Recursive Types in Kleisli Categories

Alex K. Simpson*

Department of Computer Science, University of Edinburgh,

JCMB, The King's Buildings, Edinburgh, EH9 3JZ

August 10, 1992

Abstract

We show that an enriched version of Freyd's principle of versality holds in the Kleisli category of a commutative strong monad with fixed-point object. This gives a general categorical setting in which it is possible to model recursive types involving the usual datatype constructors.

1 Introduction

One of the goals of axiomatic domain theory is to give a categorical account of datatype constructions in terms of suitable universal properties. In this paper we show how recursive types, involving the usual type constructors, can be modelled in the Kleisli category of a strong monad satisfying certain assumptions.

A datatype with n free type variables is naturally interpreted by a bifunctor $[\![\sigma]\!] : (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})^n \to \mathcal{C}$. The lack of true (covariant) functoriality makes the interpretation of recursive types quite difficult, whereas the corresponding situation for inductive types (those interpretable by covariant functors $[\![\sigma]\!] : \mathcal{C}^n \to \mathcal{C}$) is much simpler, as inductive types can be easily interpreted using initial algebras. In a series of recent articles, [5, 6, 7], Peter Freyd has shown how, under certain conditions, the simple methods for interpreting inductive types can be applied to recursive types.

Central to Freyd's approach is the requirement that initial algebras and terminal coalgebras be *canonically isomorphic*. Technically, this requirement is exactly suited to its purpose of dealing with contravariance. For intuition, Freyd justifies his requirement by appeal to what he calls the *principle of versality*: namely that datatypes should be equally good for both input and output, thus universal definitions should be equivalent to their duals. But the relation between the technical requirement and the computational intuition is not that compelling. Further, the requirement is inconsistent with many standard properties of categories such as distributivity, cartesian closure and pre-ordering, to name but three. Clearly, if the requirement is to apply, we are unlikely to be able to interpret products and sums by their categorical analogues. It is desirable to find a setting which accounts for all the usual datatypes, and in which Freyd's requirement follows from some less sweeping, more intuitive axioms.

*Supported by SERC grant no. 90311820

In this paper we fit Freyd's work into a general categorical setting for interpreting the usual type constructors. Here we build upon the work of Moggi who suggests that programs are naturally interpreted in the Kleisli category of a strong monad [14]. Moggi's approach gives an elegant account of the usual datatypes such as sums and products, for these are interpreted in the base category of the monad by their genuine categorical counterparts and then lifted by the inclusion functor into the Kleisli category. In order to incorporate recursive types, we give conditions on the monad for which it is a derived feature of the Kleisli category that the property of being an initial algebra (for suitably enriched endofunctors) is equivalent to the property of being a terminal coalgebra. The main requirement that ensures this is the existence of a *fixed-point object*, using definitions and reults due to Crole, Pitts and Mulry [3, 16].

The structure of the paper is as follows. Section 2 reviews some of the elementary properties of initial algebras, terminal coalgebras, and some of the important ideas of Freyd. In Section 3 we introduce the structure on the category and monad that we shall assume throughout. In Section 4 some useful results about fixed-points are proved. These are then applied in Section 5 to give, in an abstract enriched setting, the desired equivalence between initial algebras and terminal coalgebras. In Section 6 it is shown that the Kleisli category can be considered as a suitably enriched category and thus the results of Section 5 are applicable. We then sketch how the theory can be applied to give models of recursively typed calculi. Finally, in Section 7, we suggest possible developments, including how the theory may also be applicable to the Eilenberg-Moore category.

2 Initial algebras and terminal coalgebras

Given an endofunctor T on a category $\mathcal{C}$, a *T-algebra* is a morphism $TX \xrightarrow{x} X$ in $\mathcal{C}$. A *T-invariant object* is a T-algebra that is an isomorphism. A *T-homomorphism* from $TX \xrightarrow{x} X$ to $TY \xrightarrow{y} Y$ is a morphism $X \xrightarrow{z} Y$ such that the diagram below commutes.

$$\begin{tikzcd} TX \arrow[r, "x"] \arrow[d, "Tz"'] & X \arrow[d, "z"] \\ TY \arrow[r, "y"] & Y \end{tikzcd}$$

T-algebras and T-homomorphisms form a category with identities and composition inherited from $\mathcal{C}$. An *initial* T-algebra is just an initial object in this category. Initial T-algebras are thus determined up to isomorphism. The following well-known lemma is attributed to Lambek [1].

Lemma 2.1 *Any initial T-algebra is a T-invariant object.*

Given a functor $S : \mathcal{C}' \times \mathcal{C} \to \mathcal{C}$ such that, for every $X \in ob\ \mathcal{C}'$, the endofunctor $S(X, -)$ on $\mathcal{C}$ has an initial algebra, $S(X, A_X) \xrightarrow{\alpha_X} A_X$, a functor $S^\dagger : \mathcal{C}' \to \mathcal{C}$ can be defined as follows. The object part of $S^\dagger$ is defined by $S^\dagger X = A_X$. Given $X \xrightarrow{f} Y$ in $\mathcal{C}'$, $A_X \xrightarrow{S^\dagger f} A_Y$ is defined to be the unique morphism, given by the initial algebra property of α_X, making the diagram below

commute.

$$\begin{tikzcd}
S(X, A_X) \arrow[rr, "\alpha_X"] \arrow[d, "S(1, S^\dagger f)"'] & & A_X \arrow[d, "S^\dagger f"] \\
S(X, A_Y) \arrow[r, "S(f,1)"] & S(Y, A_Y) \arrow[r, "\alpha_Y"] & A_Y
\end{tikzcd}$$

It is routine to check that the $S^\dagger$ so defined is indeed a functor.

A *T-coalgebra* is just a $T^{\mathrm{op}} - algebra$. A *$T$-cohomomorphism* from $X \xrightarrow{x} TX$ to $Y \xrightarrow{y} TY$ is a morphism $X \xrightarrow{z} Y$ such that z^{op} is a T^{op}-homomorphism from y^{op} to x^{op}. A *terminal* T-coalgebra is a terminal object in the category of T-coalgebras and T-cohomomorphisms. Equivalently a terminal T-coalgebra is just an initial T^{op}-algebra. By duality, the analogues of lemma 2.1 holds for terminal coalgebras.

A category is called *algebraically complete* if every endofunctor on the category has an initial algebra. It is called *algebraically compact* if it is algebraically complete and, for each T, if $TX \xrightarrow{x} X$ is an initial T-algebra then $X \xrightarrow{x^{-1}} TX$ is a terminal T-coalgebra (x must be an isomorphism by lemma 2.1). Both these notions are due to Freyd [6] (although there he defines algebraic compactness in terms of a canonical isomporphism from initial algebra to terminal coalgebra). Examples of algebraically complete and compact categories are given in [7]. Algebraic compactness has two aspects: an existence aspect and a duality aspect. The duality is of interest in its own right. Accordingly we say a category is *consistently* algebraically compact if for every endofunctor, T, $TX \xrightarrow{x} X$ is an initial T-algebra if and only if $X \xrightarrow{x^{-1}} TX$ is a terminal T-coalgebra. Consistent algebraic compactness is none other than Freyd's principle of versality.

We now sketch why consistent algebraic compactness is important. The outline below is based upon work of Fiore and Plotkin which is as yet unpublished (though see [4]). Consider the category $\widehat{\mathcal{C}} = \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. This category has an involution:

$$(\cdot)^D : (f', f) \mapsto (f, f')$$

which establishes an isomorphism of categories between $\widehat{\mathcal{C}}^{\mathrm{op}}$ and $\widehat{\mathcal{C}}$. Call an endofunctor $T : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ *symmetric* if $(T(f', f))^D = T(f', f)^D$. Clearly symmetric endofunctors on $\widehat{\mathcal{C}}$ are in one-to-one correspondence with bifunctors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathcal{C}$. This correspondence enables the bifuntor representing a type constructor to be "diagonalised" to a covariant endofunctor on $\widehat{\mathcal{C}}$. Note that the notion of symmetry extends easily to functors from $\widehat{\mathcal{C}}^{\,n}$ to $\widehat{\mathcal{C}}$, and there is a similar correspondence with multi-arity bifunctors.

Now suppose $\widehat{\mathcal{C}}$ is consistently algebraically compact and T is a symmetric endofunctor. Fiore and Plotkin have shown that if T has an initial algebra then T has an initial algebra of the form:

$$T(X, X) \xrightarrow{(x^{-1}, x)} (X, X)$$

Thus X above is a canonical solution to the recursive domain equation given by the bifunctor generating T. Importantly, the above result extends so that if $S : \widehat{\mathcal{C}}^{\,n} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ is symmetric and $S^\dagger : \widehat{\mathcal{C}}^{\,n} \to \widehat{\mathcal{C}}$ exists, then $S^\dagger$ can be constructed to be symmetric.

Although stated in terms of all functors, the definitions of algebraic compactness and consistent algebraic compactness are normally to be understood in the setting of a suitable 2-category of categories and functors [7]. Examples of algebraically compact categories have been particularly

forthcoming in various 2-categories of CPO enriched categories (where CPO is to be understood flexibly enough to include subcategories of PERS [7]). We too will be dealing with enriched versions of the definitions (see Section 5).

3 A category of predomains

Throughout this paper we work with a category, $\mathcal{C}$, of "predomains" which we assume to have the following structure. $\mathcal{C}$ must be cartesian (but not necessarily cartesian closed), with a distinguished *faithful strong monad*, (T, η, μ, t), with respect to which $\mathcal{C}$ has *Kleisli exponentials*, Y_T^X, and a *fixed-point object*, $\mathbf{1} \xrightarrow{\omega} \Phi \xleftarrow{\sigma} T\Phi$. We spell out these requirements below.

We write $X \xleftarrow{\pi_{1_{XY}}} X \times Y \xrightarrow{\pi_{2_{XY}}} Y$ for the distinguished product structure, denoting pairing by $C \xrightarrow{\langle f,g \rangle} X \times Y$, the symmetry isomorphism by $X \times Y \xrightarrow{c_{XY}} Y \times X$, and the associativity isomorphism by $(X \times Y) \times Z \xrightarrow{a_{XYZ}} X \times (Y \times Z)$. We write $X \xrightarrow{!_X} \mathbf{1}$ for the terminal object and its universal morphism.

Strong monads

A *strong monad*, (T, η, μ, t), is a monad (T, η, μ) together with a natural transformation, its *strength*, $t_{XY} : X \times TY \to T(X \times Y)$, satisfying four diagrams (see [14]). This definition is due to Kock [11]. The strong monad is *faithful* if T is. Faithfulness is easily seen to be equivalent to Moggi's *mono requirement*, namely that all the components of η be monos (see [15]).

We write $\mathcal{C}_T$ for the Kleisli category with the standard adjunction $F_T \dashv G_T : \mathcal{C}_T \to \mathcal{C}$. We write $\mathcal{C}^T$ for the Eilenberg-Moore category with adjunction $F^T \dashv G^T : \mathcal{C}^T \to \mathcal{C}$. $\mathcal{C}^T$ is cartesian with products given by:

$$(X, x) \times (Y, y) \quad = \quad (X \times Y,\ (x \times y) \circ \langle T\pi_1, T\pi_2 \rangle)$$

A fourth category $\mathcal{D}$ is defined by $ob\ \mathcal{D} = ob\ \mathcal{C}^T$ and $\mathcal{D}((X, x), (Y, y)) = \mathcal{C}(X, Y)$ with identities and composition inherited from $\mathcal{C}$. In fact $\mathcal{D}$ is the co-Kleisli category of the obvious $F^T G^T$ comonad on $\mathcal{C}^T$. Clearly the forgetful functor $U : \mathcal{D} \to \mathcal{C}$ gives (using choice) an equivalence of categories between $\mathcal{D}$ and the full subcategory $\mathcal{C}'$ of $\mathcal{C}$ obtained as the image of U. The reason for preferring $\mathcal{D}$ to $\mathcal{C}'$ is that we have the co-Kleisli inclusion $I : \mathcal{C}^T \to \mathcal{D}$ which gives $\mathcal{C}^T$ as a distinguished subcategory of $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is cartesian, its products being the same as those of $\mathcal{C}^T$. Further, in the case that $\mathcal{C}$ is cartesian closed then $\mathcal{D}$ is too (in fact $\mathcal{C}'$ is an exponential ideal of $\mathcal{C}$).

