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Rational equivalence of 0-cycles on surfaces

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We will consider in this note 0-cycles on a complete non-singular algebraic surface F over the field $\boldsymbol{C}$ of complex numbers. We will use the language of schemes, and every scheme will be assumed separated and of finite type over $\boldsymbol{C}$. In a very extensive set of papers, Severi set up and investigated the concept of rational equivalence (cf. [2], [3], [4], [5] among many others). It is not however very easy to find a precise definition in Severi's work, and there was a good deal of discussion on this point at the International Congress of 1954. A much more elementary approach was worked out by Chevalley in his seminar "Anneaux de Chow" [1]. For 0-cycles on F, the most elementary definition is this:

Let Σ_n = group of permutations on n letters.

Let $S^nF = F^n/\Sigma_n$, the n^{th} symmetric power of F

$\underset{\text{as set}}{\cong} \{A \mid A$ effective 0-cycle of degree n on $F\}$.

Definition: 2 0-cycles A_1, A_2 of degrees n_1 and n_2 are *rationally equivalent* if $n_1 = n_2$ and $\exists$ a 0-cycle B of degree m such that $A_1 + B$ and $A_2 + B$ are effective, corresponding to points $x_1, x_2 \in S^{n+m}F$, and $\exists$ a morphism $f: \boldsymbol{P}^1 \to S^{n+m}F$ such that $f(0) = x_1$, $f(\infty) = x_2$.

Definiton: $A_0(F)$ = group of all zero-cycles of degree 0 on F modulo rational equivalence.

What is the structure of $A_0(F)$? Consider the following three statements:

(1) $\exists n$ such that $\forall$ 0-cycles A with $\deg(A)\geq n$,
$A \underset{\text{rat}}{\sim} A'$, A' effective.

(2) $\exists n$ such that the natural map
$S^n F \times S^n F \to A_0(F)$
$(A, B) \quad \mapsto$ class of $A-B$
is surjective.

(3) $\exists n$ such that $\forall m$, $A \in S^m F$, $\exists$ a subvariety W: $A \in W \subset S^m F$ of codimension $\leq n$ consisting of points rationally equivalent to A.

It is not hard to show that these are all equivalent to each other* and that, intuitively, they mean simply $A_0(F)$ *is finite-dimensional.* Severi, unfortunately, took it as almost evident that these statements were valid and it is often hard to discover which results in his papers depended on making this assumption at some point. For example in [2], p. 251, §13, he says:

"Ammettiamo che per ogni varieta k-dimensionale (virtuale) A di W_r esista un numero finito di caratteri numerativi $c_1, c_2, \cdots$ tali che le relazioni $c_1>0$, $c_2>0$, $\cdots$ sieno sufficienti per affermare che A e una varieta effettiva. Con questo si vuol dire che nel sistema di equivalenza $|A|$ esittono varieta totali effettive."

The purpose of this paper is to prove that if $p_g>0$, (1)-(3) are *false*. Now, after criticizing Severi like this, I have to admit the following: the method of disproof of (1)-(3) is due entirely to Severi: Severi created, in fact, a very excellent tool for analyzing

* (1)$\Longrightarrow$(2) is obvious. To show (2)$\Longrightarrow$(3), fix a base pt. $x_0 \in F$ and consider

$$Z=\{(A_1, A_2, B) \in S^n F \times S^n F \times S^m F \mid A_1 - A_2 \underset{\text{rat}}{\sim} B - m(x_0)\}.$$

By lemma 3 below, Z is countable union of closed subvarieties of $S^n F \times S^n F \times S^m F$ and by (2) one of its components Z_0 projects *onto* $S^m F$. Then for all $B \in S^m F$, take as W one of the components of $\{B' \in S^m F \mid \exists (A_1, A_2) \in S^n F \times S^n F$ such that $(A_1, A_2, B) \in Z_0$, $(A_1, A_2, B') \in Z_0\}$. To show (3)$\Longrightarrow$(1), show first that (3) implies the existence of n such that $\forall A \in S^n F$, $\forall x \in F$, $\exists A' \in S^{n-1} F$, $A \underset{\text{rat}}{\sim} A' + (x)$.

the influence of regular 2-forms on F on his systems of eqivalence. One must admit that in this case the *technique* of the Italians was superior to their vaunted intuition. This paper is certainly one of the "mémoires d'esploitation" referred to by Severi in the 1954 Congress [5], p. 541, vol. 3, but not I think part of his "prévision correspondante à mes vifs souhaits".

