On the degree of irrationality in Noether’s problem

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Abstract

Noether’s problem asks whether, for a given field $K$ and finite group $G$, the fixed field $L := K(x_h : h \in G)^G$ is a purely transcendental extension of $K$, where $G$ acts on the $x_h$ by $g \cdot x_h = x_{gh}$. The field $L$ is naturally the function field for a quotient variety $V := V(K, G)$. We study the degree of irrationality $\text{Irr}(V)$ of $V$, which is defined to be the minimal degree of a dominant rational map from $V$ to projective space. In particular, we give bounds for $\text{Irr}(V)$ in terms of the arithmetic of cyclotomic extensions $K$.

1 Introduction

The inverse Galois problem for a field $K$ and a finite group $G$ asks whether there exists a Galois extension $L/K$ with group $G$. We can embed $G$ in $\text{GL}_n(K)$ for some $n$, so $G$ acts faithfully on $V = K^n$, and there is a faithful action of $G$ on the field $K(V) = K(x_1, \ldots, x_n)$. If the fixed field $K(V)^G$ is a purely transcendental extension of $K$ (of transcendence degree $n$), then as a consequence of Hilbert Irreducibility, we obtain the existence of a field (infinitely many fields, in fact) $L/K$ with $G(L/K) = G$.

A purely transcendental extension $F$ of $K$ is said to be rational over $K$. Given a finite group $G$ and field $K$, consider the regular representation $V_G := \langle x_g \rangle_{g \in G}$ of $G$ over $K$. Noether’s problem asks whether $K(G) := K(V_G)^G$ is rational over $K$. An affirmative answer to Noether’s problem for $G$ and $K$ implies an affirmative answer to the inverse Galois problem for $G$ and $K$. Swan was the first to give an example of a group $G$ and field $K$ for which Noether’s problem had a negative answer. He proved [14] that $\mathbb{Q}(\mathbb{Z}/47\mathbb{Z})$ is not rational over $\mathbb{Q}$ by showing that Noether’s problem for $\mathbb{Z}/p\mathbb{Z}$ was equivalent to asking whether a prime ideal above $p$ in $\mathbb{Z}[\zeta_{p-1}]$ is principal, where $\zeta_{p-1}$ is a primitive $(p - 1)$st root of unity. Building on Swan’s work, Lenstra gave necessary and sufficient conditions under which Noether’s problem has an affirmative answer for all finite abelian groups over any field [7]. Saltman pioneered the study of Noether’s problem over the complex numbers, giving an example of a group $G$ of order $p^9$ for any prime $p \neq 2$ for which $\mathbb{C}(G)$ is not rational [11]. A good deal of work using techniques ranging from explicit computation to Galois cohomology and spectral sequences has been done on various cases of Noether’s problem over algebraically
closed fields, including $p$-groups [2, 10], direct products and wreath products [5], and the alternating groups $A_n$ [8]; however, there remain many open cases, such as the rationality of $\mathbb{C}(A_6)$.

Our aim in this paper is to measure the extent to which $K(A)$ may fail to be rational for an arbitrary finite abelian group $A$ and field $K$. To this end, we introduce the following quantity.

**Definition 1.** Let $V$ be a variety of dimension $n$ over a field $K$. The degree of irrationality of $V$ $\text{Irr}(V)$ is the minimal degree of a dominant rational map $V \dasharrow \mathbb{P}^n$.

In the case that $V$ is a curve, $\text{Irr}(V)$ is the gonality of $V$, a quantity that has been studied extensively and about which many questions remain. The quantity was first introduced by Heinzer and Moh in [9] in the context of function fields of one variable. Yoshihara and others have studied $\text{Irr}(V)$ when $V$ is an algebraic surface, beginning with [16]. The degree of irrationality of hypersurfaces has also been studied [1].

In our case, we fix a finite abelian group $A$ and a field $K$, and let $V = V_{K,A}$ be the variety (up to birational equivalence) with function field $K(A)$. That is, $V = A_{K,A}/A$.

**Definition 2.** For a field $K$ and a finite group $G$, define $\text{Irr}(K,G) = \text{Irr}(V_{K,G})$.

An equivalent definition of $\text{Irr}(K,G)$, with which we will work primarily, is

$$\text{Irr}(K,G) = \min_{K \subseteq L \subseteq K(G)} \{ [K(G) : L] : L/K \text{ is rational} \}.$$  

That is, $\text{Irr}(K,G)$ is the minimum degree of $K(G)$ over any field that is rational over $K$. Since our original field $K(h : h \in G)$ is finitely generated over $K$, $K(G)$ is finitely generated as well, so a transcendence basis $S$ for $K(G)$ is finite. Since $K(G)$ is finitely generated and algebraic over $K(S)$, it is finite over $K(S)$. Therefore, the quantity $\text{Irr}(K,G)$ is well defined. For example, Swan’s result that $\mathbb{Q}(\mathbb{Z}/47\mathbb{Z})$ is not rational may be written as $\text{Irr}(\mathbb{Q}, \mathbb{Z}/47\mathbb{Z}) \geq 2$.

