ON THE SHARP DISTORTION THEOREMS FOR A SUBCLASS OF
STARLIKE MAPPINGS IN SEVERAL COMPLEX VARIABLES

Xiaosong Liu* and Taishun Liu

Abstract. In this article, we first establish the sharp distortion theorems of the
Fréchet derivative for a subclass of starlike mappings on the unit ball of complex
Banach spaces and the bounded starlike circular domain in $\mathbb{C}^n$. Meanwhile, we
also obtain the sharp distortion theorems of the Jacobi determinant for a subclass
of starlike mappings on the bounded starlike circular domain in $\mathbb{C}^n$. Our derived
conclusions are the generalizations of some known results in several complex
variables and the classical results in one complex variable.

1. INTRODUCTION

In the theory of univalent functions, there is a classical and well-known theorem
as follows.

Theorem A. [6]. If $f$ is a normalized biholomorphic function on the unit disk $U$
in $\mathbb{C}$, then

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}, \quad z \in U.$$ 

However, Cartan[2] pointed out that Theorem A is fail by providing a counter exam-
ple, and he proposed that people should study the geometric properties for the subclasses
of biholomorphic mappings, such as normalized biholomorphic starlike mappings and
normalized biholomorphic convex mappings.

In several complex variables, the results concerning the distortion theorems for
convex mappings are the best. For example, the estimates of the Jacobi determinant
for convex mappings defined on the Euclidean unit ball in $\mathbb{C}^2$ were first established
due to Barnard, FitzGerald and Gong [1] in 1994, and consequently Liu and Zhang

Received May 18, 2014, accepted June 18, 2014.
Communicated by Der-Chen Chang.
2010 Mathematics Subject Classification: Primary 32A30, 32H02.
Key words and phrases: Distortion theorem, A zero of order $k + 1$, Fréchet derivative, Jacobi determinant,
Starlike mapping.
*Corresponding author.
[13] obtained a general version of the above result. As to the distortion theorem of the Fréchet derivative for convex mappings, Gong, Wang and Yu [5] first established the corresponding estimates for convex mappings, after that Gong and Liu [4], Liu and Zhang [14], Zhu and Liu [19], Chu, Hamada, Honda and Kohr [3] established various versions of the distortion theorem for convex mappings on different unit balls in complex Banach spaces. Hamada and Kohr [8] obtained the stronger upper bounds estimate of the distortion theorem for convex mappings on the unit ball of a complex Hilbert space. In contrast to the distortion theorems for convex mappings, the corresponding results of starlike mappings are rather few until now. There is not any work on this topic except the distortion theorem for starlike mappings on the unit polydisk along a unit direction was given by Liu, Wang and Lu [12]. At present, more and more people see that the problems concerning the distortion theorem for starlike mappings are extreme difficult. A natural question arouse us to pay attention to whether the distortion theorem for starlike mappings under restricted conditions holds or not. Our answer is affirmative. That is to say, we establish the sharp distortion theorems of the Fréchet derivative and the Jacobi determinant for a subclass of starlike mappings in several complex variables.

Let $X$ be a complex Banach space with the norm $||\cdot||$, $X^*$ be the dual space of $X$, $B$ be the open unit ball in $X$, and $U$ be the Euclidean open unit disk in $\mathbb{C}$. Also, let $U^n$ be the open unit polydisk in $\mathbb{C}^n$, and let $\mathbb{N}$ be the set of all positive integers. We denote by $\partial U^n$ the boundary of $U^n$, and $\partial_0 U^n$ the distinguished boundary of $U^n$. Let the symbol $'$ mean transpose. For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{ T_x \in X^* : ||T_x|| = 1, T_x(x) = ||x|| \}.$$ 

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $H(B)$ be the set of all holomorphic mappings from $B$ into $X$. We know that if $f \in H(B)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x)((y - x)^n),$$

for all $y$ in some neighborhood of $x \in B$, where $D^n f(x)$ is the $n$th-Fréchet derivative of $f$ at $x$, and for $n \geq 1$,

$$D^n f(x)((y - x)^n) = D^n f(x)(y - x, \ldots, y - x).$$

Furthermore, $D^n f(x)$ is a bounded symmetric $n$-linear mapping from $\prod_{j=1}^{n} X$ into $X$.

We say that a holomorphic mapping $f : B \to X$ is biholomorphic if the inverse $f^{-1}$ exists and is holomorphic on the open set $f(B)$. A mapping $f \in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $D f(x)$ has a bounded inverse for each $x \in B$. If $f : B \to X$ is a holomorphic mapping, then we say that $f$ is normalized
if \( f(0) = 0 \) and \( Df(0) = I \), where \( I \) represents the identity operator from \( X \) into \( X \). A domain \( \Omega \) in \( \mathbb{C}^n \) is said to be circular if \( e^{i\theta}z \in \Omega \) for any \( z \in \Omega \), and a domain \( \Omega \) in \( \mathbb{C}^n \) is said to be a complete Reinhardt domain if \((\xi_1 z_1, \xi_2 z_2, \cdots, \xi_n z_n)^t \in \Omega \) for any \( z = (z_1, z_2, \cdots, z_n)^t \in \Omega \), where \( \xi_k \in \mathbb{U} \) \((k = 1, 2, \cdots, n)\), \( i = \sqrt{-1} \) and \( \theta \) is a real number.

We say that a normalized biholomorphic mapping \( f : B \to X \) is a starlike mapping if \( f(B) \) is a starlike domain with respect to the origin.

