ω-languages*

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Contents

1 Introduction

The purpose of this paper is to give an introduction into languages of infinite strings (of order type ω), so-called ω-languages. The set of all infinite strings over a finite alphabet may be considered, as we shall see below, in a natural way as a metric space.
In general, the space of infinite strings is not only considered in topology but also in various other areas of mathematics and computer science as for example in symbolic dynamics, information theory.
In formal language theory we are mainly interested in computational aspects of ω-languages. Therefore, in this respect it is convenient to consider ω-languages together with their finite string counterparts, languages. In this paper we will not stress too much the automata-theoretic aspect of ω-languages, but rather their description of ω-languages via languages. The automata-theoretic aspects are already considered recently in detail in W. Thomas' contribution "Automata on infinite objects" [Th90] to the the Handbook of Theoretical Computer Science and in J. Engelfriet's and H. J. Hoogeboom's Fundamental Study "X-automata on ω-words" [EH93].
Therefore our consideration lays stress on the relations between classes of languages and classes of ω-languages as well as on topological aspects, in particular on mappings acting continuously on the space of infinite words.
We try to link the consideration of classes of languages with their ω-counterparts where we use the fact that both parts are defined by the same classes of devices. The investigated relationships involve operations transferring languages to ω-languages (and vice versa), and they give descriptional results for ω-languages by operations and languages in the related language class.
Most of the classes of ω-languages investigated up to now were defined by accepting devices. Adopting the topological point of view we try to consider our accepting devices (at least the deterministic ones) as continuous mappings on spaces of infinite words. To be concrete, we shall focus on the consideration of classes of ω-languages which were on the one hand investigated up to now and have, on the other hand, close relationships to classical families of languages such as regular, context-free, or recursively enumerable languages.
Therefore, we assume the reader to be familiar with the classical theory of formal languages (cf. e.g. [Gi75],[HU79],[Sa73]), in particular with the families of regular, deterministic context-free, context-free, recursive and recursively enumerable languages, which will be denoted by **R**, **DCF**, **CF**, **REK**, and **RE**, respectively, in the sequel.
Finally, we mention that there are some books as [TB70], [Ei74], [LS77], [NP85] [PP93] and, except the above mentioned papers [EH93] and [Th90], there are also some other papers surveying parts of the theory of ω-languages (cf. [CG77], [HR86], [Ni79], [St87b], [Wi93]).

Notation

The set $\{0, 1, 2, \ldots\}$ of natural numbers is denoted by $\mathbb{N}$, and for a finite alphabet X by X^* (X^ω) we denote the set of finite words (infinite sequences) on X. For a word $w \in X^*$

and a string $b \in X^* \cup X^\omega$ let $w \cdot b$ be their concatenation. This in an obvious way defines a product $W \cdot B$ as well as an n–th power W^n $(n \geq 1)$ of sets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a word $w \in X^*$ its length is $|w|$, and e denotes the empty word in X^*.
An infinite sequence $\xi \in X^\omega$ is also called an *ω-word*, and it is also understood as a mapping $\xi : \omega \to X$ where $\omega := \{1, 2, 3, \ldots\}$. Then $\xi(i)$ is the i-th letter of ξ, and $\xi(m, n] := \xi(m+1) \cdot \ldots \cdot \xi(n)$ is an interval (subword) of ξ.
For $B \in X^* \cup X^\omega$ we define the *state* B/w of B generated by the word $w \in X^*$ as $B/w := \{b : w \cdot b \in B\}$, and we call a set B *finite-state* if the number of different states B/w $(w \in X^*)$ is finite.
We introduce into $X^* \cup X^\omega$ a partial ordering

$$w \sqsubseteq b :\Leftrightarrow \exists b'(b' \in X^* \cup X^\omega \wedge w \cdot b' = b) ,$$

and we call $\mathbf{A}(b) := \{w : w \in X^* \wedge w \sqsubseteq b\}$ and $\mathbf{A}(B) := \bigcup_{b \in B} \mathbf{A}(b)$ the sets of initial words (prefixes) of $b \in X^* \cup X^\omega$ and $B \subseteq X^* \cup X^\omega$, respectively. A language $W \subseteq X^*$ is referred to as *prefix-free* provided $w, v \in W$ and $w \sqsubseteq v$ imply $w = v$, that is, no word in W is a proper prefix of an other word in W.
We extend the operations * and $^\omega$ to arbitrary subsets $W \subseteq X^*$ in the usual way :

$$\begin{aligned} W^* &:= \bigcup_{n \in \mathbb{N}} W^n \quad \text{where} \quad W^0 := \{e\} \text{ , and} \\ W^\omega &:= \{w_0 \cdot w_1 \cdot \ldots \cdot w_i \cdot \ldots : i \in \mathbb{N} \wedge w_i \in W \setminus \{e\}\} \end{aligned}$$

is the set of (infinite) sequences in X^ω formed by concatenating members of W.
We will refer to subsets of X^* and X^ω as languages or ω-languages, respectively.
A *homomorphism* $h : X^* \to Y^*$ is a mapping satisfying the relation $h(w \cdot v) = h(w) \cdot h(v)$. Thus, in order to define a homomorphism it is sufficient to assign to each letter $x \in X$ its value $h(x) \in Y^*$. If $h(w) = e$ implies $w = e$ we call the homomorphism h *e-free*.
If $h(x) \in Y$, that is, the homomorphism is length preserving, we will call h also a *projection*. Particular cases of projections are the following ones:
For the product alphabet $X \times Y$ of the alphabets define the projections into X and Y via $pr_1((x, y)) := x$ and $pr_2((x, y)) := y$, respectively.

2 Topology for languages and ω-languages

2.1 Cantor topology

Unlike the case of languages topology is a very important means to facilitate the study of ω-languages. For instance, as we shall see below several hierarchies of ω-languages are topologically based, that is, the distinctness of their classes can be proven by topological means already.
In this paper we consider X^ω as a topological space with the basis $(w \cdot X^\omega)_{w \in X^*}$. Since X is finite, this topological space is homeomorphic to the Cantor discontinuum, hence compact.

It is more convenient to define this topology via the following metric.

$$\varrho(\eta,\xi) = \inf\ \{r^{-|w|} :\ w \text{ is a common prefix of } \eta \text{ and } \xi\} \tag{2.1}$$

It is easily verified that ϱ is indeed a metric which, in addition it satisfies the ultrametric inequality.

$$\varrho(\zeta,\xi) \leq \max\{\varrho(\zeta,\eta), \varrho(\xi,\eta)\} \tag{2.2}$$

Open (in view of Eq. (2.2) they are simultaneously closed) balls in this space (X^ω, ϱ) are the above mentioned basis sets $w \cdot X^\omega$. Then open sets in X^ω are of the form $W \cdot X^\omega$ where $W \subseteq X^*$. From this it follows that a subset $F \subseteq X^\omega$ is *closed* if and only if $\mathbf{A}(\xi) \subseteq \mathbf{A}(F)$ implies $\xi \in F$, and $E \subseteq X^\omega$ is simultaneously *open and closed* iff $F = W \cdot X^\omega$ for some finite language $W \subseteq X^*$.
The *topological closure* of a subset $F \subseteq X^\omega$, that is, the smallest closed subset of X^ω containing F is denoted by $\mathcal{C}(F)$. One has $\mathcal{C}(F) = \{\xi : \mathbf{A}(\xi) \subseteq \mathbf{A}(F)\}$.
Having defined open and closed sets in X^ω, we proceed to the next classes of the Borel hierarchy (cf. [Ku66]):

$\mathbf{F}_\sigma$ is the set of countable unions of closed subsets of X^ω,

$\mathbf{G}_\delta$ is the set of countable intersections of open subsets of X^ω.

$\mathbf{F}_{\sigma\delta}$ is the set of countable intersections of $\mathbf{F}_\sigma$-subsets of X^ω,

$\mathbf{G}_{\delta\sigma}$ is the set of countable unions of $\mathbf{G}_\delta$-subsets of X^ω, and so on.[1]

To be consistent with this notation, we denote the classes of open and of closed subsets of X^ω by $\mathbf{G}$ and $\mathbf{F}$, respectively.
A subset $F \subseteq X^\omega$ is referred to as a *Borel set* if F belongs to some Borel class $\mathbf{F}_\alpha$.
It is interesting to note that ω-languages in low-level Borel classes can be characterized via operations transforming languages to ω-languages. To this end we introduce two limit operations: the first one will be referred to as *δ-limit*, and the second one introduced in [SW74], [LS77] or [BN80] is usually called the *adherence* of a language.

Definition 2.1 Let $W \subseteq X^*$. We will refer to

$$W^\delta := \{\zeta : \zeta \in X^\omega \wedge \mathbf{A}(\zeta) \cap W \text{ is infinite}\}$$

as the *δ-limit* of the language W, and to

$$\boldsymbol{ls}\, W := \{\zeta : \zeta \in X^\omega \wedge \mathbf{A}(\zeta) \subseteq \mathbf{A}(W)\}$$

as the *adherence* of the language W.

[1] Borel classes are also defined for larger countable ordinals than natural numbers, but since we will not need higher level Borel classes, we refer the interested reader to some text book on topology, as e.g. [Ku66].

The following relations hold true.

$$W^\delta \subseteq \mathbf{A}(W)^\delta = \boldsymbol{ls}\, W = \boldsymbol{ls}\, \mathbf{A}(W) \tag{2.3}$$

Other notations used for W^δ are $\overrightarrow{W}$ (as e.g. in [Th90]) or $\lim(W)$ (as e.g. in [EH93]). We prefer the first one for a reason which is given in the next proposition.

Proposition 2.1 *A subset $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set iff there is a $V \subseteq X^*$ such that $F = V^\delta$.*

Using the identity $\mathcal{C}(F) = \boldsymbol{ls}\, \mathbf{A}(F) = (\mathbf{A}(F))^\delta$ we obtain a similar proposition for $\boldsymbol{ls}$.

Corollary 2.1 *A subset $F \subseteq X^\omega$ is closed iff there is a language $W \subseteq X^*$ such that $F = \boldsymbol{ls}\, W$.*

Denote by $\mathcal{B}(\mathcal{M})$ the closure of the class of sets $\mathcal{M}$ by Boolean operations. Then it is well known that in the space (X^ω, ϱ) the following proper inclusions hold.

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & \nearrow & & \nwarrow & \\
\mathbf{F}_{\sigma\delta} & & & & \mathbf{G}_{\delta\sigma} \\
 & \nwarrow & & \nearrow & \\
 & & \mathbf{F}_{\sigma\delta} \cap \mathbf{G}_{\delta\sigma} & & \\
 & & \uparrow & & \\
 & & \mathcal{B}(\mathbf{F}_\sigma) = \mathcal{B}(\mathbf{G}_\delta) & & \\
 & \nearrow & & \nwarrow & \\
\mathbf{F}_\sigma & & & & \mathbf{G}_\delta = \{V^\delta : V \subseteq X^*\} \\
 & \nwarrow & & \nearrow & \\
 & & \mathbf{F}_\sigma \cap \mathbf{G}_\delta & & \\
 & & \uparrow & & \\
 & & \mathcal{B}(\mathbf{F}) = \mathcal{B}(\mathbf{G}) & & \\
 & \nearrow & & \nwarrow & \\
\{\boldsymbol{ls}\, W : W \subseteq X^*\} = \mathbf{F} & & & & \mathbf{G} = \{W \cdot X^\omega : W \subseteq X^*\} \\
 & \nwarrow & & \nearrow & \\
 & & \mathbf{F} \cap \mathbf{G} & &
\end{array}$$

2.2 Continuous mappings

Let X, Y be finite alphabets. In this part we consider continuous mappings from (X^ω, ϱ) to (Y^ω, ϱ).[2] We recall that a mapping $\Phi : X^\omega \to Y^\omega$ is *continuous* provided it satisfies the following property.

$$\Phi(\lim_{i\to\infty} \xi_i) = \lim_{i\to\infty} \Phi(\xi_i) \tag{2.4}$$

[2]Since there is no danger of confusion, we use the same symbol ϱ for the metrics in both spaces.

Here it is understood that the identity holds if only $\lim_{i\to\infty} \xi_i$ exists. Next we derive a relationship between continuous mappings $\Phi : X^\omega \to Y^\omega$ and word functions monotone with respect to the prefix relation.
To this end we call a function $\varphi : X^* \to Y^*$ *sequential* provided for all words $w, v \in X^*$ it holds $\varphi(w) \sqsubseteq \varphi(v)$ whenever $w \sqsubseteq v$, moreover, a mapping $\varphi : X^* \to Y^*$ is referred to as *totally unbounded* if the image $\varphi(W)$ of every infinite language $W \subseteq X^*$ is again infinite. Consequently, every totally unbounded sequential mapping $\varphi : X^* \to Y^*$ defines in a unique way an extension mapping $\overline{\varphi} : X^\omega \to Y^\omega$ via the equation:

$$\mathbf{A}(\overline{\varphi}(\xi)) = \mathbf{A}(\varphi(\mathbf{A}(\xi))) \tag{2.5}$$

Then we have the following.

Theorem 2.1 *Let $\Phi : X^\omega \to Y^\omega$. Φ is continuous if and only if there is a totally unbounded sequential mapping $\varphi : X^* \to Y^*$ such that $\Phi = \overline{\varphi}$,*

Proof. First remark that a mapping $\Phi : X^\omega \to Y^\omega$ is continuous if and only if for every infinite family of words $(w_i)_{i\in\mathbb{N}}$ with $w_i \sqsubset w_{i+1}$ it holds

$$\lim_{i\to\infty} \max\{\varrho(\Phi(\xi), \Phi(\eta)) : \xi, \eta \in w_i \cdot X^\omega\} = 0 \ .$$

Let $\Phi : X^\omega \to Y^\omega$ be continuous. Then the image $\Phi(w \cdot X^\omega)$ of a ball $w \cdot X^\omega$ is either a single point $\{\eta_w\} \subseteq Y^\omega$ or there is a smallest ball $v_w \cdot Y^\omega$ which contains $\Phi(w \cdot X^\omega)$, that is, v_w is the longest word such that $v_w \cdot Y^\omega \supseteq \Phi(w \cdot X^\omega)$. We define φ inductively as follows.

$$\begin{aligned} \varphi(e) &:= e \text{ , and} \\ \varphi(w \cdot x) &:= \begin{cases} v_{w\cdot x} & \text{, if } v_{w\cdot x} \text{ exists} \\ \eta_{w\cdot x}(0, |\varphi(w)| + 1] & \text{, otherwise} \end{cases} \end{aligned} \tag{2.6}$$

Since $\Phi(w \cdot X^\omega) \supseteq \Phi(w \cdot x \cdot X^\omega)$, the function φ is sequential.
It remains to show that φ is totally unbounded. First we show that the image of an infinite directed family of words $(w_i)_{i\in\mathbb{N}}$, $w_i \sqsubset w_{i+1}$ is infinite. Following the above remark on the continuity of Φ we observe that $\lim_{i\to\infty} |v_{w_i}| = \infty$ or, otherwise, $\Phi(w_i \cdot X^\omega) = \{\eta_{w_i}\}$ for some $i \in \mathbb{N}$. In both cases it is evident that $\{\varphi(w_i) : i \in \mathbb{N}\}$ is infinite.
Now let $(w_i)_{i\in\mathbb{N}}$ be an arbitrary infinite family of words. Assume $\{\varphi(w_i) : i \in \mathbb{N}\}$ to be finite. Without loss of generality we may assume $\varphi(w_i) = v$ for all $i \in \mathbb{N}$. Consider the family $(w_i \cdot x^\omega)_{i\in\mathbb{N}}$ for some $x \in X$ which is an infinite subfamily of X^ω. Since the metric space (X^ω, ϱ) is compact, this family contains a convergent subfamily $(w_{i_j} \cdot x^\omega)_{j\in\mathbb{N}}$. Let $\lim_{j\to\infty} w_{i_j} \cdot x^\omega = \xi \in X^\omega$. Then every prefix u of ξ is a prefix of some word w_{i_j}, hence $\varphi(u) \sqsubseteq \varphi(w_{i_j}) = v$, and $\varphi(\mathbf{A}(\xi))$ is a finite set, which was shown to be impossible.

Q.E.D.

Theorem 2.1 shows a close relationship between continuous mappings $\Phi : X^\omega \to Y^\omega$ and sequential functions $\varphi : X^* \to Y^*$. In the preceding proof we derived from Φ a particular function φ generating Φ. The structure of the set of all sequential functions generating

one continuous mapping Φ, $\{\varphi : \overline{\varphi} = \Phi\}$, is investigated in more detail in [LS77, §7.1 and §7.4].
We conclude our considerations on continuous mappings with considering again the Borel hierarchy. First we mention the following well-known preservation property of the inverse of continuous mappings.

Theorem 2.2 *A mapping $\Phi : X^\omega \to Y^\omega$ is continuous if and only if for every closed (open) subset $E \subseteq Y^\omega$ its preimage $\Phi^{-1}(E)$ is also closed (open).*

Since the inverse mapping $\Phi^{-1} : 2^{Y^\omega} \to 2^{X^\omega}$ preserves complement and arbitrary union and intersection, from Theorem 2.2 the following property holds.

Corollary 2.2 *Let $\Phi : X^\omega \to Y^\omega$ be continuous and $\mathcal{M} \subseteq 2^{Y^\omega} \cup 2^{X^\omega}$ be a Borel class. Then $E \in \mathcal{M}$ or $E \in \mathcal{B}(\mathcal{M})$ implies $\Phi^{-1}(E) \in \mathcal{M}$ or $\Phi^{-1}(E) \in \mathcal{B}(\mathcal{M})$, respectively.*

The images of Borel sets under continuous mappings are, however, not necessarily again Borel sets. To characterize them we have to introduce analytic sets, that is, projective sets of the first class.
Projections like $pr_1 : (X \times Y)^* \to X^*$ are totally unbounded sequential mappings. Their extensions like $\overline{pr}_1 : (X \times Y)^\omega \to X^\omega$ are, therefore, continuous mappings. As usually, we set $\mathcal{P}_1 := \{\overline{pr}_1(E) : E \subseteq (X \times Y)^\omega \wedge E \in \mathbf{G}_\delta\}$ and refer to $\mathcal{P}_1$ as the class of *analytic subsets* of X^ω. It holds the following.

Lemma 2.1 *Let $F \subseteq X^\omega$ be a Borel set and let $\Phi : X^\omega \to Y^\omega$ be continuous. Then $\Phi(F) \in \mathcal{P}_1$.*

In particular, all Borel sets in X^ω are analytic sets. Since (X^ω, ϱ) is compact the continuous image of a closed set is again closed. This yields a strengthening of Lemma 2.1 for the Borel classes $\mathbf{F}$ and $\mathbf{F}_\sigma$.

Lemma 2.2 *Let $E \subseteq X^\omega$ be closed or an $\mathbf{F}_\sigma$-set, and let $\Phi : X^\omega \to Y^\omega$ be continuous. Then $\Phi(E)$ is also closed or an $\mathbf{F}_\sigma$-set, respectively.*

Another relation between Borel sets and analytic sets is given by Souslin's theorem.

Theorem 2.3 *A subset $F \subseteq X^\omega$ is a Borel set if and only if F and $X^\omega \setminus F$ are analytic sets.*

2.3 Wadge's hierarchy

As we shall see later in Section 5 the hierarchy of Borel classes is too coarse to yield topologically based hierarchy results for certain classes of ω-languages. In view of Theorem 2.2 and Corollary 2.2 we may use continuous mappings to refine the Borel hierarchy in the following sense. The resulting hierarchy is called Wadge hierarchy after its inventor W. W. Wadge (cf. [Mo80], [vE86]).