For any finitely generated field $E$ over $F$, we can similarly define the degree of irrationality $\text{Irr}(E)$ of $E$ to be the minimum over all transcendence bases $S$ for $E/F$ of $[E : F(S)]$.

In the case $G = \mathbb{Z}/p\mathbb{Z}$, we may embed $G$ in the symmetric group $S_p$, which acts naturally on $W := K^p$ by permuting coordinates. The algebraic independence of the elementary symmetric polynomials gives the rationality of $K(W)^{S_p}$. The existence of this rational subfield of $K(G)$ implies $\text{Irr}(K,G) \leq (p - 1)!$.

Our main theorem is:

**Theorem 1.** Let $A$ be a finite abelian group and let $K$ be a field such that $[K(\zeta_s) : K]$ is cyclic for every prime power $s$ that divides the order of $A$ and is prime to the characteristic of $K$. Then $\text{Irr}(K,A)$ is less than an explicit quantity, which is given in the statement of Theorem 5.

In addition to Theorem 1, we discuss conditional lower bounds for the degree of irrationality of a field over which $K(A)$ is rational.
Definition 3. If $E$ is a transcendental field extension of $F$ and $S$ is a transcendence basis for $E/F$ such that $[E : F(S)] = \text{Irr}(E)$, then we will call $S$ a maximal transcendence basis for $E$ (over $F$).

For an abelian group $A$ and an algebraically closed field of characteristic prime to $|A|$, Noether’s problem is known to have an affirmative answer, due to Fischer

Theorem 2. [12] Let $A$ be an abelian group of exponent $e$ and $K$ a field of characteristic prime to $e$ that contains the $e$th roots of unity $\mu_e$. Then $K(A)$ is rational.

Proof. Let $|A| = a$. The group $A$ acts on $K(x_1, \ldots, x_a)$. Let $V = \oplus_{i=1}^a Kx_i$ be the regular representation of $A$. Since $A$ is abelian and $K$ contains $\mu_e$, $V$ can be diagonalized—i.e., $V$ has a basis $\{y_1, \ldots, y_a\}$ such that for any $g \in A$, $g \cdot y_i = \chi_i(g)y_i$, for a character $\chi_i \in \hat{A} = \text{Hom}(A, K^*)$. Let $M$ be the multiplicative free abelian group on the $y_i$, and define a group homomorphism $\psi : M \to \hat{A}$ by sending $y_i \mapsto \chi_i$. The kernel of $\psi$ is a free abelian group of rank $a$, generated, say, by $\{z_1, \ldots, z_a\}$. By construction, each $z_i \in K(A)$. If $f$ is any element of $K(y_1, \ldots, y_a)^A (= K(A))$, then since $g$ acts by scalars on each monomial term of $f$, we must have $f \in K(z_1, \ldots, z_a)$. Therefore, $K(A) = K(z_1, \ldots, z_a)$, and the $z_i$ are algebraically independent since there are $a$ of them generating a field of transcendence degree $a$.

We note that the question of irrationality in Noether’s problem leads to several other natural questions. As alluded to earlier, one can ask about the degree of irrationality of an arbitrary variety. Additionally, rather than just considering the rationality of $K(G)$, we can ask whether $K(G)$ satisfies the weaker condition of stable rationality—that is, whether $K(G)$ becomes rational upon adding finitely many indeterminates— or the even weaker condition of retract rationality (see [11] or [3] for a definition and discussion of retract rationality). These conditions have been studied for the general case of a quotient variety $V/G$ when $G$ is any linear algebraic group acting on a vector space $V$ (assuming such a quotient makes sense). See [3] for a nice survey of this.

This paper is organized as follows. In the next section, we introduce notation and review the results of Lenstra [7] that we will need for Theorem 1 (throughout, our presentation of Lenstra’s material is suitably adapted for our purposes). In Section 3, we begin by modifying Lenstra’s method to obtain Theorem 1 in the case $K = \mathbb{Q}$, $G = \mathbb{Z}/p\mathbb{Z}$. We work out this example in detail because the proof of the general case of Theorem 1 proceeds similarly to this case, which is less hampered by notation. We conclude Section 3 by adding the necessary details for the general case of Theorem 1. In Section 4, we investigate a certain class of rational subfields of $K(A)$ and give conditional lower bounds for the degree of irrationality of these fields.