**Definition 1.1.** [9]. Suppose that \( \Omega \) is a domain (connected open set) in \( X \) which contains \( 0 \). It is said that \( x = 0 \) is a zero of order \( k \) of \( f(x) \) if \( f(0) = 0, \cdots, D^{k-1}f(0) = 0 \), but \( D^k f(0) \neq 0 \), where \( k \in \mathbb{N} \).

We denote by \( S^*(B) \) (resp. \( S^*(\Omega) \)) the set of all normalized biholomorphic starlike mappings on \( B \) (resp. \( \Omega \)), and \( S^*_{k+1}(B) \) (resp. \( S^*_{k+1}(\Omega) \)) by the set of all normalized biholomorphic starlike mappings on \( B \) (resp. \( \Omega \)) and \( x = 0 \) (resp. \( z = 0 \)) is a zero of order \( k+1 \) of \( f(x) - x \) (resp. \( f(z) - z \)).

In this article, we shall establish the sharp distortion theorems of the Fréchet derivative for a subclass of starlike mappings on the unit ball of complex Banach spaces and the bounded starlike circular domain in \( \mathbb{C}^n \), and the sharp distortion theorems of the Jacobi determinant for a subclass of starlike mappings on the bounded starlike circular domain in \( \mathbb{C}^n \). Our obtained conclusions show that some known results in several complex variables are extended, and they both reduce to the classical results in one complex variable.

2. **Sharp Distortion Theorems of the Fréchet Derivative for a Subclass of Starlike Mappings**

In order to establish the desired results in this section, it is necessary to give some lemmas as follows.

**Lemma 2.1.** [18]. Let \( f \) be a normalized locally biholomorphic mapping on \( B \). Then \( f \in S^*(B) \) if and only if
\[
\Re \{ T_x[(Df(x))^{-1}f(x)] \} \geq 0, \ x \in B.
\]

**Lemma 2.2.** [10]. If \( F \in S^*_{k+1}(B) \), then
\[
\frac{\|x\|(1 - \|x\|^{k})}{1 + \|x\|^k} \leq |T_x[(DF(x))^{-1}F(x)]| \leq \frac{\|x\|(1 + \|x\|^{k})}{1 - \|x\|^k}, \ x \in B.
\]

We now begin to present the following theorems in this section.
Theorem 2.1. Let \( f : B \to \mathbb{C} \in H(B) \), \( F(x) = xf(x) \in S^*_k(B) \). Then
\[
\frac{\|x\|(1 - \|x\|^{k})}{(1 + \|x\|^{k})^{1 + \frac{2}{k}}} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|^{k})}{(1 - \|x\|^{k})^{1 + \frac{2}{k}}}, x \in B,
\]
and the above estimates are sharp.

Proof. Since \( F(x) = xf(x) \), we have
\[
(2.1) \quad DF(x)x = xf(x) + (Df(x)x)x, x \in B.
\]
Also \( F(x) = xf(x) \in S^*(B) \), then we obtain
\[
\|x\| \left(1 + \frac{\|x\|^{k}}{(1 + \|x\|^{k})^{\frac{2}{k}}} \right) \leq \|F(x)\| \leq \frac{\|x\|}{(1 - \|x\|^{k})^{1 + \frac{2}{k}}}, x \in B,
\]
from [10, Theorem 1]. Hence
\[
|f(x)| \geq \frac{1}{(1 + \|x\|^{k})^{\frac{2}{k}}}, x \in B.
\]
This implies that \( f(x) \neq 0 \) for \( x \in B \).

Straightforward computation shows that
\[
(2.2) \quad [DF(x)]^{-1}F(x) = \frac{f(x)x}{f(x) + Df(x)x}, x \in B.
\]
Note that \( \Re\{T_x[(DF(x))^{-1}F(x)]\} > 0, x \in B \setminus \{0\} \) because of \( F \in S^*(B) \). Therefore according to (2.2), it yields that
\[
f(x) + Df(x)x \neq 0, x \in B.
\]
In view of Lemma 2.2, we have
\[
\frac{\|x\|(1 - \|x\|^{k})}{1 + \|x\|^{k}} \leq |T_x[(DF(x))^{-1}F(x)]| \leq \frac{\|x\|(1 + \|x\|^{k})}{1 - \|x\|^{k}}, x \in B,
\]
On the other hand, by (2.1) and (2.2), we deduce that
\[
DF(x)x = xf(x) + (Df(x)x)x = xf(x) \left(1 + \frac{Df(x)x}{f(x)}\right) = F(x) \frac{\|x\|}{T_x[(DF(x))^{-1}F(x)]}.
\]
It yields that
\[
\frac{\|x\|(1 - \|x\|^{k})}{(1 + \|x\|^{k})^{1 + \frac{2}{k}}} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|^{k})}{(1 - \|x\|^{k})^{1 + \frac{2}{k}}}, x \in B.
\]
from Lemma 2.2 and [10, Theorem 1].

It is easy to verify that

\[ F(x) = \frac{x}{(1 - T^k_a(x))^{\frac{1}{k}}} , \quad x \in B \]

satisfies the condition of Theorem 2.1. A short computation shows that

\[ DF(x)x = \left( I + \frac{2T^k_b - 1(x)T_u(\cdot)}{1 - T^k_b(x)} \right) \frac{x}{(1 - T^k_b(x))^{\frac{1}{k}}} = \frac{x(1 + T^k_b(x))}{(1 - T^k_b(x))^{1 + \frac{1}{k}}} , \quad x \in B . \]

We set \( x = ru \) or \( x = e^{\frac{2\pi}{k}}ru(0 \leq r < 1), u \in \partial B \). Then it is shown that the estimates of Theorem 2.1 are sharp. This completes the proof.

