PROBLEMS IN LOW-DIMENSIONAL TOPOLOGY

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Introduction

Four-dimensional topology is in an unsettled state: a great deal is known, but the largest-scale patterns and basic unifying themes are not yet clear. Kirby has recently completed a massive review of low-dimensional problems [Kirby], and many of the results assembled there are complicated and incomplete. In this paper the focus is on a shorter list of "tool" questions, whose solution could unify and clarify the situation. However we warn that these formulations are implicitly biased toward positive solutions. In other dimensions tool questions are often directly settled one way or the other, and even a negative solution leads to a general conclusion (eg. surgery obstructions, Whitehead torsion, characteristic classes, etc). In contrast, failures in dimension four tend to be indirect inferences, and study of the failure leads nowhere. For instance the failure of the disk embedding conjecture in the smooth category was inferred from Donaldson's nonexistence theorems for smooth manifolds. And although some direct information about disks is now available, eg. [Kr], it does not particularly illuminate the situation.

Topics discussed are: in section 1, embeddings of 2-disks and 2-spheres needed for surgery and s-cobordisms of 4-manifolds. Section 2 describes uniqueness questions for these, arising from the study of isotopies. Section 3 concerns handlebody structures on 4-manifolds. Finally section 4 poses a triangulation problem for certain low-dimensional stratified spaces.

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1: 2-disks and spheres in 4-manifolds

The target results here are surgery and the s-cobordism theorem. In general these are reduced, via handlebody theory, to questions about disks and spheres in the middle dimension of the ambient manifold. The tool results, hence the targets, are known in the topological category for 4-manifolds when the fundamental group is "small", [FQ, FT1], but are unsettled in general.

Two n-dimensional submanifolds of a manifold of dimension $2n$ will usually intersect themselves and each other in isolated points. The "Whitney trick" uses an isotopy across an embedded 2-disk to simplify these intersections. Roughly speaking this reduces the study of n-dimensional embeddings to embeddings of 2-disks. But this is not a reduction when the dimension is 4: the 2-disks themselves are middle-dimensional, so trying to embed them encounters exactly the same problems they are supposed to solve. This is the phenomenon that separates dimension 4 from others. The central conjecture is that some embeddings exist in spite of this problem.

1.1 Disk conjecture. *Suppose A is an immersion of a 2-disk into a 4-manifold, boundary going to boundary, and there is a framed immersed 2-sphere B with trivial algebraic selfintersection and algebraic intersection 1 with A. Then there is an embedded 2-disk with the same framed boundary as A*

If this were true then the whole apparatus of high-dimensional topology would apply in dimension 4. There are very interesting generalizations, which for example ask about the minimal genus of an embedded surface with a given boundary, or in a given homology class (cf. [Kirby, Problem 4.36]). However the data in 1.1 is available in the Whitney disk applications, so its inclusion reflects the "tool" orientation of this paper.

The conjecture is very false for smooth embeddings, since it would imply existence and uniqueness results that are known to be false [Kirby Problems 4.1, 4.6]. It may be true for topological (locally flat) embeddings. The current best results are by Freedman and Teichner [FT1, FT2]. In [FT1] they show that the conjecture as stated holds if the fundamental group of the 4-manifold has "subexponential growth," while [FT2] gives a technical but useful statement about embeddings when the 4-manifold changes slightly. We briefly discuss the proofs.

For surfaces in 4-manifolds here is a correspondence between intersections and fundamental group of the image: adding an intersection point enlarges the fundamental group of the image by one free generator (if the image is connected). Freedman's work roughly gives a converse: in order to remove intersections in M, it is sufficient to kill the image of the fundamental group of the data, in the fundamental group of M. More precisely, if we add the hypothesis that $A \cap B$ is a single point, and π_1 of the image $A \cup B$ is trivial in $\pi_1 M$ then there is an embedded disk. However applications of this depend on the technology for reducing images in fundamental groups. Freedman's earlier work showed (essentially) how to change A and B so the fundamental group image becomes trivial under any $\phi\colon \pi_1 M \to G$, where G is poly-(finite or cyclic). [FT] improves this to allow G of subexponential growth. Quite a lot of effort is required for this rather minute advance, giving the impression that we are near the limits of validity of the theorem. In a nutshell, the new ingredient is the use of (Milnor) link homotopy. Reduction of fundamental group images is achieved by trading an intersection with a nontrivial loop for a great many intersections with trivial, or at least smaller, loops. The delicate point is to avoid reintroducing big loops through unwanted intersections. The earlier argument uses explicit moves. The approach in [FT1] uses a more efficient abstract existence theorem. The key is to think of a collection of disks as a nullhomotopy of a link. Selfintersections are harmless, while intersections between different components are deadly. Thus the nullhomotopies needed are exactly the ones studied by Milnor, and existence of the desired disks can be established using link homotopy invariants.