2 Notation and Lenstra’s Setup

Let $K$ be a field, $\pi$ a group of automorphisms of $K$, and $M$ a $\pi$-module that is a finitely generated free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $x_1, \ldots, x_m$. We use multiplication for the group operation
of $M$, so elements of $M$ are monomials in the $x_i$. The group ring $K[M]$ is then isomorphic to the ring of Laurent polynomials in $m$ variables over $K$—that is, $K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, and its quotient field is the rational field $K(M) = K(x_1, \ldots, x_m)$. The group $\pi$ acts on $K[M]$ by

$$(\Sigma \alpha_i m_i)^\sigma = \Sigma \alpha_i^\sigma m_i^\sigma, \alpha_i \in K, m_i \in M,$$

which extends to an automorphism of the field $K(M)$. The units of $K[M]$ are monomials—that is, $K^*M$.

We introduce a few other pieces of notation. By $\zeta_m$ we denote a primitive $m$th root of unity. For a field $K$, we have the natural injection $G(K(\zeta_m)/K) \hookrightarrow (\mathbb{Z}/m\mathbb{Z})^*$, which allows us to view $G(K(\zeta_m)/K)$ as a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. We will take the set of divisors of a positive integer $n$ to be all positive divisors of $n$. The $n$th cyclotomic polynomial will be denoted $\Phi_n$. Lastly, the function $\phi$ refers to Euler’s $\phi$ function.

We begin by studying our irrationality question in the case $K = \mathbb{Q}$, $G = \mathbb{Z}/p\mathbb{Z}$, where $p$ is prime. Let $l = \mathbb{Q}(\zeta_p)$ and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\zeta_{p-1}]$ lying above the rational prime $p$. Then $\mathfrak{p}$ is of the form $(p, \zeta_{p-1} - t)$, where $t$ is an integer that generates $(\mathbb{Z}/p\mathbb{Z})^*$ when reduced modulo $p$. Since $\mathbb{F}_p$ has all $(p-1)$st roots of unity, $p$ splits completely in $\mathbb{Z}[\zeta_{p-1}]$. Let $\pi = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. The additive group of the ideal $\mathfrak{p}$ is a free $\mathbb{Z}$-module of rank $r := \phi(p-1)$. If $x_1, \ldots, x_r$ is a $\mathbb{Z}$-basis for $\mathfrak{p}$, written multiplicatively, then $\pi$ acts on the monomials in the $x_i$ and thus on the field $l(\mathfrak{p})$ (acting on $l$ by Galois automorphisms).

Let $m$ be a divisor of $p-1$ and let $\pi'$ be the quotient group of $\pi$ of order $m$. The group $\pi'$ can be identified with $\text{Gal}(L/\mathbb{Q})$ for a subfield $L \subseteq \mathbb{Q}(\zeta_p)$. We have a ring homomorphism

$$\psi_m : \mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi'] \rightarrow \mathbb{Z}[\zeta_m],$$

where the first map is induced by the natural quotient map $\pi \rightarrow \pi'$, and the second map is defined by sending a generator of $\pi'$ to $\zeta_m$. This allows us to view any $\mathbb{Z}[\zeta_m]$-module as a $\mathbb{Z}[\pi]$-module. For any divisor $m$ of $p-1$, following Lenstra [7], we define a functor $F_m$ from the category of $\pi$-modules to the category of torsion-free $\mathbb{Z}[\zeta_m]$-modules by

$$F_m(M) = (M \otimes_\pi \mathbb{Z}[\zeta_m])/\{\text{additive torsion}\},$$

where we view $\mathbb{Z}[\zeta_m]$ as a $\pi$-module via the map $\psi_m$.

We can make $\mathbb{Z}/p\mathbb{Z}$ into a $\pi$-module by identifying $\pi$ with $(\mathbb{Z}/p\mathbb{Z})^* \cong \text{Aut}(\mathbb{Z}/p\mathbb{Z})$. There exists a unique map $\mathbb{Z}[\pi] \rightarrow \mathbb{Z}/p\mathbb{Z}$ of $\pi$-modules taking $1 \mapsto 1$. Let $J_p$ denote be the kernel of this map, a free $\mathbb{Z}$-module of rank $p-1$. Lenstra shows:

**Proposition 1.** [7, Proposition 3.6] $F_m(J_p) \cong \mathfrak{p}$ if $m = p-1$, where $\mathfrak{p}$, as above, is an ideal of $\mathbb{Z}[\zeta_{p-1}]$ above $p$, and $F_m(J_p) \cong \mathbb{Z}[\zeta_m]$ if $m \neq p-1$.

The utility of the functor $F_m$ is demonstrated in the following theorem of Lenstra, suitably adapted here for our purposes.