**Corollary 2.1.** Let \( f : B \to \mathbb{C} \in H(B), F(x) = xf(x) \in S^*(B) \). Then

\[ \frac{\|x\|(1 - \|x\|)}{(1 + \|x\|)^3} \leq \|DF(x)x\| \leq \frac{\|x\|(1 + \|x\|)}{(1 - \|x\|)^3} , \quad x \in B , \]

and the above estimates are sharp.

Let \( \Omega \) be a bounded starlike circular domain in \( \mathbb{C}^n \), and its Minkowski functional \( \rho(z) \) is a \( C^1 \) function except for a lower dimensional manifold in \( \Omega \).

**Lemma 2.3.** If \( F \in S^*_{k+1}(\Omega) \), then

\[ \frac{\rho(z)(1 - \rho^k(z))}{1 + \rho^k(z)} \leq \left| 2 \frac{\partial \rho(z)}{\partial z} (DF(z))^{-1} F(z) \right| \leq \frac{\rho(z)(1 + \rho^k(z))}{1 - \rho^k(z)} , \quad z \in \Omega . \]

**Proof:** Fix \( z \in \Omega \setminus \{0\} \), and denote \( z_0 = \frac{\rho(z)}{\rho(z)} \). Define

\[ p(\xi) = \begin{cases} 2 \frac{\partial \rho(z)}{\partial z} DF(z_0) F(z_0) & , \quad \xi \in U \setminus \{0\} , \\ 1 & , \quad \xi = 0 . \end{cases} \]

Since \( F \in S^*(\Omega) \), similar to that in the proof of [15, the case \( \alpha = 0 \) of Lemma 4], we conclude that \( p \in H(U), p(0) = 1, \) and \( \Re p(\xi) > 0, \xi \in U \setminus \{0\} \), and \( \xi = 0 \) is at least a zero of order \( k \) of \( p(\xi) - 1 \). According to [10, Lemma 3], it follows the desired result. This completes the proof.

**Theorem 2.2.** Let \( f : \Omega \rightarrow \mathbb{C} \in H(\Omega), F(z) = zf(z) \in S^*_{k+1}(\Omega) \). Then

\[ \frac{\rho(z)(1 - \rho^k(z))}{(1 + \rho^k(z))^{1 + \frac{1}{k}}} \leq \rho(DF(z)z) \leq \frac{\rho(z)(1 + \rho^k(z))}{(1 - \rho^k(z))^{1 + \frac{1}{k}}} , \quad z \in \Omega , \]
and the above estimates are sharp.

Proof. Similar to that in the proof of Theorem 2.1, we obtain

\[ DF(z)z = F(z) \frac{\rho(z)}{2 \partial \rho(z) (DF(z))^{-1} F(z)}. \]

Therefore

\[ \frac{\rho(z)(1 - \rho^k(z))}{(1 + \rho^k(z))^{1 + \frac{1}{k}}} \leq \rho(DF(z)z) \leq \frac{\rho(z)(1 + \rho^k(z))}{(1 - \rho^k(z))^{1 + \frac{1}{k}}}, \quad z \in \Omega \]

from Lemma 2.3 and [15, the case \( \alpha = 0 \) of Theorem 5].

It is not difficult to check that

\[ F(z) = \frac{z}{\left(1 - \left(\frac{z}{r}\right)^k\right)^{\frac{1}{k}}}, \quad z \in \Omega \]

satisfies the condition of Theorem 2.2, where \( r = \sup\{|z_1| : z = (z_1, 0, \cdots, 0)' \in \Omega\} \).

A direct computation shows that

\[ DF(z)z = \frac{z}{\left(1 - \left(\frac{z}{r}\right)^k\right)^{\frac{1}{k}}} + \frac{2 \left(\frac{z}{r}\right)^{k-1} \frac{z}{r} z}{\left(1 - \left(\frac{z}{r}\right)^k\right)^{1 + \frac{1}{k}}} = \frac{1 + \left(\frac{z}{r}\right)^k}{\left(1 - \left(\frac{z}{r}\right)^k\right)^{1 + \frac{1}{k}}} z, \quad z \in \Omega. \]

Taking \( z = Ru \) or \( z = e^{\frac{2\pi i}{k}} Ru(0 \leq R < 1) \), where \( u = (u_1, u_2, \cdots, u_n)' \in \partial \Omega \), \( u_1 = r \), and \( i = \sqrt{-1} \), then the estimates of Theorem 2.2 are sharp. This completes the proof.

Taking \( k = 1 \) in Theorem 2.2, we have the following corollary.

**Corollary 2.2.** Let \( f : \Omega \to \mathbb{C} \in H(\Omega) \), \( F(z) = zf(z) \in S^*(\Omega) \). Then

\[ \frac{\rho(z)(1 - \rho(z))}{(1 + \rho(z))^3} \leq \rho(DF(z)z) \leq \frac{\rho(z)(1 + \rho(z))}{(1 - \rho(z))^3}, \quad z \in \Omega, \]

and the above estimates are sharp.

