)$ and $d^{\alpha}_q(B)$ for all $q < m$, and with $B_{m-} = A_{m+}$ and both equivalent to both $t_m(A)$ and $s_m(B)$.*

Proof. It is a pasting scheme because the axioms can be checked in $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ separately.

There are no direct loops since the position of the identities dictates that if $a \triangleleft b$ for a and b in different $A_{q\alpha}$'s or $B_{q\alpha}$'s then a and b must be q-dimensional, and $\mathsf{E}(b)$ is confined to the q-dimensional "layer" containing $B_{q\alpha}$, which possibly intersects the layer of $A_{q\alpha}$, but only in cells not having incoming nor outgoing q-cells.

For any $x \in \mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$, $\mathsf{R}(x)$ is well formed since $\mathsf{R}(x)$ is completely in $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ or $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$.

Loop-freeness is proven exactly as in the proof of lemma 8.6.

Now $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ are well-formed subpasting schemes of the loop-free pasting scheme $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$. The proof of [8, Theorem 12] gives that $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ is well formed.

$\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ is a generated pasting because the correctness of the labeling can be checked in $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ separately.

In fact, the above proof only depends on the relative position of the copies of A, B, $s_q(A)$ and $t_q(A)$, and the position of the identities in between them, so if other well-formed loop-free pasting schemes are used instead of them the result is still a generated pasting, which proves the more general statement. □

The previous lemma makes it possible to choose representatives making m-source and m-target match up and making the pasting of them a well-formed loop-free pasting scheme. The next lemma tells what happens if such representatives are already given.

Lemma 9.3 *If A and B are generated pastings with $t_m(A) = s_m(B)$ such that $A \cup B$ is a loop-free pasting scheme, then $A \cup B$ is a generated pasting, which is equivalent to $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$.*

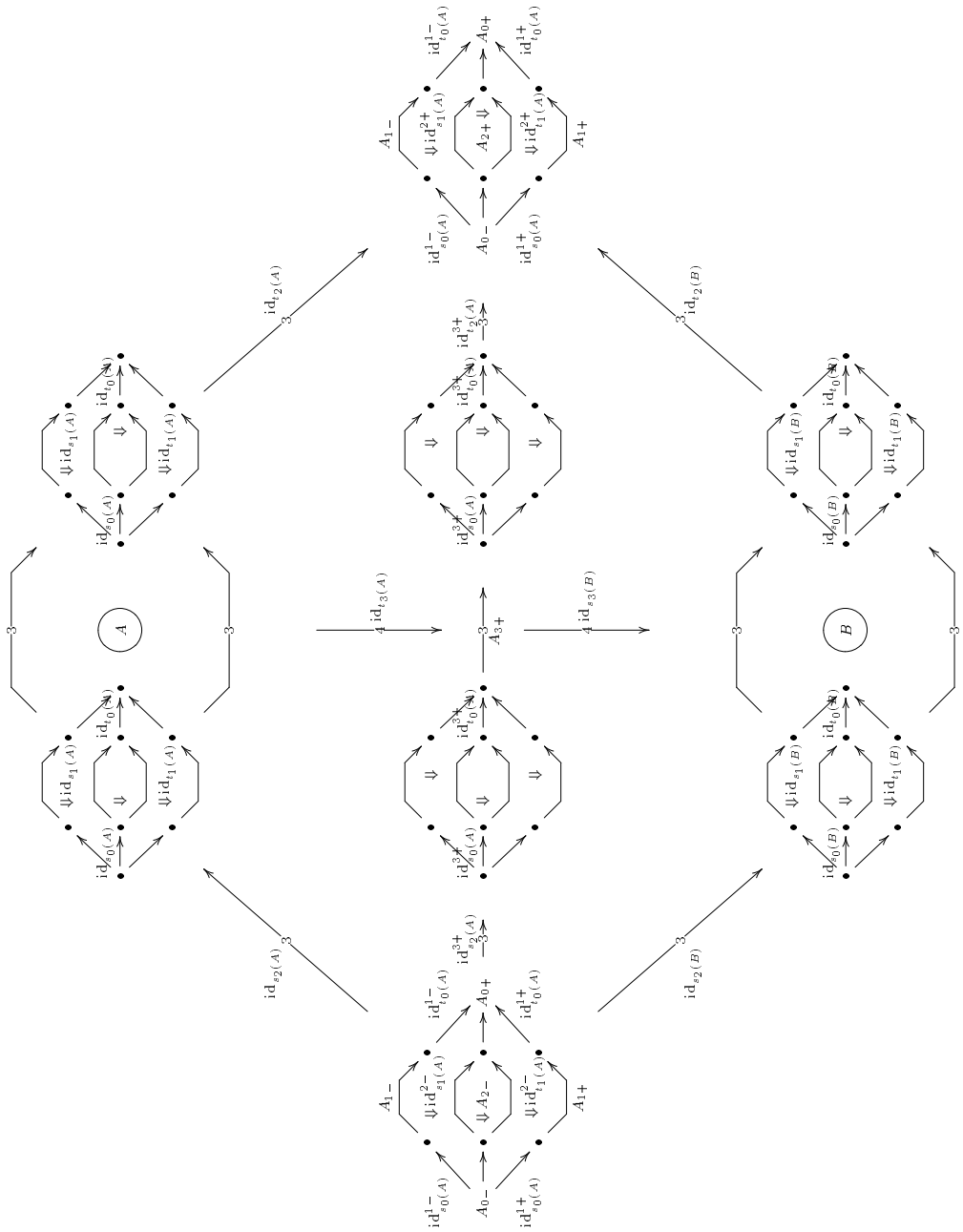


Figure 2: $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ for $m = 3$

Proof. $A \cup B$ is well formed, again because of the proof of [8, Theorem 12], and it is a generated pasting, again because this can be checked locally.