**Theorem 3.** [7, Proposition 2.4] Let $M$ be a finitely generated, projective $\pi$-module. The fields $l(M)^\pi$ and $l(\oplus_{m|p-1} F_m(M))^\pi$ are isomorphic, where $\pi$ acts separately on each direct summand of $\oplus_{m|p-1} F_m(M)$ via the maps $\psi_m$ in (1).
Remark 1. One checks that $F_m$ respects direct sums, a fact we will use hereon without further reference.

We also have:

**Proposition 2.** [7, Proposition 5.3] The field $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to a purely transcendental extension of $l(J_p)^\pi$ (of transcendence degree 1).

We wish to apply Theorem 3 to the case $M = J_p$, so we need to establish that $J_p$ is projective, which we do now:

**Proposition 3.** [13, Proposition 7.1] Let $R$ be Dedekind domain of characteristic zero and $\pi$ a finite group of order $n$. Let $I$ be an ideal of $R\pi$ such that the ideal $(R\pi : I)$ of $R\pi$ and the ideal $nR$ of $R$ are comaximal (that is, there exists $a \in (R\pi : I), b \in nR$ such that $a + b = 1$). Then $I$ is a projective $R\pi$-module.

We may apply Proposition 3 in our case since $p \in (\mathbb{Z}[\pi] : J_p)$ and $\pi$ is of order $p - 1$.

Putting together Propositions 1 and 2 with Theorem 3, we find that $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to a rational extension of the field $L_{\pi}$, where

$$L_{\pi} := l\left(\left(\bigoplus_{m|p-1, m \neq p-1} \mathbb{Z}[\zeta_m]\right) \oplus p\right).$$

From here, we can proceed as Lenstra does in [7, Theorem 2.6] to show that if $p$ is principal, then $L_{\pi}$, and thus $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})$, is rational over $\mathbb{Q}$:

Suppose that $p$ is principal, so it is a free $\mathbb{Z}[\zeta_{p-1}]$-module. One checks [7, Proposition 2.3] that $F_m(\mathbb{Z}[\pi]) = \mathbb{Z}[\zeta_m]$, for every $m$ dividing $p - 1$. Therefore, aside from the $m = p - 1$ summand, the summands of $\bigoplus_{m|p-1} F_m(\mathbb{Z}[\pi])$ and $\bigoplus_{m|p-1} F_m(J_p)$ agree, and for the $m = p - 1$ summand, since $p$ is assumed to be principal, we have $\mathbb{Z}[\zeta_{p-1}] = p$ as $\mathbb{Z}[\zeta_{p-1}]$-modules. Thus

$$\bigoplus_{m|p} F_m(J_p) \cong \bigoplus_{m} F_m(\mathbb{Z}[\pi])$$

as $\mathbb{Z}[\pi]$-modules, and by applying Theorem 3 twice, it follows that $l(J_p)^\pi \cong l(\mathbb{Z}[\pi])^\pi$. But $\mathbb{Z}[\pi]$ is a $\mathbb{Z}[\pi]$-permutation module— that is, a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis that is permuted by $\pi$— and for any finitely generated $\mathbb{Z}[\pi]$-permutation module $N$, $l(N)^\pi$ is rational over $l^\pi$ [7, Theorem 1.4]. Therefore, $l(J_p)^\pi$ is rational over $l^\pi = \mathbb{Q}$, as desired.

### 3 Bounding the Degree of Irrationality from Above

#### 3.1 The Case $K = \mathbb{Q}, G = \mathbb{Z}/p\mathbb{Z}$

Suppose now that the ideal $p$ is not principal. Lenstra [7] shows this implies that $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})$ is not rational. In this case, we wish to bound $\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$ from above. Recall from the Introduction that

$$\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) = \min\{|\mathbb{Q}(\mathbb{Z}/p\mathbb{Z}) : L| : L/\mathbb{Q} \text{ is rational}\}.$$
Let \( \mathcal{I} \) be a principal ideal of the ring \( \mathbb{Z}[\zeta_{p-1}] \) contained in the ideal \( \mathfrak{p} \). Consider the field

\[
L_\mathcal{I} := l \left( \bigoplus_{m \mid p-1, m \neq p-1} \mathbb{Z}[\zeta_m] \oplus \mathcal{I} \right),
\]
a subfield of \( L_\mathfrak{p} \). Since \( \mathcal{I} \) is a free \( \mathbb{Z}[\zeta_{p-1}] \)-module of rank 1,

\[
\bigoplus_{m \mid p-1, m \neq p-1} \mathbb{Z}[\zeta_m] \oplus \mathcal{I} \cong \bigoplus_{m \mid p-1} F_m(\mathbb{Z}[\pi])
\]
as \( \mathbb{Z}[\pi] \)-modules. Therefore, \( L_\mathcal{I}^\pi \) is isomorphic to \( l(\bigoplus_m F_m(\mathbb{Z}[\pi]))^\pi \), which, as shown in the case when \( \mathfrak{p} \) was assumed to be principal, is a rational extension of \( l^\pi = \mathbb{Q} \).

