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RATIONAL WHITEHEAD PRODUCTS AND A SPECTRAL SEQUENCE OF QUILLEN, II

Christopher Allday

Introduction. In "Rational Homotopy Theory", Quillen obtained an interesting spectral sequence relating the rational homology of a simply-connected topological space to its rational homotopy ([3], (6.9)). In order to calculate certain differentials in this spectral sequence, we were led, in Part I of this paper ([1]), to attempt to define higher order rational Whitehead products in terms of certain cycles in DG Lie algebras. Our success in [1] was limited. A satisfactory definition was found only for third and fourth order products, and the relation between these products and those defined classically was not shown to be as precise as might have been hoped.

In this paper, however, the difficulties encountered in [1] have been resolved. Given a simply-connected topological space, X, we are able to describe precisely the cycles in $\lambda(X)$, the associated DG Lie algebra defined by Quillen in [3], which represent the classical rational Whitehead products of X of all orders. This is the main result of this paper; it is given in Theorem 3.1. In addition, we give a precise calculation of certain differentials in Quillen's spectral sequence (Theorem 4.1). We use the notations and results of [1] and [3]; but, in other respects, this paper supersedes [1]; and the cycles defining the third and fourth order products here differ slightly from those used in [1].

Since the algebra of this paper can be performed without major variation in a DG Lie algebra, which is 0-reduced, but not necessarily 1-reduced, it is hoped that our results will generalize easily to certain non-simply-connected spaces, when Quillen's theory has been generalized appropriately.

1. The Minimal DG Lie Algebra. In this section we review the effect of applying Sullivan's minimal model construction ([4]) to Quillen's rational homotopy theory ([3]). The results of this section are fairly well-known, although they do not appear to

be in the literature: their proofs, which are surprisingly easy, can be obtained from the author in a preprint ([2]).

For a category B, $(GB)_r$ and $(DGB)_r$ will denote the categories of r-reduced graded and r-reduced differential graded objects over B (as in [3]). Here B will be V (rational vector spaces), L (rational Lie algebras), or C (rational coalgebras). For a model category N, N_c will denote the full subcategory of cofibrant objects, and ho N will denote the homotopy category. Thus $ho(DGL)_1$ is the category of cofibrant (i.e. free) reduced differential graded Lie algebras, and homotopy classes of maps. ([3], Final remark 6.11). ho $\mathcal{T}_2$ will denote the homotopy category of 1-connected CW-spaces localized at 0. s^{-1} (called Ω in [3]) will be the desuspension functor of graded objects, which decreases degrees by 1. For an object C of $(DGC)_2$, $JC = \operatorname{coker}[Q \to C]$. For an object α of $(DGL)_1$, $Q\alpha = \alpha/[\alpha,\alpha]$, the indecomposible quotient of α.

In [3], Quillen constructs equivalences of categories λ: ho $\mathcal{T}_2 \to ho(DGL)_1$, and $\mathcal{C}$: $ho(DGL)_1 \to ho(DGC)_2$, such that, for any 1-connected CW-space X, localized at 0, $H\lambda(X) = \mathcal{L}_*(X)$, the rational homotopy Lie algebra of X, and $H\mathcal{C}\lambda(X) = H_*(X;Q)$, the rational singular homology coalgebra of X. (For B, as above, H: $(DGB)_r \to (GB)_r$ is the homology functor.)

DEFINITION 1.1. A reduced DG Lie algebra, α, will be said to be minimal if

(a) α is free as a graded Lie algebra, and

(b) $d\alpha \subseteq [\alpha,\alpha]$, where d is the differential on α.

For any reduced DG Lie algebra, α, we may construct à la Sullivan [4], a minimal model e_α: $M(\alpha) \to \alpha$, where $M(\alpha)$ is minimal, and e_α is a weak equivalence. Choosing a minimal model for each α, with $M(\alpha) = \alpha$ and e_α = the identity, if α is minimal, then we see that the inclusion $mho(DGL)_1 \to ho(DGL)_1$ is an equivalence of categories, where $mho(DGL)_1$ is the full subcategory of $ho(DGL)_1$ generated by the minimal objects.