The strength of the monad enables the following natural transformations to be defined.

$$t'_{XY} \quad = \quad TX \times Y \xrightarrow{Tc \circ t \circ c} T(X \times Y)$$

$$\psi_{XY} \quad = \quad TX \times TY \xrightarrow{\mu \circ Tt \circ t'} T(X \times Y)$$

$$\psi'_{XY} \quad = \quad TX \times TY \xrightarrow{\mu \circ Tt' \circ t} T(X \times Y)$$

We say that the monad is *commutative* if $\psi = \psi'$.

We call a morphism $(X, x) \xrightarrow{f} (Y, y)$ in $\mathcal{D}$ *linear* if it is an algebra homomorphism from (X, x) to (Y, y), ie. if $f \circ x = y \circ Tf$, or equivalently if f is in the image of I. A morphism

$(X,x) \times (Y,y) \xrightarrow{f} (Z,z)$ is called *right-linear* if the diagram below commutes in $\mathcal{C}$.

$$\begin{tikzcd}
X \times TY \arrow[r, "t"] \arrow[d, "1 \times y"'] & T(X \times Y) \arrow[r, "Tf"] & TZ \arrow[d, "z"] \\
X \times Y \arrow[rr, "f"] & & Z
\end{tikzcd}$$

Intuitively, f is right-linear if whenever the first argument is fixed the resulting morphism on the second argument is linear. The same f is called *left-linear* if the diagram below commutes in $\mathcal{C}$.

$$\begin{tikzcd}
TX \times Y \arrow[r, "t'"] \arrow[d, "x \times 1"'] & T(X \times Y) \arrow[r, "Tf"] & TZ \arrow[d, "z"] \\
X \times Y \arrow[rr, "f"] & & Z
\end{tikzcd}$$

f is called *bilinear* if it is both right-linear and left-linear. All the above variations on the notion of linearity were defined by Kock who also gave two other equivalent definitions of bilinearity utilizing ψ and ψ' [12].

Kleisli exponentials

We say that $\mathcal{C}$ has *Kleisli exponentials* if, for all $X \in ob\ \mathcal{C}$, the functor $F_T \circ (- \times X) : \mathcal{C} \to \mathcal{C}_T$ has a right adjoint, $(-)_T^X : \mathcal{C}_T \to \mathcal{C}$.

The property of having Kleisli exponentials is essentially equivalent to that of λ_C*-model* due to Moggi [14]. However, we do not require Moggi's *equalising requirement* (this terminology is from [15]) and we prefer not to refer to any extraneous lambda-calculus. Note that if $\mathcal{C}$ is cartesian closed then $\mathcal{C}_T$ is automatically Kleisli closed, the right adjoint being $(-)^X \circ G_T$.

We write Λ_{XYZ} for the induced natural isomorphism from $\mathcal{C}(X \times Y, TZ)$ to $\mathcal{C}(X, Z_T^Y)$. We write ϵ_{XY} for the maps $Y_T^X \times X \longrightarrow TY$ in $\mathcal{C}$ giving the components of the counit in $\mathcal{C}_T$. The defining adjunction of Kleisli exponentials is now equivalent to the statement that the diagram (in $\mathcal{C}$) below commutes if and only if $f = \Lambda g$.

$$\begin{tikzcd}
Z_T^Y \times Y \arrow[r, "\epsilon"] & TZ \\
X \times Y \arrow[u, "f \times 1"] \arrow[ur, dotted, "g"'] &
\end{tikzcd}$$

We consider Kleisli exponentials, rather than making the stronger assumption of cartesian closure, for two reasons. First, it is in spirit of axiomatic domain theory to make the weakest assumptions possible for the desired results to follow. Second, we want as good a correspondence as possible between the category theory and the typed calculi modelled. Standard recursively typed calculi (such as Plotkin's metalanguage [18]) have partial function spaces (corresponding to the Kleisli exponentials of a suitable strong monad) but not total function spaces. However, it must be admitted that most real-world models are cartesian closed (with the possible exception of Rosolini's σ-domains [20]).

Fixed-point objects

A *fixed-point object* is an object Φ together with arrows $\mathbf{1} \xrightarrow{\omega} \Phi \xleftarrow{\sigma} T\Phi$ such that:

1. $T\Phi \xrightarrow{\sigma} \Phi$ is an initial T-algebra.
2. $\mathbf{1} \xrightarrow{\omega} \Phi$ equalises $\Phi \xrightarrow{1} \Phi$ and $\Phi \xrightarrow{\sigma \circ \eta} \Phi$.

Fixed-point objects were introduced by Crole and Pitts [3]. However, we follow Mulry in making ω a global element of Φ [16]. Our definition is easily shown to be equivalent to the original.

By lemma 2.1, σ is an isomorphism. Define:

$$\phi \quad = \quad T\Phi \xrightarrow{\sigma \circ \mu \circ T\sigma^{-1}} \Phi$$

A routine calculation shows that ϕ is an Eilenberg-Moore algebra. It is then easy to see that σ and σ^{-1} are morphisms in $\mathcal{C}^T$ (between $(T\Phi, \mu)$ and (Φ, ϕ)). These observations remove some of the hypotheses from theorem 3.12 of [16].

The following lemma gives an important property of the fixed-point object (which we require in the proof of Theorem 4.6).

Lemma 3.1 *For every $X \times TY \xrightarrow{y} Y$, there is a unique $X \times \Phi \xrightarrow{x} Y$ making the diagram below commute.*

$$\begin{tikzcd}
X \times T\Phi \arrow[r, "\langle \pi_1, Tx \circ t \rangle"] \arrow[d, "1 \times \sigma"'] \arrow[dr, phantom, "(1)"] & X \times TY \arrow[d, "y"] \\
X \times \Phi \arrow[r, "x"] & Y
\end{tikzcd}$$

Proof. Consider the diagrams below:

$$\begin{tikzcd}
X \times T\Phi \arrow[r, "\langle \pi_1, \mu \circ Tz \circ t \rangle"] \arrow[d, "1 \times \sigma"'] \arrow[dr, phantom, "(2)"] & X \times TY \arrow[d, "\eta \circ y"] \\
X \times \Phi \arrow[r, "z"] & TY
\end{tikzcd}$$

$$\begin{tikzcd}
T\Phi \arrow[r, "Tw"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "(3)"] & TY_T^X \arrow[d, "\Lambda(\eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ t' \rangle)"] \\
\Phi \arrow[r, "w"] & Y_T^X
\end{tikzcd}$$

We show that:

$$\eta \circ - : \{X \times \Phi \xrightarrow{x} Y \mid (1) \text{ commutes}\} \longrightarrow \{X \times \Phi \xrightarrow{z} TY \mid (2) \text{ commutes}\}$$

and:

$$\Lambda(- \circ c) : \{X \times \Phi \xrightarrow{z} TY \mid (2) \text{ commutes}\} \longrightarrow \{\Phi \xrightarrow{w} Y_T^X \mid (3) \text{ commutes}\}$$

are both well defined and bijections. Then the lemma follows, as the initial algebra property of $T\Phi \xrightarrow{\sigma} \Phi$ ensures that $\{\Phi \xrightarrow{w} Y_T^X \mid (3) \text{ commutes}\}$ is a singleton set.

First we show that x makes (1) commute if and only if $z = \eta \circ x$ makes (2) commute.

$$
\begin{array}{lrcll}
x \circ (1 \times \sigma) = y \circ \langle \pi_1, Tx \circ t \rangle & & & & \\
\text{iff} & \eta \circ x \circ (1 \times \sigma) & = & \eta \circ y \circ \langle \pi_1, Tx \circ t \rangle & (\text{'if' because } \eta \text{ is mono}) \\
 & & = & \eta \circ y \circ \langle \pi_1, \mu \circ T\eta \circ Tx \circ t \rangle & (\text{by unit law of monad}) \\
 & & = & \eta \circ y \circ \langle \pi_1, \mu \circ T(\eta \circ x) \circ t \rangle &
\end{array}
$$

Now for any z making (2) commute we have $z = \eta \circ y \circ \langle \pi_1, \mu \circ T(z) \circ t \circ \sigma^{-1} \rangle$, so z has the form $\eta \circ x$ and this x is necessarily unique (again as η is a mono). So we have established that $\eta \circ -$ is indeed a bijection between the given sets.

Similarly we show that z makes (2) commute if and only if $w = \Lambda(z \circ c)$ makes (3) commute.

$$
\begin{array}{lrcl}
\Lambda(z \circ c) \circ \sigma = \Lambda(\eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ t' \rangle) \circ T\Lambda(z \circ c) & & & \\
\text{iff} & \epsilon \circ (\Lambda(z \circ c) \times 1) \circ (\sigma \times 1) & = & \epsilon \circ (\Lambda(\eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ t' \rangle) \times 1) \circ (T\Lambda(z \circ c) \times 1) \\
\text{iff} & z \circ c \circ (\sigma \times 1) & = & \eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ t' \rangle \circ (T\Lambda(z \circ c) \times 1) \\
\text{iff} & z \circ (1 \times \sigma) & = & \eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ t' \circ (T\Lambda(z \circ c) \times 1) \rangle \circ c \\
 & & = & \eta \circ y \circ \langle \pi_2, \mu \circ T\epsilon \circ T(\Lambda(z \circ c) \times 1) \circ t' \rangle \circ c \\
 & & = & \eta \circ y \circ \langle \pi_2, \mu \circ Tz \circ Tc \circ t' \rangle \circ c \\
 & & = & \eta \circ y \circ \langle \pi_1, \mu \circ Tz \circ Tc \circ t' \circ c \rangle \\
 & & = & \eta \circ y \circ \langle \pi_1, \mu \circ Tz \circ t \rangle
\end{array}
$$

That $\Lambda(- \circ c)$ is a bijection is obvious (Λ is a bijection and c an isomorphism). $\square$

Note that the proof uses both Kleisli exponentials and that the η components are monos. If $\mathcal{C}$ is cartesian closed then the the proof is simpler (though the idea is the same) and does not rely on the mono requirement.

The property expressed by Lemma 3.1 is meaningful for any *strong* endofunctor T. In categories that are not cartesian closed it can be used as the defining property of a good *parametrised* notion of initial T-algebra for such endofunctors (in cartesian closed categories the property is automatically satisfied by the usual initial T-algebras). For example, if $\mathcal{C}$ is a distributive category then the $1 + (-)$ functor is strong. With our definition of parametrised initial $(1 + (-))$-algebra we derive the usual (parametrised) notion of natural numbers object for non cartesian-closed categories (see [19]).

Examples

The motivating example is the following. $\mathcal{C}$ is **PreDom**, the category of ωCPOs (ω-complete partial orders, possibly without least element) and ω-continuous functions. For the monad T, we take the lift functor on **PreDom** (which adds a new least element to an ωCPO), this has an associated commutative strong monad. $\mathcal{C}_T$ is now **pPreDom**, the category of ωCPOs and partial continuous functions [18]. $\mathcal{C}^T$ is the category $\mathbf{Dom}_{\bot}$, of those ωCPOs with a least element and strict (ie. least-element preserving) continuous functions. Incidentally, **pPreDom** and $\mathbf{Dom}_{\bot}$ are equivalent categories, this is not true in general for $\mathcal{C}_T$ and $\mathcal{C}^T$. $\mathcal{D}$ is the category, **Dom**, of ωCPOs and all ω-continuous functions. As is well-known, **Dom** is a cartesian closed category, and every endomorphism has a (least) fixed-point. The notions of linearity, right-linearity, left-linearity and bilinearity coincide in **Dom** with the usual notions of strictness, right-strictness, left-strictness and bistrictness. **PreDom** is cartesian closed, so it has Kleisli exponentials. It has a fixed-point object given by the vertical natural numbers [3].