While the conjecture is expected to be false for arbitrary fundamental groups, no proof is in sight. Constructing an invariant to detect failure is a very delicate limit problem. The fundamental group of the image of the data can be compressed into arbitrarily far-out terms in the lower central series of the fundamental group of M. If it could be pushed into the intersection the general conjecture would follow. (This is because it is sufficient to prove the conjecture for M with free fundamental group, eg. a regular neighborhood of the data, and the intersection of the lower central series of a free group is trivial). One approach is to develop a notion of nesting of data so that the intersection of an infinite nest gives something useful. Then in order for the theorem to fail there must be data with no properly nested subdata, and maybe this can be detected.

There is a modification of the conjecture, in which we allow the ambient manifold to change by s-cobordism. This form implies that "surgery" works, but not the s-cobordism theorem. [FQ, 6] shows that if the fundamental group of the image of the data of 1.1 is trivial in the whole manifold, then there is an embedding up to s-cobordism. This differs from the hypothesis of the version above in that $A \cap B$ is not required to be one point, just algebraically 1. The improvement of [FT2] is roughly that infinitesimal holes are allowed in the data. A regular neighborhood of the data gives a 4-manifold with boundary, and carrying certain homology classes. In the regular neighborhood the homology class is represented by a sphere, since a sphere is given in the data. The improvement relaxes this: the homology class is required to be in a certain subgroup of H_2, but not necessarily in the image of π_2. Heuristically we can drill a hole in the sphere, as long as it is small enough not to move it too far out of π_2 (technically, still in the ω term of Dwyer's filtration on H_2).

The improved version has applications, but again falls short of the full conjecture. Again it is a limit problem: we can start with arbitrary data and drill very small holes to get the image π_1 trivial in M. The holes can be made "small" enough that the resulting homology classes are in an arbitrarily far-out term in the Dwyer filtration, but maybe not in the infinite intersection.

There is still room for hope that this form of the conjecture is true, but it may require a more elaborate construction or another infinite process. A "shell game" approach would begin with arbitrary data, introduce some $S^2 \times S^2$ summands, and use them as gently as possible to represent the original data as a π_1-trivial submanifold with homology in Dwyer's ω term. The $S^2 \times S^2$'s are now messed up, and to repair this we want to represent them also with π_1-trivial submanifolds with ω-filtration homology. The new advantage is that the data is no longer random, given by an abstract existence theorem, but is obtained from an embedding by carefully controlled damage done in the first step. An infinite swindle would involve introducing infinitely many copies of $S^2 \times S^2$ and moving the damage down the line. The objective would be to do this with control on sizes, so the construction will converge in an appropriate sense (see [BFMW]). The limit should be an ANR homology 4-manifold, but this can be resolved to regain a topological manifold [Q1].

2: Uniqueness

The uniqueness question we want to address is: when are two homeomorphisms of a 4-manifold topologically isotopic? This is known for compact 1-connected 4-manifolds [Q2], but not for nontrivial groups even in the good class for surgery.

Neither is there a controlled version, not even in the 1-connected case. The controlled version may be more important than general fundamental groups, since it is the main missing ingredient in a general topological isotopy extension theorem for stratified sets [Q4].

The study of isotopies is approached in two steps. First determine if two homeomorphisms are concordant (pseudoisotopic), then see if the concordance is an isotopy. The first step still works for 4-manifolds, since it uses 5-dimensional surgery. The high-dimensional approach to the second step [HW] reduces it to a "tool" question. However the uniqueness tool question is *not* simply the uniqueness analog of the existence question. In applications Conjecture 1.1 would be used to find Whitney disks to manipulate 2-spheres. The tool question needed to analyse isotopies directly concerns these Whitney disks.

Conjecture 2.1. *Suppose A and B, are framed embedded families of 2-spheres, and V, W are two sets of Whitney disks for eliminating AB intersections. Each set of Whitney disks reduces the intersections to make the families transverse: the spheres in A and B are paired, and the only intersections are a single point between each pair. Then the sets V, W equivalent up to isotopy and disjoint replacement.*

"Isotopic" means there is an ambient isotopy that preserves the spheres A, B setwise, and takes one set of disks to the other. Note that $A \cap B$ must be pointwise fixed under such an isotopy. "Disjoint replacement" means we declare two sets to be equivalent if the only intersections are the endpoints (in $A \cap B$). Actually there are further restrictions on framings and π_2 homotopy classes, related to Hatcher's secondary pseudoisotopy obstruction [HW]. In practice these do not bother us because the work is done in a relative setting that encodes a vanishing of the high-dimensional obstruction: we try to show that a 4-dimensional concordance is an isotopy if and only if the product with a disk is an isotopy. In [Q2] this program is reduced to conjecture 2.1. The conjecture itself is proved for simply connected manifolds and A, B each a single sphere.