It is equivalent to $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$ because all identities can be inserted at the right places. More precisely, first make $\mathrm{Gl}_{m-1}(A \cup B)$ and then add two identities at $t_m(A) = s_m(B)$, i.e., take W such that no cell in $A \cup B$ ends in $\mathrm{id}_{t_m(A)}$, and that all relevant cells in A are before $\mathrm{id}_{t_m(A)}$ and all relevant cells in B are after $\mathrm{id}_{t_m(A)}$. Full insertability is immediate, and the intermediate result is a generated pasting because it is $A[\mathrm{id}_{t_m(A)} : W] \cup B$, which are generated pastings with loop-free union. □

This gives the following procedure for composing two generated pastings:

Definition 9.4 For generated pastings A and B with $\overline{t_m(A)} = \overline{s_m(B)}$, the m-composition $\overline{B} \circ_m \overline{A}$ in $\boldsymbol{\omega}(\underline{G}, \underline{R})$ is $\overline{\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})}$. ◇

Proposition 9.5 *m-composition in $\boldsymbol{\omega}(\underline{G}, \underline{R})$ is well defined. Moreover, $\overline{B} \circ_m \overline{A}$ can be calculated by* choosing *representatives A' and B' of $\overline{A}$ and $\overline{B}$ respectively such that $t_m(A') = s_m(B')$ and such that $A' \cup B'$ is a generated pasting, and then $\overline{A' \cup B'} = \overline{B} \circ_m \overline{A}$.*

Proof. As in the proof of lemma 8.8 it can be shown that replacements can be carried out locally in $\overline{\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}) \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})}$. And when starting with A equivalent to A', the result of carrying out these replacements is indeed $\overline{\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}') \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})}$.

For the second statement, such A' and B' always exist: take $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$. And if A'' and B'' also satisfy the condition, $A'' \cup B''$ is equivalent to $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}'') \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B}'')$ by lemma 9.3, which is equivalent to $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A}') \cup \mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B}')$ by well-definedness, which is equivalent to $A' \cup B'$ again by lemma 9.3. So this calculation also gives a well-defined equivalence class of generated pastings. □

9.3 Identity

The complication here is that in order to define the identity on $\overline{A}$ by $\overline{A[\mathrm{id}_A : \varnothing]}$ it is necessary that A is round since otherwise the result might fail to be a pasting scheme. The solution is to *choose* a round representative.

Definition 9.6 For a generated pasting A, the identity on $\overline{A}$, denoted $\mathrm{id}_{\overline{A}}$, is $\overline{\mathrm{Gl}_n(A)[\mathrm{id}_{s_n(\mathrm{Gl}_n(A))} : W^{\mathrm{af}}]}$. ◇

Lemma 9.7 *For every generated pasting A, $\mathrm{Gl}_n(A)[\mathrm{id}_{s_n(\mathrm{Gl}_n(A))}\colon W^{\mathrm{af}}]$ is a generated pasting.*

Proof. $\mathrm{Gl}_n(A)[\mathrm{id}_{s_n(\mathrm{Gl}_n(A))}\colon W^{\mathrm{af}}]$ is equal to $\mathrm{Gl}_n^{\mathfrak{l}}(A)$, which was shown to be a generated pasting in proposition 8.8. □

Lemma 9.8 *For round A, $A[\mathrm{id}_A\colon\varnothing]$ is a generated pasting which is equivalent to $\mathrm{Gl}_n(A)[\mathrm{id}_{s_n(\mathrm{Gl}_n(A))}\colon W^{\mathrm{af}}]$.*

Proof. This comes down to showing that the identities can be inserted in another order, and that the intermediate results are generated pastings again. Details are taken for granted. □

Proposition 9.9 *The identity is well defined. Moreover, $\mathrm{id}_{\overline{A}}$ can be calculated by* choosing *a round representative A' of $\overline{A}$, and then $\overline{A[\mathrm{id}_A\colon\varnothing]} = \mathrm{id}_{\overline{A}}$.*

Proof. Analogous to the proof of proposition 9.5. □

9.4 ω-category

Proposition 9.10 *$\omega(\underline{G},\underline{R})$, with the above defined operations s_m, t_m and $\circ_m$, is an ω-category.*

Proof. The elementary properties of s_m and t_m follow from the corresponding properties for pasting schemes.

For identity, if $\overline{s_m(B)} = \overline{t_m(A[\mathrm{id}_A\colon W])}$ with A round, make $\mathrm{Gl}_m^{\mathfrak{r}}(A[\mathrm{id}_A\colon W])\cup \mathrm{Gl}_m^{\mathfrak{l}}(\mathcal{B})$ by going from $\mathrm{Gl}_m^{\mathfrak{l}}(\mathcal{B})$ to $\mathrm{Gl}_m^{\mathfrak{r}}(\mathcal{A})\cup\mathrm{Gl}_m^{\mathfrak{l}}(\mathcal{B})$ by adding identities and doing the necessary replacements, and then adding id_A. The result is then the canonical representative of $\overline{B}\circ_m\overline{A[\mathrm{id}_A\colon W]}$, which can thus be represented by B.

For associativity, suppose $t_m(A)$ is equivalent to $s_m(B)$ and $t_m(B)$ is equivalent to $s_m(C)$. I will give representatives A', B' and C' of $\overline{A}$, $\overline{B}$ and $\overline{C}$ respectively such that $A'\cup B'$, $B'\cup C'$ and $A'\cup(B'\cup C') = (A'\cup B')\cup C'$ are all generated pastings. This will prove associativity because of proposition 9.5. Take $A' = \mathrm{Gl}_m^{\mathfrak{r}}(\mathcal{A})$, $B' = \mathrm{Gl}_m(\mathcal{B})$, and $C' = \mathrm{Gl}_m^{\mathfrak{l}}(\mathcal{C})$, where $A_{q\alpha} = B_{q\alpha} = C_{q\alpha} = d_q^{\alpha}(B)$ for $q < m$, and $A_{m+} = B_{m-} = s_m(B)$, and $B_{m+} = C_{m-} = t_m(B)$. In figure 3 a picture for $m = 1$ to clarify this.

