Path Planning to Optimize Observability in a Planar Uniform Flow Field

Brian T. Hinson, Michael K. Binder, and Kristi A. Morgansen

Abstract—This paper is concerned with path planning for under-sensed vehicles, where the vehicle has insufficient sensors to estimate its state without active control. One such system is an autonomous underwater vehicle, which typically does not have complete inertial position information. Using the condition number of the observability gramian as a cost functional, we develop an observability optimal control problem for a general nonlinear system and discuss solution methods. Two planar navigation problems for a nonholonomic vehicle in uniform currents are presented, and we derive trajectories that maximize the observability of the system and currents using limited inertial sensors. Simulation results show that the optimal trajectories yield faster estimator convergence and lower steady-state covariances than suboptimal trajectories.

I. INTRODUCTION

Knowledge of the local flow field is vital to navigation for autonomous underwater and micro air vehicles where the flow velocity magnitude may be significant when compared to the vehicle velocity. However, these vehicles often have limited sensor sets due to cost or practicality considerations. For example, underwater vehicles may operate without complete inertial position measurements due to the cumbersome requirement of an underwater localization system.

If all inertial information can be measured, then uniform flow field magnitudes and rate of change can be determined using linear Kalman filtering [1], [2]. With limited sensing, however, the flow vector field is not directly observable in the linear sense, but it can be constructed using the nonlinear motion of the vehicle. Some trajectories provide better information about the environment than others, which prompts the formulation of an optimal control problem to maximize the information obtained. This concept of deriving trajectories to maximize the amount or quality of information is commonly referred to in the literature as adaptive sampling. Adaptive sampling utilizes the time history of measurements to actively plan (or re-plan) vehicle paths to obtain a better estimate of some scalar or vector field. One approach to this problem is designing a feedback law using gradient climbing [3] where multiple vehicles are used to construct the gradient of a field and move in the steepest ascent (descent) direction to find a local maximum (minimum). Another common theme is to derive paths through numerical optimization that minimize either the uncertainty of measurements [4] or the covariances of a particular estimation routine [5]. Previous work by the authors of this paper have used the eigenvalues of the observability gramian in the cost functional to derive analytical trajectories that optimize system observability characteristics [6], [7]. Other researchers have utilized the numerical approach of the empirical observability gramian [8] to optimize observability of spatially varying flow fields using multiple vehicles [9] and to optimize sensor placement on an airfoil for flow identification [10].

In this paper, we utilize the condition number of the observability gramian as a cost functional in the optimization problem. The benefit of the observability gramian is that it is a system property and does not rely on any particular estimation scheme. Trajectories that optimize the observability gramian produce the most well-conditioned estimation problem, thus they inherently optimize the performance of any estimation algorithm. The contributions of this paper are two-fold. First, the observability optimal control problem is presented for a general nonlinear, control-affine system and the process of solving the problem is discussed. Second, two motivating navigation problems are solved to demonstrate the usefulness of this concept. The first example problem is modeled after an unmanned aerial vehicle (UAV) navigating in uniform winds with only inertial position information and no heading information. This problem builds upon previous work estimating wind fields using an unscented Kalman filter [2]. Here, the kinematic ambiguity due to unknown vehicle and wind heading is explicitly shown in the observability analysis, and no a priori knowledge of the initial vehicle heading is assumed. The second example problem is an autonomous underwater vehicle (AUV) navigating in uniform currents with inertial heading measurements and range to a single beacon. This problem builds upon the work of [11], but here we show that the observability-optimal trajectories provide faster convergence and lower estimate covariances than any other trajectory.

II. OBSERVABILITY OPTIMIZATION PROBLEM

We consider a nonlinear control affine system of the form

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)u_i, \quad y = h(x), \quad (1)$$

where $x \in \mathbb{R}^n$ are the states, and $y \in \mathbb{R}^p$ are the measurements. The first step in this optimization process is to analyze the nonlinear observability characteristics of (1), which can be accomplished using differential geometry concepts. The basics of this method are reviewed here, and the reader is referred to [12] for a more detailed account. First, some useful nomenclature is defined for the following...
analysis. The Lie derivative of the output function, \( h(x) \), with respect to a vector field, \( f_i(x) \in \mathbb{R}^n \), is defined as

\[
L_{f_i}h = \frac{\partial h}{\partial x} f_i.
\]

Repeated Lie derivatives are subsequently computed by

\[
L_{f_i}h = \frac{\partial}{\partial x}(L_{f_i}h) + L_{\frac{\partial}{\partial x} f_i} h.
\]

Mixed Lie derivatives are computed by

\[
L_{\frac{\partial}{\partial x} f_i} h = \frac{\partial}{\partial x}(L_{f_i} h) + L_{\frac{\partial}{\partial x} f_i} h.
\]

Now, define the observability Lie algebra, \( \mathcal{O} \), which is the Lie algebra of the output function, \( h \), the drift vector field, \( f_0 \), and the control vector fields, \( f_i \):

\[
\mathcal{O} = \text{span}\{L_{X_1}, L_{X_2}, \ldots, L_{X_k}, h\} \quad k = 0, 1, 2, \ldots
\]

where \( X_i \in \{f_0, f_1, \ldots, f_m\} \), for \( i \in \{1, 2, \ldots, k\} \). If the Jacobian of the observability Lie algebra, \( d\mathcal{O} \), is full rank at some state, \( x_0 \), then system (1) is said to be locally observable at \( x_0 \).