The categories above have been deliberately named to allow for easy reinterpretation. Similar examples are to be found for any lift monad on a category of predomains. These domains need not be partially-ordered, see Phoa [17] for example.

It is well-known that **pPreDom** and $\mathbf{Dom}_{\perp}$ are algebraically compact for all ωCPO-enriched functors (this is easily shown from results in [21, 5]). We will show that the considerations leading to the consistent algebraic compactness of $\mathcal{C}_T$ are in fact quite general.

4 Fixed-points in $\mathcal{D}$

The presence of the fixed-point object gives rise to well-behaved fixed-points in $\mathcal{D}$. In this section we prove some basic properties of these fixed-points. These generalise properties of least fixed-points in **Dom**. The section is based on, and inspired by, Mulry's [16]. However, the parametrisation results are new.

The basic result is that $\mathcal{D}$ has a canonical *fixed-point operator*. Let $(\cdot)^*$ be an $ob\,\mathcal{D}$-indexed family of functions:

$$(\cdot)^{*(X,x)} : \mathcal{D}((X,x),(X,x)) \longrightarrow \mathcal{D}((\mathbf{1},!),(X,x))$$

Definition 4.1 (fixed-point operator) *$(\cdot)^*$ is a fixed-point operator on $\mathcal{D}$ if, $f \circ f^* = f^*$, for every $(X,x) \xrightarrow{f} (X,x)$.*

The next definition generalises a property that has become known as Plotkin's axiom in domain theory. This algebraic generalisation is essentially a less internal version of Mulry's *algebraically strong dinaturality* [16]. We call the property *uniformity* following the terminology (in the domain-theoretic setting) in a forthcoming book by Gunter.

Definition 4.2 (uniformity) *$(\cdot)^*$ is uniform if, for every $(X,x) \xrightarrow{f} (X,x)$, $(Y,y) \xrightarrow{g} (Y,y)$, and linear $(X,x) \xrightarrow{h} (Y,y)$, if the diagram below commutes.*

$$\begin{tikzcd}
(X,x) \arrow[r, "f"] \arrow[d, "h"'] & (X,x) \arrow[d, "h"] \\
(Y,y) \arrow[r, "g"] & (Y,y)
\end{tikzcd}$$

then $h \circ f^* = g^*$.

The canonical fixed-point operator is given by the following theorem, which is essentially theorem 3.12 of [16] adapted to our situation.

Theorem 4.3 *$\mathcal{D}$ has a unique uniform fixed-point operator.*

Proof. First we prove existence. Take any morphism $(X,x) \xrightarrow{f} (X,x)$ in $\mathcal{D}$. Note that $X \xrightarrow{f} X$ and $TX \xrightarrow{x} X$ are both morphisms in $\mathcal{C}$. Let $\Phi \xrightarrow{\rho(f)} X$ be the unique morphism, given by the initial algebra property of Φ, making the diagram (in $\mathcal{C}$) below commute.

$$\begin{tikzcd}
T\Phi \arrow[r, "T\rho(f)"] \arrow[d, "\sigma"'] & TX \arrow[d, "x \circ Tf"] \\
\Phi \arrow[r, "\rho(f)"] & X
\end{tikzcd}$$

We show that $f \mapsto \rho(f) \circ \omega$ is a uniform fixed-point operator. That it is a fixed-point operator follows from the commutativity of the diagram below (the left-hand triangle commutes by the defining property of ω, the two top squares by the naturality of η, the bottom rectangle by definition of $\rho(f)$, and the right-hand triangle as (X, x) is an object of $\mathcal{C}^T$).

$$\begin{tikzcd}
 & \Phi \arrow[r, "\rho(f)"] \arrow[d, "\eta"] & X \arrow[r, "f"] \arrow[d, "\eta"] & X \arrow[d, "\eta"] \arrow[dr, "1"] & \\
\mathbf{1} \arrow[ur, dotted, "\omega"] \arrow[dr, "\omega"'] & T\Phi \arrow[r, "T\rho(f)"] \arrow[d, "\sigma"] & TX \arrow[r, "Tf"] & TX \arrow[d, "x"] & X \arrow[dl, dotted, "1"] \\
 & \Phi \arrow[rr, "\rho(f)"] & & X &
\end{tikzcd}$$

For uniformity, suppose we have $(X, x) \xrightarrow{f} (X, x)$, $(Y, y) \xrightarrow{g} (Y, y)$, and linear $(X, x) \xrightarrow{h} (Y, y)$ such that $h \circ f = g \circ h$. Then the diagram below commutes (the left-hand rectangle by the definition of $\rho(f)$, the upper square because $h \circ f = g \circ h$, and the lower square by the linearity of h).

$$\begin{tikzcd}
T\Phi \arrow[r, "T\rho(f)"] \arrow[dd, "\sigma"'] & TX \arrow[r, "Th"] \arrow[d, "Tf"] & TY \arrow[d, "Tg"] \\
 & TX \arrow[r, "Th"] \arrow[d, "x"] & TY \arrow[d, "y"] \\
\Phi \arrow[r, "\rho(f)"] & X \arrow[r, "h"] & Y
\end{tikzcd}$$

But by definition, $\rho(g)$ is the unique $\Phi \xrightarrow{z} Y$ such that $z \circ \sigma = y \circ Tg \circ Tz$. So, by the above diagram, $\rho(g) = h \circ \rho(f)$. But then $\rho(g) \circ \omega = h \circ \rho(f) \circ \omega$, which proves uniformity.

For uniqueness, let $(\cdot)^*$ be any uniform fixed-point operator. Take any $(X, x) \xrightarrow{f} (X, x)$. We must show that $f^* = \rho(f) \circ \omega$. First note that $(\Phi, \phi) \xrightarrow{\rho(f)} (X, x)$ is linear, for $\rho(f) = x \circ Tf \circ T\rho(f) \circ \sigma^{-1}$ which is a composite of linear maps. Now omitting the left-hand triangle from the first diagram in this proof we see that $\rho(f) \circ \sigma \circ \eta = f \circ \rho(f)$. So, by the uniformity of $(\cdot)^*$, $\rho(f) \circ (\sigma \circ \eta)^* = f^*$. But $(\cdot)^*$ is a fixed-point operator and, by definition, ω is the unique fixed-point of $\sigma \circ \eta$, so $(\sigma \circ \eta)^* = \omega$. Therefore $f^* = \rho(f) \circ \omega$ as required. $\square$

Henceforth we write $(\cdot)^*$ for the unique uniform fixed-point operator.

$\mathcal{D}$ also has a canonical *parametrised fixed-point operator*. Let $(\cdot)^\dagger$ be an $(ob\,\mathcal{D} \times ob\,\mathcal{D})$-indexed family of functions:

$$(\cdot)^{\dagger(X,x)(Y,y)} : \mathcal{D}((X, x) \times (Y, y), (Y, y)) \longrightarrow \mathcal{D}((X, x), (Y, y))$$

Definition 4.4 (parametrised fixed-point operator) $(\cdot)^\dagger$ *is a parametrised fixed-point op-*

erator on $\mathcal{D}$ if, for all $(X,x)\times(Y,y)\xrightarrow{f}(Y,y)$, the diagram below commutes.

$$\begin{tikzcd}
(X,x) \arrow[r, "\langle 1, f^\dagger\rangle"] \arrow[dr, "f^\dagger"'] & (X,x)\times(Y,y) \arrow[d, "f"] \\
& (Y,y)
\end{tikzcd}$$

Definition 4.5 (parametrised uniformity) *$(\cdot)^\dagger$ is parametrically uniform if, for all morphisms $(X,x)\times(Y,y)\xrightarrow{f}(Y,y)$, $(X,x)\times(Z,z)\xrightarrow{g}(Z,z)$, and right-linear $(X,x)\times(Y,y)\xrightarrow{h}(Z,z)$, if the diagram below commutes.*

$$\begin{tikzcd}
(X,x)\times(Y,y) \arrow[r, "\langle \pi_1, f\rangle"] \arrow[d, "\langle \pi_1, h\rangle"'] & (X,x)\times(Y,y) \arrow[d, "h"] \\
(X,x)\times(Z,z) \arrow[r, "g"] & (Z,z)
\end{tikzcd}$$

then $h\circ\langle 1, f^\dagger\rangle = g^\dagger$.

The canonical parametrised fixed-point operator is given by the following theorem mirroring Theorem 4.3.

Theorem 4.6 *$\mathcal{D}$ has a unique parametrically uniform parametrised fixed-point operator.*

The proof of this theorem is by relativising Theorem 4.3 to the co-Kleisli categories of comonads of the form $X\times(-)$ on $\mathcal{C}$.

Write $\mathcal{C}^X$ for the co-Kleisli category of the $X\times(-)$ comonad on $\mathcal{C}$, and I^X for the "inclusion" functor from $\mathcal{C}$ to $\mathcal{C}^X$. We must show that $\mathcal{C}^X$ has all the structure we require.

I^X is surjective on objects and preserves limits (it has a left-adjoint). So $\mathcal{C}^X$ is cartesian, moreover products are inherited from $\mathcal{C}$.

Now we define a strong monad (T^X,η^X,μ^X,t^X) on $\mathcal{C}^X$. The action of T^X on objects is inherited from T. Its action on morphisms takes the $\mathcal{C}^X$ morphism from Y to Z given by the $\mathcal{C}$ morphism $X\times Y\xrightarrow{f}Z$ to the $\mathcal{C}^X$ morphism from TY to TZ given by $X\times TY\xrightarrow{Tf\circ t}TZ$. The components of η^X, μ^X and t^X, are given by $\eta^X_Y = I^X\eta_Y$, $\mu^X_Y = I^X\mu_Y$ and $t^X_{YZ} = I^X t_{YZ}$. It is readily checked that these give natural transformations and satisfy the axioms of a strong monad. For the faithfulness of T^X it is enough to show that all the components of η^X are monos. But these are the image of the components of η (which are monos) under I^X, and I^X preserves limits and hence monos.

The Kleisli exponentials are given by $Z^Y_{T^X} = Z^Y_T$. The easiest way to see that this does indeed give a Kleisli exponential is to define the isomorphism Λ^X_{YZW} mapping $X\times(Y\times Z)\xrightarrow{f}TW$ in $\mathcal{C}^X(Y\times Z, TW)$ to $X\times Y\xrightarrow{f\circ a^{-1}}W^Z_T$ in $\mathcal{C}^X(Y,W^Z_T)$, and to define the maps $\epsilon^X_{YZ} = I^X\epsilon_{YZ}$. It is routine to check that these have the appropriate universal property giving the adjunction.

The fixed-point object is again Φ, with morphisms $\omega^X = I^X\omega$ and $\sigma^X = I^X\sigma$. That $T^X\Phi\xrightarrow{\sigma^X}\Phi$ is an initial algebra in $\mathcal{C}^X$ follows straightforwardly from Lemma 3.1. The equalising property of $I^X\omega$ follows from I^X preserving limits.

The proposition below summarises what we have sketched so far.

Proposition 4.7 *$\mathcal{C}^X$ is cartesian with a faithful strong monad with respect to which it has Kleisli exponentials and a fixed-point object.*

In fact I^X clearly preserves all the structure. So the above proposition is closely related to the functional completeness result in [3].

We now proceed to give the proof of Theorem 4.6 applying Proposition 4.7. Write $\mathcal{D}^X$ for the analogous category to $\mathcal{D}$ obtained from the T^X monad on $\mathcal{C}^X$. Note that for any object (X, x) of $\mathcal{D}$, $(X, I^X x)$ is an object of $\mathcal{D}^X$. Write $(\cdot)^{*^X}$ for the unique uniform fixed-point operator on $\mathcal{D}^X$. We now define the evident parametrised fixed-point operator on $\mathcal{D}$ by:

$$f^{\dagger(X,x)(Y,y)} \quad = \quad f^{*^X_{(Y,y)}} \circ \langle 1, ! \rangle$$

(for $(X, x) \times (Y, y) \xrightarrow{f} (Y, y)$). To see that this is parametrically uniform, suppose we have $(X, x) \times (Y, y) \xrightarrow{f} (Y, y)$, $(X, x) \times (Z, z) \xrightarrow{g} (Z, z)$, and right-linear $(X, x) \times (Y, y) \xrightarrow{h} (Z, z)$ satisfying the hypothesis of parametrised uniformity. It is easy to see that h is a linear map from $(Y, I^X y)$ to $(Z, I^X z)$ in $\mathcal{D}^X$, and that the hypothesis of uniformity is satisfied by f, g and h in $\mathcal{D}^X$. The required equation for the parametrised uniformity of $(\cdot)^\dagger$ now follows directly from the uniformity of $(\cdot)^{*^X}$.