§1. Induced differentials

We want to study the following situation:

X = non-singular variety
G = finite group acting on X
$Y = X/G$, $\pi: X \to Y$ the canonical morphism,
$\omega \in \Gamma(X, \Omega_X^q)$ a q-form invariant under G.

If G is acting freely on X, Y will be non-singular and $\omega = \pi^*(\eta)$ for a unique $\eta \in \Gamma(Y, \Omega_Y^q)$. If G is not acting freely this is false, but the following still is true: for all non-singular varieties S and morphisms $f: S \to Y$, we get a canonical induced 2-form η_f on S. We define this as follows:

Let $\widetilde{S} = (S \times_Y X)_{\mathrm{red}}$:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[f]{} & Y \end{array}$$

Then G acts on $\widetilde{S}$, and $S \cong \widetilde{S}/G$. Let $\tilde{\omega} = \tilde{f}^*(\omega) \in \Gamma(\widetilde{S}, \Omega_{\widetilde{S}}^q)$ (here $\Omega_{\widetilde{S}}^q$ is by definition Λ^q of the Kähler differentials $\Omega_{\widetilde{S}}^1$, which is not of course locally free in general). Then $\tilde{\omega}$ is G-invariant, and I claim there is one and only one $\eta_f \in \Gamma(S, \Omega_S^q)$ such that

$$(*) \qquad p^*(\eta_f) - \tilde{\omega} \text{ is torsion in } \Omega_{\widetilde{S}}^q.$$

In fact, there are non-singular open dense sets $S_0 \subset S$, $\widetilde{S}_0 = p^{-1}(S_0) \subset \widetilde{S}$ such that a suitable quotient G/H is acting freely on $\widetilde{S}_0$. Therefore there is a unique $\eta_f \in \Gamma(S_0, \Omega_S^q)$ such that $p^*(\eta_f) = res_{\widetilde{S}_0}(\tilde{\omega})$. To prove

(*) it suffices to check that this η_f is everywhere regular on S_0, and since Ω_S^q is a locally free sheaf, it suffices to check this at points of codimension 1. Let T be the normalization of $\tilde{S}$. Then the pull-back of $\tilde{\omega}$ is regular on T at all points of codimension 1, hence the pull-back of η_f is also regular on T at all points of codimension 1. But η_f times the order n of G/H is the trace of its pull-back, and since $n \neq 0$ in $\boldsymbol{C}$, η_f is regular at points of codimension 1.

More generally, suppose we have *any* diagram:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[f]{} & Y \end{array}$$

such that $\tilde{S}$ is reduced and p is a dominating morphism. Then η_f satisfies the condition: $p^*\eta_f - \tilde{f}^*\omega$ is torsion, and is characterized by this property. To say more we need the following lemma:

Lemma 1: *Let $h: X \to Y$ be a morphism of reduced schemes. Then $h^*: \Omega_Y^p \to \Omega_X^p$ takes torsion differentials to torsion differentials.*

*Proof:** We reduce this immediately to the case X and Y varieties. By blowing up the closure of $h(X)$ and normalizing, we construct a diagram:

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \text{dominating}\downarrow & & \downarrow \begin{array}{l}\text{birational} \\ \text{(not necessarily proper)}\end{array} \\ X & \longrightarrow & Y \end{array}$$

with Y' non-singular. If ω is a torsion p-form on Y, then ω dies on Y', hence it dies on X'. But if η is the pull-back of ω to X, then η dies in X'. But since the characteristic is 0, the map of meromorphic p-forms

$\Omega^p_{C(X)/C} \longrightarrow \Omega^p_{C(X')/C}$ is injective, so η dies in $\Omega^p_{C(X)/C}$ too, i.e., η is torsion. *QED*

* This proof was given to me by H. Hironaka.