Note that the terms in \( \mathcal{O} \) include Lie derivatives with respect to the drift vector field and with respect to the control vector fields. The effect of the control vector fields in the observability test for nonlinear systems is counter-intuitive to what we know from linear systems, where studying the uncontrolled system is sufficient to analyze observability. For nonlinear systems, however, the actuation and sensing can be coupled, which means that they must be analyzed simultaneously. This coupling of sensing and control in some nonlinear systems is precisely what motivates the concept of observability optimization. The observability Lie algebra tells us what, if any, control vector fields are required for full state observability. It does not, however, tell us what form of controls are “best” to achieve observability.

In order to derive controls that maximize system observability, we must turn to another analysis technique that provides a quantitative measure of observability. Previous work by Batista, et al. [11] and Krener and Ide [8] have used the observability gramian of nonlinear systems, linearized about an admissible trajectory, to determine the observability of the nonlinear system. To illustrate this method, consider the dynamics of system (1), linearized about a nominal trajectory \((x^0(t), u^0(t))\), given by

\[
\begin{align*}
\dot{x}(t) &= A(t, x^0(t), u^0(t))\dot{x}(t) + B(t)\ddot{u}(t) \\
\ddot{y}(t) &= C(t, x^0(t))\dot{x}(t),
\end{align*}
\]

where \( x^{0}(t) \) and \( u^{0}(t) \) are the nominal trajectory and corresponding controls about which the linearization is performed, \( x(t) = x(t) - x^{0}(t) \) and \( u(t) = u(t) - u^{0}(t) \) are the deviations from the nominal trajectory, and \( \ddot{y}(t) \) is the measurement deviation. The observability of linear time-varying systems can be determined by calculating the rank of the observability gramian,

\[
P(t_0, t_f) = \int_{t_0}^{t_f} \Phi^T(t, t_0)C^T(t, t_0)C(t, t_0)\Phi(t, t_0)dt,
\]

where \( \Phi(t, t_0) \in \mathbb{R}^{n \times n} \) is the state transition matrix defined by \( \Phi(t, t_0) = A(t, x^0(t), u^0(t))\Phi(t, t_0) \). System (4) is globally observable, and the corresponding nonlinear system is locally observable about \((x^0(t), u^0(t))\) if \( P(t, t_0) \) has rank \( n \). The benefit of the observability gramian is that the condition number of \( P(t, t_0) \) is a direct measurement of how well-conditioned the estimation problem is, which is denoted the estimation condition number in [8]. Also, the inverse of the minimum singular value of \( P(t, t_0) \) indicates how close to singular the matrix is, which is denoted the unobservability index in [8]. Therefore, if the estimation condition number is minimized or the unobservability index is maximized, then the estimation problem will be well-conditioned, and the states can be estimated more quickly and with lower covariances. These two parameters are used here as performance indices to select an optimal trajectory to maximize the observability.

The observability optimization problem is then given by

\[
\min_{x^0(t), u^0(t)} \kappa(P(T))
\]

subject to

\[
\begin{align*}
\dot{x}(t) &= f_0(x) + \sum_{i=1}^{m} f_i(x)u^0_i(t) \\
\Phi &= A(t, x^0(t), u^0(t))\Phi(t) \\
P &= \Phi^T(t)C^T(t, x^0(t))C(t, x^0(t))\Phi(t) \\
x^0(0) &= x_0, \\
\Phi(0) &= I, \quad P(0) = 0,
\end{align*}
\]

where \( \kappa(P(T)) \) is the condition number of \( P \) evaluated over a finite time horizon, \( T \). Alternatively, \( T \) could be used as an additional optimization variable, and a minimum time trajectory could be designed, where the estimation condition number is constrained to be less than some constant, \( \kappa(P(T)) \leq \alpha \). For a general observability gramian, \( P \), problem (6) is difficult to solve because \( \kappa(P) \) is a non-differentiable, non-convex function. Some cases do exist, however, when this problem can be readily solved. One such case is when \( (1) \) is low-dimensional (\( n \leq 3 \)) and analytically integrable. In this case, the eigenvalues of \( P(T) \) can be found analytically, and necessary conditions for a minimum can be derived by finding points where the eigenvalues are stationary. Second, if \( P(T) \) can be found analytically and is affine in parameters that describe the trajectory, then \( \kappa(P(T)) \) is a convex function of these parameters, and the global minimum can be solved for numerically [13]. In all other cases, a global minimum to (6) will be difficult to find, but insight gained from the observability Lie algebra may be used to seed a numerical method with “good” trajectories.

