The Reflection Theorem

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Summary. The goal is show that the reflection theorem holds. The theorem is as usual in the Morse-Kelley theory of classes (MK). That theory works with universal class which consists of all sets and every class is a subclass of it. In this paper (and in another Mizar articles) we work in Tarski-Grothendieck (TG) theory (see [17]) which ensures the existence of sets that have properties like universal class (i.e. this theory is stronger than MK). The sets are introduced in [15] and some concepts of MK are modeled. The concepts are: the class On of all ordinal numbers belonging to the universe, subclasses, transfinite sequences of non-empty elements of universe, etc. The reflection theorem states that if A_{ξ} is an increasing and continuous transfinite sequence of non-empty sets and class $A = \bigcup_{\xi \in On} A_{\xi}$, then for every formula H there is a strictly increasing continuous mapping $F: On \to On$ such that if \varkappa is a critical number of F (i.e. $F(\varkappa) = \varkappa > 0$) and $f \in A_{\varkappa}^{\text{VAR}}$, then $A, f \models H \equiv A_{\varkappa}, f \models H$. The proof is based on [13]. Besides, in the article it is shown that every universal class is a model of ZF set theory if ω (the first infinite ordinal number) belongs to it. Some propositions concerning ordinal numbers and sequences of them are also present.

MML Identifier: ZF_REFLE. WWW: http://mizar.org/JFM/Vol2/zf_refle.html

The articles [17], [19], [16], [2], [20], [11], [12], [18], [4], [6], [5], [7], [1], [14], [10], [15], [3], [8], and [9] provide the notation and terminology for this paper.

In this paper W denotes a universal class, H denotes a ZF-formula, x denotes a set, and X denotes a set.

Next we state several propositions:

- (1) $W \models$ the axiom of extensionality.
- (2) $W \models$ the axiom of pairs.
- (3) $W \models$ the axiom of unions.
- (4) If $\omega \in W$, then $W \models$ the axiom of infinity.
- (5) $W \models$ the axiom of power sets.
- (6) For every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $W \models$ the axiom of substitution for H.
- (7) If $\omega \in W$, then W is a model of ZF.

For simplicity, we follow the rules: E is a non empty set, F is a function, f is a function from VAR into E, A, B, C are ordinal numbers, a, b are ordinals of W, p_1 is a transfinite sequence of ordinals of W, and H is a ZF-formula.

Let us consider A, B. Let us observe that $A \subseteq B$ if and only if:

(Def. 1) For every C such that $C \in A$ holds $C \in B$.

In this article we present several logical schemes. The scheme ALFA concerns a non empty set \mathcal{A} and a binary predicate \mathcal{P} , and states that:

There exists F such that dom $F = \mathcal{A}$ and for every element d of \mathcal{A} there exists A such that A = F(d) and $\mathcal{P}[d, A]$ and for every B such that $\mathcal{P}[d, B]$ holds $A \subseteq B$

provided the following requirement is met:

• For every element d of \mathcal{A} there exists A such that $\mathcal{P}[d, A]$.

The scheme ALFA'Universe deals with a universal class \mathcal{A} , a non empty set \mathcal{B} , and a binary predicate \mathcal{P} , and states that:

There exists F such that

(i) dom $F = \mathcal{B}$, and

(ii) for every element d of \mathcal{B} there exists an ordinal a of \mathcal{A} such that a = F(d)

and $\mathcal{P}[d, a]$ and for every ordinal b of \mathcal{A} such that $\mathcal{P}[d, b]$ holds $a \subseteq b$

provided the following condition is met:

• For every element d of \mathcal{B} there exists an ordinal a of \mathcal{A} such that $\mathcal{P}[d, a]$. Next we state the proposition

(8) x is an ordinal of W iff $x \in On W$.

In the sequel p_2 denotes a sequence of ordinal numbers.

Now we present three schemes. The scheme OrdSeqOfUnivEx deals with a universal class \mathcal{A} and a binary predicate \mathcal{P} , and states that:

There exists a transfinite sequence p_1 of ordinals of \mathcal{A} such that for every ordinal a of \mathcal{A} holds $\mathcal{P}[a, p_1(a)]$

provided the following conditions are satisfied:

- For all ordinals a, b_1, b_2 of \mathcal{A} such that $\mathcal{P}[a, b_1]$ and $\mathcal{P}[a, b_2]$ holds $b_1 = b_2$, and
- For every ordinal a of \mathcal{A} there exists an ordinal b of \mathcal{A} such that $\mathcal{P}[a, b]$.

