GENERATING EQUIVALENCE RELATIONS BY HOMEOMORPHISMS

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Abstract. We give a construction of a single homeomorphism of $2^{\mathbb{N}}$ which generates the equivalence relation E_0. We then consider ways of generating this equivalence relation using homeomorphisms with nicer structural properties, and show that several such properties are impossible for a homeomorphism which generates E_0.

The equivalence relation E_0 plays a fundamental role in the theory of Borel equivalence relations, particularly in the study of hyperfiniteness. Recall that a Borel equivalence relation is hyperfinite if it is the increasing union of a countable sequence of Borel equivalence relations with finite equivalence classes; this is equivalent to being the orbit equivalence relation of a single Borel automorphism. For instance, a result of Harrington, Kechris, and Louveau [6] shows that this relation continuously embeds into any non-smooth Borel equivalence relation; similarly, any hyperfinite equivalence relation is Borel reducible to E_0 (see [1]). E_0 also occurs naturally in the study of automorphisms of the infinite binary tree, where it is also referred to as the confinality relation. It is thus useful to the study of hyperfinite equivalence relations and tree automorphisms to consider various ways of generating E_0.

We will restrict our attention here to the equivalence relation E_0. As we have noted, though, E_0 is in many ways representative of non-smooth hyperfinite equivalence relations. Hence the results here can be applied to many other hyperfinite relations, although we will omit the details.

Definition 1. The equivalence relation E_0 is defined on the Cantor space $2^{\mathbb{N}}$ by

$$x \, E_0 \, y \text{ iff } \{n : x(n) \neq y(n)\} \text{ is finite.}$$

We view $2^{\mathbb{N}}$ with the topology generated by the basic open sets $N_s = \{x : x \restriction n = s\}$ for finite sequences $s \in 2^{<\mathbb{N}}$. E_0 is then naturally generated by a continuous action of the group $\mathbb{Z}_2^{<\mathbb{N}}$ (which can be viewed as the set of finite subsets of $\mathbb{N}$ with symmetric difference as the group operation), where the action is given by $s \cdot x = x \Delta s$ (where Δ is symmetric difference, viewing $2^{\mathbb{N}}$ as the set of all subsets of $\mathbb{N}$). E_0 is also essentially generated by the

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odometer map φ_0 given by

$$\varphi_0(1^n \frown 0 \frown \alpha) = 0^n \frown 1 \frown \alpha$$
$$\varphi_0(1^\infty) = 0^\infty,$$

i.e., "adding 1 mod 2 with right carry". The orbits of φ_0 are the E_0 equivalence classes, with the exception that the E_0 classes of 0^∞ and 1^∞ are combined into one orbit.

In many categories, such as the categories of measure-preserving transformations or Borel automorphisms, this difference is inessential since we can easily modify φ_0 to split this orbit into the two equivalence classes. In a topological setting, though, this becomes non-trivial. Recent work of Gao and Jackson (see [3]) and others has shown that topological considerations can prove useful, even in the context of Borel maps. The odometer map also has many nice structural properties; in addition to being a homeomorphism, it is actually a Lipschitz automorphism, and in fact is generated by a finite automaton. It also has a shift-similarity property (defined below), and other nice features. The main purpose of this article is to consider which of these structural properties of the odometer map can be retained by a function whose orbit equivalence relation is actually E_0.

1. A homeomorphism generating E_0

We first consider whether a homeomorphism can generate E_0. The following general theorem of Giordano, Putnam, and Skau (Theorem 3.9 of [4]) implies that it can be:

Theorem. *Every AF equivalence relation is isomorphic to the orbit equivalence relation induced by the Vershik automorphism associated to some Bratteli diagram.*

An AF equivalence relation is one which can be expressed as the increasing union of compact equivalence relations E_n, where each E_n is relatively open in E_{n+1}; it is straightforward to check that E_0 is AF. Definitions of Vershik automorphisms and Bratteli diagrams may be found in [4] or [2]; we will present the necessary specifics below. Since Vershik automorphisms are homeomorphisms, this implies that every AF equivalence relation (and hence E_0) is generated by a homeomorphism.

We give here a direct construction of a homeomorphism generating E_0 which has a combinatorially simple presentation. This construction is close in spirit to the results in Proposition 1.1, Theorem 1.2, and Corollary 1.3 of [4].