To see that $(\cdot)^\dagger$ is the unique parametrically uniform parametrised fixed-point operator it is necessary to inspect the proof (given in the proof of Theorem 4.3) that $(\cdot)^{*^X}$ is the unique uniform fixed-point operator on $\mathcal{D}^X$. The proof shows that the restriction of $(\cdot)^{*^X}$ is the unique uniform fixed-point operator on any full subcategory of $\mathcal{D}^X$ containing the object $(\Phi, I^X \sigma)$. In particular this holds for the full subcategory of all objects of the form $(Y, I^X y)$ where (Y, y) is an object of $\mathcal{D}$. But the statement that the appropriate restriction of $(\cdot)^{*^X}$ is the unique uniform fixed-point operator on this full subcategory is easily seen to be equivalent to the statement that $(\cdot)^\dagger$ is the unique parametrically uniform parametrised fixed-point operator on $\mathcal{D}$. This completes the proof of Theorem 4.6.

In the sequel we shall have use for the following simple lemma relating $(\cdot)^*$ and $(\cdot)^\dagger$.

Lemma 4.8 *Given any $(Y, y) \xrightarrow{f} (Y, y)$, $(f \circ \pi_{2_{X \times Y}})^\dagger = f^* \circ !_X : (X, x) \to (Y, y)$.*

The proof is easy.

5 Algebras in $\mathcal{D}$-categories

The distinguished cartesian structure on $\mathcal{D}$ gives $\mathcal{D}$ as a (symmetric) monoidal category. We can thus consider categories enriched over $\mathcal{D}$ [10]. A $\mathcal{D}$-category, $\mathcal{K}$, is given by a class, $ob\ \mathcal{K}$, of *objects*, an object, $\mathcal{K}(A, B)$, of $\mathcal{D}$ for every $A, B \in ob\ \mathcal{K}$ and families of morphisms in $\mathcal{D}$:

$$\mathbf{1} \xrightarrow{e^{\mathcal{K}}_A} \mathcal{K}(A, A) \qquad \mathcal{K}(B, C) \times \mathcal{K}(A, B) \xrightarrow{m^{\mathcal{K}}_{ABC}} \mathcal{K}(A, C)$$

for identities and composition such that (all instances of) three coherence diagrams commute (expressing the left and right unitary properties of the identities, and the associativity of composition) [10]. The dual $\mathcal{K}^{\mathrm{op}}$ is the $\mathcal{D}$-category defined by: $ob\ \mathcal{K}^{\mathrm{op}} = ob\ \mathcal{K}$, $\mathcal{K}^{\mathrm{op}}(A, B) = \mathcal{K}(B, A)$, $e^{\mathcal{K}^{\mathrm{op}}}_A = e^{\mathcal{K}}_A$ and $m^{\mathcal{K}^{\mathrm{op}}}_{ABC} = m^{\mathcal{K}}_{CBA} \circ c$. Similarly, the product, $\mathcal{K} \times \mathcal{L}$, of $\mathcal{K}$ with another $\mathcal{D}$-category, $\mathcal{L}$, is defined as a $\mathcal{D}$-category in the obvious way.

A $\mathcal{D}$-functor, $F : \mathcal{K} \to \mathcal{L}$, from $\mathcal{K}$ to $\mathcal{L}$ is given by a function, $F : ob\ \mathcal{K} \to ob\ \mathcal{L}$, together with morphisms, $\mathcal{K}(A,B) \xrightarrow{F_{AB}} \mathcal{L}(FA,FB)$, in $\mathcal{D}$ such that two diagrams commute (expressing the functorial preservation of identities and composition) [10]. F determines an obvious $\mathcal{D}$-functor $F^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \to \mathcal{L}^{\mathrm{op}}$, and, given another $\mathcal{D}$-functor $F' : \mathcal{K}' \to \mathcal{L}'$, there is an evident $\mathcal{D}$-functor $F' \times F : \mathcal{K}' \times \mathcal{K} \to \mathcal{L}' \times \mathcal{L}$. Further, given a $\mathcal{D}$-functor $F : \mathcal{K}' \times \mathcal{K} \to \mathcal{L}$, any object A of $\mathcal{K}'$ induces an obvious $\mathcal{D}$-functor $F(A,-) : \mathcal{K} \to \mathcal{L}$.

Note that any $\mathcal{D}$-category is trivially a $\mathcal{C}$-category. Moreover, any $\mathcal{C}$-category all of whose hom-objects have an Eilenberg-Moore algebra over them can be construed (possibly in many non-equivalent ways) as a $\mathcal{D}$-category. Further, $F : \mathcal{K} \to \mathcal{L}$ is a $\mathcal{D}$-functor if and only if it is a $\mathcal{C}$-functor between the two associated $\mathcal{C}$-categories. Thus, the only point in considering $\mathcal{D}$-categories is to make use of the algebra structure associated with each object of the category. This we now do.

Each $\mathcal{K}(A,B)$ is an object of $\mathcal{D}$ and therefore an Eilenberg-Moore algebra of T on $\mathcal{C}$. We write $(\mathcal{K}(A,B), k(A,B))$ for this algebra structure (confusing the object $\mathcal{K}(A,B)$ of $\mathcal{D}$ with its underlying object in $\mathcal{C}$). We say composition in $\mathcal{K}$ is linear, left-linear, right-linear or bilinear if each $m^{\mathcal{K}}_{ABC}$ is linear, left-linear, right-linear or bilinear respectively. It is easy to see that composition in $\mathcal{K}$ is right-linear if and only if composition in $\mathcal{K}^{\mathrm{op}}$ is left-linear. Again, composition in $\mathcal{K} \times \mathcal{L}$ is right-linear if and only if composition is right-linear in both $\mathcal{K}$ and $\mathcal{L}$.

A $\mathcal{D}$-category $\mathcal{K}$ determines an underlying ordinary category $\mathcal{K}_0$ with the same objects as $\mathcal{K}$, with hom-sets given by $\mathcal{K}_0(A,B) = \mathcal{D}(\mathbf{1}, \mathcal{K}(A,B))$, with identities $e^{\mathcal{K}}_A$ and with composition defined using $m^{\mathcal{K}}_{ABC}$ in the obvious way. Similarly, each $\mathcal{D}$-functor $F : \mathcal{K} \to \mathcal{L}$ determines an ordinary functor $F_0 : \mathcal{K}_0 \to \mathcal{L}_0$ with the obvious action on morphisms. Given a $\mathcal{D}$-endofunctor $F : \mathcal{K} \to \mathcal{K}$, when we use the terms F-algebra, F-homomorphism, initial F-algebra, F-coalgebra, F-cohomomorphism, terminal F-coalgebra and F-invariant object we mean the corresponding concept for F_0. Thus, for example, an F-algebra is a morphism $\mathbf{1} \xrightarrow{\alpha} \mathcal{K}(FA,A)$ in $\mathcal{D}$. Also the category of F-algebras is not a $\mathcal{D}$-category. In the case that $\mathcal{C}$ has equalisers it is possible to make a D-category of F-algebras, but we shall not pursue this any further here.

So it makes sense to talk about initial algebras for $\mathcal{D}$-functors on $\mathcal{D}$-categories, but only as non-enriched notions. We now show that, if composition in the $\mathcal{D}$-category is right-linear, the non-enriched notion of initial algebra corresponds to a self-dual enriched property. The self-duality of the property leads to an appropriate form of consistent algebraic compactness for $\mathcal{D}$-categories in which composition is bilinear.

Let $F : \mathcal{K} \to \mathcal{K}$ be a $\mathcal{D}$-functor. Given an F-invariant object, $FA \xrightarrow{\alpha} A$, and an F-algebra, $FB \xrightarrow{\beta} B$, we write:

$$\mathcal{K}(A,B) \xrightarrow{\beta \circ F(-) \circ \alpha^{-1}} \mathcal{K}(A,B)$$

for the morphism in $\mathcal{D}$ given by the composite below.

$$\mathcal{K}(A,B) \xrightarrow{\langle \langle \beta \circ !, F \rangle, \alpha^{-1} \circ ! \rangle} (\mathcal{K}(FB,B) \times \mathcal{K}(FA,FB)) \times \mathcal{K}(A,FA) \xrightarrow{m \circ (m \times 1)} \mathcal{K}(A,B)$$

Lemma 5.1 *$(\beta \circ F(-) \circ \alpha^{-1})^*$ is an F-homomorphism from α to β.*

Proof. $(\cdot)^*$ is a fixed-point operator, so $\beta \circ F((\beta \circ F(-) \circ \alpha^{-1})^*) \circ \alpha^{-1} = (\beta \circ F(-) \circ \alpha^{-1})^*$ (using an obvious notation), and clearly $\beta \circ F((\beta \circ F(-) \circ \alpha^{-1})^*) = (\beta \circ F(-) \circ \alpha^{-1})^* \circ \alpha$ as required. $\square$

We call an F-invariant object, $FA \xrightarrow{\alpha} A$, *special* if $(\alpha \circ F(-) \circ \alpha^{-1})^* = e_A$. This definition

generalises one due to Peter Freyd [6]. The theorem below is the similar generalisation of a theorem in [6, page 502].

Theorem 5.2 *If composition in $\mathcal{K}$ is right-linear then the following are equivalent:*

1. *$FA \xrightarrow{\alpha} A$ is a special F-invariant object.*

2. *$FA \xrightarrow{\alpha} A$ is an initial F-algebra.*

Proof. Suppose that $FA \xrightarrow{\alpha} A$ is an initial F-algebra. By the Lambek lemma α is an iso. in $\mathcal{K}_0$ so α is an F-invariant object. By initiality, e_A is the unique F-homomorphism from α to α. But, by Lemma 5.1, $(\alpha \circ F(-) \circ \alpha^{-1})^*$ is such an F-homomorphism. So indeed $(\alpha \circ F(-) \circ \alpha^{-1})^* = e_A$.

Conversely, suppose that $FA \xrightarrow{\alpha} A$ is a special F-invariant object. Let $FB \xrightarrow{\beta} B$ be any F-algebra. We must show that there is a unique F-homomorphism from α to β. The existence of such an F-homomorphism, namely $(\beta \circ F(-) \circ \alpha^{-1})^*$, is given by Lemma 5.1. For uniqueness suppose $A \xrightarrow{x} B$ is an arbitrary F-homomorphism. We must show that $x = (\beta \circ F(-) \circ \alpha^{-1})^*$. Write $\mathcal{K}(A,A) \xrightarrow{x \circ -} \mathcal{K}(A,B)$ for the composite below.

$$\mathcal{K}(A,A) \xrightarrow{m \circ \langle x \circ !, 1 \rangle} \mathcal{K}(A,B)$$

That $x \circ -$ is linear follows easily from the right-linearity of m. Now, because x is an F-homomorphism, the diagram below commutes.

$$\begin{tikzcd}
\mathcal{K}(A,A) \arrow[r, "\alpha \circ F(-) \circ \alpha^{-1}"] \arrow[d, "x \circ -"'] & \mathcal{K}(A,A) \arrow[d, "x \circ -"] \\
\mathcal{K}(A,B) \arrow[r, "\beta \circ F(-) \circ \alpha^{-1}"] & \mathcal{K}(A,B)
\end{tikzcd}$$

So, by the uniformity of $*$, $(x \circ -) \circ (\alpha \circ F(-) \circ \alpha^{-1})^* = (\beta \circ F(-) \circ \alpha^{-1})^*$. However, $(\alpha \circ F(-) \circ \alpha^{-1})^* = e_A$, as α is a special F-invariant object. Therefore $x = (x \circ -) \circ e_A = (\beta \circ F(-) \circ \alpha^{-1})^*$ as required. □

Corollary 5.3 *If composition in $\mathcal{K}$ is left-linear then the following are equivalent:*

1. *$FA \xrightarrow{\alpha} A$ is a special F-invariant object.*

2. *$A \xrightarrow{\alpha^{-1}} FA$ is a terminal F-coalgebra.*

If, in addition, composition in $\mathcal{K}$ is right-linear, then the following is equivalent to the above:

3. *$FA \xrightarrow{\alpha} A$ is an initial F-algebra.*

Proof. By duality. □

In particular if composition in $\mathcal{K}$ is bilinear then $\mathcal{K}$ is consistently algebraically compact. Here we are working in the 2-category $\mathcal{D}$-**Cat** [10], understanding "all functors" to mean all $\mathcal{D}$-functors. As the bilinearity of composition is preserved by taking opposite categories and products we have also that $\mathcal{K}^{\mathrm{op}}$ and $\mathcal{K}^{\mathrm{op}} \times \mathcal{K}$ are consistently algebraically compact.