### III. Application to UAV Navigation

This application of observability optimization is inspired by the first unmanned flight across the Atlantic by the Aerosonde [14]. The Aerosonde was able to cross the Atlantic with limited sensor information and no communication with a ground station for the majority of the flight. An on-board GPS unit was used to measure inertial position and velocity during the flight and a pitot probe measured airspeed, however, no compass was installed due to payload restrictions. The standard method of wind calculation through vector subtraction, \( \mathbf{V}_w = \mathbf{V}_g - \mathbf{V}_a \), was therefore not possible due to unknown heading information. The Aerosonde instead
used ingenuitive flight path planning to sample the wind at different headings. The aircraft made 90-degree turns in flight to obtain ground speed measurements at different headings, which allowed the wind vector to be triangulated as depicted in Fig. 1. Note that if the planned trajectory is straight, constant speed path, then the measured ground velocity vectors will be identical at every point, therefore the wind velocity vector cannot be determined using straight flight paths. Interestingly, the turning motions used by the Aerosonde can be derived using nonlinear observability theory, and the observability optimization procedure can be used to derive the optimal path planning strategy to maximize the observability in the system.

A. Aerosonde Optimization

The Aerosonde can be modeled as a constant speed unicycle in a uniform flow field, with dynamics given by

$$\dot{x} = \begin{bmatrix} V \cos x_3 + x_4 \\ V \sin x_3 + x_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (7)$$

where $x = \begin{bmatrix} x_E \\ y_N \\ \theta \\ W_x \\ W_y \end{bmatrix}^T \in \mathbb{R}^5$, and $u = \omega \in \mathbb{R}$. The states $x_E$ and $y_N$ are the inertial East and North position of the vehicle, $\theta$ is the inertial orientation, $W_x$ and $W_y$ are the East and North wind velocities, $V$ is the vehicle flow-relative velocity, and the control input, $u$, is the vehicle angular velocity.

**Proposition 1:** System (7) is observable with nonzero heading control, and unobservable without heading control.

**Proof:** For (7), the observability Lie algebra is

$$dO = \frac{\partial}{\partial x} \begin{bmatrix} h \\ L_{i,t} \cdot h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -V \sin x_3 & 1 & 0 \\ 0 & 0 & V \cos x_3 & 0 & 1 \\ 0 & 0 & -V \cos x_3 & 0 & 0 \\ 0 & 0 & -V \sin x_3 & 0 & 0 \end{bmatrix}, \quad (8)$$

which is full rank for any value of $x_3$. Therefore the system is everywhere locally observable if controls in the $[f_0, f_1]$ direction are utilized. Note that higher-order Lie derivatives with respect to the drift vector field, $L_{i,t}^k h$, are zero for $k > 1$. Therefore, if zero control input is used, then the observability Lie algebra consists of only the first four rows of (8), which drops the rank of $dO$ to four.

This finding agrees with the geometric arguments shown in Fig. 1, where a straight path (i.e. no control input) yields dependent velocity measurements and an unobservable wind vector, but a curved path (i.e. nonzero control input) allows the wind vector to be observed. Control inputs corresponding to $[f_0, f_1]$ can be formed an infinitely many number of ways. The only requirement is that nonzero control input is present.

Knowing that turning motions will improve the observability of (7), we can proceed to derive an optimal trajectory. Let $u^0(t) = \omega(t)$ be an arbitrary control input sequence. Then (7) linearized about this trajectory gives

$$A(x^0(t)) = \begin{bmatrix} 0 & 0 & -V \sin(x_3^0(t)) & 1 & 0 \\ 0 & 0 & V \cos(x_3^0(t)) & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

$$C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

Since for every $\tau$ and $t$, $A(t)$ and $\int_0^t A(\sigma)d\sigma$ commute, the state transition matrix can be written as

$$\Phi(t) = \begin{bmatrix} 1 & 0 & -V \int_0^t \sin(x_3^0(\sigma))d\sigma & t & 0 \\ 0 & 1 & V \int_0^t \cos(x_3^0(\sigma))d\sigma & 0 & t \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

Although we cannot integrate $\Phi(t)$ for arbitrary $x_3^0(\sigma)$, note that the 1-3 entry of $\Phi(t)$ is simply equal to the negative of the North component of the dead-reckoning trajectory in the wind-free frame, which we will denote $z_2$. Similarly, $\Phi_{2,3}(t)$ is equal to the East component of the trajectory, denoted $z_1$.