We call a subset $E \subseteq X^\omega$ *Wadge reducible* to $F \subseteq X^\omega$ (short: $E \leq_W F$) iff $E = \Phi^{-1}(F)$ for some continuous mapping $\Phi : X^\omega \to X^\omega$, and we call E and F *Wadge equivalent* (short: $E \equiv_W F$) provided $E \leq_W F$ and $F \leq_W E$.
Clearly, the relation $\leq_W$ is reflexive and transitive, thus $\equiv_W$ is an equivalence relation. The equivalence classes of $\equiv_W$ are called *Wadge degrees*.
For a subset $F \subseteq X^\omega$ we call $\{E : E \subseteq X^\omega \wedge E \leq_W F\}$ the *Wadge class* of F. We remark that due to Corollary 2.2 each Borel class $\mathcal{M}$ contains the Wadge class $\{E : E \leq_W F\}$ whenever $F \in \mathcal{M}$.
The next propositions show that Borel sets are in some sense comparable with respect to $\leq_W$.

Theorem 2.4 (Wadge) *If $E, F \subseteq X^\omega$ are Borel sets, then $E \leq_W F$ or $F \leq_W X^\omega \setminus E$.*

Lemma 2.3 *Let $E, F \subseteq X^\omega$ be Borel sets. It holds one of the following four relations: $E <_W F$, $F <_W E$, $E \equiv_W F$, or $X^\omega \setminus E \equiv_W F$, and if $E <_W F$ then also $E <_W X^\omega \setminus F$.*

Though the differences of Borel classes $\mathbf{F} \setminus \mathbf{G}$, $\mathbf{G} \setminus \mathbf{F}$, $\mathbf{F}_\sigma \setminus \mathbf{G}_\delta$ and $\mathbf{G}_\delta \setminus \mathbf{F}_\sigma$ are Wadge degrees, already in $\mathcal{B}(\mathbf{G})$ Wadge reducibilities achieve a refinement of the Borel hierarchy in X^ω.

2.4 Joint topologies on $X^* \cup X^\omega$

Infinite words are in a natural sense least upper bounds of a directed family of finite words. This may be regarded as a limit process. Several concepts of limits are associated with topology. Therefore, it would be desirable to have a topology on a space containing X^* and X^ω and compatible with the Cantor topology an X^ω.
In [BN80] a metric extending the one from Eq. (2.1) was introduced (We denote it likewise with the same letter ϱ.).

$$\varrho(p, q) := \inf\{r^{-|w|} : w \in \mathbf{A}(p) \cup \mathbf{A}(q)\}$$

where $p, q \in X^* \cup X^\omega$.
It turns out that in this metric the space $X^* \cup X^\omega$ may be considered as a closed subset of the space $(X \cup \{d\})^\omega$ where d is a letter not in X. The function

$$f(p) := \begin{cases} p & , \quad p \in X^\omega \\ p \cdot d^\omega & , \quad p \in X^* \end{cases}$$

is a one-to-one and distance preserving embedding of $X^* \cup X^\omega$ into $X^* \cdot d^\omega \cup X^\omega \subseteq (X \cup \{d\})^\omega$.
As R. Redziejowski [Re86] pointed out, the limit in this natural extension of Cantor topology does, however, not coincide with the idea of an infinite word being a least upper bound of an infinite directed set of finite words. This drawback seems to be essential, since in the theory of automata on infinite words exactly this latter variant of limit is used. So, for example, in the extension of Cantor topology the infinite word a^ω is the limit of the sequence $(a^n \cdot b)_{n \in \mathbb{N}}$ whereas it is clearly not its least upper bound.

Redziejowski [Re86] shows a way how to avoid these inconsistencies by introducing a topology on $X^* \cup X^\omega$ whose limit coincides with the above mentioned least upper bound. This space is, however, not metrizable, and thus differs essentially from Cantor topology.
In this paper we present a different idea: We do not try to construct a topology on $X^* \cup X^\omega$, but we relate a topology on X^* to the Cantor topology on X^ω.
Since X^* is only a countable set, we cannot expect a topology on X^* to be close to a metrizable topology, in fact we will use a topology on X^* in which the Borel hierarchy collapses to the classes $\mathbf{F}$ and $\mathbf{G}$:
We define a language $W \subseteq X^\omega$ to be *open* if $W \cdot X^* = W$, thus resembling the definition of open sets in Cantor topology. Likewise, a language $V \subseteq X^*$ is *closed* iff $V = \mathbf{A}(V)$. Since $\bigcup_{i\in\mathbb{N}} \mathbf{A}(V_i) = \mathbf{A}(\bigcup_{i\in\mathbb{N}} V_i)$, the Borel classes $\mathbf{F}$ and $\mathbf{F}_\sigma$, and hence also $\mathbf{G}$ and $\mathbf{G}_\delta$, coincide in the space X^*.
We link this topology on X^* via our δ-limit to the Cantor topology on X^ω. Then every language $W \subseteq X^*$ has as its image the $\mathbf{G}_\delta$-set $W^\delta \subseteq X^\omega$. (In fact, because of the different cardinalities of the spaces X^* and X^ω, we cannot expect to obtain every subset of X^ω as an image.) One easily observes, that due to the properties $(W \cdot X^*)^\delta = W \cdot X^\omega$ and $\xi \in \mathbf{A}(V)^\delta \Leftrightarrow \mathbf{A}(\xi) \subseteq \mathbf{A}(V)$ there is a close correspondence between open or closed subsets, respectively, in both topologies. To be more specific, the image of every open (closed) language $W \subseteq X^*$ is also open (closed), and every open (closed) ω-language $F \subseteq X^\omega$ is the image of an appropriately chosen open (closed) language. Moreover, one can easily prove that $(\bigcup_{i=1}^{n}(W_i \cdot X^* \setminus V_i \cdot X^*))^\delta = \bigcup_{i=1}^{n}(W_i \cdot X^\omega \setminus V_i \cdot X^\omega)$. This shows that the presented correspondence extends to the Boolean closure of $\mathbf{G}$, $\mathcal{B}(\mathbf{G})$, in both spaces. What about the class $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$ in the Borel hierarchy of X^ω?
Above we mentioned that the Borel hierarchy in X^* collapses to the classes $\mathbf{F}$ and $\mathbf{G}$. Nevertheless, we find a class of languages which corresponds to the class $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$ in the above mentioned sense.
As in [St87a] we call a language $W \subseteq X^*$ a (σ,δ)-*subset* of X^* provided for every $\xi \in X^\omega$ either $\mathbf{A}(\xi) \cap W$ or $\mathbf{A}(\xi) \setminus W$ is a finite language. Then it holds

Theorem 2.5 ([St87a]) *If $W \subseteq X^*$ is a (σ,δ)-subset of X^* then $W^\delta \in \mathbf{F}_\sigma \cap \mathbf{G}_\delta$, and vice versa, if $F \subseteq X^\omega$ is simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set then there is a (σ,δ)-set $W \subseteq X^*$ such that $F = W^\delta$.*
If, moreover, $F = V^\delta \in \mathbf{F}_\sigma$ where V is a regular (recursive) language then there is a regular (recursive) (σ,δ)-set $W \subseteq X^*$ such that $V \subseteq W$ and $F = W^\delta$.

We summarize these relationships in the following table:

a language	corresponds to	an ω-language
$W = A(W)$, closed	$\longrightarrow$	F closed
$W = W \cdot X^*$, open	$\longrightarrow$	F open
$W \in \mathcal{B}(\mathbf{G})$	$\longrightarrow$	$F \in \mathcal{B}(\mathbf{G})$
W a (σ,δ)-set	$\longrightarrow$	$F \in \mathbf{F}_\sigma \cap \mathbf{G}_\delta$
W arbitrary	$\longrightarrow$	$F \in \mathbf{G}_\delta$

In Theorem 2.1 we derived a connection between continuous mappings Φ of the Cantor space and sequential word-functions φ. It is interesting to note that to some extent our δ-limit connects also the images or preimages of φ and $\Phi = \overline{\varphi}$.

Lemma 2.4 ([St87a]) *Let $\varphi : X^* \to Y^*$ be a totally unbounded sequential function, and let $V \subseteq Y^*$ be a (σ, δ)-subset of Y^* and $W \subseteq X^*$ be arbitrary. Then it holds*

$$\boldsymbol{ls}\,\varphi(W) = \overline{\varphi}(\boldsymbol{ls}\,W)\ , \text{ and} \tag{2.7}$$
$$(\varphi^{-1}(V))^{\delta} = \overline{\varphi}^{-1}(V^{\delta})\ . \tag{2.8}$$

Remark 2.1 In view of the identities $\mathbf{A}(\varphi(W)) = \mathbf{A}(\varphi(\mathbf{A}(W)))$ and $\boldsymbol{ls}\,W = \mathbf{A}(W)^{\delta}$ our Eq. (2.7) is in fact an equation for closed languages. Though Lemma 2.2 states that the continuous image of an $\mathbf{F}_{\sigma}$-set is again an $\mathbf{F}_{\sigma}$-set, in contrast to Eq. (2.8), an identity like $\varphi(U)^{\delta} = \overline{\varphi}(U^{\delta})$ cannot be established for (σ, δ)-subsets of X^*. For instance, let $\varphi(w) := a^{|w|}$ and $U := a^* \cdot b \subseteq \{a, b\}^*$. Then $U = \mathbf{A}(U) \cap U \cdot \{a, b\}^* \in \mathcal{B}(\mathbf{G})$, but $\varphi(U)^{\delta} = \{a^{\omega}\}$ and $\overline{\varphi}(U^{\delta}) = \emptyset$.

For more detailed investigations on the validity of equations similar to the ones of Lemma 2.4 also in case of partially defined and not necessarily totally unbounded sequential functions see [St87a].

3 The Chomsky-hierarchy of ω-languages

In formal language theory one starting point was the study of phrase-structure-grammars and the Chomsky-hierarchy. Though the study of ω-languages started from logical considerations, most of the classes of ω-languages best known today are closely related to the classes of languages in Chomsky's hierarchy. Up to now, however, it appeared that generation by grammars, except for some investigations concerning context-free ω-languages [CG77], [Ni77,78], [BN80], is of minor importance in the ω-case. Instead if one refers to language theoretic generation mainly operations which transfer languages to ω-languages like ${}^{\omega}$, $\boldsymbol{ls}$, ${}^{\delta}$ are used. This yields also descriptions of classes of ω-languages in terms of the Chomsky hierarchy.
Therefore, in this paper we will not pursue generation of ω-languages by grammars, but will mainly deal with acceptance by automata and the just mentioned transfer operations.

3.1 Acceptance of ω-languages by automata

Now let us consider in more detail the accepting process for ω-languages. Since finite automata as well as pushdown automata may be viewed as special cases of Turing machines, we start with this more general case of accepting devices. We consider Turing machines $M = (X, ?\,, Z, z_0, \boldsymbol{R})$ with an input tape on which the read-only-head moves only to right,

n working tapes, X as its input alphabet, ? as its worktape alphabet, Z the finite set of internal states, z_0 the initial state, and the relation

$$\boldsymbol{R} \subseteq Z \times X \times ?^n \times Z \times \{0,+1\} \times (? \times \{-1,0,+1\})^n$$

defining the next configuration. If $n = 0$, then M is called a finite automaton, and a pushdown automaton may be conceived as a Turing machine with only one working tape on which the head has to erase the written symbols when moving to the left.
Here $(z, x_0, x_1, \ldots, x_n; z, y_0, y_1, \ldots, y_n) \in \boldsymbol{R}$ means that when M is in state $z \in Z$, reads $x_0 \in X$ on its input tape and $x_i \in ?$ on its worktapes $(i \in \{1, \ldots, n\})$, M changes its state to $z' \in Z$, moves its head on the input tape to the right if $y_0 = +1$ or if $y_0 = 0$ does not move the head, and for $(i \in \{1, \ldots, n\})$ and $y_i = (x'_i, m_i)$ with $x'_i \in ?$ and $m_i \in \{-1, 0, +1\}$ the machine M writes x'_i instead of x_i in its i-th worktape and moves the head on this tape to the left, if $m_i = -1$, to the right, if $m_i = +1$, or does not move it, if $m_i = 0$.
Unless stated otherwise, in the sequel we shall assume that our accepting devices be fully defined, i.e. the transition relation $\boldsymbol{R}$ is to contain for every situation $(z, x_0, \ldots, x_n)$ at least one (exactly one, if the device is deterministic) move $(z, x_0, \ldots, x_n; z', y_0, \ldots, y_n)$.
Because of the complexity of notation of the accepting process for multitape Turing machines, we will describe the construction of machines and their behaviour only in an informal manner leaving the details to the reader.
Let the input of the Turing machine be some sequence $\xi \in X^\omega$. We accept ξ if the sequence of states the Turing machine runs through in its (some of its, if the machine is nondeterministic) computation with input ξ, $\Phi_{\mathcal{M}}(\xi)$, fulfills a certain condition.

Remark 3.1 It should be noted that our acceptance condition differs from the ones considered in [EH93], [CG77], [CG78a]. We do not require that the machine M read the whole input tape we only require that the device runs forever.
In the above mentioned papers an infinite word is accepted only if additionally the machine reads the whole input tape. This latter condition allows for a further possibility to reject an input ω-word ξ, thus introduces into acceptance an additional $\forall\exists$-condition. This $\forall\exists$-condition causes, in particular in nondeterministic case, the classes investigated here and in [SW78] to differ from the ones of [EH93], [CG77], [CG78a]. We shall return to this point later.

We say that an input sequence $\xi \in X^\omega$ is accepted by M according to condition Ξ if there is a sequence of states $\eta \in \Phi_M(\xi)$ such that η satisfies Ξ.
In the sequel we shall confine to conditions which are constructed along the following lines: Let $\alpha : Z^\omega \to 2^Z$ be a mapping which assigns to every ω-word $\eta \in Z^\omega$ a subset $Z' \subseteq Z$, and let $R \subseteq 2^Z \times 2^Z$ be a relation between subsets of Z. We say that a pair $(M, \mathcal{Z})$ where $\mathcal{Z} \subseteq 2^Z$ accepts an ω-word $\xi \in X^\omega$ if and only if

$$\exists Z' \exists \eta (Z' \in \mathcal{Z} \wedge \eta \in \Phi_{\mathcal{M}}(\xi) \wedge (\alpha(\eta), Z') \in R) \ .$$

Here we shall be mainly concerned with the following mappings α and relations R: For an ω-word $\eta \in Z^\omega$ let $ran(\eta) := \{z : z \in Z \wedge \exists i (i \in \mathbb{N} \setminus \{0\} \wedge \eta(i) = z)\}$ be the *range* of η

(considered as a mapping $\eta : \mathbb{N} \setminus \{0\} \to Z$), that is, the set of all letters occurring in η, and let $inf(\eta) := \{z : z \in Z \wedge \eta^{-1}(z) \text{ is infinite}\}$ be the *infinity set* of η, that is, the set of all letters occurring infinitely often in η. As relations R we shall use $=$, $\subseteq$ and $\sqcap$ where $Z' \sqcap Z'' :\Leftrightarrow Z' \cap Z'' \neq \emptyset$. Other mappings and relations used throughout the literature are $fin(\eta) := ran(\eta) \setminus inf(\eta)$, or $\supseteq$ [MY88, Ya89], and $\not\subseteq$, which is equivalent to $\sqcap$ (cf. [Wa79]). We obtain the six types of acceptance presented in the following table. For the sake of completeness we add a simple description and their correspondence to the five types originally defined by Landweber [La69] a notation which is also frequently used. Moreover, we give hints to the first appearance.

$(ran, \sqcap)$	1-acceptance	at least once	[HS67]
$(ran, \subseteq)$	1'-acceptance	everywhere	[La69]
$(ran, =)$			[SW74]
$(inf, \sqcap)$	2-acceptance	infinitely often	[Bü60]
$(inf, \subseteq)$	2'-acceptance	almost everywhere	[La69]
$(inf, =)$	3-acceptance		[Mu63]

Other acceptance conditions not based on a subset $\mathcal{Z} \subseteq 2^Z$ but rather on a subset $\mathcal{R} \subseteq 2^Z \times 2^Z$ were introduced by Rabin [Ra69] and Streett [Sr82]. A comparison of these and Büchi's [Bü60] and Muller's [Mu63] acceptance conditions can be found in [CD93] or [VW94].
The ω-language accepted by a pair $(M, \mathcal{Z})$ where $\mathcal{Z} \subseteq 2^Z$ according to mode (α, R) is defined as

$$T_R^\alpha(M, \mathcal{Z}) := \{\xi : \xi \in X^\omega \wedge \exists Z' \exists \eta (Z' \in \mathcal{Z} \wedge \eta \in \Phi_M(\xi) \wedge (\alpha(\eta), Z') \in R)\} .$$

For our six acceptance modes we have the following relationships.

Proposition 3.1 *Let $\alpha \in \{ran, inf\}$, $\mathcal{Z} \subseteq 2^Z$ and define $\hat{\mathcal{Z}} := \{Z'' : \exists Z'(Z'' \subseteq Z' \wedge Z' \in \mathcal{Z})\}$, $\check{Z} := \bigcup_{Z' \in \mathcal{Z}} Z'$ and $\check{\mathcal{Z}} := \{Z' : Z' \subseteq Z \wedge \check{Z} \cap Z' \neq \emptyset)\}$. Then*

$$T_\subseteq^\alpha(M, \mathcal{Z}) = T_\subseteq^\alpha(M, \hat{\mathcal{Z}}) = T_=^\alpha(M, \hat{\mathcal{Z}}) \ , \tag{3.1}$$
$$T_\sqcap^\alpha(M, \mathcal{Z}) = T_\sqcap^\alpha(M, \{\check{Z}\}) = T_=^\alpha(M, \check{\mathcal{Z}}) \ , \tag{3.2}$$

and if the machine M is a deterministic one, we have further

$$T_\sqcap^\alpha(M, \{\check{Z}\}) = X^\omega \setminus T_\subseteq^\alpha(M, \{Z \setminus \check{Z}\}) \ , \text{ and} \tag{3.3}$$
$$T_=^\alpha(M, \{Z'\}) = T_\subseteq^\alpha(M, \{Z'\}) \cap \bigcap_{z \in Z'} T_\sqcap^\alpha(M, \{\{z\}\}) \ . \tag{3.4}$$

For deterministic accepting devices M it is natural to consider Φ_M as a mapping $\Phi_M : X^\omega \to Z^\omega$. It is obvious that this mapping is continuous, because $\varrho(\Phi_M(\xi), \Phi_M(\eta)) \leq \varrho(\xi, \eta)$. Then we have the following.

Proposition 3.2 *Let M be a deterministic Turing machine, and let $\Phi_M : X^\omega \to Z^\omega$ be the mapping as described above. It holds*

$$T_{\subseteq}^{ran}(M,\{Z'\}) = \Phi_M^{-1}((Z')^\omega)\ , \tag{3.5}$$
$$T_{\sqcap}^{ran}(M,\{Z'\}) = \Phi_M^{-1}(Z^*\cdot Z'\cdot Z^\omega)\ , \tag{3.6}$$
$$T_{\subseteq}^{\inf}(M,\{Z'\}) = \Phi_M^{-1}(Z^*\cdot (Z')^\omega)\ , and \tag{3.7}$$
$$T_{\sqcap}^{\inf}(M,\{Z'\}) = \Phi_M^{-1}((Z^*\cdot Z')^\omega)\ . \tag{3.8}$$

Since $(Z')^\omega \in \mathbf{F}$, $Z^*\cdot Z'\cdot Z^\omega \in \mathbf{G}$, $Z^*\cdot (Z')^\omega \in \mathbf{F}_\sigma$, and $(Z^*\cdot Z')^\omega \in \mathbf{G}_\delta$, we obtain some relations between accepted classes and Borel classes for deterministic devices.