Theorem 2. *There is a homeomorphism of $2^{\mathbb{N}}$ whose orbit equivalence relation is E_0.*

Proof. Our homeomorphism φ will be the Vershik automorphism associated to a properly ordered Bratelli diagram. Our presentation of these automorphisms is slightly different from the standard presentation, as we want to represent functions on the space $2^{\mathbb{N}}$ and not on abstract spaces. We will begin by explaining a Vershik automorphism for the odometer.

Consider first the Bratteli diagram shown for the odometer φ_0 on the left in Figure 1. Each sequence in $2^{\mathbb{N}}$ corresponds to a unique path down through the diagram, starting at the top node and following the edges for the corresponding co-ordinates. For each node, we order the edges above it from left to right. For a node at level n of the diagram, we then order the finite paths which start at the top node and end at the given node as follows. For two such paths s and t of length n, we find the least $k \leq n$ such that $s \restriction k \neq t \restriction k$ but $s \restriction k$ and $t \restriction k$ end at the same node, and we set $s < t$ if the last edge of $s \restriction k$ is less than (i.e., to the left of) the last edge of $t \restriction k$.

Then, for a sequence $x \in 2^{\mathbb{N}}$ we find the least n such that the path $x \restriction n$ is not the maximum path to the corresponding node, and replace this initial sequence of x by the successor of $x \restriction n$ in the ordering given above. We leave the rest of the co-ordinates of x unchanged. The only exception is the unique maximal path 1^∞, since 1^n is the maximum path to the corresponding node for each n. This path is mapped to the unique minimal path 0^∞, preserving continuity.

So now let φ be the Vershik automorphism associated to the diagram shown on the right of Figure 1. This is defined as above, with the unique maximal path $1 \frown 0^\infty$ being sent to the unique minimal path 0^∞ (and hence changing only the first digit). We claim that this generates E_0. Note that for any path other than the maximal path $1 \frown 0^\infty$ we obtain $\varphi(x)$ by replacing an initial segment, thus changing only finitely many co-ordinates, so $\varphi(x)\, E_0\, x$ for all x.

Conversely, suppose $x, y \in 2^{\mathbb{N}}$ with $x\, E_0\, y$, and let n be such that $x(k) = y(k)$ for $k \geq n$. We consider two cases. First, if x and y contain infinitely many 1's, then the paths corresponding to x and y must eventually merge at the first level $m > n$ where $x(m) = y(m) = 1$ (since both edges labeled 1 lead to the same node) and agree (as paths) beyond that node. There will be only finitely many paths to this node at level m, and so we can apply φ to the lesser of $x \restriction n$ and $y \restriction n$ finitely many times to obtain the other. Hence x and y are in the same orbit of φ.

Otherwise, both x and y have all but finitely many co-ordinates equal to 0. Then each of these paths must eventually agree with one of the two paths 0^∞ or $1 \frown 0^\infty$, and as before must be in the corresponding φ orbit. Since the two paths 0^∞ and $1 \frown 0^\infty$ are themselves in the same φ orbit, as noted above, we conclude that x and y are in the same orbit. Hence φ generates E_0. □

We can also consider other groups besides $\mathbb{Z}$ whose actions generate E_0. Since E_0 is isomorphic to $E_0 \times E_0$ via a homeomorphism of $2^{\mathbb{N}}$ with $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, E_0 can be generated by two commuting homeomorphisms. We can, for instance, take the two homeomorphisms φ^{even} and φ^{odd} which behave like the map φ above restricted to the even coordinates and the odd coordinates

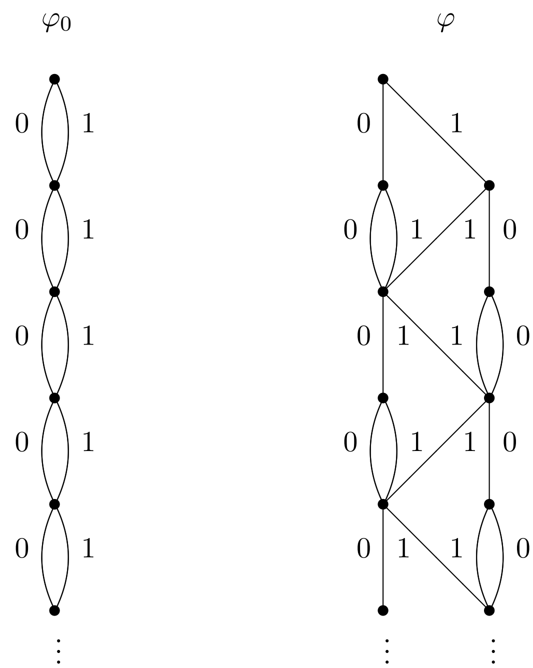

FIGURE 1. Bratelli diagrams for φ_0 and φ

of $\mathbb{N}$, respectively. Each of these homeomorphisms will have nowhere dense orbits, though, so we may ask:

Question. *Can E_0 be generated by two commuting homeomorphisms, each with dense orbits?*

This is related to a topological version of the result that any Borel equivalence relation generated by an action of $\mathbb{Z}^2$ is hyperfinite. We can also ask about other groups:

Question. *Which other groups can generate E_0 acting by homeomorphisms? Which groups can generate E_0 via homeomorphisms with dense orbits?*

2. LIPSCHITZ AUTOMORPHISMS

We now consider homeomorphisms generating E_0 with stronger structural conditions. We begin by discussing Lipschitz homeomorphisms.

Definition 3. A *Lipschitz automorphism* of $2^{\mathbb{N}}$ is a homeomorphism such that if $x \upharpoonright n = y \upharpoonright n$, then $\varphi(x) \upharpoonright n = \varphi(y) \upharpoonright n$.

Equivalently, φ is a Lipschitz automorphism if it can be expressed as the limit of a sequence of automorphisms $\varphi_n : 2^n \to 2^n$, where $\varphi_{n+1} \upharpoonright n = \varphi_n$ for all n. The Lipschitz automorphisms of $2^{\mathbb{N}}$ are precisely the isometries of $2^{\mathbb{N}}$ under the metric given by $d(x, y) = 2^{-n(x,y)}$, where $n(x, y)$ is the least n with

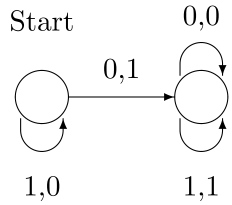

FIGURE 2. An automaton generating φ_0

$x(n) \neq y(n)$; they are also precisely the maps generated by automorphisms of the infinite binary tree. It is straightforward to check that the odometer map is a Lipschitz automorphism of $2^{\mathbb{N}}$. Note that the automorphism φ from the previous section is not Lipschitz, since if x begins with 00 then $\varphi(x)$ begins with 01, but if x begins with 01 then $\varphi(x)$ begins with 11. The action of $\mathbb{Z}_2^{<\mathbb{N}}$ given by $s \cdot x = x \Delta s$ is an action by Lipschitz automorphisms which generates E_0. We do not know whether we can find a single Lipschitz automorphism which actually generates E_0.

Question. *Can E_0 be generated by a Lipschitz automorphism?*

This is essentially question 7.5 #9 of [5]. We suspect that the answer may be yes, but the combinatorics must be non-trivial, as results below will show.

A particularly simple type of Lipschitz automorphism is one generated by a finite automaton. This means that there is a finite-state automaton which reads a single digit of the input and outputs a single digit of output at each step. An automaton generating the odometer is shown in Figure 2.

Question. *Can E_0 be generated by a finite automaton?*

Again, we do not know the answer to this question; we suspect that the answer is no. Automaton maps are particularly special; for instance, an automaton generating E_0 would have an upper bound on the number of digits changed in each sequence.

3. Continuous cocycles

We will now show that we can rule out certain structural properties of homeomorphisms generating E_0. We begin with a cocycle property.

Definition 4. Let φ be a homeomorphism of $2^{\mathbb{N}}$ generating E_0. We define the $(\mathbb{Z}_2^{<\mathbb{N}}, \mathbb{Z})$-*cocycle of* φ to be the map $n : \mathbb{Z}_2^{<\mathbb{N}} \times 2^{\mathbb{N}} \to \mathbb{Z}$ given by

$$n(s, x) = \text{the unique } k \in \mathbb{Z} \text{ such that } x \Delta s = \varphi^k(x).$$