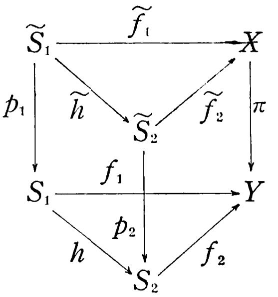
In view of the lemma, η_f has the property that for any diagram:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow[f]{} & Y \end{array}$$

such that $\widetilde{S}$ is reduced, $p^*(\eta_f)-\widetilde{f}^*(\omega)$ is torsion. We can now prove functoriality of η_f in f: suppose we are given

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & Y \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle f_2} & \\ & S_2 & \end{array}$$

with S_1 and S_2 non-singular. Then I claim $h^*(\eta_{f_2})=\eta_{f_1}$. In fact, let $\widetilde{S_1}=(S_1\times_Y X)_{\text{red}}$, $\widetilde{S_2}=(S_2\times_Y X)_{\text{red}}$, and use the diagram:



Since $p_2^*(\eta_{f_2})-\widetilde{f_2}^*(\omega)$ is torsion, it follows that

$$\widetilde{h}^*[p_2^*(\eta_{f_2})-\widetilde{f_2}^*(\omega)]=p_1^*(h^*(\eta_{f_2}))-\widetilde{f_1}^*(\omega)$$

is torsion too. Therefore $h^*(\eta_{f_2})=\eta_{f_1}$.

§2. The main theorem

We apply this machinery now to the case $X=F^n$, $G=\Sigma_n$, $Y=S^nF$. Fix for the rest of this paper a differential $\omega\in\Gamma(F,\Omega_F^2)$. Let $\omega^{(n)}=\Sigma_1^n p_i^*(\omega)\in\Gamma(F^n,\Omega_{F^n}^2)$. Then $\omega^{(n)}$ is G-invariant, and by our definitions above, for all $f:S\rightarrow S^nF$, S a non-singular variety, we obtain canonical, functorial 2-forms $\eta_f\in\Gamma(S,\Omega_S^2)$. We can now give an exact statement of our main theorem:

Theorem (*due essentially to Severi*): *For all* $f: S \to (S^n F)$ *such that all the* 0-*cycles* $f(s)$, $s \in S$, *are rationally equivalent, it follows that* $\eta_f = 0$.

Before proving the theorem, we need 2 lemmas. For the first, suppose we are given 2 morphisms

$$f: S \longrightarrow S^n F$$
$$g: S \longrightarrow S^m F.$$

Then let $f * g$ denote the composition:

$$S \xrightarrow{(f,\, g)} S^n F \times S^m F \xrightarrow{\pi} S^{n+m} F$$

where π is the obvious map.

Lemma 2: $\omega_{f*g} = \omega_f + \omega_g$.

Proof: There exists a reduced scheme $\widetilde{S}$ and a dominating morphism $p: \widetilde{S} \to S$ such that there are diagrams

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{f}} & F^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ S & \xrightarrow[f]{} & S^n F \end{array}, \qquad \begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{g}} & F^m \\ {\scriptstyle p}\downarrow & & \downarrow \\ S & \xrightarrow[g]{} & S^m F \end{array}.$$

Then $p^*(\eta_f) = \widetilde{f}^*(\omega^{(n)}) + \text{torsion}$ and $p^*(\eta_g) = \tilde{g}^*(\omega^{(m)}) + \text{torsion}$. But we get also the diagram:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{(\widetilde{f},\, \tilde{g})} & F^{n+m} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_{n+m}} \\ S & \xrightarrow[f*g]{} & S^{n+m} F \end{array}.$$