DEFINITION 1.2. Given a minimal DG Lie algebra α, and $x \in \alpha$, let $\bar{x}$ denote the image of x under the quotient map h: $\alpha \to Q\alpha$. (i.e. $\bar{x} = h(x)$.) Suppose that $dx = \Sigma_i[x_i,y_i]$, where $\deg x_i = p_i$ and $\deg y_i = q_i$.

Define Δ: $Q\alpha \to Q\alpha \otimes Q\alpha$ by the formula

$$\Delta(\bar{x}) = \Sigma_i(-1)^{p_i+1}\{\bar{x}_i \otimes \bar{y}_i + (-1)^{(p_i+1)(q_i+1)}\bar{y}_i \otimes \bar{x}_i\}.$$

Note that h: $\alpha \to Q\alpha$ is a morphism of $(DGL)_1$, if $Q\alpha$ is regarded as an abelian Lie algebra with zero differential.

The principal result is the following.

THEOREM 1.3. (1) *There is a natural transformation of functors from $(DGL)_{1c}$ to $(DGV)_1$:*

$$T: s^{-1}JC \to Q.$$

(2) *For any object α of $(DGL)_{1c}$, $T(\alpha)$: $s^{-1}JC\alpha \to Q\alpha$ is a weak equivalence.*

(3) *For any homotopy $f \simeq g$: $\alpha \to \beta$ in $(DGL)_{1c}$, $Hs^{-1}JC(f) = Hs^{-1}JC(g)$, and $HQ(f) = HQ(g)$.*

(4) *Using the coproduct Δ of Definition 1.2,*

$$HT : Hs^{-1}JC \to Q$$

is a natural isomorphism of functors from $mho(DGL)_1$ to $(GV)_1$; such that, for any minimal DG Lie algebra α, the following diagram commutes

$$\begin{tikzcd} Hs^{-1}JC(\alpha) \arrow[r, "HT(\alpha)"] \arrow[d] & Q\alpha \arrow[d, "\Delta"] \\ Hs^{-1}JC(\alpha) \otimes Hs^{-1}JC(\alpha) \arrow[r] & Q\alpha \otimes Q\alpha \end{tikzcd},$$

where the lower horizontal arrow is $HT(\alpha) \otimes HT(\alpha)$, and the left hand vertical arrow is induced by the coproduct on $C(\alpha)$.

Being interpreted, Theorem 1.3 says that, given a 1-connected CW-space X, localized at 0, we can recover directly and naturally its rational homology coalgebra from $QM\lambda(X)$, the indecomposible quotient of the minimal model of $\lambda(X)$, via the coproduct Δ of Definition 1.2.

We also have the following.

PROPOSITION 1.4. *Let X be a 1-connected CW-space localized at 0; and let h: $M\lambda(X) \to QM\lambda(X)$ be the quotient map (as above). Then passing to homology h induces a $(GL)_1$-morphism, which composed with $HT(M\lambda(X))^{-1}$ (of Theorem 1.3 (4)), is precisely the rational Hurewicz homomorphism*

$$\mathcal{L}_*(X) \to s^{-1}\tilde{H}_*(X;Q),$$

where $s^{-1}\tilde{H}_*(X;Q)$ *has trivial Lie bracket.*

REMARK. The fact that X is a CW-space localized at 0 is immaterial in the above, since any 1-connected topological space is rationally homotopy equivalent to such a space in a natural way.

2. **The Universal Example**. Let $\{S^{n_i+1} | 1 \leqslant i \leqslant k, n_i \geqslant 1\}$ be a family of spheres. Let $T = T(S^{n_1+1},\ldots,S^{n_k+1})$ be the fat wedge, and let $P = S^{n_1+1} \times \cdots \times S^{n_k+1}$ be the product. Let $N = \Sigma_{i=1}^{k}(n_i + 1)$. Let $\bar{a}_i \in \pi_{n_i+1}(T)$ be the class represented by the inclusion $S^{n_i+1} \to T$. Then P is obtained from T by attaching an N-cell, e^N; and the attaching map, $S^{N-1} \to T$, represents the universal k-th order Whitehead product $\bar{a} = [\bar{a}_1,\ldots,\bar{a}_k] \in \pi_{N-1}(T)$. Localizing at 0, $T \to T_0$, we shall denote by a_i, $1 \leqslant i \leqslant k$, and $a = [a_1,\ldots,a_k]$ the images of $\bar{a}_i$, $1 \leqslant i \leqslant k$, and $\bar{a}$ in $\pi_*(T_0)$.