We say this cocycle is *continuous* if it is continuous as a function from $\mathbb{Z}_2^{<\mathbb{N}} \times 2^{\mathbb{N}}$ to $\mathbb{Z}$, where $\mathbb{Z}_2^{<\mathbb{N}}$ and $\mathbb{Z}$ are given the discrete topology,

This definition makes sense if we just require that φ generates an equivalence relation containing E_0. In this setting, the $(\mathbb{Z}_2^{<\mathbb{N}}, \mathbb{Z})$-cocycle for the odometer is continuous, since $n(s,x)$ can be determined from $x \upharpoonright |s|$. No homeomorphism generating E_0 can have such a continuous cocycle, though:

Theorem 5. *There is no homeomorphism φ which generates E_0 such that the $(\mathbb{Z}_2^{<\mathbb{N}}, \mathbb{Z})$-cocycle of φ is continuous.*

Proof. Suppose φ generates an equivalence relation containing E_0. If the $(\mathbb{Z}_2^{<\mathbb{N}}, \mathbb{Z})$-cocycle of φ is continuous, then each of the sets $\{x : n(s,x) = k\}$ with s and k fixed is clopen, since $\mathbb{Z}$ is discrete. Let $s \neq \emptyset$, so $n(s,x) \neq 0$ for all x. Then the sets

$$\mathrm{Pos}_s = \{x : n(s,x) > 0\}$$
$$\mathrm{Neg}_s = \{x : n(s,x) < 0\}$$

are both open, and since their union is $2^{\mathbb{N}}$ they are in fact clopen. Note that the collection $\{\mathrm{Pos}_s : s \neq \emptyset\}$ has the finite intersection property, since if we take the finite subgroup of $\mathbb{Z}_2^{<\mathbb{N}}$ generated by a finite collection of s's and consider the (finite) orbit of some point x, the element of this orbit which occurs first in the φ-orbit of x will be in the intersection of the corresponding Pos_s's. Hence, by compactness we get that the set

$$\bigcap_{s \neq \emptyset} \mathrm{Pos}_s \neq \emptyset.$$

But now if $x \in \bigcap_{s\neq\emptyset} \mathrm{Pos}_s$, then for each $s \neq \emptyset$ we have that $x \Delta s$ is in the positive φ-orbit of x, $\{\varphi^k(x) : k > 0\}$, so the E_0 class of x is contained in the set $\{\varphi^k(x) : k \geq 0\}$. Hence any point in the negative φ-orbit of x must differ from x on infinitely many coordinates, and hence is not in the E_0 equivalence class of x. Hence φ does not generate E_0. ☐

4. Shift-similarity

We consider another structural property of the odometer which can not hold of any homeomorphism generating E_0.

Definition 6. We say a homeomorphism φ of $2^{\mathbb{N}}$ is *shift-similar* if for all x we have $\varphi^2(x) = x(0) \frown \varphi(x^*)$, where $x^*(n) = x(n+1)$ for all n.

We then have that for all n, $\varphi^{2^n}(x) = x \upharpoonright n \frown \varphi(x^{*n})$, where $x^{*n}(k) = x(n+k)$. Note that the odometer is shift-similar. Shift-similarity is a desirable combinatorial property to consider when seeking maps which generate E_0, since a shift-similar homeomorphism φ will generate an equivalence relation containing E_0 precisely when $x\Delta\{0\}$ is in the φ-orbit of x for each x, since the n^{th} digit can be changed (without changing other digits) by applying φ^{2^n}. Unfortunately, we will see that no shift-similar homeomorphism can generate E_0.

Lemma 7. *Let φ be a shift-similar homeomorphism of $2^{\mathbb{N}}$ with at least one dense orbit. Then φ is Lipschitz.*

Proof. Define the sets $\mathcal{F}_n$ by:

$$\mathcal{F}_n = \{x : \varphi(x)(n) \neq x(n)\}.$$

Note that $\mathcal{F}_n$ is clopen. We begin by showing that each $\mathcal{F}_n$ is φ^{2^n}-invariant. First consider $\mathcal{F}_0$. Since there is a dense orbit, we must have $\mathcal{F}_0 \neq \emptyset$ (which will be sufficient for the rest of the argument). We have that $\varphi^2(x)(0) = x(0)$, so $x \in \mathcal{F}_0$ if and only if $\varphi(x) \in \mathcal{F}_0$, so $\mathcal{F}_0$ is φ-invariant. In fact, we claim that $\mathcal{F}_0 = 2^{\mathbb{N}}$. If not, since $\mathcal{F}_0$ is clopen we could find a sequence s such that the neighborhood $N_{s\frown 0} \subseteq \mathcal{F}_0$ and $N_{s\frown 1} \cap \mathcal{F}_0 = \emptyset$ (or vice versa). Let $x \in \mathcal{F}_0$. Then one of $s \frown x$ and $\varphi^{2^{|s|}}(s \frown x) = s \frown \varphi(x)$ must be in each of these two sets since $x(0) \neq \varphi(x)(0)$, contradicting that $\mathcal{F}_0$ is φ-invariant.

We thus have that all orbits are dense, since $\varphi^{2^n}(x)(n) = \varphi(x^{*n})(0) \neq x(n)$ for all x and n. Hence, for each x we have that the size of the set

$$\left|\{k : 0 \leq k < 2^n \text{ and } \varphi^k(x) \in \mathcal{F}_n\}\right|$$

is odd, since $n \in x\Delta\varphi^{2^n}(x)$ and

$$x\Delta\varphi^{2^n}(x) = (x\Delta\varphi(x))\Delta(\varphi(x)\Delta\varphi^2(x))\Delta\cdots\Delta(\varphi^{2^n-1}(x)\Delta\varphi^{2^n}(x)).$$

Replacing x with $\varphi(x)$ we then have that

$$\left|\{k : 0 < k \leq 2^n \text{ and } \varphi^k(x) \in \mathcal{F}_n\}\right|$$

is odd, and hence $x \in \mathcal{F}_n$ if and only if $\varphi^{2^n}(x) \in \mathcal{F}_n$. So $\mathcal{F}_n$ is φ^{2^n}-invariant for all n.

Now, since the φ^{2^n}-orbit of x is dense in the neighborhood $N_{x\restriction n}$, for each $s \in 2^n$ we have that $\mathcal{F}_n = N_s$ or $\mathcal{F}_n \cap N_s = \emptyset$. Hence x's membership in $\mathcal{F}_n$ is determined by $x \restriction n$; since $\varphi(x) \restriction n$ is determined by $x \restriction n$ and x's membership in the sets $\mathcal{F}_0, \ldots, \mathcal{F}_{n-1}$, we must have that φ is Lipschitz. □

Proposition 8. *The following are equivalent for a homeomorphism φ of $2^{\mathbb{N}}$ with dense orbits:*

1. *φ is a shift-similar Lipschitz homeomorphism.*
2. *There is $\alpha \in 2^{\mathbb{N}}$ such that $\varphi(x) = x \oplus \alpha$, where $\oplus$ is addition mod 2 with right carry.*
3. *φ is in the commutant of the odometer.*

Proof. The equivalence of (2) and (3) is well-known. First, since the odometer is the map $\varphi_0(x) = x \oplus 1$ (where $1 = 1 \frown 0^\infty$) and $\oplus$ is commutative, any map $\varphi(x) = x \oplus \alpha$ commutes with the odometer. Conversely, if $\varphi(x)$ commutes with the odometer we can define the continuous map $\alpha(x)$ where $\alpha(x)$ is the unique α such that $\varphi(x) = x \oplus \alpha$. We then have

$$\varphi \circ \varphi_0(x) = x \oplus 1 \oplus \alpha(\varphi_0(x))$$
$$\varphi_0 \circ \varphi(x) = x \oplus \alpha(x) \oplus 1,$$

so $\alpha(\varphi_0(x)) = \alpha(x)$; by the ergodicity of φ_0 and continuity of α we then have that $\alpha(x)$ is constant. Note that having dense orbits for such a map is equivalent to $\alpha(0) = 1$.

