JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 1999
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Hahn-Banach Theorem

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Summary. We prove a version of Hahn-Banach Theorem.

MML Identifier: `HAHNBAN`.

WWW: `http://mizar.org/JFM/Vol5/hahnban.html`

The articles [12], [18], [5], [6], [8], [19], [2], [3], [16], [15], [14], [9], [4], [10], [7], [17], [13], [11], and [1] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For all sets x, y and for every function f such that $\langle x, y\rangle \in f$ holds $y \in \operatorname{rng} f$.

(2) For every set X and for all functions f, g such that $X \subseteq \operatorname{dom} f$ and $f \subseteq g$ holds $f{\restriction}X = g{\restriction}X$.

(3) For every non empty set A and for every set b such that $A \neq \{b\}$ there exists an element a of A such that $a \neq b$.

(4) For all sets X, Y holds every non empty subset of $X \dot{\rightarrow} Y$ is a non empty functional set.

(5) Let B be a non empty functional set and f be a function. Suppose $f = \bigcup B$. Then $\operatorname{dom} f = \bigcup\{\operatorname{dom} g : g$ ranges over elements of $B\}$ and $\operatorname{rng} f = \bigcup\{\operatorname{rng} g : g$ ranges over elements of $B\}$.

(6) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $r \leq -\infty$ holds $A = \{-\infty\}$.

(7) For every non empty subset A of $\overline{\mathbb{R}}$ such that for every *Real number* r such that $r \in A$ holds $+\infty \leq r$ holds $A = \{+\infty\}$.

(8) Let A be a non empty subset of $\overline{\mathbb{R}}$ and r be a *Real number*. If $r < \sup A$, then there exists a *Real number* s such that $s \in A$ and $r < s$.

(9) Let A be a non empty subset of $\overline{\mathbb{R}}$ and r be a *Real number*. If $\inf A < r$, then there exists a *Real number* s such that $s \in A$ and $s < r$.

(10) Let A, B be non empty subsets of $\overline{\mathbb{R}}$. Suppose that for all *Real numbers* r, s such that $r \in A$ and $s \in B$ holds $r \leq s$. Then $\sup A \leq \inf B$.

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(12)[1] Let x, y be real numbers and x', y' be *Real numbers*. If $x = x'$ and $y = y'$, then $x \leq y$ iff $x' \leq y'$.

2. Sets Linearly Ordered by the Inclusion

Let I_1 be a set. We say that I_1 is $\subseteq$-linear if and only if:

(Def. 1) For all sets x, y such that $x \in I_1$ and $y \in I_1$ holds $x \subseteq y$ or $y \subseteq x$.

Let A be a non empty set. Note that there exists a subset of A which is $\subseteq$-linear and non empty.

We now state the proposition

(13) For all sets X, Y and for every $\subseteq$-linear subset B of $X \dot{\rightarrow} Y$ holds $\bigcup B \in X \dot{\rightarrow} Y$.

3. Subspaces of a Real Linear Space

In the sequel V is a real linear space.

Next we state a number of propositions:

(14) For all subspaces W_1, W_2 of V holds the carrier of $W_1 \subseteq$ the carrier of $W_1 + W_2$.

(15) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(16) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(17) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(18) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.

(19) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.

(20) Let W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

(21) Let V_1 be a subspace of V, W_1 be a subspace of V_1, and v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(22) For all subspaces V_1, V_2, W of V and for all subspaces W_1, W_2 of W such that $W_1 = V_1$ and $W_2 = V_2$ holds $W_1 + W_2 = V_1 + V_2$.

(23) For every subspace W of V and for every vector v of V and for every vector w of W such that $v = w$ holds $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(24) Let v be a vector of V and X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(25) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.

[1] The proposition (11) has been removed.

(26) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle w, 0_V \rangle\!\rangle$.

(27) For every vector v of V and for all subspaces W_1, W_2 of V there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2\rangle} = \langle\!\langle v_1, v_2 \rangle\!\rangle$.

(28) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists a real number r such that $w_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle x, r \cdot v \rangle\!\rangle$.