Corollary 3.1 *Let $M = (X, ?, Z, z_0, \boldsymbol{R})$ be a deterministic Turing machine, and let $\mathcal{Z} \subseteq 2^Z$. Then*

$$T_{\subseteq}^{ran}(M,\mathcal{Z}) \in \mathbf{F}\ ,\quad T_{\sqcap}^{ran}(M,\mathcal{Z}) \in \mathbf{G}\ ,\qquad T_{=}^{ran}(M,\mathcal{Z}) \in \mathcal{B}(\mathbf{G})\ ,$$
$$T_{\subseteq}^{inf}(M,\mathcal{Z}) \in \mathbf{F}_\sigma\ ,\quad T_{\sqcap}^{inf}(M,\mathcal{Z}) \in \mathbf{G}_\delta\ ,\ \textit{and}\ \ T_{=}^{inf}(M,\mathcal{Z}) \in \mathcal{B}(\mathbf{G}_\delta)\ .$$

Our Eq. (3.2) shows that for acceptance via $(\alpha,\sqcap)$-mode a single final set is sufficient. Next we show that the same is true for $(\alpha,\subseteq)$-mode, but with a slight modification of the accepting Turing machine, whereas in case of $(\alpha,=)$-mode the results of Wagner [Wa79] (see also Section 5) show that it is not possible to bound the cardinality of $\mathcal{Z}$.

Lemma 3.1 *Let $\alpha = inf$ or $\alpha = ran$ and $F = T_{\subseteq}^{\alpha}(M,\mathcal{Z})$ for some Turing machine $M = (X, Z, z_0, \boldsymbol{R})$ and $\mathcal{Z} \subseteq 2^Z$. Then there is a Turing machine $M_1 = (X, S, s_0, \boldsymbol{R}_1)$ and some $S' \subseteq S$ satisfying $F = T_{\subseteq}^{\alpha}(M_1,\{S'\})$.*

Proof. We follow the line of the construction given in [SW74, Lemma 7]. Let $\mathcal{Z} = \{Z_1,\ldots,Z_k\}$. Then we set

$$\begin{aligned} S &:= Z \times 2^{\{1,\ldots,k\}} \\ s_0 &:= (z_0,\{1,\ldots,k\}) \\ S' &:= Z \times \left(2^{\{1,\ldots,k\}} \setminus \{\emptyset\}\right), \text{ and} \\ ((z,A),\vec{x};(z',A'),\vec{y}) \in \boldsymbol{R}_1 &\quad \text{iff} \quad (z,\vec{x};z',\vec{y}) \in \boldsymbol{R} \end{aligned}$$

where $\vec{x} := (x_0,x_1,\ldots,x_n)$ and $\vec{y} := (y_0,y_1,\ldots,y_n)$ are shortcuts and

$$A' := \begin{cases} A \cap \{i : z' \in Z_i\} & \text{, if } A \neq \emptyset \text{ , and} \\ \{1,\ldots,k\} & \text{, otherwise.} \end{cases}$$

Informally, the construction works as follows: The machine M_1 remembers in the 2^Z-component of its states which one of the final sets $Z_1,\ldots,Z_k$ the machine M has visited continuously. If this set of indices is empty M_1 reinitializes it as $\{1,\ldots,k\}$. This much explanation shows our assertion.

Q.E.D.

Remark 3.2 From the preceding proof it is immediate that the construction introduces only an additional deterministic finite-state control. Thus neither the principal computational power of the underlying accepting device is increased, nor nondeterminism is introduced by the construction.

Next we show that for nondeterministic machines the acceptance modes $(\mathit{inf}, \sqcap)$ and $(\mathit{inf}, =)$ are of equal power.

Lemma 3.2 *Let $F = T_{=}^{\mathit{inf}}(M, \mathcal{Z})$ for some Turing machine $M = (X, Z, z_0, \boldsymbol{R})$ and $\mathcal{Z} \subseteq 2^Z$. Then there is a Turing machine $M_1 = (X, S, s_0, \boldsymbol{R}_1)$ and some $\check{S} \subseteq S$ satisfying $F = T_{\sqcap}^{\mathit{inf}}(M_1, \{\check{S}\})$.*

Proof. Let $\mathcal{Z} = \{Z_1, \ldots, Z_k\}$. Then we set

$$\begin{aligned} S &:= Z \cup \left(\bigcup_{i=1}^{k} \{i\} \times Z_i \times \left(2^{Z_i} \setminus \{\emptyset\}\right) \right) \cup \{s\} \ , \\ s_0 &:= z_0 \ , \\ \check{S} &:= \{(i, z, Z_i) : z \in Z_i\} \ , \end{aligned}$$

and for $\vec{x} := (x_0, x_1, \ldots, x_n)$ and $\vec{y} := (y_0, y_1, \ldots, y_n)$

$$\begin{aligned} \boldsymbol{R}_1 := \ & \boldsymbol{R} \cup \{(z, \vec{x}; (i, z', \{z'\}), \vec{y}) : (z, \vec{x}; z', \vec{y}) \in \boldsymbol{R} \wedge z' \in Z_i\} \cup \\ & \{((i, z, A), \vec{x}; (i, z', A \cup \{z'\}), \vec{y}) : (z, \vec{x}; z', \vec{y}) \in \boldsymbol{R} \wedge z' \in Z_i \wedge A \neq Z_i\} \cup \\ & \{((i, z, Z_i), \vec{x}; (i, z', \{z'\}), \vec{y}) : (z, \vec{x}; z', \vec{y}) \in \boldsymbol{R} \wedge z' \in Z_i\} \cup \\ & \{((i, z, A), \vec{x}; s, \vec{y}) : (z, \vec{x}; z', \vec{y}) \in \boldsymbol{R} \wedge z' \notin Z_i\} \cup \\ & \{(s, \vec{x}; s, ((x_0, 0), (x_1, +1), \ldots, (x_n, +1)) : x_j \in X\} \ . \end{aligned}$$

Informally, machine M_1 at some appropriate time instant (when $z' \in Z_i$) starts to guess whether M will only visit states in the final set Z_i. If this is true then M_1 will accept if M infinitely often runs through the whole set Z_i. Otherwise, if M leaves Z_i, M_1 switches to the nonfinal absorbing state s.

Q.E.D.

Remark 3.3 As in the proof of Lemma 3.1 this construction, besides a nondeterministic guess, introduces only an additional finite-state control.

Using a similar idea we can also show the following.

Lemma 3.3 *Let $F = T_{\subseteq}^{\mathit{inf}}(M, \mathcal{Z})$ for some Turing machine $M = (X, Z, z_0, \boldsymbol{R})$ and $\mathcal{Z} \subseteq 2^Z$. Then there is a Turing machine $M_1 = (X, S, s_0, \boldsymbol{R}_1)$ and some $\mathcal{S} \subseteq 2^S$ satisfying $F = T_{=}^{\mathit{ran}}(M_1, \mathcal{S})$.*

3.2 Finite automata and regular ω-languages

We start the presentation of the classes of the Chomsky hierarchy of ω-languages with ω-languages defined by the simplest accepting device, finite automata. This class is the one most extensively investigated in literature, and the one with the most widespread range of applications.

As it was explained above, a finite automaton is a quadruple $M = (X, Z, z_0, \boldsymbol{R})$ where X, Z are finite sets of input letters and states, respectively, $z_0 \in Z$ is the initial state, and $\boldsymbol{R} \subseteq Z \times X \times Z$ is the transition relation. An automaton is said to be *deterministic* if for each pair $(z, x) \in Z \times X$ there is at most one state z' such that in $(z, x, z') \in \boldsymbol{R}$. In order to emphasize that an automaton M is deterministic, we shall also write $M = (X, Z, z_0, f)$ instead of $M = (X, Z, z_0, \boldsymbol{R})$, where $f : Z \times X \to Z$ is the *transition function*.

In order to formalize the accepting behaviour of finite automata, we introduce the notion of a run of an automaton as follows.

Let $\xi \in X^\omega$ be an ω-word, a *complete run* of M on ξ is a sequence $\zeta \in (X \times Z)^\omega$ of pairs $(\xi(i), z_i) \in X \times Z$ such that $\forall i(i > 0 \to (z_{i-1}, \xi(i), z_i) \in \boldsymbol{R})$. When $\zeta \in (X \times Z)^\omega$ is a complete run of M on ξ the sequence $\overline{pr}_2(\zeta) \in Z^\omega$ is called a *run* of M on ξ. The set of all runs of M on ξ is denoted by $\mathsf{run}_M(\xi)$, or simply by $\mathsf{run}(\xi)$, if there is no danger of confusion.

The notion of a run over a finite word $w = x_1 \cdot \ldots \cdot x_n \in X^*$ is introduced similarly as the finite sequence of pairs (x_i, z_i) with $(z_{i-1}, x_i, z_i) \in \boldsymbol{R}$. We say that a finite automaton $M = (X, Z, z_0, \boldsymbol{R})$ accepts a language $W \subseteq X^*$ if there is a $Z' \subseteq Z$ such that $W = \{w : \mathsf{run}_M(w)$ ends with some $z' \in Z'\}$. Languages accepted by finite automata are known as *regular* languages.

The ω-language $F \subseteq X^\omega$ accepted by a finite automaton $M = (X, Z, z_0, \boldsymbol{R})$ is defined as

$$T_R^\alpha(M, \mathcal{Z}) := \{\xi : \xi \in X^\omega \wedge \exists Z' \exists \eta (Z' \in \mathcal{Z} \wedge \eta \in \mathsf{run}_M(\xi) \wedge (\alpha(\eta), Z') \in R)\} \ .$$

where α and R are defined as above.

Denote by $\mathbf{NFA}^{(X)}(\alpha, R)$ and $\mathbf{DFA}^{(X)}(\alpha, R)$ the of all ω-languages $F \subseteq X^\omega$ accepted by nondeterministic or deterministic finite automata, respectively, and let $\mathbf{R}_\omega^{(X)}$ be the class of all ω-languages $F \subseteq X^\omega$ accepted by finite automata according to at least one of the six modes $(ran, \sqcap)$, $(ran, \subseteq)$, $(ran, =)$, $(inf, \sqcap)$, $(inf, \subseteq)$, and $(inf, =)$.

In the sequel, we will refer to sets $F \in \mathbf{R}_\omega^{(X)}$ as *regular* ω-languages.

It is interesting that for $\alpha \in \{ran,\ inf\}$ and $R \in \{\sqcap,\ \subseteq,\ =\}$ the above mentioned classes $\mathbf{NFA}^{(X)}(\alpha, R)$ and $\mathbf{DFA}^{(X)}(\alpha, R)$ can be characterized in terms of $\mathbf{R}_\omega^{(X)}$ and the Borel hierarchy (cf. [MN66], [La69], [SW74], [Wa79], [TY83]).

Theorem 3.1

$$
\begin{array}{lllllll}
\mathbf{DFA}^{(X)}(ran,\subseteq) & = & \mathbf{NFA}^{(X)}(ran,\subseteq) & = & \mathbf{F}\cap\mathbf{R}_\omega^{(X)} & & \\
\mathbf{DFA}^{(X)}(ran,\sqcap) & = & \mathbf{NFA}^{(X)}(ran,\sqcap) & = & \mathbf{G}\cap\mathbf{R}_\omega^{(X)} & & \\
\mathbf{DFA}^{(X)}(ran,=) & = & \mathbf{F}_\sigma\cap\mathbf{G}_\delta\cap\mathbf{R}_\omega^{(X)} & = & \mathcal{B}(\mathbf{G})\cap\mathbf{R}_\omega^{(X)} & & \\
\mathbf{DFA}^{(X)}(inf,\subseteq) & = & \mathbf{NFA}^{(X)}(inf,\subseteq) & = & \mathbf{NFA}^{(X)}(ran,=) & = & \mathbf{F}_\sigma\cap\mathbf{R}_\omega^{(X)} \\
\mathbf{DFA}^{(X)}(inf,\sqcap) & = & \mathbf{G}_\delta\cap\mathbf{R}_\omega^{(X)} & & & & \\
\mathbf{DFA}^{(X)}(inf,=) & = & \mathbf{NFA}^{(X)}(inf,\sqcap) & = & \mathbf{NFA}^{(X)}(inf,=) & = & \mathbf{R}_\omega^{(X)}
\end{array}
$$

Remark 3.4 The identity $\mathbf{DFA}^{(X)}(inf,=)=\mathbf{NFA}^{(X)}(inf,\sqcap)$ is McNaughton's theorem [MN66] stating that also in case of ω-languages deterministic and nondeterministic finite automata have the same accepting power. Due to this result one easily observes that $\mathbf{R}_\omega^{(X)}\subseteq\mathcal{B}(\mathbf{G}_\delta)$.
The original proof of McNaughton is quite complicated, simpler variants were developed e.g. by Choueka [Ch74], Schützenberger (cf. [Pe85]), and Thomas [Th81] (cf. also [Th90]). In [Sa88] Safra gave a sophisticated essentially optimal variant of the determinization procedure (cf. also [Tho]).

Remark 3.5 The inclusion $\mathbf{NFA}^{(X)}(ran,\sqcap)\subseteq\mathbf{G}$ is due to the fact that we consider only automata having a fully defined transition relation. For partially defined automata we have (cf. [EH93])

$$
\begin{array}{lll}
\mathbf{pDFA}^{(X)}(ran,\sqcap) & = & \{F\cap E : F\in\mathbf{F}\cap\mathbf{R}_\omega^{(X)}\wedge E\in\mathbf{G}\cap\mathbf{R}_\omega^{(X)}\}\text{ and}\\
\mathbf{pNFA}^{(X)}(ran,\sqcap) & = & \mathbf{F}_\sigma\cap\mathbf{R}_\omega^{(X)}\ .
\end{array}
$$

For the other five modes of acceptance the use of partially defined automata does not change the defined classes. This can be easily verified by the introduction of transitions leading to a nonfinal absorbing state.

Regular ω-languages were introduced by Büchi [Bü60] using $(inf,\sqcap)$-aceptance by nondeterministic automata. It was also in his paper where the following closure properties of $\mathbf{R}_\omega^{(X)}$ were mentioned.

Proposition 3.3 $\mathbf{R}_\omega^{(X)}$ *is closed under union, intersection, complement, and product with regular languages and, moreover, if* $E\in\mathbf{R}_\omega^{(X\times Y)}$ *then* $\overline{pr}_1(E)$ *is also regular.*

It should be mentioned that the nondeterministic classes and the deterministic ones are related via a projection lemma, the proof of which is easily accomplished by adding to the input letters a second coordinate which controls the branching behaviour of the nondeterministic automaton.

Lemma 3.4 (Projection lemma for finite automata) *Let* X,Y *be finite alphabets containing at least two letters. Then* $F\in\mathbf{NFA}^{(X\times Y)}(\alpha,R)$ *implies* $\overline{pr}_1(F)\in\mathbf{NFA}^{(X)}(\alpha,R)$. *Conversely, if* $E\in\mathbf{NFA}^{(X)}(\alpha,R)$ *then there are a finite alphabet* Y *and an* ω*-language* $F\in\mathbf{DFA}^{(X\times Y)}(\alpha,R)$ *such that* $E=\overline{pr}_1(F)$.

Next we are going to give a description of regular ω-languages in terms of regular languages.

Theorem 3.2 ([Bü60]) *It holds $F \in \mathbf{R}_\omega^{(X)}$ iff there are an $n \in \mathbb{N}$ and regular languages $W_i, V_i \subseteq X^*$ $(i = 1, \ldots, n)$ such that*

$$F = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega \ .$$

Proof. By standard methods of automata theory it is easy to construct a nondeterministic automaton for which $F = T_\sqcap^{inf}(M, \{\check{Z}\})$, when we are given automata accepting the languages $W_i, V_i \subseteq X^*$.

Conversely, let $F = T_\sqcap^{inf}(M, \{\check{Z}\})$ for some (nondeterministic) automaton $M = (X, Z, z_0, \boldsymbol{R})$, and let $W_z := \{x_1 \cdots x_n : n \in \mathbb{N} \wedge \exists z_1, \ldots, z_n (z_i \in Z \wedge (z_0, x_1, z_1), \ldots, (z_{n-1}, x_n, z_n) \in \boldsymbol{R} \wedge z_n = z)\}$ and $V_z^{z'} := \{x_1 \cdots x_n : n \in \mathbb{N} \wedge \exists z_1, \ldots, z_n (z_i \in Z \wedge (z, x_1, z_1), \ldots, (z_{n-1}, x_n, z_n) \in \boldsymbol{R} \wedge z_n = z \wedge z' \in \{z_1, \ldots z_n\})\}$, that is, W_z consists of all words $w \in X^*$ which move the automaton M from the initial state to state $z \in Z$, and $V_z^{z'}$ consists of all words $v \in X^*$ which move the automaton M from the state z to itself intermediately passing a designated state z'. Then the languages W_z and $V_z^{z'}$ are regular, and the assertion follows from the obvious identity $F = \bigcup_{z \in Z, z' \in \check{Z}} W_z \cdot \left(V_z^{z'}\right)^\omega$.

Q.E.D.

As an immediate consequence of Theorem 3.2 we obtain that the initial word languages of regular ω-languages are again regular.

Lemma 3.5 (Init-lemma for regular ω-languages) *If $F \subseteq X^\omega$ is regular then $\mathbf{A}(F)$ is also regular.*

Representations similar to the one given in Theorem 3.2 in terms of regular languages and the operations $^\omega$, $\boldsymbol{ls}$, and $^\delta$ can be derived also for the subclasses given in Theorem 3.1. First we remark the following property of the mentioned operations.

Lemma 3.6 *If $W \subseteq X^*$ is a regular language, then $W^\omega, \boldsymbol{ls}\, W$ and W^δ are regular ω-languages.*

Proposition 3.4 ([SW74], [TY83])

	Class	Representation	Comment
1.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{G}$	$W \cdot X^\omega$	*W regular*
2.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{F}$	$\boldsymbol{ls}\, W$	*W regular*
3.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta$	$\bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}\, V_i$	*W_i, V_i regular and W_i prefix-free*
3a.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta$	V^δ	*V regular and a (σ, δ)-set*
4.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{F}_\sigma$	$\bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}\, V_i$	*W_i, V_i regular*
5.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{G}_\delta$	$\bigcup_{i=1}^{n} W_i \cdot V_i^\omega$	*W_i, V_i regular and prefix-free*
5a.	$\mathbf{R}_\omega^{(X)} \cap \mathbf{G}_\delta$	V^δ	*V regular*

(Here e.g. line 4 reads as follows: $F \in \mathbf{R}_\omega^{(X)} \cap \mathbf{F}_\sigma$ if and only if there are an $n \in \mathbb{N}$ and regular languages W_i and V_i $(1 \le i \le n)$ such that $F = \bigcup_{i=1}^{n} W_i \cdot \boldsymbol{ls}\, V_i$.)