Every map of the form $\varphi(x) = x \oplus \alpha$ is Lipschitz, and is self-similar since

$$\varphi^2(x) = x \oplus \alpha \oplus \alpha = x \oplus 0 \frown \alpha = x(0) \frown \varphi(x^*).$$

So it remains to establish that every shift-similar Lipschitz homeomorphism with dense orbits has the form $\varphi(x) = x \oplus \alpha$, where $\alpha = \varphi(0^\infty)$. We claim that $\varphi \restriction 2^{n+1}$ is completely determined by $\varphi \restriction 2^n$ and the value of $\varphi(0^{n+1}) = \alpha \restriction (n+1)$. By density of the orbits, every $s \in 2^{n+1}$ will be $\varphi^k(0^{n+1})$ for some $0 \leq k < 2^{n+1}$. When $k = 2j$ is even we have

$$\varphi(s) = \varphi^{2j}(0^{n+1}) = 0 \frown \varphi^j(0^n),$$

and when $k = 2j+1$ is odd we have

$$\varphi(s) = \varphi^{2j+1}(0^{n+1}) = \varphi^{2j}(\alpha \restriction (n+1)) = \alpha(0) \frown \varphi^j(\alpha^*).$$

By induction on n, this shows that φ is completely determined by $\varphi(0^\infty)$, so there is a unique shift-similar Lipschitz automorphism with dense orbits satisfying $\varphi(0^\infty) = \alpha$. Since the map $\varphi(x) = x \oplus \alpha$ satisfies these conditions, it must be φ. □

We can now establish:

Theorem 9. *There is no shift-similar homeomorphism which generates E_0.*

Proof. If φ is a shift-similar homeomorphism which generates E_0, then its orbits are dense so it must be Lipschitz. Hence there is an $\alpha \in 2^{\mathbb{N}}$ such that $\varphi(x) = x \oplus \alpha$, and $\alpha(0) = 1$ since the orbits are dense. Now it is easy to see that φ can not generate E_0, since α must have only finitely many 1's in order for $\varphi(0^\infty) = \alpha E_0 0^\infty$, in which case $\varphi = \varphi_0^k$ where $k = \sum_{n \in \mathbb{N}} 2^n \alpha(n)$, so the φ orbit of each x (other than 0^∞) will be a proper subset of its E_0-class. Alternately, if we let x be the point with $x(n) = 1 - \alpha(n+1)$ then (since $\alpha(0) = 1$) we have that $\varphi(x)(n) = 1 - x(n)$ for all n, so $\varphi(x)$ is not E_0-equivalent to x. □

The above analysis of shift-similarity seems to require the assumption of a dense orbit. So we can ask:

Question. *Is every shift-similar homeomorphism of $2^{\mathbb{N}}$ Lipschitz?*

When φ is Lipschitz but does not have dense orbits, it need not be of the above form, but we can still give a suitable analysis. Suppose φ is not the identity, and let n_0 be least such that there is an x_0 with $\varphi(x_0)(n_0) \neq x_0(n_0)$. Let $s_0 = x_0 \restriction (n_0 + 1)$. Since φ is Lipschitz, we may view it as acting upon finite sequences. The orbit of a sequence t under φ must then have size 2^k for some k, and if m is least such that $\varphi^2(t)(m) \neq t(m)$, then we must have $\varphi(t)(m-1) \neq t(m-1)$. Consequently, if the orbit of t under φ^{2^m} has size $2^k > 1$, then the orbit of t under φ must have size 2^{k+m}. We claim that the orbit of every x will be dense in the neighborhood N_t, where $t = x \restriction n_0$.

Since $\varphi^{2^{n_0+k}}(x \restriction (n_0+k) \frown s_0) = x \restriction (n_0+k) \frown \varphi(s_0)$ for $k \geq 0$, we have that n_0+k+n_0 is the least j such that $\varphi^{2^{n_0+k}}(x \restriction (n_0+k) \frown s_0)(j) \neq x \restriction (n_0+k) \frown s_0(j)$ and the orbit of $x \restriction (n_0+k) \frown s_0$ under φ must have size 2^{n_0+k+1}. This requires that the orbit of $x \restriction (n_0+k)$ under φ has size 2^k, so the orbit of x meets each basic neighborhood contained in N_t. Similar to above, we then have that φ is completely determined by its values on a set of size 2^{n_0} containing one point from each of the basic open neighborhoods of length n_0.

Finally, a weakening of shift-similarity which still provides useful combinatorial properties is the following:

Definition 10. We say a homeomorphism φ of $2^{\mathbb{N}}$ is *left shift-similar* if for all $y \in 2^{\mathbb{N}}$ we have $\varphi^2(0 \frown y) = 0 \frown \varphi(y)$. Right shift-similarity is defined analogously, with 1 in place of 0.

We can then ask:

Question. *Can E_0 be generated by a left (or right) shift-similar homeomorphism?*

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