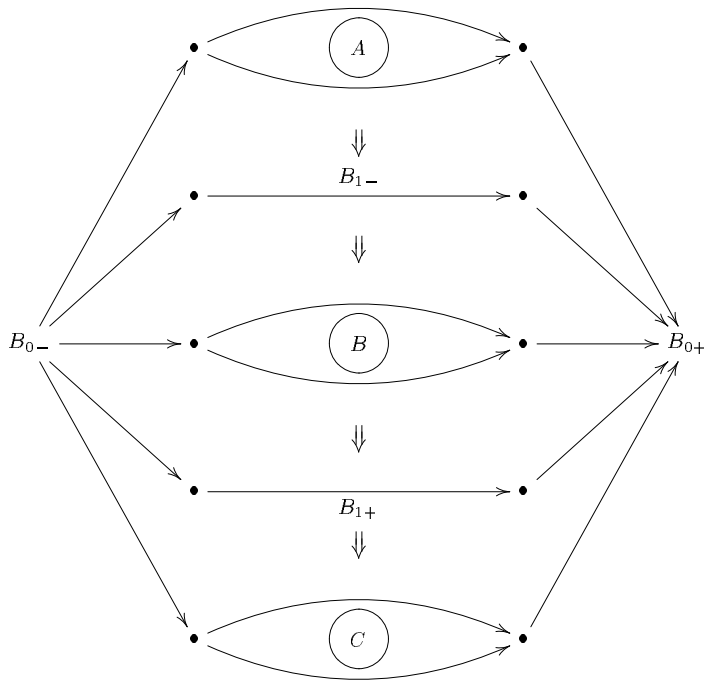


Figure 3: Associativity of composition for $m = 1$

For the interchange law something similar can be done: if $t_m(A)$ is equivalent to $s_m(B)$, $t_m(C)$ is equivalent to $s_m(D)$, and $t_{m'}(A)$ is equivalent to $s_{m'}(C)$, and $t_{m'}(B)$ is equivalent to $s_{m'}(D)$ for $m' < m$, then take $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{A})$, $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{B})$, $\mathrm{Gl}^{\mathfrak{r}}_m(\mathcal{C})$ and $\mathrm{Gl}^{\mathfrak{l}}_m(\mathcal{D})$ as representatives of $\overline{A}$, $\overline{B}\,\overline{C}$ and $\overline{D}$ respectively, where $A_{q\alpha} = B_{q\alpha} = C_{q\alpha} = D_{q\alpha} = d^{\alpha}_q(A)$ for $q < m'$, $A_{q\alpha} = B_{q\alpha} = d^{\alpha}_q(A)$, $C_{q-} = D_{q-} = t_q(A)$ and $C_{q+} = D_{q+} = t_q(C)$ for $m' \leq q < m$, and $A_{m+} = B_{m-} = t_m(A)$, and $C_{m+} = D_{m-} = t_m(C)$. In figure 4 a picture for $m = 2$ and $m' = 1$ to clarify this.

The other composition axioms follow from the corresponding properties for pasting schemes using proposition 9.5. □

10 $\omega(\underline{G}, \underline{R})$ is universal

The ω-category of equivalence classes of generated pastings satisfies a uni-

versal property: ω-functors going out from it are completely characterized by their value on the generators and the fact that they have the same value on generated pastings if there is a relation between them.

Definition 10.1 A *family of realizations* $(\underline{G}, \varphi_i)$ of a pasting presentation $(\underline{G}, \underline{R})$ in an ω-category $\mathbb{C}$ is a function assigning to every i-dimensional generator g an i-cell $\varphi_i(g)$ of $\mathbb{C}$. $\diamond$

If the set of generators of dimension n is denoted by $\underline{G}_n$, a family of realizations is thus a function $\varphi_i : \underline{G}_i \to C_i$ for every i.

I will follow an analogous procedure to Johnson's for realizations of well-formed loop-free pasting schemes to see when families of realizations behave nicely.

10.1 Extendability

Write $|\boldsymbol{\omega}(\underline{G}, \underline{R})|_n$ for the n-category of equivalence classes of generated pastings of dimension $\leq n$. Denoting the set of generators (relations) of dimension $\leq n$ by $\underline{G}_{\leq n}$ ($\underline{R}_{\leq n}$), this n-category is exactly $\boldsymbol{\omega}(\underline{G}_{\leq n}, \underline{R}_{\leq n})$. Since every generator is a generated pasting by lemma 7.3, there is a canonical function $\underline{G}_k \to |\boldsymbol{\omega}(\underline{G}, \underline{R})|_k$ for every k, and there are ω-functors embedding $|\boldsymbol{\omega}(\underline{G}, \underline{R})|_k$ in $|\boldsymbol{\omega}(\underline{G}, \underline{R})|_n$ for all $k < n$.

Definition 10.2 A family of realizations $(\underline{G}, \varphi_i)$ is *n-extendable* when there exists a unique ω-functor $\varphi : |\boldsymbol{\omega}(\underline{G}, \underline{R})|_n \to \mathbb{C}$ such that

$$\begin{tikzcd}
|\boldsymbol{\omega}(\underline{G}, \underline{R})|_k \arrow[r] & |\boldsymbol{\omega}(\underline{G}, \underline{R})|_n \arrow[d, "\varphi"] \\
\underline{G}_k \arrow[u] \arrow[r, "\varphi_k"'] & \mathbb{C}
\end{tikzcd}$$

commutes for all $k \leq n$, i.e., for every k-dimensional generator $g = (A, \mathfrak{L})$, $\varphi(\overline{A, \mathfrak{L}}) = \varphi_k(g)$.