In Lemma 2.4, Theorem 2.3 and Corollary 2.3 as follows, let \( m_l (l = 1, 2, \cdots, n) \) be a non-negative integer, \( N = m_1 + m_2 + \cdots + m_n \in \mathbb{N} \), and \( m_l = 0 \) mean the corresponding component in \( Z \) and \( F(Z) \) is omitted. We denote by \( \Omega^m_l \) (resp. \( \Omega^N \)) the bounded complete Reinhardt domain of \( \mathbb{C}^m_l (l = 1, 2, \cdots, n) \) (resp. \( \mathbb{C}^N \)). Let \( \rho_{m_l}(Z_l) \) (resp. \( \rho_N(Z) \)) be the Minkowski functional of \( \Omega^m_l \) (resp. \( \Omega^N \)). \( U^m_l \) (resp. \( U^N \)) is denoted by the unit polydisk of \( \mathbb{C}^m_l (l = 1, 2, \cdots, n) \) (resp. \( \mathbb{C}^N \)).
Lemma 2.4. Suppose that $f_l : \Omega^{m_l} \to \mathbb{C} \in H(\Omega^{m_l}), l = 1, 2, \cdots, n$, $F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))^\prime = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))^\prime$, $Z = (Z_1, Z_2, \cdots, Z_n)^\prime \in \Omega^N$. Then $F \in S^\ast(\Omega^N)$ if and only if $F_l \in S^\ast(\Omega^{m_l}), l = 1, 2, \cdots, n$.

Proof: By a simple calculation, it yields that

\[(DF(Z))^{-1} F(Z) = ((DF_1(Z_1))^{-1} F_1(Z_1), (DF_2(Z_2))^{-1} F_2(Z_2), \cdots), \]

\[(DF_n(Z_n))^{-1} F_n(Z_n))^\prime \]

and

\[(DF_l(Z_l))^{-1} F_l(Z_l) = \frac{Z_l f_l(Z_l)}{f_l(Z_l) + D f_l(Z_l) Z_l}, l = 1, 2, \cdots, n. \]

Therefore

\[2 \frac{\partial \rho_N(Z)}{\partial Z}(DF(Z))^{-1} F(Z) = \sum_{l=1}^{n} \left( 2 \frac{\partial \rho_N(Z)}{\partial Z_l} Z_l \right) \frac{f_l(Z_l)}{f_l(Z_l) + D f_l(Z_l) Z_l} \]

and

\[2 \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l} (DF_l(Z_l))^{-1} F_l(Z_l) \]

\[= \frac{f_l(Z_l)}{f_l(Z_l) + D f_l(Z_l) Z_l} \left( \sum_{k=1}^{m_l} 2 \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{lk}} Z_{lk} \right), l = 1, 2, \cdots, n \]

from (2.3) and (2.4), where $\frac{\partial \rho_N(Z)}{\partial Z} = \left( \frac{\partial \rho_N(Z)}{\partial Z_1}, \cdots, \frac{\partial \rho_N(Z)}{\partial Z_n} \right)$, $\frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l} = \left( \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{l1}}, \cdots, \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{lm_l}} \right), l = 1, 2, \cdots, n$. Note that

\[2 \partial \rho_{n}(z_{l} | z_{l}=0, z_{l}, \cdots, 0)^\prime \]

\[= \rho_{n}((0, \cdots, z_{l}, \cdots, 0)^\prime) > 0, z_{l} \neq 0, l = 1, 2, \cdots, n \]

(see the proof of [20, Theorem 2.1]).

On the one hand, in view of (2.5) and (2.7), we deduce that

\[\Re e \left( \frac{f_l(Z_l)}{f_l(Z_l) + D f_l(Z_l) Z_l} \right) > 0, Z_l \in \Omega^{m_l} \setminus \{0\}, l = 1, 2, \cdots, n \]

if $F \in S^\ast(\Omega^N)$ for $Z = (0, \cdots, Z_l, \cdots, 0)^\prime$. Hence

\[\Re e \left( 2 \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_l} (DF_l(Z_l))^{-1} F_l(Z_l) \right) \]

\[= \Re e \left( \frac{f_l(Z_l)}{f_l(Z_l) + D f_l(Z_l) Z_l} \left( \sum_{k=1}^{m_l} 2 \frac{\partial \rho_{m_l}(Z_l)}{\partial Z_{lk}} Z_{lk} \right) > 0, \]

$Z_l \in \Omega^{m_l} \setminus \{0\}, l = 1, 2, \cdots, n.$
This implies that \( F_l \in S^*(\Omega^m) \) \((l = 1, 2, \cdots, n)\) (see [11]).

On the other hand, taking into account (2.6) and (2.7), we also conclude that

\[
\Re e \left( \frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) > 0, \quad Z_l \in \Omega^m \setminus \{0\}, \quad l = 1, 2, \cdots, n
\]

if \( F_l \in S^*(\Omega^m) \). Consequently

\[
\Re e \left( 2\frac{\partial \rho_N(Z)}{\partial Z} (DF(Z))^{-1} F(Z) \right) = \sum_{l=1}^{n} \left( 2\frac{\partial \rho_N(Z)}{\partial Z_l} Z_l \right) \Re e \left( \frac{f_l(Z_l)}{f_l(Z_l) + Df_l(Z_l)Z_l} \right) > 0,
\]

\( Z \in \Omega^N \setminus \{0\}. \)

It is shown that \( F \in S^*(\Omega^N) \) (see [11]). This completes the proof.

\[\square\]

**Theorem 2.3.** Let \( f_l : U^m \to \mathbb{C} \in H(U^m) \), \( l = 1, 2, \cdots, n \), \( F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S_{k+1}^*(U^N) \). Then

\[
\frac{\|Z\|(1 - \|Z\|^k)}{(1 + \|Z\|^k)^{1 + \frac{k}{2}}} \leq \|DF(Z)Z\| \leq \frac{\|Z\|(1 + \|Z\|^k)}{(1 - \|Z\|^k)^{1 + \frac{k}{2}}}, \quad Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N,
\]

and the above estimates are sharp.