For any simply-connected topological space, X, $\lambda(X)$ is the DG Lie algebra defined by Quillen ([3]), and $M\lambda(X)$ is a minimal model for $\lambda(X)$ as above. Thus $HM\lambda(X) = \mathcal{L}_*(X)$. Then there are isomorphisms $\pi_{*+1}(X) \otimes Q \to \pi_{*+1}(X_0) \to \mathcal{L}_*(X)$, which are natural with respect to homotopy classes of maps. For $\bar{x} \in \pi_{n+1}(X)$, we shall let x denote $\bar{x} \otimes 1$ and its images in $\pi_{n+1}(X_0)$ and $\mathcal{L}_n(X)$.

Let Π denote the set of all subsets of $\{1,\ldots,k\}$. For any $\mu = \{\mu 1,\ldots,\mu p\} \in \Pi$, with $1 \leqslant \mu 1 < \cdots < \mu p \leqslant k$, and $p < k$, let β_μ denote an element of $M\lambda(T)$, which, under the composition

$$h\colon M\lambda(T) \to QM\lambda(T) \to \tilde{H}_*(T;Q), \text{ (Proposition 1.4)},$$

maps to the inclusion of the top class of $P_\mu = S^{n_{\mu 1}+1} \times \cdots \times S^{n_{\mu p}+1}$. Thus $\beta_\mu \in M\lambda(T)_{\delta(\mu)}$, where $\delta(\mu) = \Sigma_{i=1}^{p}(n_{\mu i} + 1) - 1$. Let

$$\alpha_\mu = -\Sigma(-1)^{\delta(\mu\rho)+\epsilon(\pi)}[\beta_{\mu\rho},\beta_{\mu\sigma}],$$

where the summation runs over all shuffles $\pi = (\rho,\sigma)$ of $\{1,\ldots,p\}$ of type II relative to 1 ([1], page 319, for example), and where $\epsilon(\pi)$ is defined as follows. If (ρ,σ) is an (r,s)-shuffle of $\{1,\ldots,p\}$, then $z_1 \cdots z_p = (-1)^{\epsilon(\pi)} z_{\rho 1} \cdots z_{\rho r} z_{\sigma 1} \cdots z_{\sigma s}$ in the graded rational symmetric algebra generated by $z_1,\ldots,z_p$, with $\deg(z_i) = n_{\mu i} + 1$, for $1 \leqslant i \leqslant p$. Let $\beta_i = \beta_{\{i\}}$, $\alpha_{\{i\}} = 0$, for $1 \leqslant i \leqslant k$.

By Theorem 1.3(4), since $H_*(P_\mu;Q)$ is the product of the coalgebras $H_*(S^{n_{\mu i}+1};Q)$, for $1 \leqslant i \leqslant p$, we have that

$$d\beta_\mu = \alpha_\mu + 0(3),$$

where $0(3) \in M\lambda(T)^{(3)} = [M\lambda(T),[M\lambda(T),M\lambda(T)]]$.

For the purpose of induction, we shall order Π in the following way. For $\mu, \nu \in \Pi$, $\mu = \{\mu 1,...,\mu p\}$, $\nu = \{\nu 1,...,\nu q\}$, we shall say that $\mu < \nu$ if $p < q$. If $p = q$, then we shall say that $\mu < \nu$ if $\delta(\mu) < \delta(\nu)$. If $p = q$, and $\delta(\mu) = \delta(\nu)$, then we shall say that $\mu < \nu$ if $\mu i < \nu i$, where i is the least integer such that $\mu i \neq \nu i$.

PROPOSITION 2.1. *The generators* $\beta_\mu \in M\lambda(T)_{\delta(\mu)}$ *may be chosen so that* $d\beta_\mu = \alpha_\mu$, *for all* $\mu \in \Pi' = \Pi - \{\emptyset, \{1,...,k\}\}$.

PROOF. The result is immediate for small μ. Suppose β_μ have been chosen appropriately for all $\mu < \nu$. Then a straightforward calculation, with careful application of the Jacobi identity, shows that α_ν is a cycle. Thus $d\beta_\nu = \alpha_\nu + \gamma$, where γ is a cycle in $M\lambda(T)^{(3)}$.