The application of the above results will be to model datatypes in $\mathcal{D}$-categories with bilinear composition. As long as the type-constructors yield enriched functors, the consistent algebraic compactness will be applicable. We now show that one important way of optaining one type constructor from another preserves enrichment. See the discussion on page 19 for the application.

Let $G : \mathcal{L} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{D}$-functor such that for each $B \in ob\ \mathcal{L}$ there exists a special $G(B,-)$-invariant object, $G(B, A_B) \xrightarrow{\alpha_B} A_B$. Assume furthermore that application in $\mathcal{K}$ is right-linear. Then by Theorem 5.2 there is an initial $G(B,-)$-algebra for every $B \in ob\ \mathcal{K}$. Thus the conditions are satisfied for defining $(G_0)^\dagger$ as in section 2. We now show that this functor is enriched in the sense that it is the underlying ordinary functor associated with some D-functor (which we call $G^\dagger$). For each $B, C \in ob\ \mathcal{L}$ we write:

$$\mathcal{L}(B,C) \times \mathcal{K}(A_B, A_C) \xrightarrow{(\alpha_C \circ G(-) \circ \alpha_B^{-1})} \mathcal{K}(A_B, A_C)$$

for the composite below.

$$\mathcal{L}(B,C) \times \mathcal{K}(A_B, A_C) \xrightarrow{m \circ (m \times 1) \circ \langle \langle \alpha_C \circ !, G \rangle, \alpha_B^{-1} \circ ! \rangle} \mathcal{K}(A_B, A_C)$$

Now define:

$$G^\dagger_{BC} \quad = \quad \mathcal{L}(B,C) \xrightarrow{(\alpha_C \circ G(-) \circ \alpha_B^{-1})^\dagger} \mathcal{K}(A_B, A_C)$$

Lemma 5.4 *Given any morphism $Z \xrightarrow{f} \mathcal{L}(B,C)$ in $\mathcal{D}$, there is a unique $Z \xrightarrow{x} \mathcal{K}(A_B, A_C)$ such that the diagram below commutes.*

$$\begin{tikzcd}
Z \arrow[r, "\langle f, x \rangle"] \arrow[d, "x"'] & \mathcal{L}(B,C) \times \mathcal{K}(A_B, A_C) \arrow[dl, dotted, "\alpha_C \circ G(-) \circ \alpha_B^{-1}"] \\
\mathcal{K}(A_B, A_C) &
\end{tikzcd}$$

Proof. For existence define $x = ((\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1))^\dagger$. Then:

$$\begin{aligned}
x &= ((\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1))^\dagger \\
&= (\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1) \circ \langle 1, x \rangle \quad \text{(as } \dagger \text{ is a parametrised fixed-point operator)} \\
&= (\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ \langle f, x \rangle
\end{aligned}$$

For uniqueness suppose x is any morphism making the above diagram commute. We must show that $x = ((\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1))^\dagger$. Note now that the diagram below commutes.

$$\begin{tikzcd}
Z \times \mathcal{K}(A_B, A_B) \arrow[r, "1 \times (\alpha_B \circ G(B,-) \circ \alpha_B^{-1})"] \arrow[d, "\langle \pi_1, m \circ (x \times 1) \rangle"'] & Z \times \mathcal{K}(A_B, A_B) \arrow[d, "m \circ (x \times 1)"] \\
Z \times \mathcal{K}(A_B, A_C) \arrow[r, "(\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1)"] & \mathcal{K}(A_B, A_C)
\end{tikzcd}$$

Note further that the right-linearity of $m \circ (x \times 1)$ is an easy consequence of the right-linearity of m. So the equation below follows from the parametrised uniformity of $\dagger$.

$$m \circ (x \times 1) \circ \langle 1, ((\alpha_B \circ G(B,-) \circ \alpha_B^{-1}) \circ \pi_2)^\dagger \rangle \quad = \quad ((\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1))^\dagger \qquad (1)$$

But then:

$$
\begin{array}{rcll}
x & = & m \circ \langle x, e_B \circ ! \rangle & \\
 & = & m \circ (x \times 1) \circ \langle 1, e_B \circ ! \rangle & \\
 & = & m \circ (x \times 1) \circ \langle 1, (\alpha_B \circ G(B,-) \circ \alpha_B^{-1})^* \circ ! \rangle & \text{(as } \alpha_B \text{ is a special } G(B,-)\text{-invariant object)} \\
 & = & m \circ (x \times 1) \circ \langle 1, ((\alpha_B \circ G(B,-) \circ \alpha_B^{-1}) \circ \pi_2)^\dagger \rangle & \text{(by Lemma 4.8)} \\
 & = & ((\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ (f \times 1))^\dagger & \text{(by (1) above)}
\end{array}
$$

□

Theorem 5.5 *$G^\dagger$ is a $\mathcal{D}$-functor and $(G^\dagger)_0 = (G_0)^\dagger$.*

Proof. For preservation of identity we must show that the diagram below commutes.

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "e^{\mathcal{L}}_B"] \arrow[dr, "e^{\mathcal{K}}_{A_B}"'] & \mathcal{L}(B,B) \arrow[d, "G^\dagger_{BB}"] \\
 & \mathcal{K}(A_B, A_B)
\end{tikzcd}
$$

Now, by Lemma 5.4, there is a unique $\mathbf{1} \xrightarrow{x} \mathcal{K}(A_B, A_B)$ making the following diagram commute.

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "\langle e^{\mathcal{L}}_B, x \rangle"] \arrow[d, "x"'] & \mathcal{L}(B,B) \times \mathcal{K}(A_B, A_B) \arrow[dl, dotted, "\alpha_B \circ G(-) \circ \alpha_B^{-1}"] \\
\mathcal{K}(A_B, A_B) &
\end{tikzcd}
$$

It is easy to see that the diagram commutes with $x = e^{\mathcal{K}}_{A_B}$ (as G preserves identities). So to show preservation of identity we need only prove that the diagram commutes with $x = G^\dagger_{BB} \circ e^{\mathcal{L}}_B$. This is by:

$$
\begin{array}{rcll}
G^\dagger_{BB} \circ e^{\mathcal{L}}_B & = & (\alpha_B \circ G(-) \circ \alpha_B^{-1})^\dagger \circ e^{\mathcal{L}}_B & \text{(by definition of } G^\dagger_{BB}\text{)} \\
 & = & (\alpha_B \circ G(-) \circ \alpha_B^{-1}) \circ \langle 1, G^\dagger_{BB} \rangle \circ e^{\mathcal{L}}_B & \text{(as } \dagger \text{ is a parametrised fixed-point operator)} \\
 & = & (\alpha_B \circ G(-) \circ \alpha_B^{-1}) \circ \langle e^{\mathcal{L}}_B, G^\dagger_{BB} \circ e^{\mathcal{L}}_B \rangle &
\end{array}
$$

For the peservation of composition we must show that the diagram below commutes.

$$
\begin{tikzcd}
\mathcal{L}(C,D) \times \mathcal{L}(B,C) \arrow[r, "M^{\mathcal{L}}"] \arrow[d, "G^\dagger_{CD} \times G^\dagger_{BC}"'] & \mathcal{L}(B,D) \arrow[d, "G^\dagger_{BD}"] \\
\mathcal{K}(A_C, A_D) \times \mathcal{K}(A_B, A_C) \arrow[r, "M^{\mathcal{K}}"] & \mathcal{K}(A_B, A_D)
\end{tikzcd}
$$

Again, by Lemma 5.4, there is a unique $\mathcal{L}(C,D) \times \mathcal{L}(B,C) \xrightarrow{x} \mathcal{K}(A_B, A_D)$ making the following

diagram commute.

$$\begin{tikzcd}
\mathcal{L}(C,D) \times \mathcal{L}(B,C) \arrow[r, "\langle m^{\mathcal{L}}, x\rangle"] \arrow[d, "x"'] & \mathcal{L}(B,D) \times \mathcal{K}(A_B, A_D) \arrow[dl, dotted, "\alpha_D \circ G(-) \circ \alpha_B^{-1}"] \\
\mathcal{K}(A_B, A_D) &
\end{tikzcd}$$

So for the preservation of composition it is enough to check that the diagram commutes for both $x = G^{\dagger}_{BD} \circ m^{\mathcal{L}}$ and $x = m^{\mathcal{K}} \circ (G^{\dagger}_{CD} \times G^{\dagger}_{BC})$. The former is by:

$$\begin{aligned}
G^{\dagger}_{BD} \circ m^{\mathcal{L}} &= (\alpha_D \circ G(-) \circ \alpha_B^{-1})^{\dagger} \circ m^{\mathcal{L}} && \text{(by definition of } G^{\dagger}_{BD}) \\
&= (\alpha_D \circ G(-) \circ \alpha_B^{-1}) \circ \langle 1, G^{\dagger}_{BD}\rangle \circ m^{\mathcal{L}} && \text{(as } \dagger \text{ is a parametrised fixed-point operator)} \\
&= (\alpha_D \circ G(-) \circ \alpha_B^{-1}) \circ \langle m^{\mathcal{L}}, G^{\dagger}_{BD} \circ m^{\mathcal{L}}\rangle
\end{aligned}$$

The latter is by:

$$\begin{aligned}
& m^{\mathcal{K}} \circ (G^{\dagger}_{CD} \times G^{\dagger}_{BC}) \\
&= m^{\mathcal{K}} \circ ((\alpha_D \circ G(-) \circ \alpha_C^{-1}) \circ \langle 1, G^{\dagger}_{CD}\rangle \times (\alpha_C \circ G(-) \circ \alpha_B^{-1}) \circ \langle 1, G^{\dagger}_{BC}\rangle) \\
&\qquad \text{(by expanding definitions and applying parametrised fixed-point property)} \\
&= m^{\mathcal{K}} \circ ((\alpha_D \circ G(-) \circ \alpha_C^{-1}) \times (\alpha_C \circ G(-) \circ \alpha_B^{-1})) \circ (\langle 1, G^{\dagger}_{CD}\rangle \times \langle 1, G^{\dagger}_{BC}\rangle) \\
&= (\alpha_D \circ G(-) \circ \alpha_B^{-1}) \circ (m^{\mathcal{L}} \times m^{\mathcal{K}}) \circ \langle 1, (G^{\dagger}_{CD} \times G^{\dagger}_{BC})\rangle \\
&\qquad \text{(by composing in } \mathcal{K} \text{ and applying preservation of composition of } G) \\
&= (\alpha_D \circ G(-) \circ \alpha_B^{-1}) \circ \langle m^{\mathcal{L}}, m^{\mathcal{K}} \circ (G^{\dagger}_{CD} \times G^{\dagger}_{BC})\rangle
\end{aligned}$$

It remains to show that $(G^{\dagger})_0 = (G_0)^{\dagger}$. Given a morphism $B \xrightarrow{f} C$ in $\mathcal{L}_0$, we must show that the diagram below commutes in $\mathcal{K}_0$.

$$\begin{tikzcd}
G(B, A_B) \arrow[rr, "\alpha_B"] \arrow[d, "G(1, G^{\dagger} f)"'] & & A_B \arrow[d, "G^{\dagger} f"] \\
G(B, A_C) \arrow[r, "G(f,1)"] & G(C, A_C) \arrow[r, "\alpha_C"] & A_C
\end{tikzcd}$$

However, this is an easy consequence of the definition of $G^{\dagger}_{A_B A_C}$ as a parametrised fixed-point. $\square$

6 The Kleisli category as a $\mathcal{D}$-category

In this section we apply the results of the last section by showing that the Kleisli category is a $\mathcal{D}$-category. Further, if the monad is commutative then composition in the Kleisli category is bilinear. Under these conditions the Kleisli category is consistently algebraically compact (for $\mathcal{D}$-functors) and it makes sense to interpret recursive types in this setting.