A family of realizations $(\underline{G}, \varphi_i)$ is $(n - \frac{1}{2})$-*extendable* when there exists a unique ω-functor $\underline{\varphi} : \boldsymbol{\omega}(\underline{G}_{\leq n}, \underline{R}_{\leq n-1}) \to \mathbb{C}$ such that

$$\begin{tikzcd}
|\boldsymbol{\omega}(\underline{G}, \underline{R})|_k \arrow[r] & \boldsymbol{\omega}(\underline{G}_{\leq n}, \underline{R}_{\leq n-1}) \arrow[d, "\underline{\varphi}"] \\
\underline{G}_k \arrow[u] \arrow[r, "\varphi_k"'] & \mathbb{C}
\end{tikzcd}$$

commutes for all $k < n$, and such that for every n-dimensional generator $g = (A, \mathfrak{L})$, $\underline{\varphi}(A, \mathfrak{L}) = \varphi_n(g)$.

A family of realizations $(\underline{G}, \varphi_i)$ is extendable if it is n-extendable for all n. $\diamond$

If $(\underline{G}, \varphi_i)$ is an n-extendable family of realizations in $\mathbb{C}$ with n-extension φ, then for every generated pasting A of dimension $n+1$ there is a realization (A, ψ_i^A) of A in $\mathbb{C}$: if $a \in A$ is labeled by g then $\psi_i^A(a) = \varphi_i(g)$, and if $a \in A$ is labeled by $\mathrm{id}_{\overline{X}}$ then $\psi_i^A(a) = \mathrm{id}_{\varphi(\overline{X})}$. This definition is possible because if a is labeled by $\mathrm{id}_{\overline{X}}$ then X is at most n-dimensional.

10.2 Respectability

Any family of realizations is said to *respect* 0*-labels*. Clearly a family of realizations respecting 0-labels is $(-\frac{1}{2})$-extendable: φ_0 is already an ω-functor $\boldsymbol{\omega}(\underline{G}_0, \varnothing) \to \mathbb{C}$.

Suppose that for every $(n-1)$-extendable family of realizations $(\underline{G}, \varphi_i)$ with $(n-1)$-extension $\underline{\underline{\varphi}}$ respecting n-labels holds that:

- for every generated pasting A of dimension n, the realization (A, ψ_i^A) is appropriate, and
- $(\underline{G}, \varphi_i)$ is $(n - \frac{1}{2})$-extendable with $(n - \frac{1}{2})$-extension $\underline{\varphi}$ given by

$$\begin{array}{rcll} \underline{\varphi}(A) & = & \psi^A(A) & \text{if } \dim(A) = n \\ & = & \underline{\underline{\varphi}}(\overline{A}) & \text{if } \dim(A) < n. \end{array}$$

Such a family of realizations $(\underline{G}, \varphi_i)$ is said to *respect n-relations* if for every $(A, A') \in \underline{R}_n$, $\underline{\varphi}(A) = \underline{\varphi}(A')$.

Proposition 10.3 *Under the above assumption, every family of realizations respecting n-relations is n-extendable.*

Proof. I have to check that $\underline{\varphi}$ induces a unique well-defined ω-functor $\varphi : \boldsymbol{\omega}(\underline{G}_{\leq n}, \underline{R}_{\leq n}) \to \mathbb{C}$. Thus, that if $(X, X') \in \underline{R}$, X is fully replaceable by X' in A and $A' \cong A[X'/X]$ then $\underline{\varphi}(A') = \underline{\varphi}(A)$, and if id_X is insertable in A and $A' \cong A[\mathrm{id}_X : W]$ then $\underline{\varphi}(A') = \underline{\varphi}(A)$.

In the first case the assumption on $\underline{\varphi}$ makes the situation $\dim(A) < n$ trivial, and the situation $\dim(A) = n$ amounts to showing that $\psi^A(A) = \psi^{A[X'/X]}(A[X'/X])$. To this end, consider the appropriate realizations (A, ψ_i^A)

and $(X', \psi_i^{X'}) = (X', \psi_i^{A[X'/X]}|_{X'})$. Then for every $x \in X - \widetilde{X}$, $\psi_i^A(x) = \psi_i^{X'}(x)$, so this information induces a realization $(A[X'/X], (\psi^A)'_i)$, which is equal to $(A[X'/X], \psi_i^{A[X'/X]})$. Proposition 5.12 can now be applied, giving that $\psi^{A[X'/X]}(A[X'/X]) = (\psi^A)'(A[X'/X]) = \psi^A(A)$, as required.

In the second case the assumption on $\underline{\varphi}$ makes the situation $\dim(A) < n$ trivial as well. The situation $\dim(A) = n$ amounts to showing that $\psi^A(A) = \psi^{A[\mathrm{id}_X:W]}(A[\mathrm{id}_X:W])$. To this end, consider the appropriate realization (A, ψ_i^A). This induces an appropriate realization $(A[\mathrm{id}_X:W], (\psi^A)'_i)$ which is equal to $(A[\mathrm{id}_X:W], \psi_i^{A[\mathrm{id}_X:W]})$. Proposition 6.14 can now be applied, yielding that $\psi^{A[\mathrm{id}_X:W]}(A[\mathrm{id}_X:W]) = (\psi^A)'(A[\mathrm{id}_X:W]) = \psi(A)$, as required.

Thus φ can be defined by $\varphi(\overline{A}) = \overline{\underline{\varphi}(A)}$.

Uniqueness of $\underline{\varphi}$ immediately gives uniqueness of φ. □

An n-extendable family of realizations $(\underline{G}, \varphi_i)$ with n-extension $\underline{\underline{\varphi}}$ is said to *respect* $(n+1)$*-labels* if for every generator $g = (A, \mathfrak{L})$ of dimension $n+1$, $s_n(\varphi_{n+1}(g)) = \underline{\underline{\varphi}}\left(\overline{\mathrm{dom}(A)}\right)$ and $t_n(\varphi_{n+1}(g)) = \underline{\underline{\varphi}}\left(\overline{\mathrm{cod}(A)}\right)$.