Next we give examples which show that the classes of regular ω-languages of Proposition 3.4 form a hierarchy analogous to the first levels of the Borel hierarchy. To this end we define for an alphabet $X \supseteq \{0,1\}$ the following regular ω-languages $S_1 := X^* \cdot 0 \cdot X^\omega$, $P_1 := 0^\omega$, $S_2' := X^* \cdot 0^\omega$, and $P_2' := (X \cdot 0)^\omega$. Then we have

$$S_1 \in (\mathbf{G} \setminus \mathbf{F}) \cap \mathbf{R}_\omega^{(X)} \tag{3.9}$$
$$P_1 \in (\mathbf{F} \setminus \mathbf{G}) \cap \mathbf{R}_\omega^{(X)} \tag{3.10}$$
$$0 \cdot S_1 \cup 1 \cdot P_1 \in (\mathcal{B}(\mathbf{G}) \setminus (\mathbf{F} \cup \mathbf{G})) \cap \mathbf{R}_\omega^{(X)} \tag{3.11}$$
$$S_2' \in (\mathbf{F}_\sigma \setminus \mathbf{G}_\delta) \cap \mathbf{R}_\omega^{(X)} \tag{3.12}$$
$$P_2' \in (\mathbf{G}_\delta \setminus \mathbf{F}_\sigma) \cap \mathbf{R}_\omega^{(X)} \tag{3.13}$$
$$0 \cdot S_2' \cup 1 \cdot P_2' \in \mathbf{R}_\omega^{(X)} \setminus (\mathbf{G}_\delta \cup \mathbf{F}_\sigma) \tag{3.14}$$

3.3 Context-free ω-languages

Next we deal with so called context-free ω-languages. As it was mentioned above in the introduction to this section in contrast to the language case we do not pursue the generation of ω-languages by grammars as e.g. in [CG77] or [BN80], but rather define them via their accepting devices, pushdown automata, and relate them to the class of context-free languages in a way similar to Theorem 3.2. In order to avoid confusions because of the different ways of acceptance by pushdown automata used in [CG77] and [Li76] and used here (cf. also Remark 3.1 above) we will not derive a characterization of the classes of ω-languages accepted by nondeterministic pushdown automata analogous to Theorem 3.1 in the case of regular ω-languages.

In order to facilitate our arguments we introduce the usual notation for pushdown automata.
A pushdown automaton is a sextuple $M = (X, ?, Z, z_0, ?_0, \boldsymbol{R})$ where X, Z, z_0, and $\boldsymbol{R}$ are as above in the definition of the multitape Turing machine, $?_0 \in ?$ is the initial pushdown letter. Observe that we consider, as usual, the transition relation $\boldsymbol{R}$ as a subset of $Z \times (X \cup \{e\}) \times ? \times Z \times ?^*$.
A *configuration* of M is a pair $(z, \gamma) \in Z \times ?^*$with the intuitive meaning that the pushdown automaton M is in state $z \in Z$ with a word $\gamma \in ?^*$ as the contents of its pushdown store. For $x \in (X \cup \{e\})$ and $a \in ?$ we write

$$(z, a \cdot \gamma) \xrightarrow{x} (z', \gamma' \cdot \gamma) \tag{3.15}$$

iff $(z, x, a, z', \gamma') \in \boldsymbol{R}$. The formula Eq. (3.15) is called a *move* of M, if x is the empty word e, it is called an *e-move*.
In case of a pushdown automaton $M = (X, ?, Z, z_0, ?_0, \boldsymbol{R})$ and an input $\xi \in X^\omega$ we call an infinite sequence of $((z_i, x_i, \gamma_i))_{i=1}^\infty$ where $(z_i, \gamma_i) \stackrel{x_{i+1}}{\to} (z_{i+1}, \gamma_{i+1})$, starting with the initial configuration $(z_0, ?_0)$ and satisfying $\prod_{i=1}^\infty x_i \sqsubseteq \xi$,[3] a *complete run* over ξ, and the according sequence of states $z_1 \cdot z_2 \cdots z_i \cdots \in Z^\omega$ a *run* of M over ξ. As for finite automata we use the notation $\mathsf{run}_M(\xi)$ to denote the set of all runs of M over ξ.
We define the ω-language accepted by a pushdown automaton in the same way as it was done above for finite automata, and we denote by $\mathbf{NPDA}^{(X)}(\alpha, R)$ and $\mathbf{DPDA}^{(X)}(\alpha, R)$ the classes of all ω-languages $F \subseteq X^\omega$ accepted by nondeterministic or deterministic pushdown automata, respectively.

Remark 3.6 As it was already explained above we do not require that the pushdown automaton reads the whole input tape, thus it might happen that we define – at least in the nondeterministic case – classes smaller than the corresponding ones in [CG77], [CG78b] and [Li76].
But, as e.g. Lemma 5.2 of [Li76] shows, our classes $\mathbf{DPDA}^{(X)}(\alpha, R)$ coincide with the ones given in [Li76] or [CG78b].

Utilizing Lemmas 3.2 and 3.3 we can show that in the nondeterministic case the class $\mathbf{NPDA}^{(X)}(\mathit{inf}, \sqcap) = \mathbf{NPDA}^{(X)}(\mathit{inf}, =)$ is the largest of our classes of ω-languages accepted by pushdown automata.
Then applying the same idea as in the proof of [Li76, Lemma 4.1] we obtain the following theorem which verifies that our class $\mathbf{NPDA}^{(X)}(\mathit{inf}, \sqcap) = \mathbf{NPDA}^{(X)}(\mathit{inf}, =)$ coincides with the corresponding class of [Li76] or [CG78].

Theorem 3.3 ([Li76],[CG77]) *It holds $F \in \mathbf{NPDA}^{(X)}(\mathit{inf}, \sqcap)$ iff there are an $n \in \mathbb{N}$ and context-free languages $W_i, V_i \subseteq X^*$ $(i = 1, \ldots, n)$ such that*

$$F = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega \ .$$

[3]Here the condition $\prod_{i=1}^\infty x_i \sqsubseteq \xi$ suffices, because we do not require the automaton M to read the whole input tape.

Proof. If W_i, V_i are context-free languages then it is easy to construct a pushdown automaton accepting F from the nondeterministic pushdown automata accepting the languages W_i, V_i.
To prove the converse, let $M = (X, ?, Z, z_0, ?_0, \boldsymbol{R})$ be a pushdown automaton such that $F = T_{\sqcap}^{inf}(M, \{\check{Z}\})$ for some $\check{Z} \subseteq Z$. Define for all pairs $(z,a) \in Z \times ?$ and all $z' \in Z$ the following context-free languages

$$W_{z,a} := \{x_1 \cdots x_n : n \in \mathbb{N} \wedge \exists\gamma \left(\gamma \in ?^* \wedge (z_0, ?_0) \xrightarrow{x_1} \cdots \xrightarrow{x_n} (z, a\cdot\gamma)\right)\} \text{ and}$$

$$V_{z,a}^{(z')} := \{x_1 \cdots x_n : n \in \mathbb{N} \wedge \exists\gamma,\gamma' \left(\gamma,\gamma' \in ?^* \wedge (z,a) \xrightarrow{x_1} \cdots \xrightarrow{x_l} (z', \gamma') \cdots \xrightarrow{x_n} (z, a\cdot\gamma)\right)\} \ ,$$

that is, $W_{z,a}$ consists of all words $w \in X^*$ which move the automaton M from the initial configuration to a configuration with state $z \in Z$ and $a \in ?$ as the topmost letter of the pushdown store, and$V_{z,a}^{z'}$ consists of all words $v \in X^*$ which move the automaton M from the configuration (z,a) to a configuration with same state $z \in Z$ and the same letter $a \in ?$ at the top of the pushdown store thereby intermediately passing a designated state z' (not necessarily distinct from z).
Let $\xi \in F$. Then there is a complete run $((z_i, x_i, \gamma_i))_{i=1}^{\infty}$ of M over ξ in which a state $z' \in \check{Z}$ occurs infinitely often. Since the length of the information stored in the pushdown store $|\gamma_i|$ is positive, there are infinitely many $i \in \mathbb{N}$ such that $|\gamma_i| \le |\gamma_j|$ for all $j \ge i$. Among these infinitely many time instants there are infinitely many having the property that z_{i_j} equals some fixed state $z \in Z$ and the word γ_{i_j} starts with a fixed letter $a \in ?$ and the automaton M passes the state z' in between time instants i_j and i_{j+1}.
Then, by definition, the input words w, v_j transferring M from time instant 0 to i_1 and from i_j to i_{j+1} belong to the languages $W_{z,a}$ and $V_{z,a}^{z'}$ respectively.
Now, we distinguish two cases:
First, let M read the whole input $\xi \in X^\omega$. Then infinitely many of the v_j are nonempty. Thus $\xi \in W_{z,a} \cdot (V_{z,a}^{z'})^\omega$.
Otherwise, M reads only a finite part $u \sqsubset \xi$ of the input $\xi \in X^\omega$. Then $u \in W_{z,a}$, and regardless of the tail ζ, where $u \cdot \zeta = \xi$, the automaton M accepts every ω-word starting with u. Thus $\xi \in W_{z,a} \cdot X^\omega$.
Consequently $F = \bigcup_{(z,a)\in Z\times\Gamma} W_{z,a} \cdot (U_{z,a}^{z'})^\omega$ where

$$U_{z,a}^{z'} := \begin{cases} V_{z,a}^{z'} \ , & \text{if } e \notin V_{z,a}^{z'} \text{ , and} \\ X \ \ , & \text{otherwise.} \end{cases}$$

Q.E.D.

As a consequence of our above considerations we define

$$\begin{array}{rcl} \mathbf{CF}_\omega^{(X)} & := & \mathbf{NPDA}^{(X)}(inf, =) \quad \text{, and} \\ \mathbf{DCF}_\omega^{(X)} & := & \mathbf{DPDA}^{(X)}(inf, =) \quad , \end{array}$$

and call them the classes of *context-free ω-languages* and *deterministic context-free ω-languages*, respectively.

Analogously to the case of regular languages we obtain an Init-lemma for context-free ω-languages. The first part of it is an immediate consequence of the above Theorem 3.3, whereas the second case is proved in [CG78b, Corollary 2.11].

Lemma 3.7 (Init-lemma for context-free ω-languages) *If $F \subseteq X^\omega$ is a (deterministic) context-free ω-language then $\mathbf{A}(F)$ is (deterministic) context-free language.*

As it was mentioned in Remark 3.6 for deterministic pushdown automata the results of [CG78b] and [Li76, 77] apply to our situation. We obtain characterizations of Borel subclasses of $\mathbf{DCF}_\omega^{(X)}$ which are similar to Theorem 3.1 and Proposition 3.4. We present it in a similar way as Proposition 3.4.

Proposition 3.5 ([CG78b], [Li77])

	Class	$\mathbf{DPDA}(\alpha, R)$	Representation	Comment
1.	$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{G}$	$\mathbf{DPDA}(ran, \sqcap)$	$W \cdot X^\omega$	$W \in \mathbf{DCF}$
2.	$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{F}$	$\mathbf{DPDA}(ran, \subseteq)$	$\boldsymbol{ls}\, W$	$W \in \mathbf{DCF}$
3.	$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{G}_\delta$	$\mathbf{DPDA}(inf, \sqcap)$	W^δ	$W \in \mathbf{DCF}$
4.	$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{F}_\sigma$	$\mathbf{DPDA}(inf, \subseteq)$	$X^\omega \setminus W^\delta$	$W \in \mathbf{DCF}$

Next we are going to derive a projection lemma for context-free ω-languages. To this end we start with some language theoretic observations.
A proof of the following technical lemma can be found e.g. in [WW86, Theorem 12.4].

Lemma 3.8 *For every context-free language $W \subseteq X^*$ there is pushdown automaton M without e-moves and having a single accepting state which accepts W. Moreover every word $w \notin \mathbf{A}(W)$ leads M to a nonfinal absorbing state.*

Labeling the last letter of every word and using an additional input coordinate controlling the nondeterministic behaviour of the automaton in Lemma 3.8 yields.

Lemma 3.9 *For every context-free language $W \subseteq X^* \setminus \{e\}$ there are an alphabet Y and a deterministic pushdown automaton M without e-moves accepting a prefix-free language $V \subseteq (X \times Y)^*$ such that $W = pr_1(V)$. Moreover, this same automaton (using a different set of final states) accepts also $\mathbf{A}(V)$.*

Utilizing Lemma 3.9 and the characterization of Theorem 3.3 we can prove the following.

Lemma 3.10 (Projection lemma for context-free ω-languages) *If $F \subseteq X^\omega$ is a context-free ω-language then there are an alphabet Y and a deterministic context-free ω-language $E \subseteq (X \times Y)^\omega$ such that $F = \overline{pr}_1(E)$. In particular, the ω-language $E \subseteq (X \times Y)^\omega$ can be shown to be in the form $E = \bigcup_{i=1}^n W_i \cdot V_i^\omega$ where W_i, V_i are prefix-free deterministic context-free languages, thus $E \in \mathbf{DCF}_\omega^{(X)} \cap \mathbf{G}_\delta$.*

In a similar way, using additionally Proposition 3.5.2 and Eq. (2.7), one proves an analogous statement for closed context-free ω-languages. Its implication 3. $\to$ 1. was shown already in [BN80].

Lemma 3.11 (Projection lemma for closed context-free ω-languages) *If $F \subseteq X^\omega$ is an ω-language then the following conditions are equivalent:*

1. $F \in \mathbf{F} \cap \mathbf{CF}_\omega^{(X)}$.
2. *There are an alphabet Y and a closed deterministic context-free ω-language $E \subseteq (X \times Y)^\omega$ such that $F = \overline{pr}_1(E)$.*
3. $E = \boldsymbol{ls}\, W$ *for some context-free language $W \subseteq X^*$.*

Lemma 3.11.3 and Proposition 3.5.2 state that $\boldsymbol{ls}\, W$ is a (deterministic) context-free ω-language provided only W is a (deterministic) context-free language. What concerns the operations ${}^\omega$ and ${}^\delta$, so Theorem 3.3 and Proposition 3.5.3 show that $W^\omega \in \mathbf{CF}_\omega^{(X)}$ for $W \in \mathbf{CF}$ and $V^\delta \in \mathbf{DCF}_\omega^{(X)}$ for $V \in \mathbf{DCF}$.
Using the same technique which proves nonclosure of **DCF** under * it is shown in [CG78b] that there is $V \in \mathbf{DCF}$ such that $V^\omega \notin \mathbf{DCF}_\omega^{(X)}$. Next we give an example showing that the δ-limit of a context-free language need not be context-free.

Example 3.1 ([Li76]) Let $\#(w, 1)$ be the number of occurrences of the letter 1 in the word w and define $W := \{w \cdot 1 \cdot 0^{\#(w,1)} \cdot 1 : w \in \{0,1\}^*\}$. Then $W^\delta \neq \emptyset$ (e.g. $\prod_{i=0}^{\infty} 1 \cdot 0^i \in W^\delta$) but does not contain an ultimately periodic ω-word. According to Theorem 3.3 $W^\delta \notin \mathbf{CF}_\omega^{(X)}$.

Though the class $\{W^\delta : W \subseteq X^* \wedge W \in \mathbf{CF}\}$ is not contained in $\mathbf{CF}_\omega^{(X)}$, its prefix languages are in some sense related to the class of context-free languages.

Proposition 3.6 ([Li76]) *Let $X \supseteq \{a, b, d\}$. Then $W \subseteq \{a, b\}^*$ and $W \cdot d^\omega = V^\delta$ for some $V \in \mathbf{CF}$ imply that $W \in \mathbf{CF}$.*

From the Init-lemma for context-free ω-languages and the well-known closure properties of the families **CF** and **DCF** we obtain the following useful proposition similar to the preceding one.

Proposition 3.7 *Let $X \supseteq \{a, b, d\}$ and $W \subseteq \{a, b\}^*$. Then the following equivalences hold.*

$$W \in \mathbf{CF} \quad (\mathbf{DCF}) \quad \Leftrightarrow \quad W \cdot d \cdot X^\omega \in \mathbf{CF}_\omega^{(X)} \cap \mathbf{G} \quad \left(\mathbf{DCF}_\omega^{(X)} \cap \mathbf{G}\right)$$

$$W \in \mathbf{CF} \quad (\mathbf{DCF}) \quad \Leftrightarrow \quad \boldsymbol{ls}(W \cdot d^*) \in \mathbf{CF}_\omega^{(X)} \cap \mathbf{F} \quad \left(\mathbf{DCF}_\omega^{(X)} \cap \mathbf{F}\right)$$

Utilizing this proposition we get that well-known results from Language theory allow not only to show the proper inclusion $\mathbf{DCF}_\omega^{(X)} \subset \mathbf{CF}_\omega^{(X)}$ but also the following more detailed ones involving Borel subclasses.

Corollary 3.2

$$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{G} \;\subset\; \mathbf{CF}_\omega^{(X)} \cap \mathbf{G} \quad , \qquad \mathbf{DCF}_\omega^{(X)} \cap \mathbf{G}_\delta \;\subset\; \mathbf{CF}_\omega^{(X)} \cap \mathbf{G}_\delta$$

$$\mathbf{DCF}_\omega^{(X)} \cap \mathbf{F} \;\subset\; \mathbf{CF}_\omega^{(X)} \cap \mathbf{F} \quad , \textit{ and } \quad \mathbf{DCF}_\omega^{(X)} \cap \mathbf{F}_\sigma \;\subset\; \mathbf{CF}_\omega^{(X)} \cap \mathbf{F}_\sigma$$

To conclude this section we mention some closure and nonclosure properties of the families $\mathbf{CF}_\omega^{(X)}$ and $\mathbf{DCF}_\omega^{(X)}$ which may be derived from the preceding results and well-known closure and nonclosure properties of the corresponding families of languages.

Proposition 3.8 *1.* $\mathbf{CF}_\omega^{(X)}$ *is closed under union, multiplication with context-free languages and intersection with regular ω-languages, but not under intersection and complementation.*

2. $\mathbf{DCF}_\omega^{(X)}$ *is closed under complementation, union and intersection with regular ω-languages, and multiplication with prefix-free deterministic context-free languages, but not under union, intersection and multiplication with arbitrary deterministic context-free languages*

3.4 ω-languages accepted by Turing machines

In contrast to the case of finite automata and pushdown automata in this part we will not refer explicitly on the model of Turing machines, but rather immediately give a description of those accepting devices in terms of mappings $\Phi : X^\omega \to Z^\omega$. This description has several advantages. So we avoid the cumbersome notation of (multitape) Turing machines and its infinite behaviour. Instead we can make use of the concise description and notation introduced in recursion theory and predicate calculus. Moreover, we may base on several results obtained there.
In order to describe the classes of ω-languages accepted by deterministic Turing machines we start with introducing the Arithmetical hierarchies of languages and ω-languages, respectively (cf. [Ro67], [SW78]):
We say that a language $W \subseteq X^*$ belongs to the class Σ_n iff $W = \{w : \exists a_1 \ldots \mathbf{Q}_n a_n : (a_1, \ldots, a_n, w) \in \boldsymbol{R}_W\}$, and $F \subseteq X^\omega$ belongs to the class $\Sigma_n^{(X)}$ iff $F = \{\xi : \exists a_1 \ldots \mathbf{Q}_n a_n : (a_1, \ldots, a_{n-1}, \xi(0, a_n]) \in \boldsymbol{R}_F\}$, where $\boldsymbol{R}_W \subseteq (\mathbb{N})^n \times X^*$ and $\boldsymbol{R}_F \subseteq (\mathbb{N})^{n-1} \times X^*$ are recursive relations and $\mathbf{Q}_i$ is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order).
The classes Π_n and $\Pi_n^{(X)}$ are defined as $\Pi_n := \{X^* \setminus W : W \in \Sigma_n\}$ and $\Pi_n^{(X)} := \{X^\omega \setminus F : F \in \Sigma_n^{(X)}\}$. In particular, $\Pi_1 \cap \Sigma_1$ and Σ_1 are the classes of recursive or recursively enumerable languages, respectively.