If $p_1: F^{n+m} \to F^n$, and $p_2: F^{n+m} \to F^m$ are the projections onto the first n and last m factors respectively, then $\omega^{(n+m)} = p_1^* \omega^{(n)} + p_2^* \omega^{(m)}$, and we calculate:

$$\begin{aligned} p^*(\omega_f + \omega_g) &= \widetilde{f}^*(\omega^{(n)}) + \tilde{g}^*(\omega^{(m)}) + \text{torsion} \\ &= (\widetilde{f}, \tilde{g})^*(p_1^* \omega^{(n)} + p_2^* \omega^{(m)}) + \text{torsion} \end{aligned}$$

$$= (\widetilde{f}, \tilde{g})^* \omega^{(n+m)} + \text{torsion}$$
$$= p^*(\omega_{f*g}) + \text{torsion}. \qquad QED$$

Lemma 3: *$S^nF \times S^nF$ contains a countable set $Z_1, Z_2, \cdots$ of closed subvarieties, such that if $(A, B) \in S^nF \times S^nF$, then*

$$(*) \qquad A \underset{\text{rat}}{\sim} B \Longleftrightarrow (A, B) \in \bigcup_{i=1}^{\infty} Z_i .$$

For each i, there is a reduced scheme W_i and a set of morphisms

$$e_i : W_i \longrightarrow Z_i$$
$$f_i : W_i \longrightarrow S^mF$$
$$g_i : W_i \times \boldsymbol{P}^1 \longrightarrow S^{n+m}F$$

such that we get the equations between 0-cycles:

$$(**) \qquad \begin{aligned} g_i(w, 0) &= p_1(e_i(w)) + f_i(w) \\ g_i(w, \infty) &= p_2(e_i(w)) + f_i(w) \end{aligned}$$

all $w \in W_i$, and e_i is surjective.

Proof: We divide up the pairs $(A, B) \in S^nF \times S^nF$ such that $A \underset{\text{rat}}{\sim} B$ into sets $S_{m,p}$, where

$(A, B) \in S_{m,p} \Longleftrightarrow \exists C \in S^mF$

$\exists$ 1-cycle $\Sigma n_i E_i$ on $S^{m+n}F$ such that

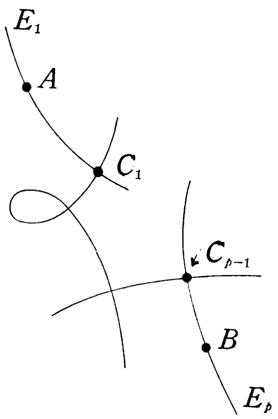
(a) $\bigcup E_i$ connected

(b) E_i normalized is $\boldsymbol{P}^1$

(c) $\Sigma_1^k n_i[\text{degree } (E_i)] = p$

(d) $A + C, B + C \in \bigcup E_i$.

Then it is easy to check using the Chow variety of $S^{m+n}F$ that $S_{m,p}$ is Zariski-closed and we may take the Z_i to be the components of the $S_{m,p}$'s. Next recall that if 2 0-cycles $A, B \in S^kF$ are joined by a chain of p rational curves $E_1, \cdots, E_p$ thus:



then $A+C_1+\cdots+C_{p-1}$ and $C_1+\cdots+C_{p-1}+B$ in $S^{pk}F$ are joined by a single rational curve, whose degree can be bounded by the degree of the E_i's. Therefore, there is an m and a p such that for all $(A, B)\in Z_i$, there exists $C\in S^m F$ and an irreducible rational curve $E\subset S^{m+n}F$ of degree $\leq p$ connecting $A+C$ and $B+C$. Then let

$$\mathrm{Hom}^{\leq p}(\boldsymbol{P}^1, S^{n+m}F)$$

be the scheme of all morphisms $g:\boldsymbol{P}^1\to S^{n+m}F$ such that degree $(g(\boldsymbol{P}^1))\leq p$. Define

$$W\subset Z_i\times S^m F\times \mathrm{Hom}^{\leq p}(\boldsymbol{P}^1, S^{n+m}F)$$
$$W_i=\{((A, B), C, g)\mid g(0)=A+C, g(\infty)=B+C\}.$$