Looking at $\mathcal{L}_{\delta(\nu)-1}(T)$, we see that there must exist rational numbers r_i, s_τ such that

$$\gamma = \Sigma_i r_i [\beta_i, \beta_i] + d(\Sigma_\tau s_\tau \beta_\tau) + d\eta,$$

where Σ_i runs over all $i \in \{1,...,k\}$ with $2n_i = \delta(\nu) - 1$, Σ_τ runs over all $\tau \in \Pi$ with $\delta(\tau) = \delta(\nu)$, and $\eta \in M\lambda(T)^{(2)}$. Thus $\Sigma_\tau s_\tau \alpha_\tau = \Sigma_i - r_i[\beta_i,\beta_i] + \xi$, where $\xi \in M\lambda(T)^{(3)}$. Clearly, however, the $[\beta_i,\beta_i]$, if non-zero, and the α_τ are linearly independent. Hence all $r_i = 0$ and all $s_\tau = 0$, and so $\gamma = d\eta$.

Replacing β_ν by $\beta_\nu - \eta$, we have the result.

Now $M\lambda(P)$ may be constructed from $M\lambda(T)$ according to the next proposition.

PROPOSITION 2.2. *$M\lambda(P)$ may be constructed as a direct sum $M\lambda(T) \amalg L(\beta)$, such that*

(i) *the differential restricted to $M\lambda(T)$ is that of $M\lambda(T)$;*

(ii) *$h(\beta)$ is the top class of $\widetilde{H}_*(P;Q)$, where $h\colon M\lambda(P) \to s^{-1}\widetilde{H}_*(P;Q)$ is the natural map; and*

(iii) *$d\beta = \alpha \in M\lambda(T)_{N-2}$, where α represents the universal k-th order Whitehead product $a \in \mathcal{L}_{N-2}(T)$.*

PROOF. It is clear that $M\lambda(P) \cong M\lambda(T) \amalg L(\beta)$, satisfying (i) and (ii), with $d\beta = r\alpha$, for some non-zero rational number r. The fact that $r = 1$ follows from a detailed examination of Quillen's functors and isomorphisms ([3]) applied to the cocartesian

square

$$\begin{array}{ccc} S^{N-1} & \longrightarrow & D^N \\ \downarrow & & \downarrow \\ T & \longrightarrow & P \end{array},$$

where $S^{N-1} \to T$ represents $\bar{a}$.

We may now state and prove the main result of this section.

THEOREM 2.3. *$M\lambda(P)$ may be constructed as $M\lambda(T) \amalg L(\beta)$ such that*

(a) *(i), (ii) and (iii) of Proposition 2.2 are satisfied,*

and

(b) *$d\beta = -\Sigma_{\pi=(\mu,\nu)}(-1)^{\delta(\mu)+\epsilon(\pi)}[\beta_\mu,\beta_\nu]$, where the summation runs over all shuffles of $\{1,\ldots,k\}$ of type II relative to 1, and β_μ, for $\mu \in \Pi'$, is the inclusion of the class given by Proposition 2.1.*

PROOF. Let α' denote the summation of (b), and suppose that β satisfies (a). Again by Theorem 1.3(4), we see that $d\beta = \alpha' + \gamma$, where $\gamma \in M\lambda(P)^{(3)}$, and indeed $\gamma \in M\lambda(T)^{(3)}$. By Proposition 2.1, however, α' is a cycle, and it follows that $\gamma = d\xi$ for some $\xi \in M\lambda(T)^{(2)}$, as in the proof of Proposition 2.1. Replacing β by $\beta - \xi$, and α by $\alpha - d\xi$, we have the result.

3. Rational Whitehead Products. Throughout this section X will be a simply-connected pointed topological space; and we shall retain the special notations of Section 2. Let $x_i \in \mathcal{L}_{n_i}(X)$, $1 \leqslant i \leqslant k$, be rational homotopy classes.