The classes Σ_n and Π_n, and likewise $\Sigma_n^{(X)}$ and $\Pi_n^{(X)}$ satisfy the inclusion relations indicated in the following diagram (All inclusions are proper and other than the indicated ones do not hold.):

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & \nearrow & & \nwarrow & \\
\Sigma_{n+1} & & & & \Pi_{n+1} \\
 & \nwarrow & & \nearrow & \\
 & & \Sigma_{n+1} \cap \Pi_{n+1} & & \\
 & & \uparrow & & \\
 & & \mathcal{B}(\Sigma_n) = \mathcal{B}(\Pi_n) & & \\
 & & \uparrow & & \\
 & & \Sigma_n \cup \Pi_n & & \\
 & \nearrow & & \nwarrow & \\
\Sigma_n & & & & \Pi_n \\
 & \nwarrow & & \nearrow & \\
 & & \vdots & &
\end{array}
$$

This diagram resembles in some sense the diagram of the classes of the Borel hierarchy. Indeed, the classes $\Sigma_n^{(X)}$ and $\Pi_n^{(X)}$ are properly contained in the respective classes of the Borel hierarchy (The proper inclusions hold already for cardinality reasons.):

$$
\begin{array}{ccccccc}
\Sigma_1 & \subset & \mathbf{G} & \qquad & \Pi_1 & \subset & \mathbf{F} \\
\Sigma_2 & \subset & \mathbf{F}_\sigma & & \Pi_2 & \subset & \mathbf{G}_\delta \\
\Sigma_3 & \subset & \mathbf{G}_{\delta\sigma} & & \Pi_3 & \subset & \mathbf{F}_{\sigma\delta} \\
 & & & \vdots & & &
\end{array}
$$

Moreover, like their Borel counterparts, the classes of the Arithmetical hierarchy are closed under union and intersection.
Next we exhibit ω-languages S_n and P_n which are in some sense most complex in the classes $\Sigma_n^{(X)}$ and $\Pi_n^{(X)}$, respectively.
To this end we introduce a Cantor pairing function, that is, a computable bijection $\langle\cdot,\cdot\rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Let $\boldsymbol{l} : \mathbb{N} \to \mathbb{N}$ and $\boldsymbol{r} : \mathbb{N} \to \mathbb{N}$ be the left and right inverse functions of $\langle\cdot,\cdot\rangle$, respectively, defined via the equations

$$\boldsymbol{l}(\langle k, m\rangle) := k \quad \text{and} \quad \boldsymbol{r}(\langle k, m\rangle) := m \ .$$

For the sake of convenience let $\langle\cdot,\cdot\rangle$ be monotone in both arguments. Then $\boldsymbol{l}(k) \leq k$ as well as $\boldsymbol{r}(k) \leq k$.
As usually the Cantor pairing function $\langle\cdot,\cdot\rangle$ is extended to the n arguments ($n \geq 1$):

$$\langle k_1\rangle := k_1 \ \text{ and } \ \langle k_1, \ldots, k_n\rangle := \langle k_1, \langle k_2, \ldots, k_n\rangle\rangle \ .$$

Let $\{0,1\} \subseteq X$. Then

$$P_n \ := \ \{\xi : \xi \in X^\omega \wedge \forall k_1 \exists k_2 \ldots \mathbf{Q}_n k_n(\xi(\langle k_1, \ldots, k_n\rangle) = 0) \ , \text{ and} \tag{3.16}$$
$$S_n \ := \ \{\xi : \xi \in X^\omega \wedge \exists k_1 \forall k_2 \ldots \mathbf{Q}_n k_n(\xi(\langle k_1, \ldots, k_n\rangle) = 0) \ . \tag{3.17}$$

Remark 3.7 Here we require that in both definitions the sequence of quantifiers has to alternate. Observe that $S_n \in \Sigma_n^{(X)}$ and $P_n \in \Pi_n^{(X)}$.

Let $\varphi : X^* \to X^*$ be a recursive totally unbounded sequential function. Then the continuous mapping $\overline{\varphi} : X^\omega \to X^\omega$ is called a *recursive operator*. The following theorem proves that the ω-languages S_n and P_n are most complex elements with respect to reducibility by recursive operators in the classes $\Sigma_n^{(X)}$ and $\Pi_n^{(X)}$ (cf. [SW78]).

Theorem 3.4 *Let $\Psi : X^\omega \to X^\omega$ be a recursive operator, and let $F \in \Sigma_n^{(X)}$ and $E \in \Pi_n^{(X)}$. Then $\Psi^{-1}(F) \in \Sigma_n^{(X)}$ and $\Psi^{-1}(E) \in \Pi_n^{(X)}$.*
Moreover, there are recursive operators $\Phi_F, \Phi_E : X^\omega \to X^\omega$ such that $F = \Phi_F^{-1}(S_n)$ and $E = \Phi_E^{-1}(P_n)$.

Remark 3.8 A similar theorem holds for continuous mappings Φ and Borel classes of the first countable ordinal involving the same sets S_n and P_n. For a proof stressing the parallel treatment of Arithmetical and Borel hierarchies see e.g. [St86a].
In case $n = 2$ Theorem 3.4 holds also with $S_2(P_2)$ replaced by the regular ω-languages $S_2' = X^* \cdot 0^\omega$ $(P_2' = (X^* \cdot 0)^\omega)$ introduced at the end of Section 3.2.

Similar to the low level Borel classes, for the low level classes of the Arithmetical hierarchy of ω-languages we have also some characterizations in terms of recursive or recursively enumerable languages (cf. [St86a]).

Lemma 3.12

1. *A set $E \subseteq X^\omega$ is in $\Sigma_1^{(X)}$ iff there is a recursive (or, equivalently, a recursively enumerable) language $W \subseteq X^*$ such that $E = W \cdot X^\omega$.*
2. *A set $F \subseteq X^\omega$ is in $\Pi_1^{(X)}$ if and only if F is closed in the Cantor topology of X^ω and $X^* \setminus \mathbf{A}(F)$ is recursively enumerable.*
3. *A set $E \subseteq X^\omega$ is in $\Pi_2^{(X)}$ iff there is a recursive (or, equivalently, a recursively enumerable) language $W \subseteq X^*$ such that $E = W^\delta$.*

Proof. We give only a proof for the most complicated Assertion 3. That $W^\delta \in \Pi_2^{(X)}$ for a recursively enumerable language $W \subseteq X^*$ follows by a simple application of the Tarski-Kuratowski-algorithm to the formula

$$\xi \in W^\delta \Leftrightarrow \forall i \exists n \exists m (n \geq i \wedge (m, \xi(0,n]) \in \boldsymbol{R}_W) \ .$$

Conversely, let $F \in \Pi_2^{(X)}$: Then there is a recursive relation $\boldsymbol{R}_F$ such that $F = \{\xi : \forall k \exists m ((k, \xi(0,m]) \in \boldsymbol{R}_F)\}$. First observe that without loss of generality we may assume that $(k, w) \in \boldsymbol{R}_F$ implies $(k, u) \notin \boldsymbol{R}_F$ for all proper prefixes $u \sqsubset w$. Using a Cantor pairing function as defined above we define the relation $\boldsymbol{R} := \{(k, v) : \exists w (w \sqsubseteq v \wedge (k, w) \in \boldsymbol{R}_F \wedge |v| = \langle k, |w| \rangle)\}$.

Then, since the quantifier $\exists w$ is bounded, $\boldsymbol{R}$ is also a recursive relation. Moreover, both relations $\boldsymbol{R}$ and $\boldsymbol{R}_F$ share the property that given $k \in \mathbb{N}$ and $\xi \in X^\omega$ there is at most one word $v \sqsubset \xi$ or $w \sqsubset \xi$ such that $(k, v) \in \boldsymbol{R}$ or $(k, w) \in \boldsymbol{R}_F$, respectively. The purpose of our construction is to guarantee that k is recursively bounded by $|v|$ for $\boldsymbol{R}$, more precisely, for every $v \sqsubset \xi$ such that $(k, v) \in \boldsymbol{R}$ it holds $k = \boldsymbol{l}(|v|)$.
The following items show some relationships between $\boldsymbol{R}$ and $\boldsymbol{R}_F$:

1. If $(k, v) \in \boldsymbol{R}$ then $k = \boldsymbol{l}(|v|)$ and there is a $w \sqsubseteq v$ such that $(k, w) \in \boldsymbol{R}_F$.
2. If $(k, w) \in \boldsymbol{R}_F$ and $w \sqsubset \xi$ then there is a v such that $w \sqsubseteq v \sqsubset \xi$ and $(k, v) \in \boldsymbol{R}$.
3. If $(k, v), (k, v') \in \boldsymbol{R}$ and $v \sqsubseteq v'$ then $v = v'$.

Now 1. and 2. yield $F = \{\xi : \forall k \exists m((k, \xi(0, m]) \in \boldsymbol{R})\}$.
Define $V := \{v : (\boldsymbol{l}(|v|), v) \in \boldsymbol{R} \wedge \forall k(k \leq \boldsymbol{l}(|v|) \rightarrow \exists u(u \sqsubseteq v \wedge (k, u) \in \boldsymbol{R}))\}$. The quantifier $\exists u$ is bounded, hence V is a recursive language.
If $\xi \in V^\delta$ there are infinitely many $v \in \mathbf{A}(\xi) \cap V$. In virtue of the above Item 3 we have $\boldsymbol{l}(|v|) \neq \boldsymbol{l}(|v'|)$ for different words $v, v' \in \mathbf{A}(\xi) \cap V$. In particular, the set $\{\boldsymbol{l}(|v|) : v \in \mathbf{A}(\xi) \cap V\}$ is infinite. By the construction of V this implies that for all $k \in \mathbb{N}$ there is a $v \sqsubset \xi$ such that $(k, v) \in \boldsymbol{R}$.
Let now $\xi \in F$. Then for every $k \in \mathbb{N}$ there is a $v_k \sqsubset \xi$ such that $(k, v_k) \in \boldsymbol{R}$. By the monotonicity of the Cantor pairing function, $|v_k| \geq k$, and the family $(v_k)_{k \in \mathbb{N}}$ is a family of prefixes of ξ of unbounded lengths. Consequently, there are infinitely many l such that $v_i \sqsubseteq v_l$ for all $i \leq l$. Since $(i, v_i) \in \boldsymbol{R}$, we have $v_l \in V$ and also $\xi \in V^\delta$.

Q.E.D.

Since on the one hand the Turing machine mappings Φ_M considered in connection with Proposition 3.2 are recursive operators and on the other hand for a recursive language $W \subseteq X^*$ it is easy to construct Turing machines M_1, M_2 and sets Z_1, Z_2 such that $T_\sqcap^{ran}(M_1, \{Z_1\}) = W \cdot X^\omega$ and $T_\sqcap^{inf}(M_2, \{Z_2\}) = W^\delta$, we obtain as a corollary to Lemma 3.12 the following characterization of the classes of ω-languages accepted by deterministic Turing machines.
To this end the class of all ω-languages $F \subseteq X^\omega$ accepted by deterministic (nondeterministic) Turing machines according to mode (α, R) is denoted by $\mathbf{DT}^{(X)}(\alpha, R)$ (or $\mathbf{NT}^{(X)}(\alpha, R)$, respectively).

Corollary 3.3

$$\begin{aligned} \mathbf{DT}^{(X)}(ran, \subseteq) &= \Pi_1^{(X)} \quad , \quad \mathbf{DT}^{(X)}(ran, \sqcap) = \Sigma_1^{(X)} \\ \mathbf{DT}^{(X)}(inf, \subseteq) &= \Sigma_2^{(X)} \quad , \quad \mathbf{DT}^{(X)}(inf, \sqcap) = \Pi_2^{(X)} \end{aligned}$$

Since Turing machines allow for a parallel composition, we obtain from Eq. (3.4) a characterization for the classes $\mathbf{DT}^{(X)}(\alpha, =)$.

Corollary 3.4

$$\begin{aligned} \mathbf{DT}^{(X)}(ran, =) &= \mathcal{B}(\Pi_1^{(X)}) \\ \mathbf{DT}^{(X)}(inf, =) &= \mathcal{B}(\Pi_2^{(X)}) \end{aligned}$$

Next we prove a lemma which relates the nondeterministic to the deterministic classes in a simple way.

Lemma 3.13 (Projection lemma for Turing machines) *$F \in \mathbf{NT}^{(X\times Y)}(\alpha, R)$ implies that $\overline{pr}_1(F) \in \mathbf{NT}^{(X)}(\alpha, R)$.*
Conversely, for every $E \in \mathbf{NT}^{(X)}(\alpha, R)$ there is an $F \in \mathbf{DT}^{(X\times Y)}(\alpha, R)$ such that $E = \overline{pr}_1(F)$.

Proof. The first assertion is clear because nondeterministic classes are closed under projection.
Let $M = (X, ?, Z, z_0, \boldsymbol{R})$ be a nondeterministic Turing machine. Then for every tuple $(z, x_0, \cdots, x_n) \in Z \times X \times ?^n$ there are at most k_M tuples $(z, x_0, \ldots, x_n, z', y_0, \ldots, y_n) \in \boldsymbol{R}$. Put $Y := \{1, \ldots, k_M\}$ and define a deterministic Turing machine $M' = (X \times Y, ?, Z', z'_0, \boldsymbol{R'})$ in such a way that the second component of the input $\eta \in (X \times Y)^\omega$ controls the choice of the transition whenever M has to choose its next action nondeterministically.

Q.E.D.

The following lemma shows that, similar to the case of finite automata, several classes of ω-languages accepted by nondeterministic Turing machines coincide with the corresponding "deterministic" classes.

Lemma 3.14 *Let $F \subseteq (X \times Y)^\omega$. Then $F \in \Sigma_2^{(X\times Y)}, F' \in \Pi_1^{(X\times Y)}$ and $E \in \Sigma_1^{(X\times Y)}$ imply that $\overline{pr}_1(F) \in \Sigma_2^{(X)}, F' \in \Pi_1^{(X)}$ and $E \in \Sigma_1^{(X)}$, respectively.*

But, similar to the case of Borel classes, also the projections of sets from the second Π-class lead beyond the range of the Arithmetical hierarchy. Here we have to introduce the first Σ-class, $\Sigma_1^{1(X)}$, of the analytical hierarchy of ω-languages (cf. [Ro67, SW78]):

$$F \in \Sigma_1^{1(X)} \text{ iff } F = \{\xi : \exists \eta (\eta \in \{0,1\}^\omega \wedge \forall n \exists m\, ((n, \eta(0,m], \xi(0,m]) \in \boldsymbol{R}_F))\},$$

for some recursive relation $\boldsymbol{R}_F$, that is,

$$\Sigma_1^{1(X)} = \{\overline{pr}_1(F) : F \in \Pi_2^{(X\times\{0,1\})}\} \ . \tag{3.18}$$

Analogously, $W \in \Sigma_1^1$ iff $W = \{w : \exists \eta (\eta \in \{0,1\}^\omega \forall n \exists m\, ((n, \eta(0,m], w) \in \boldsymbol{R}_W))\}$, where $\boldsymbol{R}_W$ is an appropriately chosen recursive relation.
The following facts are known (cf. [Ro67]):

$$\Sigma_n^{(X)} \cup \Pi_n^{(X)} \subset \Sigma_1^{1(X)}, \text{ and}$$

$$\text{if } F \in \Sigma_1^{1(X\times Y)} \text{ then } \overline{pr}_1(F) \in \Sigma_1^{1(X)} \ .$$

We obtain the recursive analogue to Theorem 3.1.

Theorem 3.5

$$\begin{array}{lllllll}
\mathbf{DT}^{(X)}(ran,\subseteq) & = & \mathbf{NT}^{(X)}(ran,\subseteq) & = & \Pi_1^{(X)} & & \\
\mathbf{DT}^{(X)}(ran,\sqcap) & = & \mathbf{NT}^{(X)}(ran,\sqcap) & = & \Sigma_1^{(X)} & & \\
\mathbf{DT}^{(X)}(ran,=) & = & \mathcal{B}(\Sigma_1^{(X)}) & & & & \\
\mathbf{DT}^{(X)}(inf,\subseteq) & = & \mathbf{NT}^{(X)}(inf,\subseteq) & = & \mathbf{NT}^{(X)}(ran,=) & = & \Sigma_2^{(X)} \\
\mathbf{DT}^{(X)}(inf,\sqcap) & = & \Pi_2^{(X)} & & & & \\
\mathbf{DT}^{(X)}(inf,=) & = & \mathcal{B}(\Sigma_2^{(X)}) & & & & \\
\mathbf{NT}^{(X)}(inf,\sqcap) & = & \mathbf{NT}^{(X)}(inf,=) & = & \Sigma_1^{1(X)} & &
\end{array}$$

For the initial word languages we get the following results. It is shown in [St86b] that, in general, we cannot improve on the classes achieved for $\mathbf{A}(F)$. To this end let $\widehat{\mathbf{A}}(\mathcal{F})$ be the class of languages generated by $\{\mathbf{A}(F) : F \in \mathcal{F}\}$ and closed under intersection with regular languages and removal of the last letter (end marker).

Lemma 3.15 (Init-lemma for recursive ω-languages) *Let d be a letter not in X. Then it holds.*

$$\begin{array}{rcccrcccccc}
\widehat{\mathbf{A}}(\Pi_2^{(X\cup\{d\})}) & \cap & 2^{X^*} & = & \widehat{\mathbf{A}}(\Sigma_1^{1(X\cup\{d\})}) & \cap & 2^{X^*} & = & \Sigma_1^1 & \cap & 2^{X^*} \\
\widehat{\mathbf{A}}(\mathcal{B}(\Sigma_1^{(X\cup\{d\})})) & \cap & 2^{X^*} & = & \widehat{\mathbf{A}}(\Sigma_2^{(X\cup\{d\})}) & \cap & 2^{X^*} & = & \Sigma_2 & \cap & 2^{X^*} \\
 & & & & \widehat{\mathbf{A}}(\Pi_1^{(X\cup\{d\})}) & \cap & 2^{X^*} & = & \Pi_1 & \cap & 2^{X^*} \\
 & & & & \widehat{\mathbf{A}}(\Sigma_1^{(X\cup\{d\})}) & \cap & 2^{X^*} & = & \Sigma_1 & \cap & 2^{X^*}
\end{array}$$

Moreover, unlike the cases of finite automata and deterministic pushdown automata, the classes of ω-languages accepted by Turing machines are not characterized in a topological way. We mention the following generalization of Theorem 15-XI of [Ro67].

Lemma 3.16 ([Ro67,St86b]) *Let X be an alphabet and d be a letter not in X. Then for a language $W \subseteq X^*$ the following equivalence is true:*

$$W \in \Sigma_n \Longleftrightarrow W \cdot d \cdot (X \cup \{d\})^\omega \in \Sigma_n^{(X\cup\{d\})}\ , \textit{ and}$$

$$W \in \Pi_n \Longleftrightarrow X^\omega \cup W \cdot d^\omega \in \Pi_n^{(X\cup\{d\})}\ .$$

As a consequence of the strictness of the Arithmetical hierarchy of languages we obtain that every Σ-class of the Arithmetical hierarchy of ω-languages contains open sets which are not contained in the preceding Σ-class, and every Π-class contains closed sets which are not contained in the preceding Π-class.