Figure 1:

\[
\begin{array}{ccc}
\mathbb{Q}(\mathbb{Z}/p\mathbb{Z}) & \cong & L_\mathfrak{p}^\pi(T) \\
\text{rational deg 1} & d & \text{d}
\end{array}
\]

\[
\begin{array}{ccc}
l(J_\mathfrak{p})^\pi & \cong & L_\mathfrak{p}^\pi \\
| & d & |
\end{array}
\]

\[
\begin{array}{c}
L_\mathcal{I}^\pi \\
\text{rational deg 1}
\end{array}
\]

\[
\begin{array}{c}
\mathbb{Q}
\end{array}
\]

Our next task is to give an upper bound for \( [L_\mathfrak{p}^\pi : L_\mathcal{I}^\pi] := d \), which will serve as our upper bound for \( \text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \) (see Figure 1). Since \( l(J_\mathfrak{p}) \cong L_\mathfrak{p}^\pi \), by Proposition 2, we can identify \( \mathbb{Q}(\mathbb{Z}/p\mathbb{Z}) \) with \( L_\mathfrak{p}^\pi(T) \) for an indeterminate \( T \). Since \( \pi \) acts faithfully on \( L_\mathcal{I}^\pi \), from elementary field theory, we have \( [L_\mathfrak{p}^\pi : L_\mathcal{I}^\pi] = [L_\mathfrak{p}^\pi : L_\mathcal{I}] \).

Lemma 1. Let \( M \) and \( N \) be free \( \mathbb{Z} \)-modules with \( N \subseteq M \) and \( |M : N| < \infty \). Let \( K \) be a field. Then \( [K(M) : K(N)] = |M : N| \).

Proof. By induction on \( |M : N| \), we may assume that \( M/N \) is cyclic of prime order \( p \). The result will follow from the following two claims.

- **Claim I** If \( \phi \) is an automorphism of \( M \), then \( [K(M) : K(N)] = [K(M) : K(\phi(N))] \).

  **Proof** We may extend \( \phi \) by linearity to an automorphism of \( K(M) \). We have \( \phi(K(N)) = K(\phi(N)) \), and since \( \phi \) sends elements of \( K(M) \) linearly independent over \( K(N) \) to elements of \( K(M) \) linearly independent over \( \phi(K(N)) \), we have \( [K(M) : K(N)] \leq [K(M) : \phi(K(N))] \). We obtain the reverse inequality using \( \phi^{-1} \).
\textbf{Claim II} Let $N, N'$ be submodules of $M$ of index $p$ and let $\phi : N \to N'$ be an isomorphism. Then $\phi$ may be extended to an automorphism $\tilde{\phi}$ of $M$.

\textbf{Proof} We can do this by picking an element $x \in M$ that lies in neither $N$ nor $N'$ and setting $\tilde{\phi}(x) = x$. These two claims allow us to assume that if $M$ is generated by elements $x_1, x_2, \ldots, x_r$, then $N$ is generated by $x_1^p, x_2, \ldots, x_r$, and in this case it is clear that $[K(M) : K(N)] = p$. \hfill $\Box$

From Lemma 1 and the discussion preceding it, we see that $\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$ can be bounded above by

$$\min_{\mathfrak{I} \text{ principal}, \mathfrak{I} \subseteq \mathfrak{p}} |\mathfrak{p} : \mathfrak{I}|.$$ Bounding this quantity is our next task.

We take $\mathfrak{I} = (\zeta_{p-1} - t)$. By [4], for $p$ satisfying $\log_2 (p - 1) \geq 24$, there exists a primitive root $t$ modulo $p$ such that

$$|t| \leq \frac{1}{2}p^{1/2}, \quad (2)$$

and if $g(p)$ denotes the least primitive root modulo $p$ in absolute value, then

$$g(p) = O(p^{1/4}). \quad (3)$$

We have $|\mathfrak{p} : \mathfrak{I}| = N(\mathfrak{I})/N(\mathfrak{p}) = N(\mathfrak{I})/p$, where $N = \text{Norm}_{\mathbb{Q}(\gamma)/\mathbb{Q}}$. Let $\gamma = G(\mathbb{Q}(\zeta_{p-1})/\mathbb{Q})$. We have

$$N(\mathfrak{I}) = \prod_{\sigma \in \gamma} (t - \zeta_{p-1}^{\sigma}) \leq (t + 1)^{\phi(p-1)} \leq \left(\frac{1}{2}p^{1/2} + 1\right)^{\frac{p-1}{2}},$$

where we are using that $\phi(p-1) \leq \frac{p-1}{2}$. We thus obtain a numerical version of Theorem 1 for $p$ satisfying $\log_2 p - 1 \geq 24$:

$$\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \leq \frac{1}{p} \left(\frac{1}{2}p^{1/2} + 1\right)^{\frac{p-1}{2}}.$$ If we use (3), we obtain

$$\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \leq O(p^{\frac{p-2}{8}}).$$

We close this subsection with a series of remarks.