The scheme UOS Exist deals with a universal class \mathcal{A} , an ordinal \mathcal{B} of \mathcal{A} , a binary functor \mathcal{F} yielding an ordinal of \mathcal{A} , and a binary functor \mathcal{G} yielding an ordinal of \mathcal{A} , and states that:

There exists a transfinite sequence p_1 of ordinals of \mathcal{A} such that

(i) $p_1(\mathbf{0}_{\mathcal{A}}) = \mathcal{B},$

(ii) for all ordinals a, b of \mathcal{A} such that $b = p_1(a)$ holds $p_1(\operatorname{succ} a) = \mathcal{F}(a, b)$, and

(iii) for every ordinal a of \mathcal{A} and for every sequence p_2 of ordinal numbers such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and $p_2 = p_1 \upharpoonright a$ holds $p_1(a) = \mathcal{G}(a, p_2)$

for all values of the parameters.

The scheme Universe Ind deals with a universal class \mathcal{A} and a unary predicate \mathcal{P} , and states that:

For every ordinal a of \mathcal{A} holds $\mathcal{P}[a]$

provided the following requirements are met:

- $\mathcal{P}[\mathbf{0}_{\mathcal{A}}],$
- For every ordinal a of \mathcal{A} such that $\mathcal{P}[a]$ holds $\mathcal{P}[\operatorname{succ} a]$, and
- Let a be an ordinal of \mathcal{A} . Suppose $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and for every ordinal b of \mathcal{A} such that $b \in a$ holds $\mathcal{P}[b]$. Then $\mathcal{P}[a]$.

Let f be a function, let W be a universal class, and let a be an ordinal of W. The functor $\bigcup_a f$ yields a set and is defined as follows:

(Def. 2)
$$\bigcup_a f = \text{Union}(W \upharpoonright (f \upharpoonright \mathbf{R}_a)).$$

We now state several propositions:

- (9) $\bigcup_{a} f = \text{Union}(W \upharpoonright (f \upharpoonright \mathbf{R}_{a})).$
- (10) For every transfinite sequence L and for every A holds $L |\mathbf{R}_A|$ is a transfinite sequence.
- (11) For every sequence L of ordinal numbers and for every A holds $L \upharpoonright \mathbf{R}_A$ is a sequence of ordinal numbers.
- (12) Union p_2 is an ordinal number.
- (13) Union $(X \upharpoonright p_2)$ is an ordinal number.
- (14) $\operatorname{On}(\mathbf{R}_A) = A.$
- (15) $p_2 \upharpoonright \mathbf{R}_A = p_2 \upharpoonright A.$

Let p_1 be a sequence of ordinal numbers, let W be a universal class, and let a be an ordinal of W. Then $\bigcup_a p_1$ is an ordinal of W.

The following proposition is true

- (17)¹ For every transfinite sequence p_1 of ordinals of W holds $\bigcup_a p_1 = \text{Union}(p_1 \restriction a)$ and $\bigcup_a (p_1 \restriction a) = \text{Union}(p_1 \restriction a)$.
 - Let W be a universal class and let a, b be ordinals of W. Then $a \cup b$ is an ordinal of W. Let us consider W. Note that there exists an element of W which is non empty. Let us consider W. A subclass of W is a non empty subset of W. Let F be a function. We say that F is non-empty if and only if:
- $(\text{Def. } 4)^2 \quad \emptyset \notin \operatorname{rng} F.$

Let us consider W and let I_1 be a transfinite sequence of elements of W. We say that I_1 is non empty set yielding if and only if:

(Def. 5) dom
$$I_1 = \operatorname{On} W$$
.

Let us consider W. One can verify that there exists a transfinite sequence of elements of W which is non empty set yielding and non-empty.

Let us consider W. A transfinite sequence of non empty sets from W is a non-empty non empty set yielding transfinite sequence of elements of W.

Next we state the proposition

 $(21)^3$ Every non empty element of W is a subclass of W.

Let us consider W and let L be a transfinite sequence of non empty sets from W. Then Union L is a subclass of W. Let us consider a. Then L(a) is a non empty element of W.

In the sequel L is a transfinite sequence of non empty sets from W and f is a function from VAR into L(a).

We now state several propositions:

- (22) If $X \in W$, then $\overline{\overline{X}} < \overline{\overline{W}}$.
- (23) $a \in \operatorname{dom} L$.
- (24) $L(a) \subseteq \text{Union } L.$
- (25) $\mathbb{N} \approx \text{VAR and } \overline{\overline{\text{VAR}}} = \overline{\mathbb{N}}.$

¹The proposition (16) has been removed.

²The definition (Def. 3) has been removed.

³The propositions (18)-(20) have been removed.

- $(27)^4 \quad \sup X \subseteq \operatorname{succ} \bigcup \operatorname{On} X.$
- (28) If $X \in W$, then $\sup X \in W$.
- (29) Suppose that
- (i) $\omega \in W$,
- (ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$, and
- (iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds L(a) =Union $(L \upharpoonright a)$.

Let given H. Then there exists p_1 such that

- (iv) p_1 is increasing and continuous, and
- (v) for every a such that $p_1(a) = a$ and $\emptyset \neq a$ and for every f holds Union L, $(\text{Union } L)[f] \models H$ iff $L(a), f \models H$.

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⁴The proposition (26) has been removed.

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Received August 10, 1990

Published May 12, 1999