**Proof.** Let \( F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))' \). By a simple computation, we have

\[
DF(Z)Z = (DF_1(Z_1)Z_1, DF_2(Z_2)Z_2, \cdots, DF_n(Z_n)Z_n)'
\]

and

\[
(DF(Z))^{-1} F(Z) = ((DF_1(Z_1))^{-1} F_1(Z_1), (DF_2(Z_2))^{-1} F_2(Z_2), \cdots, (DF_n(Z_n))^{-1} F_n(Z_n))'
\]

Therefore it is shown that

\[
F \in S^*(U^N) \iff F_l \in S^*(U^m), \quad l = 1, 2, \cdots, n
\]

from Lemma 2.4, and \( Z_l = 0 \) is at least a zero of order \( k + 1 \) of \( F_l(Z) - Z_l \) for each \( l = 1, 2, \cdots, n \) if \( Z = 0 \) is a zero of order \( k + 1 \) of \( F(Z) - Z \). We easily know that \( \frac{t(1 + k)}{(1 - t^k)^{1 + \frac{k}{2}}} \) is an increasing function on interval \([0, 1] \) with respect to \( t \). Also by applying the facts \( \|DF(Z)Z\| = \max_{1 \leq l \leq n} \|DF_l(Z_l)Z_l\| \) and

\[
\frac{\|Z_l\|(1 - \|Z_l\|^k)}{(1 + \|Z_l\|^k)^{1 + \frac{k}{2}}} \leq \|DF(Z_l)Z_l\| \leq \frac{\|Z_l\|(1 + \|Z_l\|^k)}{(1 - \|Z_l\|^k)^{1 + \frac{k}{2}}}, \quad Z_l \in U^m, \quad l = 1, 2, \cdots, n
\]
(the case $B = U^{m_1}$ of Theorem 2.1), we see that

$$
\|DF(Z)Z\| = \max_{1 \leq j \leq n} \{\|DF(Z_i)Z_i\|\} \leq \frac{\|Z\|(1 + \|Z\|^{k})}{(1 - \|Z\|^{k})^{1 + \frac{2}{k}}}, \quad Z \in U^N
$$

and

$$
\|DF(Z)Z\| \geq \|DF_j(Z_j)\| \geq \frac{\|Z_j\|(1 - \|Z_j\|^{k})}{(1 + \|Z_j\|^{k})^{1 + \frac{2}{k}}}, \quad Z \in U^N,
$$

where $\|Z\|_{m_1}$ (resp. $\|Z\|_N$) is briefly written as $\|Z\|$ (resp. $\|Z\|$), and $j$ satisfies $\|Z_j\| = \|Z\| = \max_{1 \leq l \leq n} \{\|Z_l\|\}$. The sharpness of the estimates of Theorem 2.3 is similar to that in the proof of Theorem 2.2, we only pay attention to the right side equality of the estimates of Theorem 2.3 holds for $Z_1 = (r, 0, \cdots, 0)(0 \leq r < 1), Z_l = (0, 0, \cdots, 0)(l = 2, \cdots, n)$, the details are omitted here. This completes the proof. $\blacksquare$

Letting $k = 1$ in Theorem 2.3, we directly obtain

**Corollary 2.3.** Let $f_l : U^{m_l} \to \mathbb{C} \in H(U^{m_l}), l = 1, 2, \cdots, n$, $F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*(U^N)$. Then

$$
\frac{\|Z\|(1 - \|Z\|)}{(1 + \|Z\|)^{3}} \leq \|DF(Z)Z\| \leq \frac{\|Z\|(1 + \|Z\|)}{(1 - \|Z\|)^{3}}, \quad Z = (Z_1, Z_2, \cdots, Z_n)' \in U^N,
$$

and the above estimates are sharp.

**Theorem 2.4.** Suppose that $F(z) = (F_1(z), F_2(z), \cdots, F_n(z)) \in H(U^n)$, and $z = 0$ is a zero of order $k + 1(k \in \mathbb{N})$ of $F(z) - z$. If $\Re e \frac{DF_j(z)z}{F_j(z)} > 0, z \in U^n \setminus \{0\}$, where $j$ satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, then

$$
\frac{\|z\|(1 - \|z\|^{k})}{(1 + \|z\|^{k})^{1 + \frac{2}{k}}} \leq \|DF(z)z\| \leq \frac{\|z\|(1 + \|z\|^{k})}{(1 - \|z\|^{k})^{1 + \frac{2}{k}}}, \quad z \in U^n,
$$

and the above estimates are sharp.

**Proof.** Fix $z \in U^n \setminus \{0\}$, and denote $z_0 = \frac{z}{\|z\|}$. Let

$$
(2.8) \quad h_j(\xi) = \frac{z_j}{\|z\|} F_j(\xi z_0), \quad \xi \in U,
$$

where $j$ satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n} \{|z_l|\}$. Straightforward computation shows that

$$
\Re e \left( \frac{h_j'(\xi)\xi}{h_j(\xi)} \right) = \Re e \frac{DF_j(\xi z_0)\xi z_0}{F_j(\xi z_0)} > 0, \xi \in U \setminus \{0\}
$$
This implies that $\xi = 0$ is at least a zero of order $k + 1$ of $h_j(\xi) - \xi$.