Lemma 10.4 *Let $(\underline{G}, \varphi_i)$ be an n-extendable family of realizations with n-extension $\underline{\underline{\varphi}}$ respecting $(n+1)$-labels. Then for every generated pasting A of dimension $n+1$, (A, ψ_i^A) is appropriate, and for B be a well-formed subpasting scheme of A of dimension $m < n+1$, $\psi^A(B) = \underline{\underline{\varphi}}(B)$.*

Proof. I will prove the following induction hypothesis: the realization (A, ψ_i^A) is m-appropriate with m-extension satisfying $\psi^A(B) = \underline{\underline{\varphi}}(\overline{B})$ for all well-formed subpasting schemes B of A of dimension $m < n+1$.

Realizations are always 0-appropriate, so suppose it is m-appropriate for some $m \leq n$, and consider $a \in A_{m+1}$. In case it is labeled by a generator g and $m < n$ then

$$\begin{array}{rcll}
s_m(\psi_{m+1}^A(a)) & = & s_m(\varphi_{m+1}(g)) & \\
& = & s_m(\underline{\underline{\varphi}}(g)) & \text{because } \underline{\underline{\varphi}} \text{ extends } \varphi_i \\
& = & \underline{\underline{\varphi}}\left(\overline{s_m(\mathsf{R}(g))}\right) & \text{because } \underline{\underline{\varphi}} \text{ is an } \omega\text{-functor} \\
& = & \underline{\underline{\varphi}}\left(\overline{s_m(\mathsf{R}(a))}\right) & \text{by the condition on } \mathrm{dom}(\mathsf{R}(a)) \\
& = & \psi^A\left(s_m(\mathsf{R}(a))\right) & \text{by the formula for the } m\text{-extension,}
\end{array}$$

and if $m = n$ then

$$
\begin{aligned}
s_m(\psi^A_{m+1}(a)) &= s_m(\varphi_{m+1}(g)) \\
&= \underline{\underline{\varphi}}\left(\overline{s_m(\mathsf{R}(g))}\right) && \text{because } (\underline{G}, \varphi_i) \text{ respects } (n+1)\text{-labels} \\
&= \underline{\underline{\varphi}}\left(\overline{s_m(\mathsf{R}(a))}\right) && \text{by the condition on } \mathrm{dom}(\mathsf{R}(a)) \\
&= \psi^A(s_m(\mathsf{R}(a))) && \text{by the formula for the } m\text{-extension.}
\end{aligned}
$$

If it is labeled by id_X then

$$
\begin{aligned}
s_m(\psi^A_{m+1}(a)) &= s_m\left(\mathrm{id}_{\underline{\underline{\varphi}}(\overline{X})}\right) \\
&= \underline{\underline{\varphi}}(\overline{X}) \\
&= \underline{\underline{\varphi}}\left(\overline{s_m(\mathsf{R}(a))}\right) && \text{by the condition on } \mathrm{dom}(\mathsf{R}(a)) \\
&= \psi^A(s_m(\mathsf{R}(a))) && \text{by the formula for the } m\text{-extension.}
\end{aligned}
$$

This proves $(m+1)$-appropriateness of (A, ψ^A_i).

For the second statement,

$$
\begin{aligned}
\psi^A(B) &= \psi^B(B) && \text{by uniqueness of extensions} \\
&= \underline{\underline{\varphi}}(B) && \text{by the formula for the } (m-\tfrac{1}{2})\text{-extension of } (\underline{G}, \varphi_i).
\end{aligned}
$$

□

Proposition 10.5 *Every n-extendable family of realizations $(\underline{G}, \varphi_i)$ with n-extension $\underline{\underline{\varphi}}$ respecting $(n+1)$-labels is $(n+\frac{1}{2})$-extendable with $(n+\frac{1}{2})$-extension $\underline{\varphi}$ given by*

$$
\begin{aligned}
\underline{\varphi}(A) &= \psi^A(A) && \textit{if } \dim(A) = n+1 \\
&= \underline{\underline{\varphi}}(\overline{A}) && \textit{if } \dim(A) \leq n.
\end{aligned}
$$

Proof. I have to show that $\underline{\varphi}$ is an ω-functor, that it extends φ_k, and that it is unique with these properties.

For a generated pasting A of dimension $n+1$,

$$
\begin{aligned}
s_m(\underline{\varphi}(A)) &= s_m(\psi^A(A)) \\
&= \psi^A(s_m(A)) && \text{because } \psi^A \text{ is an } \omega\text{-functor} \\
&= \underline{\underline{\varphi}}\left(\overline{s_m(A)}\right) \\
&= \underline{\varphi}(s_m(A)),
\end{aligned}
$$

for a generated pasting A of dimension n with A round,

$$\begin{array}{rcl}
\underline{\varphi}(\mathrm{id}_{\overline{A}}) & = & \underline{\varphi}(A[\mathrm{id}_A\colon\varnothing]) \\
& = & \psi^{A[\mathrm{id}_A:\varnothing]}(A[\mathrm{id}_A\colon\varnothing]) \\
& = & \psi_{n+1}^{A[\mathrm{id}_A:\varnothing]}(\mathrm{id}_A) \\
& = & \mathrm{id}_{\underline{\underline{\varphi}}(\overline{A})} \\
& = & \mathrm{id}_{\underline{\varphi}(A)},
\end{array}$$

and for generated pastings A and B of dimension $n+1$ such that $t_m(A) = s_m(B)$ and such that $A \cup B$ is a generated pasting,

$$\begin{array}{rcll}
\underline{\varphi}(A \cup B) & = & \psi^{A\cup B}(A \cup B) & \\
& = & \psi^{A\cup B}(B) \circ_m \psi^{A\cup B}(A) & \\
& = & \psi^{B}(B) \circ_m \psi^{A}(A) & \text{because of uniqueness of extensions} \\
& = & \underline{\varphi}(B) \circ_m \underline{\varphi}(A). &
\end{array}$$

Together with $\underline{\underline{\varphi}}$ being an ω-functor this proves that $\underline{\varphi}$ is an ω-functor.