Now the observability gramian can be assembled as

$$P(T) = \int_0^T \begin{bmatrix} 1 & 0 & -z_2 & t & 0 \\ 0 & 1 & z_1 & 0 & t \\ t & 0 & -t z_2 & t^2 & 0 \\ 0 & t & t z_1 & 0 & t^2 \end{bmatrix} dt, \quad (11)$$

where the time dependence of $z(t)$ is dropped for brevity. Each of the entries in $P(T)$ represent interesting geometric properties of the trajectory in the wind-free frame. Specifically, $\int_0^T z_i(t)dt = T\bar{z}_i$, where $\bar{z}_i$, $i \in \{1, 2\}$, is the trajectory centroid in the $z_i$ direction, $\int_0^T z_1(t)^2 + z_2(t)^2dt = I_z$, which is the polar inertia of the trajectory, and $\int_0^T t z_j(t)dt = S_{z_j}$, where $S_{z_j}$, $j \in \{1, 2\} \neq i$, is the first moment of the trajectory with mass $t$ about the $z_j$ axis. The observability
gramian is then
\[
P(T) = \begin{bmatrix}
T & 0 & -T \bar{z}_2 & \frac{1}{2} T^2 & 0 \\
0 & T & T \bar{z}_1 & 0 & \frac{1}{2} T^2 \\
-\frac{1}{2} T^2 & 0 & -S_{z1} & -\frac{1}{2} T^3 & S_{z2} \\
0 & -\frac{1}{2} T^2 & S_{z2} & 0 & \frac{1}{2} T^3 \\
\end{bmatrix}
dt. \quad (12)
\]

The trajectory geometric properties in (12) cannot all be independently specified. However, note that wind vector magnitude and direction can be determined through simple vector subtraction if all inertial information is known, which leads to the assumption that if all inertial information can be optimally observed, then wind speed and direction are also optimally observed.

**Assumption 1:** Trajectories that maximize the observability of the inertial position and heading also maximize the observability of the wind speed and direction.

Under this assumption, only the first three rows and columns of (12) need to be considered, which we denote \( P_{1:3} \). Since \( P_{1:3} \) is symmetric and at least positive semidefinite, the eigenvalues are equal to the singular values which are given by
\[
\lambda = \begin{bmatrix}
\frac{I_z + T}{2} + \frac{1}{2} \sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2} \\
\frac{I_z + T}{2} - \frac{1}{2} \sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2} \\
\end{bmatrix}.
\quad (13)
\]

This set of eigenvalues has two possible minimum eigenvalues: \( T \) and \( \frac{I_z + T}{2} - \frac{1}{2} \sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2} \). Since \( T \) will always increase with the estimation time horizon, only the second possibility is of interest. The minimum eigenvalue will achieve a maximum when it is stationary with respect to the trajectory variables, \( \bar{z}_1 \), \( \bar{z}_2 \), and \( I_z \). Calculating the derivatives with respect to each variable gives:
\[
\lambda_{\text{min}} = \begin{bmatrix}
\frac{\partial}{\partial \bar{z}_1}\lambda^* \\
\frac{\partial}{\partial \bar{z}_2}\lambda^* \\
\frac{\partial}{\partial I_z}\lambda^* \\
\end{bmatrix} = \begin{bmatrix}
\bar{z}_1 \\
\bar{z}_2 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{\bar{z}_1}{\sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2}} \\
\frac{\bar{z}_2}{\sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2}} \\
\frac{1}{2} \frac{\sqrt{T^2 - 2 I_z T + T^2 + 4 \bar{z}_1^2 + 4 \bar{z}_2^2}}{I_z} \\
\end{bmatrix}
\quad (14)
\]

The critical point of (13) where the derivative vanishes occurs when \( \bar{z}_1 = \bar{z}_2 = 0 \). Therefore, the trajectory that maximizes the minimum singular value of \( P_{1:3} \) is symmetric about the starting point of the trajectory in the \((z_1, z_2)\) plane. Substituting \( \bar{z}_1 = \bar{z}_2 = 0 \) into (13) yields the optimal eigenvalues
\[
\lambda^* = \begin{bmatrix} T & T & I_z \end{bmatrix}^T.
\quad (15)
\]

Since \( I_z \) is a free parameter in the problem, it can be selected to minimize the condition number of \( P_{1:3} \), which yields \( I_z = T \). We now have the geometric characteristics that define the optimal trajectory, and the optimal control inputs can be derived.

**Theorem 1 (Optimal Trajectory):** The control input:
\[
u^*(t) = \begin{cases}
\frac{4 \pi}{T} & t < T/2 \\
-\frac{4 \pi}{T} & t \geq T/2 \\
\end{cases}
\quad (16)
\]

will produce a trajectory that maximizes the observability of wind speed and direction for (7).