(29) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, x_1, x_2 be vectors of X, and r_1, r_2 be real numbers. If $(w_1)_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle x_1, r_1 \cdot v \rangle\!\rangle$ and $(w_2)_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle x_2, r_2 \cdot v \rangle\!\rangle$, then $(w_1 + w_2)_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle\!\rangle$.

(30) Let v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, x be a vector of X, and t, r be real numbers. If $w_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle x, r \cdot v \rangle\!\rangle$, then $(t \cdot w)_{\langle W, \mathrm{Lin}(\{y\})\rangle} = \langle\!\langle t \cdot x, t \cdot r \cdot v \rangle\!\rangle$.

4. Functionals

Let V be an RLS structure.

(Def. 2) A function from the carrier of V into $\mathbb{R}$ is said to be a functional in V.

Let us consider V and let I_1 be a functional in V. We say that I_1 is subadditive if and only if:

(Def. 3) For all vectors x, y of V holds $I_1(x + y) \leq I_1(x) + I_1(y)$.

We say that I_1 is additive if and only if:

(Def. 4) For all vectors x, y of V holds $I_1(x + y) = I_1(x) + I_1(y)$.

We say that I_1 is homogeneous if and only if:

(Def. 5) For every vector x of V and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

We say that I_1 is positively homogeneous if and only if:

(Def. 6) For every vector x of V and for every real number r such that $r > 0$ holds $I_1(r \cdot x) = r \cdot I_1(x)$.

We say that I_1 is semi-homogeneous if and only if:

(Def. 7) For every vector x of V and for every real number r such that $r \geq 0$ holds $I_1(r \cdot x) = r \cdot I_1(x)$.

We say that I_1 is absolutely homogeneous if and only if:

(Def. 8) For every vector x of V and for every real number r holds $I_1(r \cdot x) = |r| \cdot I_1(x)$.

We say that I_1 is 0-preserving if and only if:

(Def. 9) $I_1(0_V) = 0$.

Let us consider V. One can check the following observations:

* every functional in V which is additive is also subadditive,
* every functional in V which is homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also positively homogeneous,
* every functional in V which is semi-homogeneous is also 0-preserving,
* every functional in V which is absolutely homogeneous is also semi-homogeneous, and
* every functional in V which is 0-preserving and positively homogeneous is also semi-homogeneous.

Let us consider V. Note that there exists a functional in V which is additive, absolutely homogeneous, and homogeneous.

Let us consider V. A Banach functional in V is a subadditive positively homogeneous functional in V. A linear functional in V is an additive homogeneous functional in V.

One can prove the following four propositions:

(31) For every homogeneous functional L in V and for every vector v of V holds $L(-v) = -L(v)$.

(32) For every linear functional L in V and for all vectors v_1, v_2 of V holds $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(33) For every additive functional L in V holds $L(0_V) = 0$.

(34) Let X be a subspace of V, f_1 be a linear functional in X, v be a vector of V, and y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be a real number. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction$the carrier of $X = f_1$ and $p_1(y) = r$.

5. Hahn-Banach Theorem

The following propositions are true:

(35) Let V be a real linear space, X be a subspace of V, q be a Banach functional in V, and f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq q(v)$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$the carrier of $X = f_1$ and for every vector x of V holds $p_1(x) \leq q(x)$.

(36) For every real normed space V holds the norm of V is an absolutely homogeneous subadditive functional in V.

(37) Let V be a real normed space, X be a subspace of V, and f_1 be a linear functional in X. Suppose that for every vector x of X and for every vector v of V such that $x = v$ holds $f_1(x) \leq \|v\|$. Then there exists a linear functional p_1 in V such that $p_1 \restriction$the carrier of $X = f_1$ and for every vector x of V holds $p_1(x) \leq \|x\|$.

References

[1] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/supinf_1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[4] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[5] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[7] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfunct_3.html.

[8] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/anal_1.html.

[9] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/normsp_1.html.

[10] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[11] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/Axiomatics/tarski.html.

[13] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[14] Wojciech A. Trybulec. Operations on subspaces in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlsub_2.html.

[15] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlsub_1.html.

[16] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[17] Wojciech A. Trybulec. Basis of real linear space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rlvect_3.html.

[18] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/boole.html.

[19] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received July 8, 1993

Published May 12, 1999