The object of $\mathcal{D}$ corresponding to $\mathcal{C}_T(X, Y)$ will be an algebra over the Kleisli exponential Y_T^X. The structure map, $TY_T^X \xrightarrow{y_t^x} Y_T^X$, of this algebra is obtained by applying Λ to the composite:

$$TY_T^X \times X \xrightarrow{\mu \circ T\epsilon \circ t'} TY$$

Proposition 6.1 *y_t^x is an Eilenberg-Moore algebra.*

Proof. We must show that the diagrams below commute.

$$\begin{tikzcd} Y_T^X \arrow[r, "\eta"] \arrow[dr, "1"'] & TY_T^X \arrow[d, "y_t^x"] \\ & Y_T^X \end{tikzcd} \qquad \begin{tikzcd} T^2Y_T^X \arrow[r, "Ty_t^x"] \arrow[d, "\mu"'] & TY_T^X \arrow[d, "y_t^X"] \\ TY_T^X \arrow[r, "y_t^x"] & Y_T^X \end{tikzcd}$$

That the left-hand of these diagrams commutes is shown by the top diagram in Figure 1 in the appendix. The middle and bottom diagrams in Figure 1 show respectively that $Ty_t^X \circ y_t^X$ and $\mu \circ y_t^X$ are both Λ applied to:

$$T^2Y_T^X \times X \xrightarrow{\mu \circ \mu \circ T^2\epsilon \circ Tt' \circ t'} TY$$

Therefore the right-hand diagram above commutes. $\square$

We now define the identity and composition maps. Define $1 \xrightarrow{e_X} X_T^X$ to be Λ applied to:

$$\mathbf{1} \times X \xrightarrow{\eta \circ \pi_2} TX$$

Define $Z_T^Y \times Y_T^X \xrightarrow{m_{XYZ}} Z_T^X$ to be Λ applied to:

$$(Z_T^Y \times Y_T^X) \times X \xrightarrow{\mu \circ T\epsilon \circ t \circ (1 \times \epsilon) \circ a} TZ$$

Proposition 6.2 *The above data gives a $\mathcal{D}$-category whose underlying ordinary category is (isomorphic to) $\mathcal{C}_T$.*

Proof. For the data to give a $\mathcal{D}$-category we must show that the diagrams below commute.

$$\begin{tikzcd} Y_T^X \times X_T^X \arrow[dr, "m"'] & Y_T^X \arrow[l, "\langle 1, e\circ! \rangle"'] \arrow[r, "\langle e\circ!, 1 \rangle"] \arrow[d, "1"] & Y_T^Y \times Y_T^X \arrow[dl, dotted, "m"] \\ & Y_T^X & \end{tikzcd}$$

$$\begin{tikzcd} (W_T^Z \times Z_T^Y) \times Y_T^X \arrow[rr, "m \times 1"] \arrow[d, "a"'] & & W_T^Y \times Y_T^X \arrow[d, "m"] \\ W_T^Z \times (Z_T^Y \times Y_T^X) \arrow[r, "1 \times m"] & W_T^Z \times Z_T^X \arrow[r, "m"] & W_T^X \end{tikzcd}$$

We just give the proof that the lower diagram commutes. The proofs for the two triangles forming the upper diagram are easier. To show the lower diagram commutes we show that both sides of the diagram are Λ applied to:

$$((W_T^Z \times Z_T^Y) \times Y_T^X) \times X \xrightarrow{\mu \circ T\epsilon \circ \mu \circ Tt \circ T(1 \times \epsilon) \circ Ta \circ t \circ (1 \times \epsilon) \circ a} TW$$

The first diagram in Figure 2 in the appendix shows this for the top leg, and the second diagram shows it for the bottom leg.

It remains to show that $\mathcal{C}_T$ is isomorphic to the underlying ordinary category of the established $\mathcal{D}$-category. The isomorphism between hom-sets is given by:

$$\mathcal{C}_T(X,Y) \;=\; \mathcal{C}(X,TY) \xrightarrow{\Lambda(-\circ\pi_2)} \mathcal{C}(\mathbf{1},Y_T^X) \;=\; \mathcal{D}((\mathbf{1},!),(Y_T^X,y_t^x))$$

It is now easy to check that the identities and compostition are as required. □

Henceforth we refer to the $\mathcal{D}$-category as $\mathcal{C}_T$, confusing the enriched category with its underlying ordinary category.

Proposition 6.3 *Composition in $\mathcal{C}_T$ is right-linear.*

Proof. We must show that $m\circ(1\times y_t^x) = z_t^x\circ Tm\circ t : Z_T^Y\times TY_T^X \longrightarrow Z_T^X$. The first and second diagrams in Figure 3 in the appendix show respectively that the left-hand and right-hand sides of the equation are both Λ applied to:

$$(Z_T^Y\times TY_T^X)\times X \xrightarrow{\mu\circ T\epsilon\circ\mu\circ Tt\circ t\circ(1\times T\epsilon)\circ(1\times t')\circ a} TZ$$

They are therefore equal. □

Proposition 6.4 *If the monad is commutative then composition in $\mathcal{C}_T$ is left-linear.*

Proof. We must show that $m\circ(z_t^y\times 1) = z_t^x\circ Tm\circ t' : TZ_T^Y\times Y_T^X \longrightarrow Z_T^X$. The first diagram in Figure 4 in the appendix shows that the left-hand side of the equation is Λ applied to:

$$(TZ_T^Y\times Y_T^X)\times X \xrightarrow{\mu\circ T\epsilon\circ\psi'\circ(1\times\epsilon)\circ a} TZ$$

The second diagram shows that the right-hand side is Λ applied to:

$$(TZ_T^Y\times Y_T^X)\times X \xrightarrow{\mu\circ T\epsilon\circ\psi\circ(1\times\epsilon)\circ a} TZ$$

From these, the equality of the two sides is immediate by commutativity. □

Thus if the monad is commutative then composition in $\mathcal{C}_T$ is bilinear and so (by the results of section 5) $\mathcal{C}_T$, $\mathcal{C}_T^{\mathrm{op}}$ and $\mathcal{C}_T^{\mathrm{op}}\times\mathcal{C}_T$ are all consistently algebraically compact.

Assume then that the monad is commutative. We now sketch how to define recursive types involving the standard type constructors. Products in $\mathcal{C}$, the monad functor T and Kleisli exponentials lift naturally to functors $\times_T : \mathcal{C}_T\times\mathcal{C}_T\to\mathcal{C}_T$, $T_T : \mathcal{C}_T\to\mathcal{C}_T$ and $\to_T : \mathcal{C}_T^{\mathrm{op}}\times\mathcal{C}_T\to\mathcal{C}_T$ respectively. $\times_T$ is not in general a cartesian product on $\mathcal{C}_T$, but computationally it is a genuinely useful "smash product". Categorically, $\times_T$ is a tensor product, inheriting its monoidal structure from that of the cartesian product on $\mathcal{C}$ (a folklore result, see [9, Theorem 4.4]). Incidentally, the functor $\times_T$ only exists in general for commutative strong monads. So, if we are to consider type constructors as (bi)functors on $\mathcal{C}_T$, the assumption of commutativity is more or less forced upon us. Computationally, T_T is a "lifting" constructor and $\to_T$ gives "computational function spaces". In order to have sums, we have to demand that $\mathcal{C}$ have finite coproducts.[1] These lift trivially (F_T is surjective on objects and preserves colimits) to finite coproducts on $\mathcal{C}_T$.

[1] It is interesting that the distributivity of $\mathcal{C}$ does not follow, however it does if we impose Moggi's *equalising requirement* [15].

Importantly, all the above (bi)functors, $\times_T$, T_T, $\rightarrow_T$, $+$, are enriched over $\mathcal{C}$ and hence over $\mathcal{D}$, as is easily shown (we give the enrichment of T_T below). So all type constructors lift to multi-arity $\mathcal{D}$-functors on the consistently algebraically compact $\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T$.

Having made the above observations, it is easy to see how the Kleisli category can be used to model recursively typed call-by-value calculi (such as Plotkin's metalanguage [18]). A type σ with (at most) n free type variables will be modelled by a symmetric $\mathcal{D}$-functor:

$$[\![\sigma]\!] : (\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T)^n \rightarrow \mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T$$

Now if σ has a free type variable V (and n other variables) then its denotation will be:

$$[\![\sigma]\!] : (\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T)^n \times (\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T) \rightarrow \mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T$$

(where the behaviour on the second argument models instantiations of V). To model recursive types we want to define:

$$[\![\mu V.\sigma]\!] = [\![\sigma]\!]^{\dagger} : (\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T)^n \rightarrow \mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T$$

which by the remarks on page 3 is symmetric, and by Theorem 5.5 is a $\mathcal{D}$-functor as required. Thus in order to model recursive types it is necessary for $\mathcal{C}_T^{\mathrm{op}} \times \mathcal{C}_T$ to have enough initial algebras (equivalently terminal coalgebras) for the $[\![\sigma]\!]^{\dagger}$ functor to always be definable. One could demand that it be algebraically compact with respect to $\mathcal{D}$-functors (the example of $\mathcal{C} = \mathbf{PreDom}$ shows that this is consistent). However, in general it is sufficient to require only those initial algebras necessary for modelling all syntactically definable types.

Our approach leads to the possibility of having essentially algebraic notions of model for languages like Plotkin's metalanguage. It would be interesting to fully develop the induced equational calculus for such a language. One merit of our characterisation of initial algebras (and terminal coalgebras) as special invariant-objects (Theorem 5.2), is that the property of being an initial algebra is thereby reduced to three equations (two expressing the isomorphism of the invariant object, and one expressing the defining property of being special). As well as the usual real-world models (**pPreDom** for example [18]), there should also be an initial model and also a fully abstract closed-term model obtained by quotienting by operational equivalence.