On the other hand, we conclude that

$$\frac{|\xi|(1 - |\xi|^k)}{(1 + |\xi|^k)^{1+\frac{1}{n}}} \leq |DF_j(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1+\frac{1}{n}}}, \quad z_0 \in \partial U^n$$

from [7]. It is obvious to deduce that

$$\frac{|\xi|(1 - |\xi|^k)}{(1 + |\xi|^k)^{1+\frac{1}{n}}} \leq |DF_j(\xi z_0)\xi z_0| \leq \|DF(\xi z_0)\xi z_0\|, \quad z_0 \in \partial U^n.$$  

Taking $\xi = \|z\|$, we obtain

$$\|DF(z)z\| \geq \|z\|(1 - \|z\|^k), \quad z \in U^n.$$  

When $z_0 \in \partial_0 U^n$, it is shown that

$$|DF_l(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1+\frac{1}{n}}}, \quad l = 1, 2, \ldots, n$$

from (2.9). Noticing that $w(z) = DF_j(\xi z)\xi z$ is a holomorphic function on $\overline{U^n}$, in view of the maximum modulus theorem of holomorphic functions on the unit polydisk, it yields that

$$|DF_l(\xi z_0)\xi z_0| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1+\frac{1}{n}}}, \quad z_0 \in \partial U^n, \quad l = 1, 2, \ldots, n.$$  

This implies that

$$\|DF(\xi z_0)\xi z_0\| \leq \frac{|\xi|(1 + |\xi|^k)}{(1 - |\xi|^k)^{1+\frac{1}{n}}}, \quad z_0 \in \partial U^n.$$  

Letting $\xi = \|z\|$, we have

$$\|DF(z)z\| \leq \|z\|(1 + \|z\|^k), \quad z \in U^n.$$  

It follows the desired results from (2.10) and (2.11).

We set $k = 1$ in Theorem 2.4. It is immediately shown that

**Corollary 2.4.** Suppose that $F(z) = (F_1(z), F_2(z), \ldots, F_n(z)) \in H(U^n)$. If $\Re e \frac{DF_j(z)}{F_j(z)} > 0, z \in U^n \setminus \{0\}$, where $j$ satisfies the condition $|z_j| = \|z\| = \max_{1 \leq l \leq n} |z_l|$, then

$$\frac{\|z\|(1 - \|z\|)}{(1 + \|z\|)^\beta} \leq \|DF(z)z\| \leq \frac{\|z\|(1 + \|z\|)}{(1 - \|z\|)^\beta}, \quad z \in U^n,$$
and the above estimates are sharp.

**Remark 2.1.** We see that Theorem 2.1 (the case \( X = \mathbb{C}^n, B = U^n \)) is the special case of Theorem 2.4, and Theorem 2.3 (the case \( m_1 = n, m_l = 0, l = 2, \ldots, n \) or \( m_l = 1, l = 1, 2, \ldots, n \)) is also the special case of Theorem 2.4.

**Remark 2.2.** In view of Theorems 2.1, 2.2 and 2.3, the forms of distortion theorems of the Fréchet derivative for a subclass of starlike mappings are similar to each other. We may pose the following open problem.

**Open problem 2.1.** If \( F \in S_n^+(B) \), then
\[
\|x\|(1 - \|x\|)^3 \leq |\det JF(z)x| \leq \frac{\|x\|(1 + \|x\|)^3}{(1 - \|x\|)^3}, x \in B,
\]
and the above estimates are sharp.

3. DISTORTION THEOREMS OF THE JACOBI DETERMINANT FOR A SUBCLASS OF STARLIKE MAPPINGS

In this section, \( J_F(z) \) is denoted by the Jacobi matrix of the holomorphic mapping \( F(z) \), and \( \det J_F(z) \) is denoted by the Jacobi determinant of the holomorphic mapping \( F(z) \). Also, let \( B \) be the unit ball of \( \mathbb{C}^n \) with arbitrary norm \( \| \cdot \| \), and \( I_n \) be the unit matrix of \( \mathbb{C}^n \). Vectors in \( \mathbb{C}^n \) are usually written as column vectors in this section.

Suppose that \( \Omega \) is a bounded starlike circular domain in \( \mathbb{C}^n \), and its Minkowski functional \( \rho(z) \) is a \( C^1 \) function except for a lower dimensional manifold in \( \Omega \).

**Theorem 3.1.** Let \( f : \Omega \to \mathbb{C} \in H(\Omega) \), and \( F(z) = zf(z) \in S^{*}_{k+1}(\Omega) \). Then
\[
\frac{1 - \rho^k(z)}{(1 + \rho^k(z))^\frac{2n}{k+1} + 1} \leq |\det JF(z)| \leq \frac{1 + \rho^k(z)}{(1 - \rho^k(z))^\frac{2n}{k+1} + 1}, z \in \Omega,
\]
and the above estimates are sharp.

**Proof.** Since \( F(z) = zf(z) \), we obtain
\[
J_F(z)z = zf(z) + (J_f(z))z, z \in \Omega.
\]
With the same arguments of the proof in Theorem 2.1, it yields that \( f(z) \neq 0 \) for \( z \in \Omega \), and
\[
f(z) + J_f(z)z \neq 0, z \in B.
\]
A direct calculation shows that
\[
(J_F(z))^{-1}F(z) = \frac{f(z)z}{f(z) + J_f(z)z}, z \in \Omega
\]
A simple calculation shows that
\[ J_F(z) = f(z)(I_n + z(J_f(z))') = f(z) \left( I_n + \frac{z(J_f(z))'}{f(z)} \right). \]
Also
\[ |\det J_F(z)| = |f(z)|^n \left| 1 + \frac{J_f(z)z}{f(z)} \right| = |f(z)|^n \frac{\rho(z)}{2 - \left| \frac{\partial \rho(z)}{\partial z}((J_F(z)^{-1}F(z)) \right|} \]
from (3.1). Therefore, it is shown that
\[ \frac{1 - \rho^k(z)}{(1 + \rho^k(z))^{2n+1}} \leq |\det J_F(z)| \leq \frac{1 + \rho^k(z)}{(1 - \rho^k(z))^{2n+1}}, z \in \Omega \]
from Lemma 2.3 and [15, the case \( \alpha = 0 \) of Theorem 5].