DEFINITION. A subset $\langle x_1,\ldots,x_k\rangle \subseteq \mathcal{L}_{N-2}(X)$ is defined inductively as follows. First choose $\xi_i \in M\lambda(X)_{n_i}$ to represent x_i for $1 \leqslant i \leqslant k$. Now suppose that, for all $\mu \in \Pi'$, it has been possible to choose $\xi_\mu \in M\lambda(X)_{\delta(\mu)}$ such that $d\xi_\mu = -\Sigma_{\pi=(\rho,\sigma)}(-1)^{\delta(\mu\rho)+\epsilon(\pi)}[\xi_{\mu\rho},\xi_{\mu\sigma}]$, where $\mu = \{\mu 1,\ldots,\mu p\}$, $1 \leqslant \mu 1 < \cdots < \mu p \leqslant k$, and the summation runs over all shuffles, $\pi = (\rho,\sigma)$, of $\{1,\ldots,p\}$ of type II relative to 1. Set $\zeta = -\Sigma_{\pi=(\mu,\nu)}(-1)^{\delta(\mu)+\epsilon(\pi)}[\xi_\mu,\xi_\nu]$, the resulting cycle in $M\lambda(X)_{N-2}$, where the summation runs over all shuffles, $\pi = (\mu,\nu)$, of $\{1,\ldots,k\}$ of type II relative to 1.

Now let $\langle x_1,\ldots,x_k\rangle$ be the set of all classes in $\mathcal{L}_{N-2}(X)$, which are represented by elements $\zeta \in M\lambda(X)_{N-2}$, which can be defined in the above way.

Clearly, $\langle x_1,\ldots,x_k\rangle \neq \emptyset$ only if $0 \in \langle x_{\mu 1},\ldots,x_{\mu p}\rangle$, for all μ such that $1 \leqslant$

$\mu 1 < \cdots < \mu p \leqslant k$, and $1 < p < k$. And, for $1 \leqslant i < j \leqslant k$, $\langle x_i, x_j \rangle = \{ (-1)^{n_i+1} [x_i, x_j] \}$. The sign is consistent with the standard Whitehead product, in view of the definition of [,] in $\mathcal{L}_*(X)$. (See [3], page 210.)

The subset $\langle x_1,\ldots,x_k \rangle$ will be called the Whitehead product of $x_1,\ldots,x_k$. The theorem below will demonstrate that this terminology is consistent with the classical definition of the (higher order rational) Whitehead product, which we recall now.

DEFINITION. Let $S = S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ be the (lean) wedge, and let $i: S \to T$ be the inclusion. Let $g: S \to X_0$ be the map, which is uniquely determined up to homotopy, by the $x_i \in \mathcal{L}_{n_i}(X) = \pi_{n_i+1}(X_0)$ for $1 \leqslant i \leqslant k$. Let G be the set of all maps $\hat{g}_0: T_0 \to X_0$, such that $\hat{g}_0 i_0 = g_0: S_0 \to X_0$. Then the (classical higher order rational) Whitehead product of $x_1,\ldots,x_k$ is $[x_1,\ldots,x_k] = \{ \mathcal{L}_*(\hat{g}_0)(a) \in \mathcal{L}_{N-2}(X) | \hat{g}_0 \in G \}$.

THEOREM 3.1. *For any $x_i \in \mathcal{L}_{n_i}(X)$, $1 \leqslant i \leqslant k$,*

$$\langle x_1,\ldots,x_k \rangle = [x_1,\ldots,x_k].$$

PROOF. Let $x = \mathcal{L}_*(\hat{g}_0)(a) \in [x_1,\ldots,x_k]$, and let $\varphi = M\lambda(\hat{g}_0): M\lambda(T) \to M\lambda(X)$. Clearly $\varphi(\beta_\mu)$, for $\mu \in \Pi'$, will satisfy the conditions for ξ_μ in the definition of $\langle x_1,\ldots,x_k \rangle$, yielding $\zeta = \varphi(\alpha)$, by Theorem 2.3. Hence $x \in \langle x_1,\ldots,x_k \rangle$.

Now let $x \in \langle x_1,\ldots,x_k \rangle$ be represented by $\zeta = -\Sigma_{(\mu,\nu)}(-1)^{\delta(\mu)+\epsilon(\pi)}[\xi_\mu,\xi_\nu]$. Define $\varphi: M\lambda(T) \to M\lambda(X)$ by $\varphi(\beta_\mu) = \xi_\mu$ for $\mu \in \Pi'$. φ is well-defined by Proposition 2.1. By the minimal model construction of Section 1, φ gives rise to a homotopy class of maps $[f]: T_0 \to X_0$, such that $fi_0 \simeq g_0$. Since i_0 is a cofibration, there is $f' \in [f]$, such that $f'i_0 = g_0$. $M\lambda(f') \simeq \varphi$, and so $\mathcal{L}_*(f')(a) = H(\varphi)(a) = x$. Thus $x \in [x_1,\ldots,x_k]$.