We conclude this section with an application of the results of this paper to yield further information about Φ. The functor T_T referred to above inherits its behaviour on objects (of $\mathcal{C}_T$) from the behaviour of T (on objects of $\mathcal{C}$). On morphisms, T_T maps a morphism from X to Y in $\mathcal{C}_T$ given by $X \xrightarrow{f} TY$ (in $\mathcal{C}$) to the morphism from TX to TY given by $TX \xrightarrow{\eta \circ \mu \circ Tf} T^2Y$. It is routine to check that this is indeed a functor on $\mathcal{C}_T$. Moreover T_T is a $\mathcal{D}$-functor on $\mathcal{C}_T$. Its enrichment is given by morphisms $Y_T^X \xrightarrow{T_T} TY_T^{TX}$, defined as Λ applied to:

$$Y_T^X \times TX \xrightarrow{\eta \circ \mu \circ T\epsilon \circ t} T^2Y$$

Lemma 6.5 *$T\Phi \xrightarrow{F_T\sigma} \Phi$ is an initial T_T algebra in $\mathcal{C}_T$.*

Proof. Take any morphism $TX \xrightarrow{f} X$ in $\mathcal{C}_T$. Thus $TX \xrightarrow{f} TX$ in $\mathcal{C}$. Consider the diagrams

below, the first in $\mathcal{C}_T$, the second in $\mathcal{C}$.

$$\begin{tikzcd} T\Phi \arrow[r, "T_T x"] \arrow[d, "F_T\sigma"'] \arrow[dr, phantom, "(1)"] & TX \arrow[d, "f"] \\ \Phi \arrow[r, "x"] & X \end{tikzcd} \qquad \begin{tikzcd} T\Phi \arrow[r, "Tx"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "(2)"] & T^2X \arrow[d, "f \circ \mu"] \\ \Phi \arrow[r, "x"] & TX \end{tikzcd}$$

We now show that x makes (1) commute if and only if it makes (2) commute (the reason for the type mismatch in the diagrams is that in (1) x is considered qua Kleisli morphism).

$$\begin{array}{lrcll} f \circ T_T x = c \circ F_T\sigma & & & & \text{in } \mathcal{C}_T \\ \quad \text{iff} & \mu \circ Tf \circ \eta \circ \mu \circ Tx & = & \mu \circ Tx \circ \eta \circ \sigma & \text{in } \mathcal{C} \quad \text{(translation of above)} \\ \quad \text{iff} & \mu \circ \eta \circ f \circ \mu \circ Tx & = & \mu \circ \eta \circ x \circ \sigma & \text{in } \mathcal{C} \quad \text{(by naturality of } \eta) \\ \quad \text{iff} & f \circ \mu \circ Tx & = & x \circ \sigma & \text{in } \mathcal{C} \quad \text{(by monad unit law)} \end{array}$$

It follows (from the initiality of $T\Phi \xrightarrow{\sigma} \Phi$ in $\mathcal{C}$) that there is a unique x making (1) commute. $\square$

Theorem 6.6 $\Phi \xrightarrow{\sigma^{-1}} T\Phi$ *is a terminal T-coalgebra in $\mathcal{C}$.*

Proof. By the lemma, $T\Phi \xrightarrow{F_T\sigma} \Phi$ is an initial T_T algebra in $\mathcal{C}_T$. So, by consistent algebraic compactness for $\mathcal{D}$-functors, $\Phi \xrightarrow{F_T\sigma^{-1}} T\Phi$ is a terminal T_T coalgebra in $\mathcal{C}_T$.

To prove the theorem, take any morphism $X \xrightarrow{f} TX$ in $\mathcal{C}$. Now consider the two diagrams below, the first in $\mathcal{C}$, the second in $\mathcal{C}_T$.

$$\begin{tikzcd} TX \arrow[r, "Tx"] \arrow[dr, phantom, "(1)"] & T\Phi \\ X \arrow[r, "x"] \arrow[u, "f"] & \Phi \arrow[u, "\sigma^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "T_T y"] \arrow[dr, phantom, "(2)"] & T\Phi \\ X \arrow[r, "y"] \arrow[u, "F_T f"] & \Phi \arrow[u, "F_T\sigma^{-1}"'] \end{tikzcd}$$

We show that:

$$F_T : \{X \xrightarrow{x} \Phi \text{ in } \mathcal{C} \mid (1) \text{ commutes}\} \longrightarrow \{X \xrightarrow{y} \Phi \text{ in } \mathcal{C}_T \mid (2) \text{ commutes}\}$$

is a well-defined bijection. Then the terminality of $\Phi \xrightarrow{\sigma^{-1}} T\Phi$ in $\mathcal{C}$ follows from that of $\Phi \xrightarrow{F_T\sigma^{-1}} T\Phi$ in $\mathcal{C}_T$.

First, note that $F_T Tx = T_T F_T x$. So if x makes (1) commute then $y = F_T x$ makes (2) commute, as (2) is the image of (1) under F_T. So F_T is indeed a well-defined function between the two sets. Also the function is injective, as the faithfulness of F_T follows easily from the faithfulness of T.

It remains to show that any y making (2) commute is obtained as $y = F_T x$ for some x making (1) commute. Note that $T_T y = F_T(\mu \circ Ty)$ (where on the right-hand side we consider $X \xrightarrow{y} T\Phi$ qua morphism in $\mathcal{C}$). So if y makes (2) commute then:

$$\begin{array}{rcl} y & = & F_T\sigma \circ T_T y \circ F_T f \\ & = & F_T\sigma \circ F_T(\mu \circ Ty) \circ F_T f \\ & = & F_T(\sigma \circ \mu \circ Ty \circ f) \end{array}$$

Now setting $x = \sigma \circ \mu \circ Ty \circ f$, we have (by the assumption that (2) commutes) that $F_T(Tx \circ f) = F_T(\sigma^{-1} \circ x)$. So, by the faithfulness of F_T, $Tx \circ f = \sigma^{-1} \circ x$, and (1) does indeed commute. $\square$

It is known that for any strong monad, T, any initial T-algebra $T\Phi \xrightarrow{\sigma} \Phi$ for which σ^{-1} is a terminal T-coalgebra gives Φ as a fixed-point object (see [2]). The above theorem shows that, when the monad is commutative and T faithful, any fixed-point object arises in such a way. Perhaps, in the light of Freyd's work, it would be more natural to define the notion of fixed-point object using the terminal coalgebra requirement rather than the global element ω.

7 Conclusions

We have given conditions on a category with a strong monad under which Freyd's principle of versality holds (for suitably enriched endofunctors) in the Kleisli category. This mean that, assuming enough initial algebras exist, one can model recursively typed calculi in the Kleisli category.

Although the use of the Kleisli category is in keeping with Moggi's approach to semantics [14], there are also reasons to be interested in the Eilenberg-Moore category. Under certain conditions the Eilenberg-Moore category is bicartesian, symmetric monoidal closed, with a comonad and thus models intuitionistic linear logic (an observation due to Gordon Plotkin and Bart Jacobs, see [9] for details). So the Eilenberg-Moore category has the potential to model quite sophisticated type-systems. In order to model recursive types we would like to apply the analysis of Section 5 to the Eilenberg-Moore category. The work of Kock [13] leads us to believe that, whenever $\mathcal{C}$ is cartesian closed with finite limits and T is commutative, then $\mathcal{C}^T$ can be construed as a $\mathcal{D}$-category with bilinear composition. Thus under these conditions (which are in any case necessary for obtaining the model of intuitionistic linear logic), a treatment along the lines of that of Section 6 should also be possible for $\mathcal{C}^T$.

An interesting open problem is to find general conditions that will ensure that the initial algebras used to model recursive types are constructable as the colimits of ω-chains starting with an initial object (as in the the classical case of **O**-categories [21]). Ideally one would then like to obtain consistent algebraic compactness from a limit/colimit coincidence along the lines of [21, Theorem 2].

One further question is whether there are general conditions that will ensure the existence of enough initial algebras to solve recursive type equations. If initial algebras are obtainable as ω-colimits, then it is sufficient to require enough cocompleteness (or dually, completeness for terminal coalgebras). In fact, for internally small complete categories (with respect to some topos) the question has a rather trivial answer. For abstract reasons, any small complete category is (bi)algebraically complete (that is all internal endofunctors have both initial algebras and terminal coalgebras) [8]. In this setting a quite simple requirement suffices to obtain algebraic compactness: it is enough that the hom-"sets" of the category are objects of the topos for which every endomorphism has a fixed-point. This approach should be applicable to the Eilenberg-Moore categories of suitable internal strong monads on small complete categories (for abstract reasons the Eilenberg-Moore categories will also be small complete) such as the lift monad on a small complete category of predomains. So, in such a situation, (internal) algebraic compactness can be obtained without resorting to the analysis of this paper.

Acknowledgements

I am indebted to Bart Jacobs for particularly helpful discussions about strong monads, to Marcelo Fiore for discussions about algebraic compactness, and to Martin Hofmann who drew Lemma 6.5 to my attention. Thanks also to Philip Mulry, Gordon Plotkin and David Pym. Paul Taylor's diagram macros were used throughout.

References

[1] M. Barr. Coequalisers and free triples. *Math Z.*, 116:307–322, 1970.

[2] R. L. Crole. *Programming Metalogics with a Fixpoint Type.* PhD. Thesis, University of Cambridge, 1991. Available as Cambridge University Computer Laboratory Technical Report No. 247.

[3] R. L. Crole and A. M. Pitts. New foundations for fixpoint computations. In *Proceedings of 5th Annual Symposium on Logic in Computer Science*, pages 489–497, 1990.

[4] M. Fiore and G. D. Plotkin. On compactness and **cpo**-enriched categories. In G. Winskel, editor, *Proceedings of the CLICS Workshop (23-27 March 1992)*, pages 571–584. Computer Science Department, Aarhus University, 1992.

[5] P. J. Freyd. Recursive types reduced to inductive types. In *Proceedings of 5th Annual Symposium on Logic in Computer Science*, pages 498 – 507, 1990.

[6] P. J. Freyd. Algebraically complete categories. In A. Carboni, M. C. Pedicchio, and G. Rosolini, editors, *Category Theory, Proceedings Como 1990*, volume 1488 of *Lecture Notes in Mathematics.* Springer-Verlag, 1991.

[7] P. J. Freyd. Remarks on algebraically compact categories. In M. P. Fourman, P. T. Johnstone, and A. M. Pitts, editors, *Applications of Categories in Computer Science*, number 177 in London Mathematical Society Lecture Note Series, pages 95–106, 1992.

[8] J. M. E. Hyland. A small complete category. *Annals of Pure and Applied Logic*, 40:135 – 165, 1988.

[9] B.P.F. Jacobs. Semantics of weakening and contraction. *Submitted manuscript*, forthcoming, 1992.

[10] G. M. Kelly. *Basic Concepts of Enriched Category Theory.* Number 64 in Lecture Notes. London Math. Soc., 1982.

[11] A. Kock. Monads on symmetric monoidal closed categories. *Archiv der Mathematik*, 21:1 – 10, 1970.

[12] A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, pages 161–174, 1971.

[13] A. Kock. Closed categories generated by commutative monads. *J. Australian Math. Soc.*, pages 405–424, 1971.

[14] E. Moggi. Computational lambda-calculus and monads. In *Proceedings of 4th Annual Symposium on Logic in Computer Science*, pages 14–23, 1989.

[15] E. Moggi. Notions of computation and monads. *Information and Computation*, 93(1), 1991.

[16] P. S. Mulry. Strong monads, algebras and fixed points. In M. P. Fourman, P. T. Johnstone, and A. M. Pitts, editors, *Applications of Categories in Computer Science*, number 177 in London Mathematical Society Lecture Note Series, pages 202–216, 1992.

[17] W. K-S. Phoa. From term models to domains. In *Proceedings of Theoretical Aspects of Computer Software, Sendai 1991*, number 526 in Lecture Notes in Computer Science. Springer Verlag, 1991. Final version to appear in Information and Computation.

[18] G. D. Plotkin. *Denotational semantics with partial functions.* Lecture notes, C.S.L.I. Summer School, 1985.

[19] L. Román. On recursive principles in cartesian categories. *Aportaciones Matemáticas Comunicaciones*, 8:117–125, 1990.

[20] G. Rosolini. *Continuity and Effectivity in Topoi.* PhD thesis, University of Oxford, 1986.

[21] M.B. Smyth and G. D. Plotkin. The category theory solution of recursive domain equations. *SIAM Journal of Computing*, 11:761 – 783, 1982.

Appendix

This appendix contains the diagrams used in the proofs of propositions from Section 6. For the basic properties of strong monads the reader is referred to [14]. The list below gives keys to explanations for the individual commuting squares.