Concluding remarks

The reader might have expected that in connection with the title "The Chomsky Hierarchy of ω-languages" we consider also a class of ω-languages related to context-sensitive

languages. Because of the infinite length of the inputs, it is, however, difficult to define classes of ω-languages by means of space or time complexity.
For instance, Theorem 7.16 of [CG78a] shows that for every recursively enumerable language $V \subseteq X^*$ there is a deterministic context-sensitive language W such that $V \cdot X^\omega = W \cdot X^\omega$ and $V^\delta = W^\delta$. In [SN86] this result was extended to languages W accepted in linear time and logarithmic space on Turing machines.
In Remark 3.5 we mentioned that for partially defined devices the class of ω-languages accepted according to the mode $(ran, \sqcap)$ might be larger than for fully defined devices, whereas in the remaining five cases the accepted classes do not change. (In fact, the hint given there was concerned only finite automata, but it is valid also in case of most other accepting devices.)
For partially defined Turing machines one can prove analogously to Remark 3.5 that

$$\begin{aligned}\mathbf{pDT}^{(X)}(ran, \sqcap) &= \{E \cap F : E \in \Sigma_1^{(X)} \wedge F \in \Pi_1^{(X)}\} \text{ and}\\ \mathbf{pNT}^{(X)}(ran, \sqcap) &= \Sigma_2^{(X)} .\end{aligned}$$

In Remark 3.1 we claimed that the accepting power of devices is increased when we additionally demand that the device reads the whole input tape. For finite automata this results, obviously, in the same classes as for partially defined automata, but in case of more powerful devices we obtain results which show that the requirement to read the whole input offers stronger possibilities to reject an input ω-word than that to be merely partially defined. For nondeterministic devices this is explained by blind counter arguments in Theorem 3.10 and 3.12 of [EH93]. (If, however, the accepting device is not forced to read the whole input, a blind counter appears to be useless.)
We mention still that in case of deterministic Turing machines the classes $\overline{\mathbf{DT}}^{(X)}(\alpha, R)$ of ω-languages accepted by fulfilling the acceptance condition (α, R) *and* reading the whole input can be described as follows.

$$\begin{aligned}\overline{\mathbf{DT}}^{(X)}(ran, \subseteq) &= \{F : F \in \mathbf{F} \wedge \mathbf{A}(F) \in \Pi_2\}\\ \overline{\mathbf{DT}}^{(X)}(ran, \sqcap) &= \{E \cap F : E \in \Sigma_1^{(X)} \wedge F \in \overline{\mathbf{DT}}^{(X)}(ran, \subseteq)\}\\ \overline{\mathbf{DT}}^{(X)}(ran, =) &= \{E \cap F : E \in \mathcal{B}(\Pi_1^{(X)}) \wedge F \in \overline{\mathbf{DT}}^{(X)}(ran, \subseteq)\}\\ \overline{\mathbf{DT}}^{(X)}(inf, \subseteq) &= \{E \cap F : E \in \Sigma_2^{(X)} \wedge F \in \overline{\mathbf{DT}}^{(X)}(ran, \subseteq)\}\\ \overline{\mathbf{DT}}^{(X)}(inf, \sqcap) &= \Pi_2^{(X)}\\ \overline{\mathbf{DT}}^{(X)}(inf, =) &= \mathcal{B}(\Pi_2^{(X)})\end{aligned}$$

Since $\{F : F \in \mathbf{F} \wedge \mathbf{A}(F) \in \Pi_2\} \not\subseteq \Sigma_2^{(X)}$ (cf. [St86a]), the first four classes contain properly their counterparts $\mathbf{DT}^{(X)}(\alpha, R)$, and we have also $\overline{\mathbf{DT}}^{(X)}(ran, \sqcap) \supset \mathbf{pDT}^{(X)}(ran, \sqcap)$.

4 Languages and ω-languages

In this section we are going to investigate classes of ω-languages which are generated from classes of languages using the operations described in the introduction. In particular we

are interested in the closure properties which are inherited from the underlying classes of languages. We choose an alphabet to be large enough (a countably infinite one, $\mathbb{N}$ say, will suffice) to encompass all finite alphabets encountered in the sequel. As usually, we call a family $\mathcal{W} \subseteq \{W : \exists X(X \subseteq \mathbb{N} \wedge X \text{ is finite } \wedge W \subseteq X^*)\}$ containing at least one nonempty language a *family of languages*. Similarly, $\mathcal{F} \subseteq \{F : \exists X(X \subseteq \mathbb{N} \wedge X \text{ is finite } \wedge F \subseteq X^\omega)\}$ will be referred to as a *family of ω-languages* provided it contains a nonempty ω-language. In the theory of languages several families, such as the families of regular, context-free, recursive and recursively enumerable languages, are shown to be closed under natural operations like (non erasing) homomorphisms, inverse homomorphisms, intersection with regular languages, union, product and Kleene-star. Such families were called *abstract families of languages (AFLs)* (cf. [Gi75]); families of languages being closed only under the first three operations were called *trios* (cf. [Gi75]).
In the previous section we have seen that the families of ω-languages in the Chomsky hierarchy follow two patterns of generation by families of languages. Here we show that by these two patterns several closure properties of the underlying family of languages are preserved.
The first one is the one from the representation theorems of regular and context-free ω-languages which was called ω-Kleene closure in [CG77], the second one uses projection and the δ-limit as in the representation theorem for recursive ω-languages.

4.1 ω-Kleene closure

In Theorems 3.2 and 3.3 we observed that ω-languages accepted by finite or pushdown automata are in the same way derived from the underlying classes of languages accepted by the same types of automata. In this part we are going to study common properties of classes of ω-languages derived in this way form classes of languages. In particular, we are interested which closure properties are inherited from the classes of languages to their counterparts in the range of ω-languages.
First we recall the notion of ω-Kleene closure of a family of languages from [CG77].

Definition 4.1 For a family of languages $\mathcal{W}$ we call

$$\mathcal{K}_\omega(\mathcal{W}) := \left\{ \bigcup_{i=1}^{n} W_i \cdot V_i^\omega : n \in \mathbb{N} \wedge W_i, V_i \in \mathcal{W} \right\}$$

its *ω-Kleene closure*

We obtain our main result in this part.

Theorem 4.1 *Let $\mathcal{W}$ be an abstract family of languages. Then its ω-Kleene closure $\mathcal{K}_\omega(\mathcal{W})$ is closed under union, intersection with regular ω-languages, e-free homomorphism, inverse homomorphism and multiplication with languages from $\mathcal{W}$.*
If $\mathcal{W}$ is additionally closed under arbitrary homomorphism, then so is $\mathcal{K}_\omega(\mathcal{W})$.

Proof. It is easy to see that $\mathcal{K}_\omega(\mathcal{W})$ is closed under union, e-free (arbitrary) homomorphism and multiplication with languages from $\mathcal{W}$.
Next we show that $\mathcal{K}_\omega(\mathcal{W})$ is closed under inverse homomorphism. Let $\xi \in h^{-1}(W \cdot V^\omega)$ where $W, V \in \mathcal{W}$. Then there is an $\eta \in W \cdot V^\omega$ such that $\eta = h(\xi)$ and η has a factorization $\eta = h(\xi(1)) \cdot \ldots \cdot h(\xi(i)) \cdot \ldots$ according to the finite language $h(X)$.
On the other hand, η has also a factorization $\eta = w' \cdot v'_1 \cdot \ldots \cdot v'_i \cdot \ldots$ with respect to $W \cdot V^\omega$. Since the length of $h(x)$ $(x \in X)$ is bounded, one can find a word u with $|u| < \max\{|h(x)| : x \in X\}$ for which $\eta = w \cdot u \cdot v_1 \cdot u \cdot \ldots \cdot v_i \cdot u \cdot \ldots$ such that $w \cdot u \in W \cdot V^*$ and $v_i \cdot u \in V^*$. This shows $\xi \in h^{-1}(W \cdot V^* \cdot u) \cdot h^{-1}((V^*/u) \cdot u)^\omega$. Consequently,

$$h^{-1}(W \cdot V^\omega) = \bigcup_{|u| \le \max\{|h(x)|:x \in X\}} h^{-1}(W \cdot V^* \cdot u) \cdot h^{-1}((V^*/u) \cdot u)^\omega \ .$$

Since $\mathcal{W}$ is an AFL, we have $h^{-1}(W \cdot V^* \cdot u), h^{-1}((V^*/u) \cdot u) \in \mathcal{W}$. On the other hand, the inclusion $h^{-1}(W \cdot V^* \cdot u) \cdot h^{-1}((V^*/u) \cdot u)^\omega \subseteq h^{-1}(W \cdot V^\omega)$ is obvious, hence $h^{-1}(W \cdot V^\omega) \in \mathcal{K}_\omega(\mathcal{W})$.
Now consider intersection with regular ω-languages. Let $F \in \mathbf{R}_\omega^{(X)}$ Then there are a finite automaton $M = (X, Z, z_0, \boldsymbol{R})$ and a $\check{Z} \subseteq Z$ such that $F = T_\sqcap^{inf}(M, \{\check{Z}\})$. According to the proof of Theorem 3.2 we have $F = \bigcup_{z \in Z, z' \in \check{Z}} W_z \cdot (V_z^{z'})^\omega$.
Consider $\xi \in W \cdot V^\omega \cap F$. Because Z is finite, there is a factorization $\xi = w \cdot v_1 \cdot \ldots \cdot v_i \cdot \ldots$ where $w \in W \cdot V^* \cap W_z$ and all $v_i \in V^* \cap V_z^{z'}$ for some $z \in Z, z' \in \check{Z}$. This shows $W \cdot V^\omega \cap F \subseteq \bigcup_{z \in Z, z' \in \check{Z}} (W \cdot V^* \cap W_z) \cdot (V^* \cap V_z^{z'})^\omega$. The converse inclusion is obvious. Since W_z and $V_z^{z'}$ are regular languages, $W \cdot V^* \cap W_z, V^* \cap V_z^{z'} \in \mathcal{W}$, and hence $W \cdot V^\omega \cap F \in \mathcal{K}_\omega(\mathcal{W})$.
Q.E.D.

Since either $card(V^\omega) \le 1$ and $V^\omega \subseteq \{w\}^\omega$ or otherwise $card(V^\omega) = 2^{\aleph_0}$, Definition 4.1 implies the following property of $\mathcal{K}_\omega(\mathcal{W})$.

Corollary 4.1 *If $F \in \mathcal{K}_\omega(\mathcal{W})$ is at most countable then F contains only ultimately periodic ω-words, and if, moreover, F is finite then F is already regular.*

Next we consider the ω-Kleene closures of the families of languages mentioned in the introduction. To this end we denote by $\mathbf{R}_\omega := \bigcup\{\mathbf{R}_\omega^{(X)} : X \subseteq \mathbb{N} \wedge X \text{ is finite}\}$, $\mathbf{DCF}_\omega := \bigcup\{\mathbf{DCF}_\omega^{(X)} : X \subseteq \mathbb{N} \wedge X \text{ is finite}\}$, $\mathbf{CF}_\omega := \bigcup\{\mathbf{CF}_\omega^{(X)} : X \subseteq \mathbb{N} \wedge X \text{ is finite}\}$, and $\mathbf{REK}_\omega := \bigcup\{\Sigma_1^{1(X)} : X \subseteq \mathbb{N} \wedge X \text{ is finite}\}$ the families of regular, deterministic context-free, context-free, and recursive ω-languages, respectively, introduced in the preceding section.

Corollary 4.2 *It holds*

$$\begin{array}{lclclcl} \mathcal{K}_\omega(\mathbf{R}) & = & \mathbf{R}_\omega & , & \mathcal{K}_\omega(\mathbf{DCF}) & \supset & \mathbf{DCF}_\omega \\ \mathcal{K}_\omega(\mathbf{CF}) & = & \mathbf{CF}_\omega & , \textit{ and} & \mathcal{K}_\omega(\mathbf{RE}) & \subset & \mathbf{REK}_\omega \ . \end{array}$$

Moreover, we have

$$\mathcal{K}_\omega(\mathbf{DCF}) \subset \mathbf{CF}_\omega \subset \mathcal{K}_\omega(\mathbf{REK}) \ .$$

Proof. The identities follow from Theorems 3.2 and 3.3. The proper inclusion for deterministic context-free ω-languages is proved in [CG78b, Theorem 3.2]. Inclusion can be verified along the lines of the proof of Theorem 3.3 above, and the properness follows from the fact that $\mathcal{K}_\omega(\mathbf{DCF})$ is closed under union, whereas $\mathbf{DCF}_\omega$ is not. In the case of $\mathbf{REK}_\omega$ inclusion is obtained by usual recursion-theoretic means, and the properness follows from the fact that for example $\{\prod_{i\in\mathbb{N}}(1\cdot 0^i)\}\in \mathbf{REK}_\omega\setminus\mathcal{K}_\omega(\mathbf{RE})$ (cf. Corollary 4.1 above).
The inclusion $\mathcal{K}_\omega(\mathbf{DCF})\subset \mathbf{CF}_\omega$ is also in [CG78b, Theorem 3.2], and the other one follows from $\mathbf{CF}\subset\mathbf{REK}$ and the Init-lemma below.

Q.E.D.

It is, however, not known whether the inclusion $\mathcal{K}_\omega(\mathbf{REK})\subseteq\mathcal{K}_\omega(\mathbf{RE})$ is proper or not.
We conclude this part with a look to the initial word languages of an ω-Kleene closure.

Lemma 4.1 (Init-lemma for the ω-Kleene closure) *Let $\mathcal{W}$ be an AFL, and let $\mathcal{W}_\mathbf{A}$ be the trio generated by the family of initial word languages $\{\mathbf{A}(F): F\in\mathcal{K}_\omega(\mathcal{W})\}$. Then $\mathcal{W}_\mathbf{A}\supseteq\mathcal{W}$ and $\mathcal{W}_\mathbf{A}$ is an AFL. If $\mathcal{W}$ is closed under arbitrary homomorphism then $\mathcal{W}_\mathbf{A}=\mathcal{W}$.*

4.2 ω-power Languages

In the definition of the ω-Kleene closure the ω-power of a language $W\subseteq X^*$ plays a crucial role. Unlike the other two operations transferring languages to ω-languages, $\boldsymbol{ls}$ and $^\delta$, we have no nice characterization of W^ω in terms of Borel classes.
We have up to now only some obvious facts. If a language W is prefix-free then $W^\omega=(W^*)^\delta$ is a $\mathbf{G}_\delta$-set, it is even closed if W is finite. In [LT86b] a thorough characterization of ω-powers of finite languages is given. One might hope that this continues to some general relationship between the operations $^\delta$ and $^\omega$.
In fact this is not the case. If we consider the infinite intersection $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$ the assumption $W^\omega=\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$ is tempting, but in general W^ω and $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$ do not coincide, even if W^ω is a $\mathbf{G}_\delta$-set (cf. [St86c]). It holds only the obvious inclusion

$$W^\omega\subseteq\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega\subseteq(W^*)^\delta\ . \tag{4.1}$$

A tighter relation between W^ω and $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$ is given by the following lemma.

Lemma 4.2 *Let $v\cdot w^\omega$ be an ultimately periodic sequence in $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$. Then $v\cdot w^\omega\in W^\omega$.*

Proof. Let $v\cdot w^\omega\in\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega$. Then for every $i\in\mathbb{N}$ there is a prefix $u_1\cdots u_i$ of $v\cdot w^\omega$ such that $u_j\in W^*$. Let u_1 be longer than v and let $i>|w|$. Then there are $j,k\le i$ with $j<k$ such that $|u_1\cdots u_k|-|u_1\cdots u_j|$ is divisible by $|w|$. Hence $v\cdot w^\omega=u_1\cdots u_j\cdot(u_{j+1}\cdots u_k)^\omega$.

Q.E.D.

Since a regular ω-language is uniquely specified by its ultimately periodic ω-words, we have the following corollary.

Corollary 4.3 *Let W^ω be regular. Then $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i\cdot X^\omega = W^\omega$ if and only if $\bigcap_{i\in\mathbb{N}}(W\setminus\{e\})^i \cdot X^\omega$ is also regular.*

In [DL95], [Lt91a,b], [LT87] and [St86c] several properties of the operation $^\omega$ were considered. In particular, the properties of W^ω were investigated under the assumption that W is a *code*, that is, every word $w \in W^*$ uniquely factorizes into a product $w = w_1 \cdots w_n$ where $w_1, \ldots, w_n \in W$.
First we derive an example which shows that even for a regular code $V \subseteq X^*$ its ω-power V^ω may not be a $\mathbf{G}_\delta$-set.

Example 4.1 (Wagner) Let $V := \{a, ba\} \cup \{ab, ac\}^* \cdot aca$. In [St86c] it is shown that $V^\omega \subset \bigcap_{i\in\mathbb{N}}(V \setminus \{e\})^i \cdot X^\omega \subset (V^*)^\delta$, and that V^ω is not even a $\mathbf{G}_\delta$-set.

Finally, we quote a result from [LT87] about ω-powers of regular languages.

Theorem 4.2 ([LT87]) *Let $F = W^\omega$ be a regular ω-language. Then there is a regular language V such that $F = V^\omega$.*
Moreover, among all W such that $F = W^\omega$ there is only a finite number of maximal with respect to "$\subseteq$" ones and all of them are regular.

In contrast to our theorem, it is easy to see that for the regular ω-language $(a^* \cdot b)^\omega$ the languages $V_f := \bigcup_{i\in\mathbb{N}} a^i \cdot b \cdot (a^* \cdot b)^{f(i)}$ where $f : \mathbb{N} \to \mathbb{N}$ is an arbitrary function are $2^{\aleph_0}$ many minimal languages satisfying $V_f^\omega = (a^* \cdot b)^\omega$.

4.3 a-transducers, gsm-mappings, and ω-transductions

In the theory of formal languages the concept of a-transducer plays an important role, particularly in connection with closure properties of classes of languages. We utilize this concept for its facility to encompass several natural language theoretic operations (cf. [Gi75]).

Definition 4.2 An a-transducer is a sextuple $\mathcal{M} := (Z, X, Y, H, z_0, Z')$ where

1. Z, X, Y are finite nonempty sets of states, input letters and output letters, respectively,
2. H is a finite subset of $Z \times X^* \times Y^* \times Z$, the set of transitions,
3. $z_0 \in Z$ is the initial state, and
4. $Z' \subseteq Z$ is the set of accepting states.