Consider the boundary arcs of the disks V and W, on A and B. These fit together to form circles and arcs: each intersection point in $A \cap B$ is an endpoint of exactly one arc in each of $V \cap A$ and $W \cap A$ unless it is one of the special intersections left at the end of one of the deformations. Thus there is exactly one arc on each sphere. The proof of [Q1] works on the arcs. Focus on a single pair of spheres. The 1-connectedness is used to merge the circles into the arc. Intersections among Whitney disks strung out along the arc are then "pushed off the end" of the arc. This makes the two sets of disks equivalent in the sense of 2.1, and allows them to be cancelled from the picture. Finitely many pairs can be cancelled by iterating this, but this cannot be done with control since each cancellation will greatly rearrange the remaining spheres. To get either nontrivial fundamental groups or control will require dealing directly with the circles of Whitney arcs.

3: 4-DIMENSIONAL HANDLEBODIES

Handlebody structures on 4-manifolds correspond exactly to smooth structures. The targets in studying handlebody structures are therefore the detection and manipulation of smooth structures. However these are much more complicated than in other dimensions, and we are not yet in a position to identify tool questions that might unravel them. Consequently the questions in this section suggest useful directions rather than specific problems.

The first problem concerns detection of structures. The Donaldson and Seiberg-Witten invariants are defined using global differential geometry. But since a handlebody structure determines a smooth structure, these invariants are somehow encoded in the handle structure. There can be no direct topological understanding of these structures until we learn to decode this.

3.1: Problem. *Find a combinatorially-defined topological quantum field theory that detects exotic smooth structures.*

Three-dimensional combinatorial field theories were pioneered by Reshetikhin and Turaev [RT]. They attracted a lot of attention for a time but have not yet led to anything really substantial. Four-dimensional attempts have not gotten anywhere, cf. [CKY]. The Donaldson and Seiberg-Witten invariants do not satisfy the full set of axioms currently used to define a "topological quantum field theory", so there is no guarantee that working in this framework will ever lead anywhere. Nonetheless this is currently our best hope, and a careful exploration of it will probably be necessary before we can see something better.

4-dimensional handlebodies are described by their attaching maps, embeddings of circles and 2-spheres in 3-manifolds. The dimension is low enough to draw explicit pictures of many of these. Kirby developed notations and a "calculus" of such pictures for 1- and 2-handles, cf. [HKK]. This approach has been used to analyse specific manifolds; a good example is Gompf's identification of some homotopy spheres as standard [Gf]. However this approach has been limited even in the study of examples because:

(1) it only effectively tracks 1- and 2-handles, and Gompf's example shows one cannot afford to ignore 3-handles;
(2) it is a non-algorithmic "art form" that can hide mistakes from even skilled practitioners; and
(3) there is no clue how the pictures relate to effective (eg. Donaldson and Seiberg-Witten) invariants.

The most interesting possibility for manipulating handlebodies is suggested by the work of Poenaru on the 3-dimensional Poincaré conjecture. The following is suggested as a test problem to develop the technique:

3.2 Conjecture. *A 4-dimensional (smooth) s-cobordism without 1-handles is a product.*

Settling this would be an important advance, but a lot of work remains before it would have profound applications. To some extent it would show that the real problem is getting rid of 1-handles ([Kirby Problems 4.18, 4.88, 4.89]). It might have some application to this: if we can arrange that some subset of the 2-handles together with the 1-handles forms an s-cobordism, then the dual handlebody structure has no 1-handles and the conjecture would apply. Replacing these 1- and 2-handles with a product structure gives a new handlebody without 1-handles. The problem encountered here is control of the fundamental group of the boundary above the 2-handles. The classical manipulations produce a homology s-cobordism (with $\mathbb{Z}[\pi_1]$ coefficients), but to get a genuine s-cobordism we need for the new boundary to have the same π_1. Thus to make progress we would have to understand the relationship between things like Seiberg-Witten invariants and restrictions on fundamental groups of boundaries of sub-handlebodies.

To analyse the conjecture consider the level between the 2- and 3-handles in the s-cobordism. The attaching maps for the 3-handles are 2-spheres, and the dual spheres of the 2-handles are circles. The usual manipulations arrange the algebraic intersection matrix between these to be the identity. In other dimensions the next step is to realize this geometrically: find an isotopy of the circles so each has exactly one point of intersection with the family of spheres. But the usual methods fail miserably in this dimension. V. Poenaru has attacked this problem in the special case of $\Delta \times I$, where Δ is a homotopy 3-ball, [P, Gi]. The rough idea is an infinite process in which one repeatedly introduces new cancelling pairs of 2- and 3-handles, then damages these in order to fix the previous ones. The limit has an infinite collections of circles and spheres with good intersections. Unfortunately this limit is a real mess topologically, in terms of things converging to each other. The goal is to see that, by being incredibly clever and careful, one can arrange the spheres to converge to a singular lamination with control on the fundamental groups of the complementary components. As an outline this makes a lot of sense. Unfortunately Poenaru's manuscript is extremely long and complicated, and as a result of many years of work without feedback from the rest of the mathematical community, is quite idiosyncratic. It would probably take years of effort to extract clues from this on how to deal with the difficult parts.