1. Naturality of η.
2. Naturality of μ.
3. Unit law of monad.
4. Associative law of monad.
5. Naturality of t or t'.
6. Basic property of t or t'.
7. Derivable property of t and t'.
8. Definition of m.
9. Definition of y_t^x, z_t^x or z_t^y.

$$
\begin{tikzcd}
Y_T^X \times X \arrow[rrr, "\epsilon"] & & & TY \arrow[d, "1"] \\
TY_T^X \times X \arrow[u, "y_t^x \times 1"] \arrow[r, "t'"] \arrow[rr, phantom, shift right=4ex, "9"] & T(Y_T^X \times X) \arrow[r, "T\epsilon"] & T^2Y \arrow[r, "\mu"] & TY \arrow[d, "1"] \\
Y_T^X \times X \arrow[u, "\eta \times 1"] \arrow[r, "1"] \arrow[rr, phantom, shift right=4ex, "6"] & Y_T^X \times X \arrow[u, "\eta"] \arrow[r, "\epsilon"] \arrow[rr, phantom, shift right=4ex, "1"] & TY \arrow[u, "\eta"] \arrow[r, "1"] \arrow[rr, phantom, shift right=4ex, "3"] & TY
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y_T^X \times X \arrow[rrr, "\epsilon"] & & & TY & T^2Y \arrow[l, "\mu"'] \\
TY_T^X \times X \arrow[u, "y_t^x \times 1"] \arrow[r, "t'"] \arrow[rr, phantom, shift right=4ex, "9"] & T(Y_T^X \times X) \arrow[r, "T\epsilon"] & T^2Y \arrow[u, "\mu"] \arrow[rr, phantom, shift right=4ex, "4"] & & T^3Y \arrow[u, "\mu"] \arrow[l, "T\mu"'] \\
T^2Y_T^X \times X \arrow[u, "Ty_t^x \times 1"] \arrow[r, "t'"] \arrow[rr, phantom, shift right=4ex, "5"] & T(TY_T^X \times X) \arrow[u, "T(y_t^x \times 1)"] \arrow[rr, "Tt'"] \arrow[rr, phantom, shift right=4ex, "9"] & & T^2(Y_T^X \times X) \arrow[u, "T^2\epsilon"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y_T^X \times X \arrow[rrr, "\epsilon"] & & & TY \\
TY_T^X \times X \arrow[u, "y_t^x \times 1"] \arrow[rr, "t'"] \arrow[rrr, phantom, shift right=4ex, "9"] & & T(Y_T^X \times X) \arrow[r, "T\epsilon"] \arrow[rr, phantom, shift right=4ex, "2"] & T^2Y \arrow[u, "\mu"] \\
T^2Y_T^X \times X \arrow[u, "\mu \times 1"] \arrow[r, "t'"] \arrow[rr, phantom, shift right=4ex, "6"] & T(TY_T^X \times X) \arrow[r, "Tt'"] & T^2(Y_T^X \times X) \arrow[u, "\mu"] \arrow[r, "T^2\epsilon"] & T^3Y \arrow[u, "\mu"]
\end{tikzcd}
$$

Figure 1: Diagrams for proof of Proposition 6.1.

$$
\begin{tikzcd}
\cdot \arrow[r, "a"] \arrow[d, "(m\times 1)\times 1"'] \arrow[dr, phantom, "m\times 1"] & \cdot \arrow[r, "1\times\epsilon"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "m\times 1"] & \cdot \arrow[r, "t"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "5"] & \cdot \arrow[r, "Ta"] \arrow[d, "T(m\times 1)"] & \cdot \arrow[r, "T(1\times\epsilon)"] & \cdot \arrow[d, "Tt"] \\
\cdot \arrow[r, "a"] \arrow[dd, "m\times 1"'] \arrow[ddr, phantom, "8"] & \cdot \arrow[r, "1\times\epsilon"] \arrow[d, "m\times 1"'] & \cdot \arrow[r, "t"] \arrow[d, "m\times 1"'] & \cdot \arrow[d, "T\epsilon"] \arrow[drr, phantom, "8"] & & \\
 & & & \cdot & \cdot \arrow[l, "T\mu"'] & \cdot \arrow[l, "T^2\epsilon"'] \arrow[d, "\mu"] \\
 & & & \cdot \arrow[d, "\mu"] \arrow[drr, phantom, "4"] & & \cdot \arrow[d, "\mu"] \arrow[drr, phantom, "2"] & & \cdot \arrow[d, "\mu"] \\
\cdot \arrow[rrr, "\epsilon"] & & & TW & \cdot \arrow[l, "\mu"'] & \cdot \arrow[l, "T\epsilon"'] & & \cdot
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[rr, "a"] \arrow[d, "a\times 1"'] \arrow[drr, phantom, "6"] & & \cdot \arrow[r, "1\times\epsilon"] \arrow[d, "a"] & \cdot \arrow[r, "t"] \arrow[d, "a"] & \cdot \arrow[d, "Ta"] \\
\cdot \arrow[r, "a"] \arrow[dd, "(1\times m)\times 1"'] & \cdot \arrow[r, "1\times a"] \arrow[dd, "1\times(m\times 1)"] & \cdot \arrow[r, "1\times(1\times\epsilon)"] \arrow[dr, phantom, "8"] & \cdot \arrow[r, "1\times t"] \arrow[d, "1\times T\epsilon"'] \arrow[dr, phantom, "5"] & \cdot \arrow[d, "T(1\times\epsilon)"] \\
 & & & \cdot \arrow[r, "t"] \arrow[d, "1\times\mu"] & \cdot \arrow[dd, "Tt"] \\
\cdot \arrow[r, "a"] \arrow[d, "m\times 1"'] & \cdot \arrow[rr, "1\times\epsilon"] \arrow[drr, phantom, "8"] & & \cdot \arrow[d, "t"] \arrow[dr, phantom, "6"] & \\
\cdot \arrow[rr, "\epsilon"] & & TW & \cdot \arrow[l, "\mu"'] & \cdot \arrow[l, "T\epsilon"'] & \cdot \arrow[l, "\mu"']
\end{tikzcd}
$$

Figure 2: Diagrams for proof of Proposition 6.2.

$$
\begin{tikzcd}
(Z_T^Y \times TY_T^X) \times X \arrow[r, "(1 \times y_t^x) \times 1"] \arrow[d, "a"] & (Z_T^Y \times Y_T^X) \times X \arrow[r, "m \times 1"] \arrow[d, "a"] & Z_T^X \times X \arrow[dddd, "\epsilon"] \\
Z_T^Y \times (TY_T^X \times X) \arrow[r, "1 \times (y_t^x \times 1)"] \arrow[d, "1 \times t'"] & Z_T^Y \times (Y_T^X \times X) \arrow[dddd, "1 \times \epsilon"] & \\
Z_T^Y \times T(Y_T^X \times X) \arrow[d, "1 \times T\epsilon"] \arrow[r, phantom, "9"] & {} \arrow[r, phantom, "8"] & {} \\
Z_T^Y \times T^2 Y \arrow[r, "1 \times \mu"] \arrow[d, "t"] \arrow[r, phantom, "6"] & Z_T^Y \times TY \arrow[d, "t"] & \\
T(Z_T^Y \times TY) \arrow[r, "Tt"] & T^2(Z_T^Y \times Y) \arrow[r, "\mu"] & T(Z_T^Y \times Y) \arrow[r, "T\epsilon"] & T^2 Z \arrow[r, "\mu"] & TZ
\end{tikzcd}
$$

$$
\begin{tikzcd}
(Z_T^Y \times TY_T^X) \times X \arrow[r, "t \times 1"] \arrow[d, "a"] & T(Z_T^Y \times Y_T^X) \times X \arrow[r, "Tm \times 1"] \arrow[d, "t'"] \arrow[r, phantom, "5"] & TZ_T^X \times X \arrow[r, "z_t^x \times 1"] \arrow[d, "t'"] & Z_T^X \times X \arrow[dddd, "\epsilon"] \\
Z_T^Y \times (TY_T^X \times X) \arrow[d, "1 \times t'"] \arrow[r, phantom, "7"] & T((Z_T^Y \times Y_T^X) \times X) \arrow[r, "T(m \times 1)"] \arrow[d, "Ta"] & T(Z_T^X \times X) \arrow[r, phantom, "9"] \arrow[d, "T\epsilon"] & {} \\
Z_T^Y \times T(Y_T^X \times X) \arrow[r, "t"] \arrow[ddd, "1 \times T\epsilon"'] \arrow[r, phantom, "5"] & T(Z_T^Y \times (Y_T^X \times X)) \arrow[ddd, "T(1 \times \epsilon)"] \arrow[r, phantom, "8"] & T^2 Z \arrow[r, "\mu"] & TZ \\
{} & {} & T^3 Z \arrow[u, "T\mu"] \arrow[r, "\mu"] \arrow[r, phantom, "4"] & T^2 Z \arrow[u, "\mu"'] \\
{} & {} & T^2(Z_T^Y \times Y) \arrow[u, "T^2\epsilon"] \arrow[r, "\mu"] \arrow[r, phantom, "2"] & T(Z_T^Y \times Y) \arrow[u, "T\epsilon"'] \\
Z_T^Y \times T^2 Y \arrow[r, "t"] & T(Z_T^Y \times TY) \arrow[r, "Tt"] & {} &
\end{tikzcd}
$$

Figure 3: Diagrams for proof of Proposition 6.3.

$$
\begin{tikzcd}
(TZ_T^Y \times Y_T^X) \times X \arrow[r, "(z_t^y \times 1) \times 1"] \arrow[d, "a"] & (Z_T^Y \times Y_T^X) \times X \arrow[r, "m \times 1"] \arrow[d, "a"] & Z_T^X \times X \arrow[dddd, "\epsilon"] \\
TZ_T^Y \times (Y_T^X \times X) \arrow[r, "z_t^y \times 1"] \arrow[d, "1 \times \epsilon"] & Z_T^Y \times (Y_T^X \times X) \arrow[d, "1 \times \epsilon"] & \\
TZ_T^Y \times TY \arrow[r, "z_t^y \times 1"] \arrow[d, "t"] \arrow[dr, phantom, "5"] & Z_T^Y \times TY \arrow[d, "t"] \arrow[dr, phantom, "8"] & \\
T(TZ_T^Y \times Y) \arrow[r, "T(z_t^y \times 1)"] \arrow[d, "Tt'"] \arrow[dr, phantom, "9"] & T(Z_T^Y \times Y) \arrow[d, "T\epsilon"] & \\
T^2(Z_T^Y \times Y) \arrow[r, "T^2\epsilon"] \arrow[d, "\mu"] \arrow[dr, phantom, "2"] & T^3 Z \arrow[r, "T\mu"] \arrow[d, "\mu"] \arrow[dr, phantom, "4"] & T^2 Z \arrow[d, "\mu"] \\
T(Z_T^Y \times Y) \arrow[r, "T\epsilon"] & T^2 Z \arrow[r, "\mu"] & TZ \arrow[r, "1"] & TZ
\end{tikzcd}
$$

$$
\begin{tikzcd}
(TZ_T^Y \times Y_T^X) \times X \arrow[r, "t' \times 1"] \arrow[dddd, "a"] \arrow[dr, phantom, "6"] & T(Z_T^Y \times Y_T^X) \times X \arrow[r, "Tm \times 1"] \arrow[d, "t'"] \arrow[dr, phantom, "5"] & TZ_T^X \times X \arrow[r, "z_t^x \times 1"] \arrow[d, "t'"] \arrow[dr, phantom, "9"] & Z_T^X \times X \arrow[dddd, "\epsilon"] \\
 & T((Z_T^Y \times Y_T^X) \times X) \arrow[r, "T(m \times 1)"] \arrow[d, "Ta"] & T(Z_T^X \times X) \arrow[d, "T\epsilon"] & \\
TZ_T^Y \times (Y_T^X \times X) \arrow[r, "t'"] \arrow[dddd, "1 \times \epsilon"] \arrow[dr, phantom, "5"] & T(Z_T^Y \times (Y_T^X \times X)) \arrow[dddd, "T(1 \times \epsilon)"] \arrow[dr, phantom, "8"] & T^2 Z \arrow[r, "\mu"] & TZ \\
 & & T^3 Z \arrow[r, "\mu"] \arrow[u, "T\mu"] \arrow[dr, phantom, "4"] & T^2 Z \arrow[u, "\mu"] \\
 & & T^2(Z_T^Y \times Y) \arrow[r, "\mu"] \arrow[u, "T^2\epsilon"] \arrow[dr, phantom, "2"] & T(Z_T^Y \times Y) \arrow[u, "T\epsilon"] \\
TZ_T^Y \times TY \arrow[r, "t'"] & T(Z_T^Y \times TY) \arrow[r, "Tt"] & T^2(Z_T^Y \times Y) \arrow[r, "\mu"] & T(Z_T^Y \times Y)
\end{tikzcd}
$$

Figure 4: Diagrams for proof of Proposition 6.4.