**Proof:** The control inputs yield a trajectory in the \((z_1, z_2)\) plane:
\[
z_1^*(t) = \frac{V T}{4 \pi} \sin \left( \frac{4 \pi t}{T} \right) \quad t < T/2
\]
\[
z_2^*(t) = \frac{V T}{4 \pi} \left( 1 - \cos \left( \frac{4 \pi t}{T} \right) \right) \quad t \geq T/2,
\]

which has a centroid of \( \bar{z}_1^* = \bar{z}_2^* = 0 \). Therefore it is a stationary point of (13), and the minimum eigenvalue of \( P_{1:3} \) is maximized. Furthermore, if the estimation time horizon is selected to be \( T = \frac{2 \sqrt{2} \pi}{V} \), then the trajectory polar moment of inertia is given by \( I_z^* = \int_0^T (z_1^*(t))^2 + (z_2^*(t))^2 = 2 \sqrt{2} \pi \), and the optimal eigenvalues (15) are all equal to \( \frac{2 \sqrt{2} \pi}{V} \). Therefore, the condition number of \( P_{1:3} \) is one, and the heading angle can be computed with arbitrary accuracy. Finally, under Assumption 1, the wind speed and direction observability are maximized.

The optimal trajectory takes the form of a “figure-8” in the \((z_1, z_2)\) plane, as depicted in Fig. 2. If the estimation time horizon is selected as \( T = \frac{2 \sqrt{2} \pi}{V} \), then the optimal control magnitude is \( \omega = \sqrt{2} V \), which yields a turning radius of \( r = \frac{1}{\sqrt{2}} \). If this value of \( \omega \) is greater than the maximum allowable control input, then the optimal input consists of bang-bang controls (switching between maximum and minimum magnitudes). The bang-bang control scheme still maximizes the minimum singular value of \( P_{1:3} \), and the condition number of \( P_{1:3} \) is equal to \( \frac{8 \pi^2}{V^2} \) for \( \frac{4 \pi}{T} = \omega \leq \sqrt{2} V \), which approaches 1 as \( T \) approaches \( \frac{2 \sqrt{2} \pi}{V} \).

**B. Properties of the Optimal Trajectory**

The figure-8 trajectory just derived has some interesting properties, including time optimality and invariance under transformations. Due to length limitations, only a sketch of the proofs of these properties will be provided and the reader is referred to [6] for more detailed and rigorous proofs.

**Theorem 2 (Time Optimality):** The figure-8 trajectory is a time-optimal solution to the estimation condition number minimization problem for (7).

**Proof:** Recall from the formulation of the wind observability optimization problem that the following conditions must hold for a trajectory to minimize the estimation...

Preprint submitted to 2013 American Control Conference.
Received September 17, 2012.
condition number:
\[
0 = \int_0^T z_1(t) \, dt \\
0 = \int_0^T z_2(t) \, dt \\
T = \int_0^T (z_1(t)^2 + z_2(t)^2) \, dt.
\]
(17)

To find a trajectory that satisfies these conditions in minimum time, an optimal control problem is formulated:

\[
\min_{z(t), u(t), T} T \\
\text{subject to} \\
\dot{z} = \begin{bmatrix} V \cos z_3 \\ V \sin z_3 \\ u \\ z_1 \\ z_2 \\ z_1^2 + z_2^2 \end{bmatrix}, \\
z(0) = 0 \\
\Psi(z(T), T) = \begin{bmatrix} z_4(T) \\ z_5(T) \\ z_6(T) - T \end{bmatrix} = 0 \\
|u(t)| \leq \bar{u},
\]

where 
\[
z = [z_1 \ z_2 \ \theta \ \bar{z}_1 \ \bar{z}_2 \ \bar{I}_z]^T
\]
and \(\bar{u} = V/\rho_{\text{min}}\) is the maximum turn rate for a minimum turn radius of \(\rho_{\text{min}}\). The first-order necessary conditions for a minimum are found using Pontryagin’s Minimum Principle [15]. Assuming the control input given by (16), the dynamics are integrated and the first-order necessary conditions are satisfied when \(T^* = 4\pi/\bar{u}\) and \(\bar{u} = \sqrt{2}V\). Thus, the figure-8 described by (16) is the minimum time solution to the estimation condition number problem.

One readily notices that the optimal trajectory \(z : [0, T] \to \mathbb{R}^2\) defined by (16) remains optimal under certain operations, such as multiplying it by \(-1\), effectively reversing the orientation. In fact, the optimality conditions defined by (17) are invariant under planar rotation and reflection, of which reversal of orientation is a special case.

Theorem 3 (Rotational Invariance): Let \(z : [0, T] \to \mathbb{R}^2\) be an optimal trajectory satisfying the conditions given in (17) and let \(R : \mathbb{R}^2 \to \mathbb{R}^2\) be defined by
\[
R := \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}.
\]

Then the trajectory \(\zeta(t) := Rz(t)\) is optimal.

Proof: Since the trajectory is centered at the origin, rotational transformation does not affect the computation of \((\bar{z}_1, \bar{z}_2)\). Likewise, the polar inertia is invariant to rotational change of coordinates. Therefore any rotation of the optimal figure-8 trajectory is also optimal.