4: Stratified spaces

A class of stratified spaces with a relatively weak relationship between the strata has emerged as the proper setting for purely topological stratified questions, see eg. [Q3, W]. The analysis of these sets, to obtain results like isotopy extension theorems, uses a great deal of handlebody theory, etc., so often requires the assumption that all strata have dimension 5 or greater. This restriction is acceptable in some applications, for example in group actions, but not in others like smooth singularity theory, algebraic varieties, and limit problems in differential geometry. The suggestion here is that many of the low-dimensional issues can be reduced to (much easier) PL and differential topology. The conjecture, as formulated, is a tool question for applications of stratified sets. After the statement we discuss it's dissection into topological tool questions.

4.1: Conjecture. *A three-dimensional homotopically stratified space with manifold strata is triangulable. A 4-dimensional space of this type is triangulable in the complement of a discrete set of points.*

As stated this implies the 3-dimensional Poincaré conjecture. To avoid this assume either that there are no fake 3-balls below a certain diameter, or change the statement to "obtained from a polyhedron by replacing sequences of balls converging to the 2-skeleton by fake 3-balls." The "Hauptvermutung" for 3-dimensional polyhedra [Papa] asserts that homeomorphisms are isotopic to PL homeomorphisms. This reduces the 3-dimensional version to showing that stratified spaces are locally triangulable. The 2-skeleton and its complement are both triangulable, so the problem concerns how the 3-dimensional part approaches neighborhoods of points in the 2-skeleton. Consider a manifold point in the skeleton; a neighborhood in the skeleton is isomorphic with $\mathbb{R}^n$ for $n = 0, 1$, or 2. Near this the 3-stratum looks locally homotopically like a fibration over $\mathbb{R}^n$ with fiber a Poincaré space of dimension $3 - n - 1$. We can reduce to the case where the fiber is connected by considering components of the 3-stratum one at a time. If $n = 2$ then the fiber is a point, and the union of the two strata is a homology 3-manifold with $\mathbb{R}^2$ as boundary. Thus the question: is this union a manifold, or equivalently, is the $\mathbb{R}^2$ collared in the union? If $n = 1$ then the fiber is S^1, and the union gives an arc homotopically tamely embedded in the interior of a homology 3-manifold. Is it locally flat? Finally if $n = 0$ then the fiber is a surface (2-dimensional Poincaré spaces are surfaces, [EL]). This is an end problem: if a 3-manifold has a tame end homotopic to $S \times \mathbb{R}$, S a surface, is the end collared? Answers to these are probably known. The next step is to consider a point in the closure of strata of three different dimensions. There are three cases: $(0, 1, 3)$, $(0, 2, 3)$ and $(1, 2, 3)$. Again each case can be described quite explicitly, and should either be known or accessible to standard 3-manifold techniques.

Now consider 4-dimensional spaces. 4-manifolds are triangulable in the complement of a discrete set, so again the question concerns neighborhoods of the 3-skeleton. In dimension 4 homeomorphism generally does not imply PL isomorphism, so this does not immediately reduce to a local question. However the objective is to construct bundle-like structures in a neighborhood of the skeleton, and homeomorphism of total spaces of bundles in most cases will imply isomorphism of bundles. So the question might be localized in this way, or just approached globally using relative versions of the local questions. As above we start with manifold points in the skeleton. If the point has a 2- or 3-disk neighborhood then the question reduces to local flatness of boundaries or 2-manifolds in a homology 4-manifold, see [Q2, FQ 9.3A]. If the point has a 1-disk neighborhood then a neighborhood looks homotopically like the mapping cylinder of a surface bundle over $\mathbb{R}$. This leads to the question: is it homeomorphic to such a mapping cylinder? If the surface fundamental group has subexponential growth then this probably can be settled by current techniques, but the general case may have to wait on solution of the conjectures of section 1. Finally neighborhoods of isolated points in the skeleton correspond exactly to tame ends of 4-manifolds. Some of these are known not to be triangulable, so these would have to be among the points that the statement allows to be deleted. From here the analysis progresses to points in the closure of strata of three or four different dimensions. Again there are a small number of cases, each of which has a detailed local homotopical description.

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