\textbf{Remark 2.} The upper bound for $\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$ given by finding the principal ideal of minimal index inside $\mathfrak{p}$ tends to infinity with $p$ since for any ideal $\mathfrak{I} \subseteq \mathfrak{p}$, we can write $\mathfrak{p} = \mathfrak{I} \mathfrak{I}'$ for some ideal $\mathfrak{I}'$, so that $|\mathfrak{p} : \mathfrak{I}| = N(\mathfrak{I})/N(\mathfrak{p}) = N(\mathfrak{I}') \geq p$ since the norm of a prime ideal $\mathfrak{q}$ lying over a rational prime $q$ is $q^f$, where $q^f \equiv 1 \pmod{p-1}$.

\textbf{Remark 3.} As mentioned in the introduction, we also have the weaker bound

$$\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \leq (p - 1)!,$$

given by the field $\mathbb{Q}(x_1, \ldots, x_p)^{S_p}$.

\textbf{Remark 4.} The extension $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})/\mathbb{Q}(x_1, \ldots, x_p)^{S_p}$ is not Galois. Let $L = L^\mathbb{Q}_x(T)$, where $T$ is an indeterminate, as in Figure 1. This is the field that gives the bound in Theorem 1. It is unknown to the author for which $\mathfrak{p}$ the extension $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})/L$ might be Galois, or whether there necessarily exists any field $L'$ for which $\mathbb{Q}(\mathbb{Z}/p\mathbb{Z})/L'$ is finite Galois.
3.2 The General Case

We now extend our results about $\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$ to $\text{Irr}(K, A)$, where $K$ is an arbitrary field and $A$ an arbitrary abelian group. Following Lenstra, if $\text{char} k > 0$, write

$$A = P \oplus B,$$

where the order of $P$ is a power of $\text{char} k$ and the order of $B$ is prime to $\text{char} k$, and write

$$B \cong \oplus_{s \in \Omega} \mathbb{Z}/s\mathbb{Z},$$

where $\Omega$ is the set of prime powers giving the elementary divisor decomposition of $B$ (with possible repetitions). Let $e$ be the exponent of $B$ and put $L = K(\zeta_e)$, and $\pi = G(L/K)$. From hereon we make the following assumption:

For every prime power $s$ dividing $|B|$, $\pi_s := G(K(\zeta_s)/K)$ is cyclic.

The only situation this precludes is $s = 2^n, n \geq 3$ and $\text{char} K \neq 2$. We have a map $\pi_s \hookrightarrow \text{Aut}(\mathbb{Z}/s\mathbb{Z})$, which we can use to make $\mathbb{Z}/s\mathbb{Z}$ into a $\pi_s$-module. In analogy to the group $J_p$ defined in Section 2, we define

$$J_s = \ker \psi : \mathbb{Z}[\pi_s] \to \mathbb{Z}/s\mathbb{Z},$$

where $\psi$ is the $\mathbb{Z}[\pi_s]$-module map sending $1 \mapsto 1$. In [7], there is a more general version of Proposition 2 stating that $K(A)$ is a rational extension of $L(I)^\pi$, where

$$I = \oplus_{s \in \Omega'} J_s, \text{ and } \Omega' = \{ s \in \Omega, s \text{ is not a power of } 2 \}.$$

The group $\pi$ acts on $I$ via the quotient maps $\pi \to \pi_s$.

For a given prime power $s = l^n$ dividing $|B|$, let

$$m_s = [K(\zeta_s) : K].$$

Recall the definition of the functor $F_m$ from Section 2. In the case that $\pi$ is not cyclic, we must modify our definition (which will agree with Lenstra’s in [7]). For a divisor $m'_s$ of $m_s$, we define the action of $\pi$ on $\mathbb{Z}[\zeta_{m'_s}]$ to be given via $\mathbb{Z}[\pi] \to \mathbb{Z}[\pi_{m'_s}] \to \mathbb{Z}[\zeta_{m'_s}]$, where $\pi_{m'_s}$ is the quotient of $\pi_s$ of order $m'_s$ (with this notation, $\pi_s = \pi_{m_s}$).