REMARKS. (1). It follows that $\langle x_1,\ldots,x_k \rangle \neq \emptyset$ if and only if $0 \in \langle x_{\mu 1},\ldots,x_{\mu p} \rangle$ for all $\mu \in \Pi'$ with $p \geqslant 2$.

(2) If π is a permutation of $\{1,\ldots,k\}$, then

$$\langle x_{\pi 1},\ldots,x_{\pi k} \rangle = (-1)^{\epsilon(\pi)} \langle x_1,\ldots,x_k \rangle,$$

as can be seen by direct computation and induction.

4. Differentials in Quillen's Spectral Sequence. For X, as above, let $E^r(X)$ denote Quillen's spectral sequence ([3], (6.9)) with $E^1(X) = S(\Sigma\mathcal{L}_*(X)) \Rightarrow H_*(X;Q)$. With $x_i \in \mathcal{L}_{n_i}(X)$, $1 \leqslant i \leqslant k$, we have the following theorem.

THEOREM 4.1. *If $x \in \langle x_1,\ldots,x_k \rangle$, then $\Sigma x_1 \cdots \Sigma x_k$ survives to $E^{k-1}(X)$; and,*

denoting the resulting element of $E^{k-1}(X)$ *by* $[\Sigma x_1 \cdots \Sigma x_k]^{(k-1)}$, *we have that*

$$d^{k-1}[\Sigma x_1 \cdots \Sigma x_k]^{(k-1)} = -[\Sigma x]^{(k-1)}.$$

PROOF. By [1], Proposition 3.4, there is a non-zero rational number r, dependent only on $n_1,\ldots,n_k$, such that

$$d^{k-1}[\Sigma x_1 \cdots \Sigma x_k]^{(k-1)} = r[\Sigma x]^{(k-1)}.$$

To show that $r = -1$ we may choose X appropriately; that is, we may choose a DG Lie algebra λ to suit our purpose. To this end, let L be the free DG Lie algebra generated by elements $\eta_1,\ldots,\eta_k,\eta_{12},\eta_{123},\ldots,\eta_{12\cdots k-1}$, with

$$d\eta_i = 0,\ 1 \leqslant i \leqslant k,$$

and $d\eta_{12\cdots j} = (-1)^{\delta_{12\cdots j-1}}[\eta_{12\cdots j-1},\eta_j]$, where $\delta_{12\cdots j-1} = \deg(\eta_{12\cdots j-1}) = \Sigma_{i=1}^{j-1}(n_i + 1) - 1$; and $\deg(\eta_i) = n_i$, for $1 \leqslant i \leqslant k$. Let J be the ideal of L generated by $[\eta_i,\eta_j]$, for all $\{i,j\} \neq \{1,2\}$, $[\eta_{12\cdots j},\eta_t]$, for all $t \neq j+1$, $[\eta_{12\cdots i},\eta_{12\cdots j}]$, for all $i \geqslant 2$ and $j \geqslant 2$, and $[L[L,L]]$. Let $\lambda = L/J$, and continue to denote by η_i, $1 \leqslant i \leqslant k$, and $\eta_{12\cdots j}$, $2 \leqslant j \leqslant k-1$, the corresponding images in λ.

$H(\lambda)$ is generated as a rational vector space by x_i, $1 \leqslant i \leqslant k$, and x, where x_i is represented by η_i, $1 \leqslant i \leqslant k$, and x is represented by $\eta = -(-1)^{\delta_{12\cdots k-1}}[\eta_{12\cdots k-1},\eta_k]$. Thus $\{x\} = \langle x_1,\ldots,x_k\rangle$.

Now we need to compute $d(\Sigma\eta_1\Sigma\eta_2 \cdots \Sigma\eta_k)$ in $C\lambda$, in the manner of [1]. Owing to the elementary nature of λ, it follows at once that $d(\Sigma\eta_1\Sigma\eta_2 \cdots \Sigma\eta_k) = -\Sigma\eta$. Thus $r = -1$.

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University of Hawaii
Honolulu, Hawaii 96822

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