Theorem 4 (Reflection Invariance): Consider the optimal trajectory \(z(t)\) as above, and let
\[
A := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
which represents a reflection across the horizontal axis in the plane. Then \(\zeta(t) := Az(t)\) is optimal.

Proof: By similar geometric arguments, \(\zeta\) and \(I_z\) are unchanged under reflection. Since a reflection about any line through \((0, 0)\) may be represented by the composition of \(A\) and \(R(\phi)\), the result follows.

Hence, we see that the orientation of the planar curve does not affect the optimality conditions, and any initial heading angle will produce an optimal figure-8 trajectory.

IV. APPLICATION TO AUV NAVIGATION

The second example problem is inspired by underwater vehicle navigation using range to a single beacon. This problem is relevant to AUVs due to the requirement for an underwater localization system to precisely track points of interest. It is not always practical to have multiple underwater range measurements as in [16], particularly for long-range operations. A single beacon can be used instead, either through an anchored platform or a moving vessel. In this case, turning motions must be used to obtain an inertial position estimate, as was shown in [11]. The coupling between sensing and control again prompts the formulation of an observability optimization problem, whose solution will yield a trajectory that gives the best information about the environment.

A. Observability Optimization

The AUV can be modeled similarly to the UAV problem through constant speed unicycle dynamics in a uniform flow field. In this case, however, since the heading angle is directly measured, we can remove the heading angle state and rewrite the system with two constrained control inputs:

\[
x = \begin{bmatrix} x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_2
\]
\(y = x_1^2 + x_2^2
\)
\(u = \begin{bmatrix} V \cos \theta \\ V \sin \theta \end{bmatrix}
\]

where \(x = [x_E \ y_N \ W_x \ W_y]^T\) are the states, \(u_1\) is the East flow-relative velocity, and \(u_2\) is the North flow-relative velocity. The nonholonomic velocity constraint is now accounted for explicitly in two control inputs, rather than through the state equations.

Proposition 2: System (19) is observable using nonzero heading control and is unobservable otherwise.

Proof: The observability Lie algebra for (19) is

\[
dO = \frac{\partial}{\partial x} \begin{bmatrix} h \\ L_{\bar{b}} h \\ L_{\bar{g}} h \end{bmatrix} = \begin{bmatrix} 2x_1 & 2x_2 & 0 & 0 \\ 0 & 0 & 4x_3 & 4x_4 \\ 2 & 0 & 0 & 0 \end{bmatrix},
\]

which is full rank when the vehicle is not coincident with the beacon (i.e. \(x_1 = 0\) and \(x_2 = 0\)), so the system is almost everywhere locally observable. Note that higher-order Lie derivatives with respect to the drift, \(L_{\bar{b}} h\), are identically
zero for \( k > 2 \). Therefore, cyclic inputs in the \( f_1 \) (or \( f_2 \)) direction are required to obtain full state observability. Since the controls are constrained, cyclic inputs in either \( f_1 \) or \( f_2 \) correspond to nonzero heading rate input, and the system is observable if and only if nonzero heading control is utilized.

This finding agrees with that in [11], where sinusoidal heading input was used for source localization.

With knowledge of the controls necessary for observability, (19) can be linearized about an arbitrary trajectory. Let \((x_1^0(t), x_2^0(t))\) be the nominal trajectory. Then the linearized dynamics are given by

\[
A(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \tag{21}
\]

\[
C(x^0(t)) = \begin{bmatrix}
2x_1^0(t) & 2x_2^0(t) & 0 & 0
\end{bmatrix}.
\]

Since \( A \) is a constant matrix, the state transition matrix is

\[
\Phi(t) = \exp[At] = \begin{bmatrix}
1 & 0 & t & 0 \\
0 & 1 & 0 & t \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \tag{22}
\]

Computing the observability gramian gives

\[
P(T) = 4 \int_0^T \begin{bmatrix}
x_1^2 & x_1 x_2 & t x_1^2 & t x_1 x_2 \\
x_2^2 & x_2 x_1 & t x_2^2 & t x_2 x_1 \\
x_1 x_2 & t x_1 x_2 & t^2 x_1^2 & t^2 x_1 x_2 \\
x_2 x_1 & t x_2 x_1 & t^2 x_2^2 & t^2 x_2 x_1
\end{bmatrix} dt, \tag{23}
\]

where the explicit dependence on time is dropped for brevity. Similar to the first example, the elements of the observability gramian are interesting geometric properties of the trajectory. \( P(T) \) is represented by a block structure of inertia matrices

\[
P(T) = \begin{bmatrix}
[I] & [I]_t \\
[I]_t & [I]_t^2
\end{bmatrix}, \tag{24}
\]

where

\[
[I] = \begin{bmatrix}
x_{x_2 x_2} & x_{x_1 x_2} \\
x_{x_1 x_2} & x_{x_1 x_1}
\end{bmatrix}, \tag{25}
\]

is the inertia matrix of the trajectory of uniform unit mass, \([I]_t\) is the trajectory inertia matrix with mass distribution \( t \), and \([I]_t^2\) is the trajectory inertia matrix with mass distribution \( t^2 \).