QED

We can now prove the theorem. Given $f:S\to S^n F$ such that all the 0-cycles $f(s)$ are rationally equivalent, fix a base point $A_0\in$ Image (f). It follows from lemma 3 that there is a non-singular variety T, a dominating morphism $e:T\to S$, and morphisms $g:T\to S^m F$, $h:T\times\boldsymbol{P}^1\to S^{n+m}F$ such that:

$$h(t, 0)=g(t)+f(e(t))$$
$$h(t, \infty)=g(t)+A_0$$

all $t\in T$. In other words, if $A_0:T\to S^n F$ is the constant map with image A_0, then

$$h|_{T\times(0)}=g*(f\circ e)$$
$$h|_{T\times(\infty)}=g*A_0\,.$$

By Lemma 2, it follows that

$$\eta_h|_{T\times(0)}=\eta_g+e^*(\eta_f)$$
$$\eta_h|_{T\times(\infty)}=\eta_g+\eta_{A_0}\,.$$

Now η_h is a regular 2-form on $T\times \boldsymbol{P}^1$. Since

$$\Omega^2_{T\times \boldsymbol{P}^1}\cong p_1^*(\Omega^2_T)+p_1^*(\Omega^1_T)\otimes p_2^*(\Omega^1_{\boldsymbol{P}^1})$$

and $\Omega^1_{\boldsymbol{P}^1}$ has no global sections, it follows that $\eta_h=p_1^*(\eta)$ for some $\eta\in\Gamma(T,\Omega^2_T)$. Therefore $\eta_h|_{T\times(0)}=\eta_h|_{T\times(\infty)}$, and since $\eta_{A_0}=0$, we find $e^*(\eta_f)=0$, hence $\eta_f=0$. *QED*

To apply the theorem, let

$$(S^nF)_0=\{A=\Sigma_1^n x_i\,|\,x_1,\cdots,x_n \text{ are distinct, and } \omega(x_i)\in\Lambda^2(m_{x_i}/m_{x_i}^2) \text{ is not } 0\}$$

For all $A\in(S^nF)_0$, $(S^nF)_0$ is non-singular at A, the projection $\pi_n:F^n\to S^nF$ is etale over A, and the 2-form $\omega^{(n)}$ is non-degenerate at all points over A, i.e., defines a non-degenerate skew-symmetric form in the tangent space to F^n. Therefore, there is a 2-form on $(S^nF)_0$:

$$\omega_0^{(n)}\in\Gamma((S^nF)_0,\Omega^2_{S^nF})$$

such that $\pi_n^*(\omega_0^{(n)})=\omega^{(n)}$, and $\omega_0^{(n)}$ is everywhere a non-degenerate 2-form. In particular, the maximal isotropic subspaces of $\omega_0^{(n)}(A)$ have dimension n, i.e., if

$$W\subset T_{A,S^nF}=\text{tangent space to } S^nF \text{ at } A$$

is a subspace such that $\omega_0^{(n)}|_W$ vanishes, then $\dim W\leq n$. But if S is a non-singular subvariety of $(S^nF)_0$, and $i:S\to(S^nF)_0$ is the inclusion morphism, then $\eta_i=\mathrm{res}_S(\omega_0^{(n)})$. We conclude:

Corollary: *Let F be a non-singular surface with $p_g>0$ and let $(S^nF)_0$ be defined as above. Then if $S\subset(S^nF)_0$ is a subvariety consisting of rationally equivalent 0-cycles, it follows that* $\dim S\leq n$.

This disproves (3) of the introduction, hence also (1) and (2).

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