For $g = (A, \mathfrak{L})$ a generator of dimension $n+1$,

$$\begin{array}{rcll}
\underline{\varphi}(A) & = & \psi^{A}(A) & \\
& = & \psi_{n+1}^{A}(g) & \text{because } \psi^A \text{ extends } \psi_i^A \\
& = & \varphi_{n+1}(g) & \text{by definition of } \psi_i^A.
\end{array}$$

Together with $\underline{\underline{\varphi}}$ extending φ_k this proves that $\underline{\varphi}$ extends φ_k.

This leaves uniqueness. So suppose $\underline{\varphi}'$ is an ω-functor $\boldsymbol{\omega}(\underline{G}_{\leq n+1}, \underline{R}_{\leq n}) \to \mathbb{C}$ extending φ_k. Its restriction to $|\boldsymbol{\omega}(\underline{G}, \underline{R})|_n$ satisfies the properties making $\underline{\underline{\varphi}}$ unique, so up to dimension n it must be equal to $\underline{\varphi}$. For a generated pasting A of dimension $n+1$, $\underline{\varphi}'$ can be restricted to the subω-category A of $\boldsymbol{\omega}(\underline{G}, \underline{R})$, giving an ω-functor $A \to \mathbb{C}$ satisfying the properties making ψ^A unique. Thus this restriction must be equal to ψ^A, in particular, $\underline{\varphi}'(A) = \psi^A(A)$. Thus indeed $\underline{\varphi}'$ equals $\underline{\varphi}$. □

A family of realizations is said to be *respectable* if it respects n-labels and n-relations for all n.

10.3 Generalized pasting theorem

Theorem 10.6 *A respectable family of realizations $(\underline{G}, \varphi_i)$ in $\mathbb{C}$ is extendable, i.e., it induces a unique ω-functor $\varphi : \boldsymbol{\omega}(\underline{G}, \underline{R}) \to \mathbb{C}$ extending φ_i. Conversely, every ω-functor $\varphi : \boldsymbol{\omega}(\underline{G}, \underline{R}) \to \mathbb{C}$ is the extension of a unique respectable family of realizations in $\mathbb{C}$.*

Proof. For "$\Rightarrow$", use induction on n and proposition 10.3, lemma 10.4 and proposition 10.5. Uniqueness of all its n-restrictions gives that φ itself is unique.

For "$\Leftarrow$", φ induces a family of realizations by taking for every generator $g = (A, \mathfrak{L})$ of dimension n, $\varphi_n(g) = \varphi(A)$. This is obviously the only family of realizations which can have φ as its extension, and well-definedness of φ and it being an ω-functor imply, via an induction argument, that it respects relations and labels, i.e., it is respectable. $\square$

Thus, $\boldsymbol{\omega}(\underline{G}, \underline{R})$ is the universal ω-category for respectable families of realizations of $(\underline{G}, \underline{R})$.

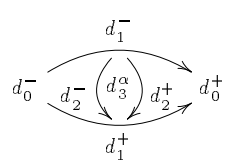
11 The standard presentation of an ω-category

Every ω-category can be described by a pasting presentation. Intuitively, this means that the composition information of the ω-category can be codified in relations of the presentation, so that composition in the ω-category becomes pasting.

Given an ω-category $\mathbb{C}$, I will describe a pasting presentation $(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$ such that $\boldsymbol{\omega}(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}}) \cong \mathbb{C}$.

11.1 Generators

Denote the n-dimensional globular pasting scheme by 2_n. It has one cell d_n of dimension n and two cells d_i^+ and d_i^- of dimension i for every $i < n$, and for $j < i$ the relation E_j^i is only non-empty for $j = i - 1$, in which case it is $\{(d_i^\alpha, d_{i-1}^+) | \alpha = \pm\}$. $\mathcal{P}(2_n)$ is the free ω-category on one n-dimensional cell. For $n = 3$, the pasting scheme 2_n can be pictured as



.

For every n-dimensional $c \in \mathbb{C}$ there is to be a generator $(2_n, \mathfrak{L}_c)$ of dimension n, where d_i^α is labeled by $d_i^\alpha(c)$.

11.2 Relations

As already remarked, the relations will codify composition in $\mathbb{C}$.

For every n-dimensional $c \in \mathbb{C}$ there is to be a relation between $(2_{n+1}, \mathfrak{L}_{\mathrm{id}_c})$ and $(2_{n+1}, \mathfrak{L}_c^{\mathrm{id}})$, where the latter pasting scheme is labeled as the first except for the top-dimensional cell, which is labeled by the *formal* identity $\mathrm{id}_{(2_n, \mathfrak{L}_c)}$ instead of by the *actual* identity id_c. This relation is possible because the latter labeled pasting scheme is indeed a generated pasting, 2_n is fully replaceable by 2_n in 2_n and the lower dimensional labels coincide.

For every appropriate realization (A, f_i) of an n-dimensional *round* pasting scheme A in $\mathbb{C}$ there is to be a relation between (A, f_i), seen as a labeled pasting scheme, and (A_t, f_i^t), where f_i^t labels A_t as A except for a_t which is to be labeled by the composite of A. This relation is possible because if this relation is imposed for lower dimensions then A and A_t are generated pastings, for which lower dimensional labels coincide. Indeed, this follows from the more complex assertion

Lemma 11.1 *For every appropriate realization (A, f_i) of a well-formed loop-free pasting scheme A, (A, f_i) is a generated pasting. If moreover A is round then A_t is a generated pasting. If A is round and A and A_t are related and relations in lower dimensions are defined as above, then for* all *A, (A, f_i) and $(2_n, \mathfrak{L}_{f(A)})$ are equivalent.*

Proof. I will prove this assertion by induction on n, the dimension of A. For 0-dimensional A there is nothing to do since a singleton pasting scheme is 2_0.