**Remark 1:** These inertia matrices are the adjoint of the typical inertia matrix used in mechanics.

The inertia parameters of the trajectory are not independent, but we will again assume that we desire a trajectory that maximizes the observability of inertial information, namely, Assumption 1. Therefore, we only need to consider the first block of (24), which is given by (25). The eigenvalues of (25) are the two principle moments of inertia of the trajectory. Therefore, a trajectory with equal principle inertias will yield an estimation condition number of one.

**Theorem 5 (Optimal AUV Trajectory):** A trajectory of the form:

\[
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} = \begin{bmatrix}
\rho \cos(\omega t) \\
\rho \sin(\omega t)
\end{bmatrix}, \quad t \in [0, 2\pi/\omega]
\]

will minimize the estimation condition number of the uniform currents estimation problem (19).

**Proof:** The trajectory gives the observability gramian

\[
P(2\pi/\omega) = \begin{bmatrix}
\pi \rho^2 / \omega & 0 \\
0 & \pi \rho^2 / \omega
\end{bmatrix}
\]

which has a condition number of one, and maximizes the observability of the inertial position given range measurements to a single beacon. Therefore, under Assumption 1, the observability of the inertial position given range measurements is optimized.

The optimal trajectory is a somewhat unintuitive result. A circular trajectory centered at the ranging beacon will yield constant measurements, which would seem to provide poor information about the vehicle position. However, with knowledge of the circular trajectory, the vehicle position can be uniquely determined with the radius of the circle (range measurement) and the vehicle heading. Unlike the first example, this path planning scheme requires trajectory following in inertial coordinates, not the wind-free coordinates. A feedback control law must be implemented to actively sense and compensate for the currents.

**B. Feedback Control for Orbit Following in Currents**

The control law implemented here is based upon that introduced by Paley [17] for multiple vehicle orbit coordination in constant uniform currents. The control law is presented briefly here for completeness, and the reader is referred to [17] for more information. In this control scheme, we model the vehicle in the complex plane, with \( r = x_w + iy_w \in \mathbb{C} \) defining the inertial position relative to the flow axes, where the \( x_w \) coordinate is aligned with the flow direction, \( \theta \) defining the inertial orientation relative to the flow axes, and \( V_w = \beta V, |\beta| < 1 \), defining the flow velocity magnitude, as shown in Fig. 3. The dynamics of this system are

\[
\begin{bmatrix}
\dot{r} \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
V(e^{i\theta} + \beta) \\
\beta V
\end{bmatrix}, \tag{26}
\]

where \( u \) is the heading control input. The angle \( \gamma \) between the vehicle inertial velocity and \( x_w \)-axis is defined by

\[
\gamma = \tan^{-1}\left(\frac{\sin \theta}{\beta + \cos \theta}\right), \tag{27}
\]

and the magnitude of the vehicle inertial velocity is

\[
v = V \left( \beta \cos \gamma + \sqrt{1 - \beta^2 \sin^2 \gamma} \right). \tag{28}
\]

Now define, \( \nu = \dot{\gamma} = (1 - \beta \nu^{-1} \cos \gamma) u \), as a new control variable which allows us to rewrite (26) as

\[
\begin{bmatrix}
\dot{r} \\
\dot{\gamma}
\end{bmatrix} = \begin{bmatrix}
ev e^{i\gamma} \\
u
\end{bmatrix}. \tag{29}
\]

Then a controller that will asymptotically converge to a circular orbit about point \( c_{des} \in \mathbb{C} \) with radius \( \rho > 0 \) is given by [17, Corollary 1]:

\[
\nu = \frac{V}{\rho} \left( v + K \left( e^{i\gamma}, c - c_{des} \right) \right), \tag{30}
\]
where \( K > 0 \) is a chosen gain, \( c = r + (i \rho e^{i \gamma})/(V |e^{i \gamma}|) \) is the center of the actual circle being traced by the vehicle, and \( \langle \cdot \rangle \) denotes the standard inner product on \( \mathbb{C} \) given by \((x, y) = \text{Re}\{xy\}\). The controller (30) can be used in feedback with the estimated vehicle position and orientation relative to the currents frame and the estimated currents magnitude and direction.

V. SIMULATION RESULTS

Numerical simulations were run to demonstrate the effectiveness of the optimal trajectories derived in Sections III and IV in conditioning the estimation problems. Since both systems considered here are unobservable when linearized about an equilibrium point, a nonlinear estimator is used to construct an estimate of the flow vector field. Here, an unscented Kalman filter (UKF) is used in simulation to estimate flow speed and direction. The UKF is implemented in discrete time form in MATLAB according to the formulation presented in [18].