We now define, for a $\pi$-module $M$,

$$F_{m'_s}(M) = M \otimes \pi \mathbb{Z}[\zeta_{m'_s}].$$

**Remark 5.** We have abused notation since the $m'_s$ in both $F_{m'_s}$ and $\pi_{m'_s}$ is actually representing a particular quotient of $\pi$ of order $m'_s$, rather than just the number $m'_s$. We will maintain this convention throughout.
Proposition 1 can be strengthened to say that for $m'_s$ dividing $m_s$, $F_{m'_s}(J_s) \cong \mathbb{Z}[\zeta_{m'_s}]$ if $m'_s \neq m_s$ and $F_{m_s}(J_s)$ is of the form $(\zeta_{m_s} - t, l)$, with $t$ being an integer whose reduction modulo $l$ generates $G(K(\zeta_l)/K)$, where we are viewing $G(K(\zeta_l)/K)$ as a subgroup of $(\mathbb{Z}/l\mathbb{Z})^\times$.

Let $a_s = F_{m_s}(J_s)$, and write $s = l^u$. The polynomial $\Phi_{s}(s)$ splits into $\phi(l - 1)$ distinct irreducible factors over $\mathbb{F}_l$. We also know that the ramification degree of $l$ in $\mathbb{Q}(\zeta_{l^{u-1}(l-1)})$ is $l^{u-2}(l - 1)$. Therefore, the ideal $a_s = (\zeta_{m_s} - t, l)$, which is a prime ideal lying above $l$ in $\mathbb{Z}[\zeta_{m_s}]$, has norm $l$.

For odd $s$, we have from [7, Proposition 3.3] that $J_s$ is a projective $\pi$-module. Let $O'$ be a subset of $\Omega'$ consisting of, for each odd prime $p$ dividing $e$, the largest power of $p$ dividing $e$. From [7, Corollary 2.5], a corollary to our Theorem 3, we obtain

$$L(I)^\pi \cong L(\oplus_{s \in O', m'_s|m_s} F_{m'_s}(I))^\pi.$$  \hfill (5)

**Remark 6.** The index of the direct sum on the right hand side of (5) should technically be in bijection with all cyclic quotients of $\pi$; however, it follows from [7, Proposition 3.6] that we only need to consider quotients of $\pi$ corresponding to subfields of $L$ that are contained in $K(\zeta_\pi)$ for prime powers $s$ dividing $e$, and we only need to consider odd prime powers by Proposition 4 below.

Important Notational Point: Because we only wish to count each such quotient of $\pi$ once, indices of the form $\{m'_s|m_s, s$ in some subset of $\Omega\}$ are understood to include the integer 1 exactly once across all $s$, as opposed to including 1 as a divisor for each $s$.

Using [7, Proposition 2.1] and [7, Proposition 3.6], and the fact that $K(\zeta_\pi) \cap K(\zeta_b) = K$ if $\gcd(a, b) = 1$, we obtain the following.

**Proposition 4.** For $s \in O'$, let $N_s$ denote either $J_s$ or $\pi_{m_s}$.

1. If $\gcd(r, s) = 1$ and $m'_s \neq 1$, then $F_{m'_s}(N_r) = 0$.

2. If $m'_s|m_s$, and $m'_s \neq m_s$, then $F_{m'_s}(N_s) = 0$.

3. $F(I)(N_s) = \mathbb{Z}$ for all $s \in \Omega'$.

Thus we have

$$\oplus_{s \in O', m'_s|m_s} F_{m'_s}(I) = \oplus_{s \in O'} \oplus_{m'_s|m_s} F_{m'_s}(\oplus_{r \in \Omega'} \mathcal{J}_r) \cong \oplus_{r \in \Omega'} \left( \oplus_{m'_s|m_r, m'_s \neq m_r} (\mathbb{Z}[\zeta_{m'_s}] \oplus a_r) \right).$$ \hfill (6)

Note that as a $\mathbb{Z}$-module, $I$ has rank $\sum_{s \in \Omega'} m_s$ and $\oplus_{m'_s|m_s, s \in O'} F_{m'_s}(I)$ has rank $\sum_{s \in \Omega'} \sum_{m'_s|m_s} [\mathbb{Q}(\zeta_{m'_s}) : \mathbb{Q}]$. These ranks are equal since $m_s = \sum_{m'_s|m_s} [\mathbb{Q}(\zeta_{m'_s}) : \mathbb{Q}]$.

We may now proceed as we did in Section 2, still working under the assumption that $\pi_s$ is cyclic for every prime power $s$ dividing $|B|$, to conclude that $K(A)$ is rational over $K$ if
the ideal $a_s$ is principal for each $s$ dividing $|B|$. For we have

$$L(I)^s \cong L(\oplus_{s \in \Omega'} m'_s | m_s, m'_s \neq m_s \sum_{r \in \Omega'} Z[\pi_r]) \oplus a_s)^s,$$

where the first isomorphism is from (5) and the second is a consequence of Proposition 4, line (6), and [7, Proposition 2.3].