Now take $(n+1)$-dimensional A, and suppose the statement proven for lower dimensions. To show that A is a generated pasting I must show that for every i-cell in A, labeled by $(2_i, \mathfrak{L}_{f_i(a)})$, the generated pastings $\left(\mathrm{dom}(\mathsf{R}(a)), f_i|_{\mathrm{dom}(\mathsf{R}(a))}\right)$ and $(2_{i-1}, \mathfrak{L}_{s_{i-1}(f_i(a))})$ are equivalent. But because of the induction hypothesis $\left(\mathrm{dom}(\mathsf{R}(a)), f_i|_{\mathrm{dom}(\mathsf{R}(a))}\right)$ is equivalent to $\left(2_{i-1}, \mathfrak{L}_{f\left(\mathrm{dom}(\mathsf{R}(a))\right)}\right)$ and by appropriateness $f\left(\mathrm{dom}(\mathsf{R}(a))\right) = s_{i-1}(f_i(a))$. If A is round then the only thing left to show that A_t is a generated pasting is that a_t is labeled correctly. This comes down to showing that $(\mathrm{dom}(A), f_i|_{\mathrm{dom}(A)})$ is equivalent to $\left(2_n, \mathfrak{L}_{f(\mathrm{dom}(A))}\right)$. But this is immediate from the induction hypothesis. Note, by the way, that I use that the extension of a restriction to a subpasting scheme of an appropriate realization is equal to the restriction to the subpasting scheme of its extension.

So this leaves me to prove that for all $(n+1)$-dimensional A, (A, f_i) and $(2_{n+1}, \mathfrak{L}_{f(A)})$ are equivalent. I will do this, roughly, as follows. First I will globularize A to ensure that all s_m and t_m are round. The cells labeled $\mathrm{id}_{\overline{X}}$ in this

globularization can also be labeled by $\mathrm{id}_{f(X)}$ since by the induction hypothesis applied to X, the generated pastings $(X, f_i|_X)$ and $(2_p, \mathfrak{L}_{f(X)})$ are equivalent, and they can be replaced by eachother since they are on the outside of the pasting scheme. This makes that the resulting generated pasting still comes from an appropriate realization of a pasting scheme. This pasting scheme, $A^{n.1}$ say, can be replaced by $A^{n.1}_t$, giving $A^{n.2}$ say. Then replace $t_{n-1}(A^{n.2})$ by $t_{n-1}(A^{n.2})_t$ resulting in $A^{(n-1).1}$, replace $s_{n-1}(A^{(n-1).1})$ by $s_{n-1}(A^{(n-1).1})_t$, and continue this for lower dimensions, ending with 2_{n+1}, labeled by $\mathfrak{L}_{f(A)}$. More precisely, define $A^{j.1} = A^{(j+1).2}[t_j(A^{(j+1).2})_t/t_j(A^{(j+1).2})]$ and $A^{j.2} = A^{j.1}[s_j(A^{j.1})_t/s_j(A^{j.1})]$. Then $A^{j.2}$ consists of $s_{j-1}(A^{n.1}) \cup t_{j-1}(A^{n.1})$ and for every $q \geq j$ two cells 2^-_q and 2^+_q of dimension q which begin in every cell of $\widetilde{s_{j-1}(A^{n.1})}$ and end in every cell of $\widetilde{t_{j-1}(A^{n.1})}$, and $A^{j.1}$ can be described likewise. Now for all j, $t_j(A^{(j+1).2})$ is elementary replaceable by $t_j(A^{(j+1).2})_t$ in $A^{(j+1).2}$ and $s_j(A^{j.1})$ is elementary replaceable by $s_j(A^{j.1})_t$ in $A^{j.1}$. Indeed, for conditions 1 and 2, a is a cell 2^α_{j+1}, and 1 is immediate and 2 is void, and condition 3 holds because A was globularized first. A similar argument works for $s_j(A^{2j})$. Finally, the $A^{j.1}$ and $A^{j.2}$ are loop free, which also implies full replaceability. Indeed, there are no direct loops of high dimension since there are particularly few cells there, there are no direct loops of low dimension since they are direct loops in $A^{n.1}$ as well, and condition (iv) of loop-freeness holds because if x is one of the 2^α_q then $s_{j'}(\mathsf{R}(x)) = s_{j'}(A^{n.1})$ and the argument is the same as in the proof that A_t is loop free for round A, and if x is lower dimensional then the conclusion follows as in $A^{n.1}$. So indeed the generated pastings (A, f_i) and $(2_{n+1}, \mathfrak{L}_{f(A)})$ are equivalent for all A. $\square$

11.3 The ω-category $\boldsymbol{\omega}(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$

Proposition 11.2 $(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$ *is a pasting presentation of* $\mathbb{C}$.

Proof. There are two ways to prove this: directly, by giving an explicit isomorphism between $\boldsymbol{\omega}(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$ and $\mathbb{C}$, or by showing that $\mathbb{C}$ satisfies the universal property for $\boldsymbol{\omega}(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$.

First way: every generated pasting is equivalent to a generator by lemma 11.1, and because equivalent generated pastings, seen as appropriate realizations in $\mathbb{C}$, have the same composite, equivalent generators are equal. Thus it is possible to take the generators as standard representatives of the equivalence classes. Write out s_m, t_m, composition and identity in terms of these standard representatives. It appears that they are exactly the operations in $\mathbb{C}$.