Corollary 3.1. Let \( f : B \to \mathbb{C} \in H(B) \), and \( F(z) = zf(z) \in S^*_k(B) \). Then
\[ \frac{1 - \|z\|^k}{(1 + \|z\|)^{2n+1}} \leq |\det J_F(z)| \leq \frac{1 + \|z\|^k}{(1 - \|z\|)^{2n+1}}, z \in B, \]
and the above estimates are sharp.

When \( \Omega = B \), \( \rho(z) = \|z\| \), we immediately obtain the following corollary.

Corollary 3.2. Let \( f : \Omega \to \mathbb{C} \in H(\Omega) \), and \( F(z) = zf(z) \in S^*(\Omega) \). Then
\[ \frac{1 - \rho(z)}{(1 + \rho(z))^{2n+1}} \leq |\det J_F(z)| \leq \frac{1 + \rho(z)}{(1 - \rho(z))^{2n+1}}, z \in \Omega, \]
and the above estimates are sharp.

Remark 3.1. Let

\[ S_n^*(B^n) = \left\{ F(z) = z \prod_{j=1}^{n} \left( \frac{f_j(z_j)}{z_j} \right)^{\lambda_j} : f_j \in S^*(U), \lambda_j \geq 0, j = 1, 2, \cdots, n, \right\} \]

\[ \sum_{j=1}^{n} \lambda_j = 1, z = (z_1, z_2, \cdots, z_n) \in B^n \]

and

\[ SS^*(B^n) = \{ F(z) = zf(z) \in S^*(B^n) : f : B_n \to \mathbb{C} \in H(B^n) \}. \]

It is apparent to see that

\[ S_n^*(B^n) \subset SS^*(B^n) \subset S^*(B^n) \]

from the fact that \( F(z) = z + \sum_{m=2}^{\infty} \frac{D^m F(0)(z^m)}{m!} \in S^*(B^n) \) if \( \| D^m F(0) \| (m = 2, 3, \cdots) \)

are small enough (see [16]). Therefore, Corollary 3.2 is a general version of the corresponding theorem in [17].

In the following theorems and corollaries, let \( m_l(l = 1, 2, \cdots, n) \) be a non-negative integer, \( N = m_1 + m_2 + \cdots + m_n \in \mathbb{N} \), and \( m_l = 0 \) means the corresponding component in \( Z \) and \( F(Z) \) are omitted. \( \Omega^{m_l} \) (resp. \( \Omega^N \)) is denoted by the bounded complete Reinhardt domain of \( C^{m_l}(l = 1, 2, \cdots, n) \) (resp. \( \mathbb{C}^N \)). Let \( \rho_{m_l}(Z_l) \) (resp. \( \rho_{N}(Z) \)) be the Minkowski functional of \( \Omega^{m_l} \) (resp. \( \Omega^N \)). We denote by \( U^{m_l} \) (resp. \( U^N \)) the unit polydisk of \( C^{m_l}(l = 1, 2, \cdots, n) \) (resp. \( \mathbb{C}^N \)).

Theorem 3.2. Let \( f_l : \Omega^{m_l} \to \mathbb{C} \in H(\Omega^{m_l}), l = 1, 2, \cdots, n, \) \( F(Z) = (Z_1 f_1(Z_1), Z_2 f_2(Z_2), \cdots, Z_n f_n(Z_n))' \in S^*_{k+1}(\Omega^N) \). Then

\[ \frac{(1 - \rho^k_{N}(Z))^n}{(1 + \rho^k_{N}(Z))^n} \leq |\det J_F(Z)| \leq \frac{(1 + \rho^k_{N}(Z))^n}{(1 - \rho^k_{N}(Z))^n}, Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N. \]

Proof. Let \( F(Z) = (F_1(Z_1), F_2(Z_2), \cdots, F_n(Z_n))' \). According to the hypothesis of Theorem 3.2, for any \( Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega^N, \) it yields that

\[ J_F(Z) = \begin{pmatrix}
J_{F_1}(Z_1) & 0 & \cdots & 0 \\
0 & J_{F_2}(Z_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & J_{F_n}(Z_n)
\end{pmatrix} \]

(3.2)
and
\[(J_F(Z))^{-1}F(Z) = ((J_{F_1}(Z_1))^{-1}F_1(Z_1), (J_{F_2}(Z_2))^{-1}F_2(Z_2), \ldots, (J_{F_n}(Z_n))^{-1}F_n(Z_n))'.\]