The transducer $\mathcal{M}^{-1} := (Z, Y, X, H^{-1}, z_0, Z')$, where $H^{-1} := \{(z', v, w, z) : (z, w, v, z') \in H\}$ is the *inverse* of the a-transducer $\mathcal{M} = (Z, X, Y, H, z_0, Z')$.
An a-transducer $\mathcal{M}$ effects an operation on languages as follows [Gi75].
A finite sequence of transductions of is called a *computation* of $\mathcal{M}$, if it has the form

$$(z_0, w_1, v_1, z_1), (z_1, w_2, v_2, z_2), \ldots, (z_{\ell-2}, w_{\ell-1}, v_{\ell-1}, z_{\ell-1}), (z_{\ell-1}, w_\ell, v_\ell, z_\ell)$$

with $z_\ell \in Z'$. The words $w := w_1 \cdot \ldots \cdot w_\ell$ and $v := v_1 \cdot \ldots \cdot v_\ell$ are called the input and the output of the computation, respectively. We abbreviate the fact that w is an input and v a corresponding output word of a computation of the a-transducer $\mathcal{M}$ by $w \vdash_{\mathcal{M}} v$. We set

$$\mathcal{M}(w) := \{v : v \in Y^* \wedge w \vdash_{\mathcal{M}} v\}$$

and $\mathcal{M}(W) := \bigcup_{w \in W} \mathcal{M}(w)$ for $W \subseteq X^*$.
We call an a-transducer *e-output bounded*, provided there is a $k \in \mathbb{N}$ such that $(z_1, w_2, e, z_2), \ldots, (z_\ell, w_\ell, e, z_{\ell+1})$ implies $\ell \leq k$, that is, the a-transducer has at most k consecutive transitions having the empty word e as output.
An a-transducer $\mathcal{M} := (Z, X, Y, H, z_0, Z')$ such that $Z' = Z$ and $\mathcal{M}$ as well as its inverse $\mathcal{M}^{-1}$ are e-output bounded will be referred to as an *ω-transducer*.
The infinite behaviour of an a-transducer $\mathcal{M}$ can be regarded as the limit of the finite behaviour, that is, we call an infinite sequence of transitions

$$(z_0, w_1, v_1, z_1), (z_1, w_2, v_2, z_2), \ldots, (z_{\ell-1}, w_\ell, v_\ell, z_\ell) \ldots$$

an *ω-computation*, iff it has infinitely many prefixes being a computation of $\mathcal{M}$.
Analogously, we denote the input-output relation also by $\xi \vdash_{\mathcal{M}} \eta$. Then $\mathcal{M}(\xi) := \{\eta : \eta \in Y^\omega \wedge \xi \vdash_{\mathcal{M}} \eta\}$ and $\mathcal{M}(F) := \bigcup_{\xi \in f} \mathcal{M}(\xi)$ for $F \subseteq X^\omega$.
We may consider computations and ω-computations of an a-transducer $\mathcal{M}$ as subsets of H^* or H^ω, respectively. Then it holds

Lemma 4.3 *The set $V_{\mathcal{M}} \subseteq H^*$ of computations is a regular language, and the set $E_{\mathcal{M}} \subseteq H^\omega$ is a regular ω-language in the Borel class $\mathbf{G}_\delta$. More precisely, we have $E_{\mathcal{M}} = V^\delta$ for some regular language $V \subseteq V_{\mathcal{M}}$.*

Using techniques known from automata theory we obtain from Lemma 4.3 the following decomposition of a-transducers.

Lemma 4.4 *Let $\mathcal{M} := (Z, X, Y, H, z_0, Z')$ be an (e-output bounded) a-transducer. Then there are homomorphisms h and g and a regular language W such that $\mathcal{M}(F) = h(W^\delta \cap g^{-1}(F))$ for every $F \subseteq X^\omega$.*
If, moreover $\mathcal{M}$ is e-output bounded then h, g and W may be chosen in such a way that h is e-free.

The proof of Theorem 4.1 now yields

Corollary 4.4 *If $F \subseteq X^\omega$ is regular then $\mathcal{M}(F)$ is also regular.*

By standard techniques of automata theory, similar to the ones in the language-case (see e.g. [Gi75]), we can prove that a-transducer-mappings are closed under composition in the case of ω-languages, too.

Lemma 4.5 *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be a-transducers. Then there is an a-transducer $\mathcal{M}$ such that $\mathcal{M}_2(\mathcal{M}_1(F)) = \mathcal{M}(F)$ for every $F \subseteq X^\omega$. If, moreover, $\mathcal{M}_1$ and $\mathcal{M}_2$ are e-output bounded then $\mathcal{M}$ is also e-output bounded.*

Our lemma has the following consequence.

Corollary 4.5 *Let $\mathcal{F}$ be a family of ω-languages. Then the family of ω-languages $\{\mathcal{M}(F) : F \in \mathcal{F}$ and $\mathcal{M}$ is an (e-output bounded) a-transducer$\}$ contains $\mathbf{R}_\omega$ and is closed under (e-free) homomorphism, inverse homomorphism and intersection with regular ω-languages.*

If is readily seen from Lemma 4.4 that the infinite behaviour of a-transducers, $\{(\zeta,\eta) : \eta \in M(\zeta)\}$, is a binary infinitary rational relation in the sense of Gire and Nivat [GN84]. It was observed by Thomas [Th92] that there are several subclasses of the class of infinitary rational relations defined by logical means and via acceptance by certain multitape finite automata. We will not pursue this direction any farther here, instead we return to special cases of a-transducers. A first one is known as generalized sequential machine.

Definition 4.3 A *generalized sequential machine* (or short: *gsm*) is a sextuple $\mathcal{A} = (X, Y, Z, f, g, z_o)$ where

X and Y are the finite input and output alphabets resp.,
Z is the finite set of states,
$z_0 \in Z$ is the initial state,
$f : Z \times X \to Z$ is the next state funtion, and
$g : Z \times X \to Y^*$ is the output function.

As usual f and g may be extended to $Z \times X^*$ via

$$\begin{aligned} f(z,e) &:= z\ , \quad f(z, w\cdot v) := f(f(z,w),v)\ , \text{ and} \\ g(z,e) &:= e\ , \quad g(z, w\cdot v) := g(z,w)\cdot g(f(z,w),v)\ . \end{aligned}$$

A generalized sequential machine $\mathcal{A}$ defines a sequential mapping $\varphi_{\mathcal{A}} : X^* \to Y^*$ by $\varphi_{\mathcal{A}}(w) := g(z_0, w)$. Particularly interesting for our further investigation are totally unbounded gsms, or in other words, finite automata realizing continuous mappings $\overline{\varphi}_{\mathcal{A}} : X^\omega \to Y^\omega$.
A related notion is the notion of ω-transductions introduced by Gire [Gi84] which are simple compositions of totally unbounded gsms and their inverses.

Definition 4.4 *An ω-transduction is a relation $\tau \subseteq X^\omega \times Y^\omega$ for which there ore two totally unbounded gsm-mappings $\varphi : X^\omega \to Z^\omega$ and $\psi : Z^\omega \to Y^\omega$ such that $\tau = \varphi \circ \psi^{-1}$.*

As usual we write $\tau(\zeta) := \{\eta : (\zeta,\eta) \in \tau\}$ and $\tau^{-1}(\eta) := \{\zeta : (\zeta,\psi) \in \tau\}$, and we shall consider ω-transductions also as mappings $\tau : X^\omega \to 2^{Y^\omega}$.

Lemma 4.6 ([Tr62]) *Let $F \subseteq X^\omega$ be a nonempty closed and regular ω-language. Then there is a sequential machine $\mathcal{A}$ with $g : X \to X$ such that $\overline{\varphi}_{\mathcal{A}}(X^\omega) = F$ and $\overline{\varphi}_{\mathcal{A}}(\zeta) = \zeta$ for $\zeta \in F$.*

In [LT90] a class of a-transducers realizing ω-transductions is characterized.

Lemma 4.7 *A mapping $\tau : X^\omega \to 2^{Y^\omega}$ is a rational ω-transduction iff there is an ω-transducer $\mathcal{M}$ such that $\mathcal{M}(\zeta) = \tau(\zeta)$ for all $\zeta \in X^\omega$.*

From our definition of ω-transductions it follows immediately that $\tau^{-1} = \psi \circ \varphi^{-1}$ is also an ω-transduction. With the help of Lemmas 4.5 and 4.7 one can derive the result of [Gi84] that ω-transductions are closed under composition.
As a consequence of Lemma 4.6 we obtain that for every nonempty closed and regular ω-language $F \subseteq X^\omega$ there is a totally unbounded gsm-mapping $\varphi_F : X^* \to X^*$ such that $\varphi_F \circ \varphi_F^{-1}(E) = F \cap E$ for all $E \subseteq X^\omega$. A strengthening of this fact has been proved in [LT86a].

Proposition 4.1 *For every closed and regular ω-language $F \subseteq X^\omega$ there are e-free homomorphisms h_1, h_2, h_3 such that $h_1 \circ h_2^{-1} \circ h_3(E) = F \cap E$ for all $E \subseteq X^\omega$.*

This yields the following representation of rational ω-transductions in terms of e-free homomorphisms.

Theorem 4.3 ([LT90]) *Let τ be a rational ω-transduction. Then there are e-free homomorphisms h_1, h_2, h_3 and g_1, g_2, g_3 such that $\tau = h_1 \circ h_2^{-1} \circ h_3 = g_1^{-1} \circ g_2 \circ g_3^{-1}$.*

From the very definition of ω-transductions and from Corollary 2.2, Lemma 2.2 and Corollary 4.4 we obtain that the class of closed regular ω-languages $\mathbf{F} \cap \mathbf{R}_\omega$ is the smallest class of ω-language closed under ω-transductions. Other classes closed under ω-transductions are $\mathbf{F}_\sigma \cap \mathbf{R}_\omega$ and $\mathbf{R}_\omega$ itself, which latter in virtue of Corollary 4.5 is the smallest class of ω-languages closed under arbitrary a-transducer-mappings. In [Ti90] it is shown indeed these three ore the only classes of regular languages closed under ω-transductions.

4.4 Limit-closure

Now we deal with the pattern of generation of ω-languages from languages which applies to ω-languages accepted by Turing machines. In virtue of Theorem 3.5 and Lemma 3.12.3 we have

$$\mathbf{REK}_\omega = \{h(W^\delta) : W \in \mathbf{REK}\} = \{h(W^\delta) : W \in \mathbf{RE}\} \tag{4.2}$$

where h denotes a projection, that is, a length preserving homomorphism.
Analogously to Eq. (4.2) we define the *limit-closure* $\mathcal{L}_\omega(\mathcal{W})$ of a family of languages $\mathcal{W}$ as

$$\mathcal{L}_\omega(\mathcal{W}) := \{h(W^\delta) : W \in \mathcal{W} \wedge h \text{ is a projection }\} \tag{4.3}$$

Along with this limit closure we define.

$$\mathbf{\Delta}\mathcal{W} := \{W^\delta : W \in \mathcal{W}\} \ . \tag{4.4}$$

Next, we are going to derive which closure properties of $\mathcal{L}_\omega(\mathcal{W})$ inherits from $\mathcal{W}$. We refer to [St87a, Section 6] where closure properties of $\mathbf{\Delta}\mathcal{W}$ inherited from those of $\mathcal{W}$ are derived.
To this end we have to introduce the language theoretic operation of continuation. We set for $U \subseteq X^*$

$$w \rhd U := \{u : u \in U \cap w \cdot X^* \wedge \forall v (w \sqsubset v \sqsubset u \to v \notin U)\} \ , \tag{4.5}$$

that is, $w \triangleright U$ is the set of smallest with respect to the prefix relation "$\sqsubseteq$" words $u \in U$ which have w as prefix. One may think of them as the least continuations of the word w to words in U, hence the term continuation.
We define the *continuation* of a language W to a language U as

$$W \triangleright U := \bigcup_{w \in W} w \triangleright U$$

Then in [St87a, Eq. (20)] it is shown that

$$(W \triangleright U)^\delta = W^\delta \cap U^\delta \ . \tag{4.6}$$

Thus the language-theoretic operation of continuation resembles in some sense intersection. Indeed, the next lemma shows that trios, that is, families of languages closed under e-free homomorphism, inverse homomorphism and intersection with regular languages are also closed under continuation to regular languages.

Lemma 4.8 *Let $U \subseteq X^*$ be a regular language. Then there is a e-free a-transducer $\mathcal{M}_U := (S, X, X, H, s_0, S')$ satisfying $\mathcal{M}_U(W) = W \triangleright U$ for arbitrary $W \subseteq X^*$.*

Proof. Let $M := (X, Z, f, z_0, Z')$ be a deterministic finite automaton accepting the language U. We take a dual state set $\widehat{Z} := \{\widehat{z} : z \in Z\}$ disjoint to Z and define
$S := Z \cup \widehat{Z}$,
$s_0 := z_0$,
$S' := Z' \cup \{\widehat{z}' : z' \in Z'\}$, and
H consisting of transitions of the following form
$(x, z, f(z, x), x)$, where $x \in X$ and $z \in Z$,
$(x, z, \widehat{f(z, x)}, x)$, where $x \in X$ and $z \in Z \setminus Z'$, and
$(e, \widehat{z}, \widehat{f(z, x)}, x)$, where $x \in X$ and $z \in Z \setminus Z'$.
Informally, the transducer works in two phases:
First by rules of the first kind it transfers the input word to the output, randomly switching by applying one rule of the second kind to the second phase.
In this second phase it adds in a nondeterministic way by transitions of the third kind a tail such that the resulting output word is in U. Here the requirement that $\widehat{z} \notin \{\widehat{z} : z \in Z'\}$ guarantees that $\mathcal{M}_U$ stops its output when reaching a first word in U which continues the part of the input word read until phase two.
Assume the transducer $\mathcal{M}_U$ reads the whole input word w and reaches a final state. Then either if $w \in U$ (in which case it never uses transitions of the third kind) or $\mathcal{M}_U$ adds a suffix v to w such that $w \cdot v \in U$ and no intermediate word u, $w \sqsubseteq u \sqsubset w \cdot v$ belongs to U. Thus $\mathcal{M}_U(w) = w \triangleright U$.

Q.E.D.

Trios are by no means the only classes of languages closed under continuation to regular languages. Closure under continuation to regular languages can be proved also for many classes of languages defined by deterministic accepting devices, e. g. for deterministic context-free languages, which are not closed under projection. Informally said, if an

accepting device for the language W may communicate with a deterministic finite automaton accepting the regular language U then using a simple protocol of communication, as e.g. described in [St87a, Section 6], one may construct a device accepting $W \triangleright U$.
Utilizing the operation of continuation the following closure property of the family $\mathbf{\Delta}\mathcal{W}$ has been shown in [St87a, Lemma 44].

Lemma 4.9 *If a family of languages is closed under inverse gsm-mapping, multiplication with finite languages from the right and under continuation to regular languages, then* $\mathbf{\Delta}\mathcal{W}$ *is closed under inverse gsm-mapping and intersection with ω-languages from* $\mathbf{G}_\delta \cap \mathbf{R}_\omega$.

From this lemma we obtain the following closure property of $\mathcal{L}_\omega(\mathcal{W})$.

Theorem 4.4 *If* $\mathcal{W}$ *is a trio then* $\mathcal{L}_\omega(\mathcal{W})$ *is closed under e-free homomorphism, inverse homomorphism, and intersection with regular ω-languages.*

Proof. In virtue of Lemmas 4.4 and 4.5 it suffices to show that $\mathcal{L}_\omega(\mathcal{W}) = \{\mathcal{M}(W^\delta) : W \in \mathcal{W} \wedge \mathcal{M}$ is an e-free a-transducer$\}$. From Lemma 4.4 we obtain that $\mathcal{M}(W^\delta) = h(g^{-1}(W^\delta) \cap V^\delta)$ for homomorphisms h, g, where h is e-free, and a regular language V. Now utilizing Lemma 4.9 we obtain a $W', W'' \in \mathcal{W}$ such that $\mathcal{M}(W^\delta) = h((W')^\delta \cap V^\delta) = h((W'')^\delta)$. Finally, we observe that every e-free homomorphism h may be represented as $h_1 \circ h_2$ where h_1 is a projection and h_2 maps its input alphabet X in a one-to-one manner onto a prefix-free language C. It is readily seen that $h_2((W'')^\delta) = h_2(W'')^\delta$, and the assertion is proved.
Q.E.D.

Now, by standard techniques of language theory one proves the following corollaries to Theorem 4.4.

Corollary 4.6 *If* $\mathcal{W}$ *is closed under union then* $\mathcal{L}_\omega(\mathcal{W})$ *is also closed under union, and if* $\mathcal{W}$ *is a trio closed under product then* $\mathcal{L}_\omega(\mathcal{W})$ *is closed under multiplication with languages* $W \in \mathcal{W}$ *from the left.*

Corollary 4.7 *If* $\mathcal{W}$ *is an* **AFL**, *then*

$$\mathcal{L}_\omega(\mathcal{W}) \supseteq \mathcal{K}_\omega(\mathcal{W}).$$

Applying our mapping $\mathcal{L}_\omega$ to the families of languages considered up to now we obtain

$$\mathcal{L}_\omega(\mathbf{R}) = \mathcal{K}_\omega(\mathbf{R}) = \mathbf{R}_\omega, \tag{4.7}$$
$$\mathcal{L}_\omega(\mathbf{DCF}) = \mathbf{CF}_\omega \supset \mathcal{K}_\omega(\mathbf{DCF}), \tag{4.8}$$
$$\mathcal{L}_\omega(\mathbf{CF}) \supset \mathcal{K}_\omega(\mathbf{CF}) = \mathbf{CF}_\omega \text{ , and} \tag{4.9}$$
$$\mathcal{L}_\omega(\mathbf{RE}) = \mathcal{L}_\omega(\mathbf{REK}) = \mathbf{REK}_\omega. \tag{4.10}$$

The proper inclusion in Eq. (4.9) is explained by Example 3.1 which shows that already $\mathbf{\Delta CF} \not\subseteq \mathbf{CF}_\omega$. We obtain also an Init-lemma.

Lemma 4.10 (Init-lemma for the Limit-closure) *Let* $\mathcal{V}$ *be a trio and let* $\mathcal{V}_L$ *be the trio generated by* $\{\mathbf{A}(F) : F \in \mathcal{L}_\omega(\mathcal{V})\}$. *Then* $\mathcal{V}_L \supseteq \mathcal{V}$.

Unlike the Init-lemma for the ω-Kleene-closure we cannot expect that $\mathcal{V}_L = \mathcal{V}$ even if $\mathcal{V}$ is an **AFL** closed under arbitrary homomorphism, as the example $\mathcal{V} = \mathbf{RE}$ shows. Here $\mathcal{V}_L = \Sigma_1^1 \supset \mathbf{RE}$.

5 Wagner's hierarchy

Our Theorem 3.1 shows that the Borel subclasses of $R_\omega^{(X)}$ correspond to the automata theoretic defined subclasses. In 1979 K. Wagner proved that this correspondence between topological and automata theoretic properties of regular ω-languages can be extended much farther to a countably infinite hierarchy of classes of regular ω-languages. This hierarchy which will be called henceforth *Wagner's hierarchy*. Utilizing results of [BL69], [vW76], [Ba92] and Wagner's original paper [Wa79], it was shown (cf. e. g. [Se95a]) that Wagner's hierarchy is indeed the Wadge hierarchy restricted to regular subsets of X^ω, thus showing that the Wagner (or Wadge) hierarchy for regular ω-languages can be also defined in automata theoretic terms.
Fragments of Wagner's hierarchy, were investigated as the Boolean hierarchy over $\mathbf{R}_\omega^{(X)} \cap \mathbf{G}_\delta$ in [Wa77] and [Ka85] and as the Hausdorff-Kuratowski hierarchy for regular ω-languages in [Ba92].
In this section we shall give a general outline of Wagner's approach. For a more detailed treatment the reader is referred to the above mentioned papers, in particular [Wa79] or [Se95a], or to [WY95] where a polynomial-time algorithm computing the Wagner degree of a regular ω-language F from an automaton accepting F is presented.