Second way: this amounts to show that a respectable family of realizations $(\underline{G}, \varphi_i)$ of $(\underline{G}_{\mathbb{C}}, \underline{R}_{\mathbb{C}})$ in $\mathbb{D}$ corresponds to an ω-functor $f : \mathbb{C} \to \mathbb{D}$. Define, for $c \in \mathbb{C}$, $f(c) = \varphi_i(2_i, \mathfrak{L}_c)$. This preserves s_m and t_m because $(\underline{G}, \varphi_i)$ respects labels:

$$\begin{array}{rcll} \mathrm{dom}(f(c)) & = & \mathrm{dom}(\varphi_i(2_i, \mathfrak{L}_c)) & \\ & = & \varphi\left(\overline{(2_{i-1}, \mathfrak{L}_{\mathrm{dom}(c)})}\right) & \text{because } \varphi_i \text{ respects labels} \\ & = & \varphi_{i-1}(2_{i-1}, \mathfrak{L}_{\mathrm{dom}(c)}) & \text{because } \varphi \text{ extends } (\underline{G}, \varphi_i) \\ & = & f(\mathrm{dom}(c)), & \end{array}$$

it preserves composition and identity because $(\underline{G}, \varphi_i)$ respects relations:

$$\begin{array}{rcll} f(c) \circ_m f(c') & = & \varphi_i(2_i, \mathfrak{L}_c) \circ_m \varphi_{i'}(2_{i'}, \mathfrak{L}_{c'}) & \\ & = & \varphi(2_i \cup_{2_m} 2_{i'}, \mathfrak{L}_{c,c'}) & \text{because } \varphi \text{ extends } (\underline{G}, \varphi_i) \\ & = & \varphi(2_{\max\{i,i'\}}, \mathfrak{L}_{c \circ_m c'}) & \text{because } (\underline{G}, \varphi_i) \text{ respects relations and hence } \varphi \text{ is well-defined} \\ & = & \varphi_{\max\{i,i'\}}(2_{\max\{i,i'\}}, \mathfrak{L}_{c \circ_m c'}) & \text{again because } \varphi \text{ extends } (\underline{G}, \varphi_i) \\ & = & f(c \circ_m c') & \end{array}$$

and

$$\begin{array}{rcll} f(\mathrm{id}_c) & = & \varphi_{i+1}(2_{i+1}, \mathfrak{L}_{\mathrm{id}_c}) & \\ & = & \varphi(2_{i+1}, \mathfrak{L}_{\mathrm{id}_c}) & \\ & = & \varphi(2_{i+1}, \mathfrak{L}_c^{\mathrm{id}}) & \text{because } (\underline{G}, \varphi_i) \text{ respects relations} \\ & = & \mathrm{id}_{\varphi(2_i, \mathfrak{L}_c)} & \text{because } \varphi \text{ is an } \omega\text{-functor} \\ & = & \mathrm{id}_{\varphi_i(2_i, \mathfrak{L}_c)} & \\ & = & \mathrm{id}_{f(c)}\,. & \end{array}$$

Conversely, starting with an ω-functor $f : \mathbb{C} \to \mathbb{D}$, define a family of realizations by $\varphi_i(2_i, \mathfrak{L}_c) = f(c)$. This family of realizations is respectable exactly because f is an ω-functor. $\square$

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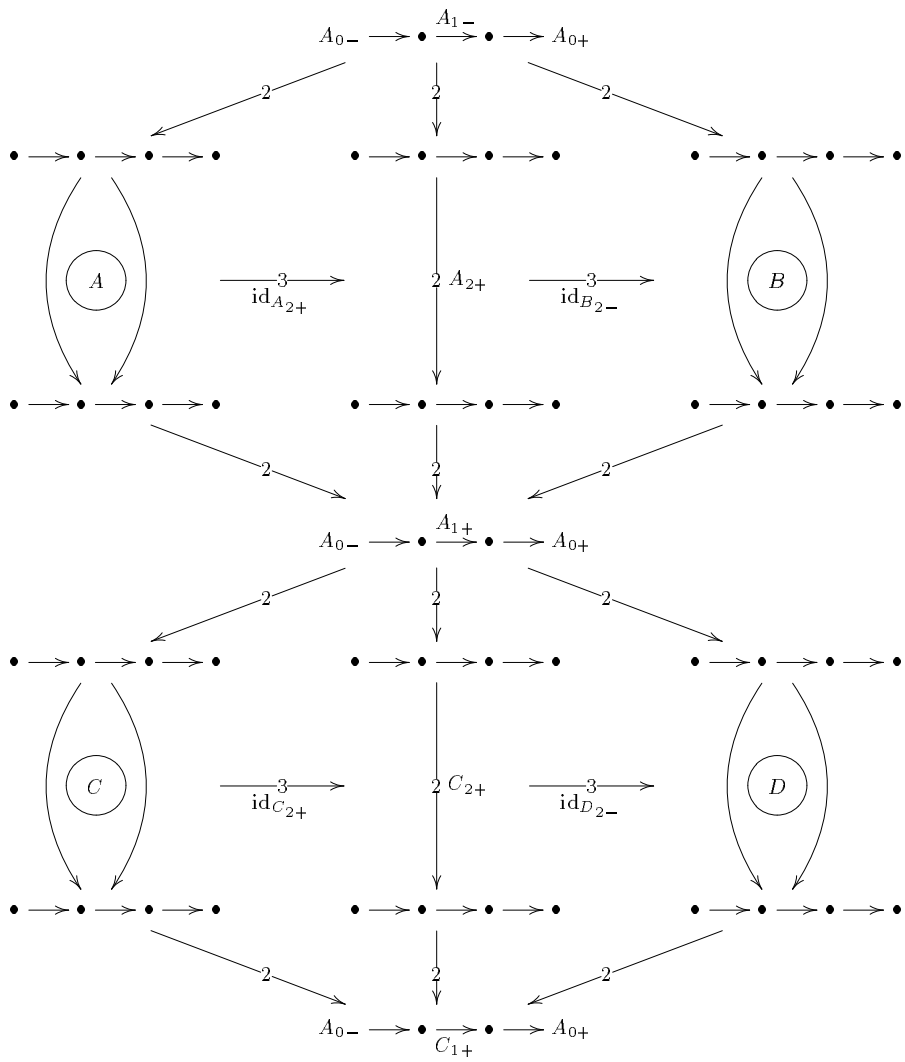

Figure 4: The interchange law for $m = 2$ and $m' = 1$

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