Consequently, we see that
\[F \in S^*(\Omega^N) \Leftrightarrow F_i \in S^*(\Omega^{m_i}), i = 1, 2, \ldots, n\]
from Lemma 2.4. It is not difficult to check that
\[1 - \rho_{m_i}(Z_i) \leq |\det J_{F_i}(Z_i)| \leq \frac{1 + \rho_{m_i}(Z_i)}{2^{n/m_i} + 1}, Z_i \in \Omega^{m_i}\]
(the case \(\Omega = \Omega^{m_i}\) of Theorem 3.1), this implies that
\[\frac{(1 - \rho_N(Z))^n}{(1 + \rho_N(Z))^{2N + n}} \leq |\det J_F(Z)| \leq \frac{(1 + \rho_N(Z))^n}{(1 - \rho_N(Z))^{2N + n}}, Z = (Z_1, Z_2, \ldots, Z_n) \in \Omega^N\]
from (3.2). It follows the result, as desired. This completes the proof. \[\blacksquare\]

When \(k = 1\), we immediately obtain

**Corollary 3.3.** Let \(f_1 : \Omega^{m_i} \to \mathbb{C} \in H(\Omega^{m_i}), l = 1, 2, \ldots, n, F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \ldots, Z_nf_n(Z_n))' \in S^*(\Omega^N).\) Then
\[\frac{(1 - \rho_N(Z))^n}{(1 + \rho_N(Z))^{2N + n}} \leq |\det J_F(Z)| \leq \frac{(1 + \rho_N(Z))^n}{(1 - \rho_N(Z))^{2N + n}}, Z = (Z_1, Z_2, \ldots, Z_n) \in \Omega^N.\]

**Remark 3.2.** We do not know the sharpness of estimates of Theorem 3.2. However, we obtain the following corollary if \(\Omega = U^N\). Moreover, the estimates of Corollary 3.4 are sharp.

**Corollary 3.4.** Let \(f_1 : U^{m_i} \to \mathbb{C} \in H(U^{m_i}), l = 1, 2, \ldots, n, F(Z) = (Z_1f_1(Z_1), Z_2f_2(Z_2), \ldots, Z_nf_n(Z_n))' \in S^*_{k+1}(U^N).\) Then
\[\frac{(1 - \|Z\|^k)^n}{(1 + \|Z\|^k)^{2N + n}} \leq |\det J_F(Z)| \leq \frac{(1 + \|Z\|^k)^n}{(1 - \|Z\|^k)^{2N + n}}, Z = (Z_1, Z_2, \ldots, Z_n) \in U^N,\]
(3.3)
and the above estimates are sharp.

Proof. Let \( \Omega^N = U^N \), \( \Omega^m = U^m \), then (3.3) holds from Theorem 3.2 immediately. Consider

\[
F(Z) = \left( \frac{Z_1}{1 - Z_{11}^k}, \frac{Z_2}{1 - Z_{21}^k}, \cdots, \frac{Z_n}{1 - Z_{n1}^k} \right)', Z = (Z_1, Z_2, \cdots, Z_n)' \in \Omega_N.
\]

Then \( F \) satisfies the condition of Corollary 3.4, where \( Z_l = (Z_{l1}, Z_{l2}, \cdots, Z_{lm_l})' \in U^m, l = 1, 2, \cdots, n \). Straightforward computation shows that

\[
J_F(Z) = \begin{pmatrix}
J_{F_1}(Z_1) & 0 & \cdots & 0 \\
0 & J_{F_2}(Z_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_{F_n}(Z_n)
\end{pmatrix},
\]

where

\[
J_{F_l}(Z_l) = \begin{pmatrix}
\frac{1 + Z_{l1}^k}{(1 - Z_{l1}^k)^{\frac{k+1}{2}}} & 0 & \cdots & 0 \\
\frac{2Z_{l2}Z_{l1}}{(1 - Z_{l1}^k)^{\frac{k+1}{2}}} & \frac{1}{(1 - Z_{l1}^k)^{\frac{k}{2}}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{2Z_{lm_l}Z_{l1}}{(1 - Z_{l1}^k)^{\frac{k+1}{2}}} & 0 & \cdots & \frac{1}{(1 - Z_{l1}^k)^{\frac{k+1}{2}}}
\end{pmatrix}, l = 1, 2, \cdots, n.
\]

Taking \( Z_l = (e^{\pi i r}, 0, \cdots, 0)' \) or \( (r, 0, \cdots, 0)' \) \((0 \leq r < 1)\), \( i = \sqrt{-1}, l = 1, 2, \cdots, n \), then the estimates of Corollary 3.4 are sharp. This completes the proof. \( \Box \)

Remark 3.3. According to Theorems 3.1, 3.2 and Corollary 3.4, the most likely form of the sharp distortion theorems of Jacobi determinant for a subclass of starlike mappings is the same as Corollary 3.4 (the case \( m_l = 1, l = 1, 2, \cdots, n, k = 1 \)). Hence we propose the following open problem.

Open problem 3.1. If \( F \in S^*(U^n) \), then

\[
\frac{(1 - \|z\|)^n}{(1 + \|z\|)^{3n}} \leq |\det J_F(z)| \leq \frac{(1 + \|z\|)^n}{(1 - \|z\|)^{3n}}, z \in U^n,
\]

and the above estimates are sharp.
ACKNOWLEDGMENTS

This work was supported by the Key Program of National Natural Science Foundation of China (Grant No. 11031008), the National Natural Science Foundation of China (Grant No. 11061015).

REFERENCES


Xiaosong Liu  
School of Mathematics and Computation Science  
Zhanjiang Normal University  
Zhanjiang, Guangdong 524048  
P. R. China  
E-mail: lxszhjnc@163.com

Taishun Liu  
Department of Mathematics  
Huzhou Teachers College  
Huzhou, Zhejiang 313000  
P. R. China  
E-mail: lts@ustc.edu.cn