A. UAV Simulation Results

Three simulation cases were examined to quantify the usefulness of the optimal trajectory. Case IA uses a straight-line trajectory, which results in an unobservable system. Case IIA uses a cyclic control input of \( u(t) = 4\pi/T \sin(4\pi t/T) \), which results in an observable system that is not the optimal trajectory. Case IIIA uses the figure-8 control input defined by (30), the AUV simulation is run using \( \beta = 2\pi/\sqrt{2} \). In all cases, the vehicle has unit velocity, the true wind vector is \( V_w = [0.5, -0.3]^T \), and the initial wind estimate is set to \( \dot{V}_w(0) = [0, 0]^T \).

Simulation results of these three cases are shown in Fig. 4. Notice that in case IA the filter does not converge to the correct wind values, as expected, since the wind velocity vector is unobservable with straight paths. In cases IIA and IIIA, the filter converges because the vehicle makes turning motions to allow the wind vector to be triangulated. Note, however, that when the optimal trajectory is used in case III, the filter converges faster than the suboptimal case II. If we define the convergence time as \( t_c = \min\{t|\dot{x}_i(\tau) - \hat{x}_i(\tau) \leq 0.02|\dot{x}_i(\tau)|, \forall \tau \geq t, i \in \{4, 5\}\} \) (i.e. the 2% settling time of the estimate), then the estimate for the optimal input has a 54% faster convergence time than the suboptimal case.

B. AUV Simulation Results

Due to the need for flow direction in the feedback control defined by (30), the AUV simulation is run using

\[
\begin{bmatrix}
\dot{x}_E \\
\dot{y}_N \\
\dot{\theta}
\end{bmatrix} = \begin{bmatrix}
V \cos \theta + V_w \cos \psi_w \\
V \sin \theta + V_w \sin \psi_w \\
u
\end{bmatrix},
\]

where \( V_w \) and \( \psi_w \) are the unknown constant currents magnitude and direction. The control input \( u \) is computed by \( u = \nu/(1 - \beta v^{-1} \cos \gamma) \), where \( \nu \) is given by (30) with \( \beta = V_w|V|, r = (\dot{x}_E \cos \psi_w + \dot{y}_N \sin \psi_w) + i(-\dot{x}_E \sin \psi_w + \dot{y}_N \cos \psi_w) \), and \( \theta = \theta - \psi_w \), where \( \cdot \) denotes the estimated state. The desired orbit center \( c_{des} \) is set to zero to coincide with the beacon and the gain is set to \( K = 0.5 \). A UKF is used to estimate the system states in three simulation cases. Case IB uses zero control input, which results in a straight path and an unobservable system. Case IIB utilizes a feedforward sinusoidal control input defined by \( u(t) = (V/\rho) \sin(\pi t/\rho) \) with \( \rho = 100 \) m. Case IIIB utilizes the feedback law just described with \( \rho = 100 \).
m. In all cases, the vehicle is moving with flow-relative velocity $V = 5$ m/s with a flow velocity of $V_w = 1$ m/s at $\psi_w = 45$ deg, the initial condition of the system is $\dot{x}(0) = [100 \ 0 \ \pi/6 \ 1 \ \pi/4]^T$ and the initial estimate is $\dot{\hat{x}}(0) = [100 \ 0 \ \pi/6 \ 0 \ 0]^T$.

Simulation results for cases IB – IIIB are shown in Fig. 5. As expected, in case IB the estimator does not converge to the correct values for the flow magnitude and direction since the system is not observable with no control input. In case IIIB, the filter is able to successfully converge to the true current vector field due to the turning motions utilized, however, the estimation convergence is rather slow when compared to case IIIB. In the final case IIIB, the optimal feedback trajectory gives the best performance, where the convergence time is 90% faster and the $\psi_w$ 3σ-bounds are 78% lower than the suboptimal case IIIB.

VI. CONCLUSION

In this paper we presented a method for optimizing the observability of nonlinear under-sensed systems using the condition number of the observability gramian as a cost metric. This procedure was applied to two example problems modeled after realistic navigation problems.

The first problem addressed UAV navigation in uniform winds without inertial heading measurements. Observability analysis showed that turning motions are required to make the system observable. An optimal trajectory in the form of a figure-8 was derived and was proven to minimize the estimation condition number in minimum time. Simulation results confirmed the solution, showing that the optimal trajectory yielded faster estimator convergence time.

A second example problem was modeled after an AUV navigating in uniform currents with a single beacon. Observability analysis again showed that turning motions are required for observability, and the optimal trajectory was proven to be a circle in the inertial frame, centered at the beacon. A feedback controller was implemented in simulation to track circular trajectories using estimated currents. Simulation results confirmed the optimality of the circular trajectory, with significant improvements in estimator convergence time and steady-state covariances.

REFERENCES