5.1 Wagner classes

Next we introduce the Wagner classes $\hat{C}_m^n, \hat{D}_m^n, \hat{E}_m^n$ of [Wa77,79]:
We say that a state $z' \in Z$ of an finite automaton $A = (X, Z, z_0, f)$ is *reachable* from another state $z \in Z$ provided there is a nonempty word $w \in X^* \setminus \{e\}$ such that $z' = f(z, w)$. A *loop* in a finite automaton A is a set $Z' \subseteq Z$ such that each two states $z, z' \in Z'$ are likewise reachable. Given $\mathcal{Z} \subseteq 2^Z$ the set $\mathbf{L}$ of all loops in A is partitioned into the set $\mathbf{P} = \mathbf{L} \cap \mathcal{Z}$ of *positive* loops and the set $\mathbf{N} := \mathbf{L} \setminus \mathbf{P}$ of *negative* loops. The set of all maximal loops (with respect to set inclusion) is denoted by $\mathbf{M}$. (Notice that a maximal loop is the same as what is known as a 'strongly connected component' in the underlying graph.)
An *alternating chain* of length n is of the form

$$L_1 \subset L_2 \subset L_3 \subset \cdots \subset L_n,$$

where $L_i \in \mathbf{P}$ iff $L_{i+1} \in \mathbf{N}$ for $1 \le i < n$. A *positive chain* starts with a positive loop $L_1 \in \mathbf{P}$, and a *negative chain* starts with a negative loop.
Notice that all loops of a chain are contained in a maximal loop $M \in \mathbf{M}$.
Let M be a maximal loop. In every alternating chain contained in M of maximal length one can replace the last element by M itself. Therefore all alternating chains of maximal length contained in M have the same sign. The length of a maximal alternating chain in M is denoted by $\ell(M)$ and the sign of these chains is denoted by $s(M)$.
The *first invariant* of a pair $(A, \mathcal{Z})$ is the maximum of all the values $\ell(M)$, that is, the maximum length of all alternating chains in $(A, \mathcal{Z})$. It is denoted by $m(A, \mathcal{Z})$.
A *top loop* is a maximal loop M that contains a chain of length $\ell(M) = m(A, \mathcal{Z})$. A top loop is positive or negative depending on whether it occurs in a longest chain that is a

positive or negative. (Notice that if a longest chain has even length then a positive top loop is a negative loop.)
An *alternating superchain* is a sequence $L_1, \ldots, L_n$ of top loops where for every i with $1 \leq i < n$ the loop L_{i+1} is reachable from L_i and L_i is a positive top loop iff L_{i+1} is not. The alternating superchain is called positive if L_1 is a positive top loop and negative otherwise. (Notice that one loop cannot occur at two distinct places in a superchain.)
The *second invariant* of a pair $(A, \mathcal{Z})$ is the maximal length of all alternating superchains. It is denoted by $n(A, \mathcal{Z})$. A *longest superchain* is an alternating superchain of length $n(A, \mathcal{Z})$. The pair $(A, \mathcal{Z})$ is called *prime* if all longest superchains have the same sign. Denote by $s(A, \mathcal{Z})$ the sign of the longest superchain if $(A, \mathcal{Z})$ is prime and $s(A, \mathcal{Z}) := 0$ if $(A, \mathcal{Z})$ is nonprime. First we recall Wagner's fundamental independence result.

Theorem 5.1 ([Wa77,79]) *Let $A = (X, Z, z_0, f)$ and $A' = (X, S, s_0, g)$ be two deterministic automata and let $\mathcal{Z} \subseteq 2^Z$ and $\mathcal{S} \subseteq 2^S$ such that $T^{inf}_{=}(A, \mathcal{Z}) = T^{inf}_{=}(A', \mathcal{S})$. Then $s(A, \mathcal{Z}) = s(A', \mathcal{S})$, $m(A, \mathcal{Z}) = m(A', \mathcal{S})$ and $n(A, \mathcal{Z}) = n(A', \mathcal{S})$.*

Theorem 5.1 allows us to define the following classes of regular ω-languages.

Definition 5.1

$$
\begin{aligned}
C^n_m &:= \{T^{inf}_{=}(A, \mathcal{Z}) : s(A, \mathcal{Z}) = +1 \wedge m(A, \mathcal{Z}) = m \wedge n(A, \mathcal{Z}) = n\} \\
D^n_m &:= \{T^{inf}_{=}(A, \mathcal{Z}) : s(A, \mathcal{Z}) = -1 \wedge m(A, \mathcal{Z}) = m \wedge n(A, \mathcal{Z}) = n\} \\
E^n_m &:= \{T^{inf}_{=}(A, \mathcal{Z}) : s(A, \mathcal{Z}) = 0 \wedge m(A, \mathcal{Z}) = m \wedge n(A, \mathcal{Z}) = n\} \\
\hat{C}^n_m &:= C^n_m \cup \{T^{inf}_{=}(A, \mathcal{Z}) : m(A, \mathcal{Z}) < m \vee (m(A, \mathcal{Z}) = m \wedge n(A, \mathcal{Z}) < n)\} \\
\hat{D}^n_m &:= D^n_m \cup \{T^{inf}_{=}(A, \mathcal{Z}) : m(A, \mathcal{Z}) < m \vee (m(A, \mathcal{Z}) = m \wedge n(A, \mathcal{Z}) < n)\} \\
\hat{E}^n_m &:= E^n_m \cup \hat{C}^n_m \cup \hat{D}^n_m
\end{aligned}
$$

Observe that, $C^n_m = \hat{C}^n_m \setminus \hat{D}^n_m$ and $D^n_m = \hat{D}^n_m \setminus \hat{C}^n_m$.
Moreover, we have the following complementation property

$$F \in \hat{D}^n_m \quad \text{iff} \quad X^\omega \setminus F \in \hat{C}^n_m. \tag{5.1}$$

The classes $\hat{C}^n_1$, $\hat{D}^n_1$, $\hat{C}^1_m$, $\hat{D}^1_m$ can be represented as Boolean expressions over regular ω-languages in $\mathbf{G}$ or $\mathbf{G}_\delta$, respectively. In view of Eq. (5.1) we may confine here to the representation results for $\hat{C}^n_1$ and $\hat{C}^1_m$.

Theorem 5.2 ([Ba92, Wa77,79]) *For $m, n \geq 1$ it holds*

$$
\begin{aligned}
\hat{C}^{2n-1}_1 &= \{\bigcup_{i=1}^{n-1} (F_i \setminus F'_i) : F_i, F'_i \in \mathbf{G} \cap \mathbf{R}^{(X)}_\omega\} \\
&= \{\bigcup_{i=1}^{n-1} (F_i \setminus F'_i) : F_i, F'_i \in \mathbf{G}\} \ \cap \ \mathbf{R}^{(X)}_\omega \ ,
\end{aligned}
$$

$$\begin{aligned}
\hat{C}_1^{2n} &= \{F \cup \bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F, F_i, F_i' \in \mathbf{G} \cap \mathbf{R}_\omega^{(X)}\} \\
&= \{F \cup \bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F, F_i, F_i' \in \mathbf{G}\} \ \cap \ \mathbf{R}_\omega^{(X)} \ , \ \textit{and} \\
\hat{C}_{2m+1}^1 &= \{\bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F_i, F_i' \in \mathbf{G}_\delta \cap \mathbf{R}_\omega^{(X)}\} \\
&= \{\bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F_i, F_i' \in \mathbf{G}_\delta\} \ \cap \ \mathbf{R}_\omega^{(X)} \ , \ \textit{and} \\
\hat{C}_{2m}^1 &= \{F \cup \bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F, F_i, F_i' \in \mathbf{G}_\delta \cap \mathbf{R}_\omega^{(X)}\} \\
&= \{F \cup \bigcup_{i=1}^{n-1}(F_i \setminus F_i') : F, F_i, F_i' \in \mathbf{G}_\delta\} \ \cap \ \mathbf{R}_\omega^{(X)} \ .
\end{aligned}$$

In particular, we have $\hat{C}_1^1 = \{\emptyset\}$, $\hat{C}_1^2 = \mathbf{G} \cap \mathbf{R}_\omega^{(X)}$, $\hat{C}_2^1 = \mathbf{G}_\delta \cap \mathbf{R}_\omega^{(X)}$.
Before proceeding to a topological characterization of the classes $\hat{C}_m^n$ and $\hat{D}_m^n$ when $m, n \geq 2$, let us consider superchains of length $\ell \geq 2$:
Assume we have a finite automaton $A = (X, Z, z_0, f)$ and two maximal loops $Z_1, Z_2 \subseteq Z$, $Z_1 \cap Z_2 = \emptyset$ such that Z_2 is reachable from Z_1. Let further $\mathcal{Z}_i$ be a nonempty set of loops in Z_i $(i = 1, 2)$. Then $F_i := T_{=}^{\mathit{inf}}(A, \mathcal{Z}_i)$ are nonempty regular ω-languages.
Consider the topological closures of F_1 and F_2, $\mathcal{C}(F_1)$ and $\mathcal{C}(F_2)$, in Cantor space. Then it holds

$$\mathcal{C}(F_1) \cap F_2 = \emptyset \quad \text{and} \quad F_1 \subseteq \mathcal{C}(F_2) \ . \tag{5.2}$$

This consideration transforming reachability of chains to a topological argument might facilitate the understanding of the following result.

Theorem 5.3 ([Wa77,79]) *Let $n \geq 1$ and $m \geq 2$. Then it holds*

1. *$E \in \hat{C}_m^n$ iff there are $E_1, \ldots, E_n$ such that $E = \bigcup_{i=1}^n E_i$, $E_i \in \hat{C}_1^n$ for i odd, $E_i \in \hat{D}_1^n$ for i even, $\mathcal{C}(E_i) \cap E_{i+1} = \emptyset$, and $E_i \subseteq \mathcal{C}(E_{i+1})$.*

2. *$F \in \hat{D}_m^n$ iff there are $F_1, \ldots, F_n$ such that $F = \bigcup_{i=1}^n F_i$, $F_i \in \hat{D}_1^n$ for i odd, $F_i \in \hat{C}_1^n$ for i even, $\mathcal{C}(F_i) \cap F_{i+1} = \emptyset$, and $F_i \subseteq \mathcal{C}(F_{i+1})$.*

5.2 gsm-reducibility

So far we have characterized Wagner's classes $\hat{C}_m^n$ and $\hat{D}_m^n$. Next we introduce the type of reducibility introduced in [Wa77,79] for regular ω-languages.

Definition 5.2 An ω-language $F \subseteq X^\omega$ is called DA-*reducible* (short: $F \leq_{\mathrm{DA}} E$) to an ω-language $E \subseteq X^\omega$ if there is a totally unbounded gsm-mapping $\varphi : X^* \to X^*$ such that $F = \overline{\varphi}^{-1}(E)$.

Totally unbounded gsm-mappings are totally unbounded sequential mappings. So Wagner's DA-reducibility is a particular case of the Wadge reducibility introduced in Section 2.3. It is remarkable that due to results of [BL69], [vW78], [Wa79], [Ba92], and [Se95a] Wadge-reducibility and DA-reducibility coincide for regular ω-languages.

Theorem 5.4 *Let $E, F \subseteq X^\omega$ be regular. Then $E \leq_W F$ implies $E \leq_{DA} F$.*

Therefore, if a Wadge-degree $\{E : E \equiv_W F\}$ contains a regular ω-language we shall refer to $\{E : E \equiv_W F\} \cap \mathbf{R}_\omega^{(X)}$ as a *Wagner degree*.
From Theorem 5.2 we obtain

Lemma 5.1 *The sets $C_m^n = \hat{C}_m^n \setminus \hat{D}_m^n$ and $D_m^n = \hat{D}_m^n \setminus \hat{C}_m^n$ are Wagner degrees.*

It turns out, however, that the sets E_m^n are not single Wagner degrees but, in fact, countable unions of Wagner degrees. We derive an example that E_2^1 is not a single degree.

Example 5.1 Consider the ω-language $F := 0\cdot\{0,1\}^*\cdot 0^\omega \cup 1\cdot(0^*\cdot 1)^\omega \subseteq \{0,1\}^\omega$ introduced at the end of Section 3.2. It is obvious that $F \in E_2^1$. Moreover, if we consider the ω-language $F' := 0^\omega \cup 0^* \cdot 1 \cdot F$ it turns out that $F' \in E_2^1$, too.
Assume $F' \leq_W F$, that is, there is a continuous mapping $\Phi : X^\omega \to X^\omega$ such that $F' = \Phi^{-1}(F)$. Then $\Phi(0^\omega) \in F$. We have to consider two cases: $\Phi(0^\omega) \in F \cap 0 \cdot \{0,1\}^\omega$ and $\Phi(0^\omega) \in F \cap 1 \cdot \{0,1\}^\omega$.
In the first case, since Φ is continuous, there is a $k \in \mathbb{N}$ satisfying $\Phi(0^k \cdot 1 \cdot 0 \cdot \{0,1\}^\omega) \subseteq 0 \cdot \{0,1\}^\omega$. Now, $F' = \Phi^{-1}(F)$ implies that $\Phi(0^k \cdot 1 \cdot (0^* \cdot 1)^\omega) \subseteq 0 \cdot \{0,1\} \cdot 0^\omega$ and $\Phi(0^k \cdot 1 \cdot \{0,1\} \cdot 0^\omega) \subseteq 0 \cdot (0^* \cdot 1)^\omega$ which is impossible. In the second case we derive likewise the contradiction $\Phi(0^k \cdot 1 \cdot \{0,1\} \cdot 0^\omega) \subseteq 1 \cdot (0^* \cdot 1)^\omega$ and $\Phi(0^k \cdot 1 \cdot (0^* \cdot 1)^\omega) \subseteq 1 \cdot \{0,1\} \cdot 0^\omega$.

In order to solve this problem, Wagner introduced a procedure which he called derivation. It relies on the following separation principle explicitly utilized by Selivanov [Se95a,b]:
Let $F \in E_m^n$ and consider the set W_F of all words w such that $F \cap w \cdot X^\omega \in \hat{C}_m^n$ and the set V_F of all words v such that $F \cap w \cdot X^\omega \in D_m^n$. It turns out that for regular ω-languages F the following separation procedure yields an ω-language in a lower class $\hat{E}_{m'}^{n'}$ $(m' < m)$: First remove from F the set $W_F \cdot X^\omega$ and then add the whole set $V_F \cdot X^\omega$ that is, to speak in terms of Wagner's paper:
The *derivative* of an ω-language $F \in E_m^n$ is $\partial F := (F \setminus W_F \cdot X^\omega) \cup V_F \cdot X^\omega$. This results in the following procedure for automata:
The *derivative* $\partial(A, \mathcal{Z})$ of a non-prime pair $(A, \mathcal{Z})$ with $A = (X, Z, z_0, f)$ is a new pair $((X, S, s_0, g), \mathcal{S})$, which is described in what follows.
The set of all states from which a positive longest superchain is reachable is denoted by $\partial^+ Z$. Likewise, the set of all states from which a negative longest superchain is reachable is denoted by $\partial^- Z$. Their intersection is denoted by ∂Z. Observe that $z_0 \in \partial Z$ if $\partial Z \neq \emptyset$. Now S is the union of ∂Z with two new states s^+ and s^-, $s_0 := z_0$ if $\partial Z \neq \emptyset$ and $s_0 := s^+$ otherwise. The transition function is defined by

$$g(s,x) := \begin{cases} f(s,x) & \text{if } f(s,x) \in \partial Z \\ s^+ & \text{if } f(s,x) \in \partial^+ Z \setminus \partial^- Z \\ s^- & \text{if } f(s,x) \notin \partial^+ Z \end{cases} \qquad \text{for } s \in \partial Z,$$

$$g(s^+,x) := s^+,$$
$$g(s^-,x) := s^- ,$$

and $\mathcal{S} := \{L \in \mathcal{Z} : L \subseteq \partial Z\} \cup \{\{s^+\}\}$.
Notice that $m(\partial(A,\mathcal{Z})) < m(A,\mathcal{Z})$ whenever $m(A,\mathcal{Z}) > 1$.
Finally, we give Wagner's naming procedure.
Let $(A,\mathcal{Z})$ be a pair such that $A = (X, Z, z_0, f)$ is a deterministic automaton, $\mathcal{Z} \subseteq 2^Z$, $m := m(A,\mathcal{Z})$, and $n = n(A,\mathcal{Z})$. We now associate with $(A,\mathcal{Z})$ a name of a Wadge degree, $\mathbf{W}(A,\mathcal{Z})$, constructed recursively as described below.

1. If $(A,\mathcal{Z})$ is prime and all longest superchains are positive, then $\mathbf{W}(A,\mathcal{Z}) := C_m^n$.
2. If $(A,\mathcal{Z})$ is prime and all longest superchains are negative, then $\mathbf{W}(A,\mathcal{Z}) := D_m^n$.
3. If $(A,\mathcal{Z})$ is non-prime and $m = 1$, then $\mathbf{W}(A,\mathcal{Z}) := E_1^n$.
4. If $(A,\mathcal{Z})$ is non-prime and $m > 1$, then $\mathbf{W}(A,\mathcal{Z}) := E_m^n \mathbf{W}(\partial(A,\mathcal{Z}))$.

As the above invariance theorem proves, this procedure assigns to a regular ω-language $F \subseteq X^\omega$ independently of the pair $(A,\mathcal{Z})$ which accepts F a unique name of the form

$$\mathbf{W}(F) := \mathbf{W}(A,\mathcal{Z}) = E_{m_1}^{n_1} \dots E_{m_k}^{n_k} H_{m_{k+1}}^{n_{k+1}} ,$$

where $m_1 > m_2 > \dots > m_k > m_{k+1}$, $H \in \{C, D, E\}$ and $m_{k+1} = 1$ if $H = E$.
Moreover, it is shown in [Wa79] that for every admissible name $E_{m_1}^{n_1} \dots E_{m_k}^{n_k} H_{m_{k+1}}^{n_{k+1}}$ there is a regular ω-language $F \subseteq X^\omega$ such that $\mathbf{W}(F) = E_{m_1}^{n_1} \dots E_{m_k}^{n_k} H_{m_{k+1}}^{n_{k+1}}$.
Now the $\leq_W$-relation can be easily read off from the names of the Wagner degrees: Let $F, F' \subseteq X^\omega$ be regular ω-languages. Then $F \leq_W F'$ iff for the corresponding names $\mathbf{W}(F) = E_{m_1}^{n_1} \dots E_{m_k}^{n_k} H_{m_{k+1}}^{n_{k+1}}$ and $\mathbf{W}(F') = E_{m'_1}^{n'_1} \dots E_{m'_l}^{n'_l} {H'}_{m'_{l+1}}^{n'_{l+1}}$ there is an index $j \leq \min\{k+1, k'+1\}$ such that $m_i = m'_i$ and $n_i = n'_i$ for $1 \leq i \leq j$ and either of the following conditions holds:

1. $j = k+1 \leq k'+1$ and $H' = E$ or $H = H'$.
2. $j < \min\{k+1, k'+1\}$ and $m_{j+1} < m'_{j+1}$ or $(m_{j+1} = m'_{j+1} \wedge n_{j+1} < n'_{j+1})$.

Concluding remark

Besides the reducibility via totally unbounded gsm-mappings in [Wa79] also a reducibility via sequential machine mappings, that is, length preserving gsm-mappings was considered. Here also the results of [BL69], [vW76], and Wagner's original paper [Wa79] show that here the corresponding degrees, Wagner's so-called DS-degrees, coincide with the restriction of so-called Lifschitz degrees (cf. [vW76]) to regular ω-languages.

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