Suppose $a_s$ is principal. Then $a_s \cong Z[\zeta_{m_s}] \cong F_{m_s}(Z[\pi_s])$. Consider the $Z[\pi]$-module $M := \oplus_{s \in \Omega'} Z[\pi_s]$. From [7, Corollary 2.5] and [7, Proposition 2.1], we conclude that $L(M)^s \cong L(\oplus_{s \in \Omega'} m'_s | m_s Z[\zeta_{m'_s}])^s$. Using (7) and Proposition 4, we find that $L(I)^s \cong L(M)^s$. The $Z[\pi]$-module $M$ is a $\pi$-permutation module, from which it follows that $L(M)^s$, and thus $L(I)^s$, is rational over $l$ [7, Proposition 1.4]. At last we obtain:

**Theorem 4 ([7]).** Using the notation above, assume that $\pi_s$ is cyclic for every prime power $s$ dividing $|B|$. If $a_s$ is principal for all $s \in \Omega'$, then $K(A)/K$ is rational.

**Remark 7.** Lenstra’s version of Theorem 4 is stronger than what we have written; what is actually true is that if $a_s^{n_s}$ is principal for all $s \in \Omega'$, then $K(A)/K$ is rational, where $n_s$ is the multiplicity of $s$ in $\Omega'$. This strengthening ultimately comes from the fact that $a_s^{n_s}$ is principal ideal in $Z[\zeta_{m_s}]$ if and only if $a_s \oplus \cdots \oplus a_s$ ($n_s$ summands) is a free $Z[\zeta_{m_s}]$-module [6].

We can now bound $\text{Irr}(K, A)$ analogously to the way we bounded $\text{Irr}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z})$. The field $K(A)$ is isomorphic to a purely transcendental extension of $L(I)^s$, so we can write $K(A) \cong L(I)^s(T_1, \ldots, T_k)$, where $T_1, \ldots, T_k$ are indeterminates and $k$ depends on $K$ and $A$. If $\mathfrak{J}_s$ is a principal ideal contained in $a_s$, then it follows from Lemma 1 and the discussion of the case $K = \mathbb{Q}$, $A = \mathbb{Z}/p\mathbb{Z}$ that

$$[L(\oplus_{s \in \Omega'} m'_s | m_s, m'_s \neq m_s \sum_{r \in \Omega'} Z[\pi_r]) \oplus a_s)^s : L(\oplus_{s \in \Omega'} m'_s | m_s, m'_s \neq m_s \sum_{r \in \Omega'} Z[\pi_r]) \oplus \mathfrak{J}_s)^s = \prod_{s \in \Omega'} |a_s : \mathfrak{J}_s|.$$

The following Theorem now follows in analogy with the $K = \mathbb{Q}$, $A = \mathbb{Z}/p\mathbb{Z}$ case.

**Theorem 5.** Let $A$ be an abelian group and $K$ a field for which $K(\zeta_s)/K$ is cyclic for every prime power $s$ that divides $|A|$ and is prime to the characteristic of $K$. Then, using the notation above,

$$\text{Irr}(K, A) \leq \prod_{s \in \Omega'} \min \{|a_s : \mathfrak{J}_s|,$$

where the minimum is taken over all principal ideals $\mathfrak{J}_s \subseteq a_s$ for each $s \in \Omega'$.

**Remark 8.** In the course of proving Theorem 5, we have shown that

$$\text{Irr}(l(I)^s) \leq \prod_{s \in \Omega'} \min \{|a_s : \mathfrak{J}_s|.$$
3.3 An Example of Theorem 5

As an example of Theorem 5, we translate the result into a numerical bound for the case $K = \mathbb{Q}$ (note that if $\text{Irr}(\mathbb{Q}, A) = d$, and $K$ is a number field, then $\text{Irr}(K, A) \leq d$ because if $E$ is a rational field over $\mathbb{Q}$ with $[\mathbb{Q}(A) : E] = d$, then $K \otimes E$ is a rational field over $K$ with $[K(A) : K \otimes E] = d$). The Sylow-$l$ subgroups of $A$ for a prime $l$ dividing $|A|$ can be dealt with independently. Thus we take a prime $l$, and for the remainder of Section 3.3, we assume that $A$ is an $l$-group. We write

$$A = (\mathbb{Z}/l^{u_1} \mathbb{Z})^{v_1} \oplus \cdots \oplus (\mathbb{Z}/l^{u_b} \mathbb{Z})^{v_b}$$

with $u_i < u_{i+1}$, so $l^{u_i}$ appears with multiplicity $v_i$ in $\Omega'$. Let $s = l^{u_i}$. We have $m_{l^{u_i}} = \phi(s) = l^{u_i-1}(l-1)$. For each $s \in \Omega'$, we take $\mathfrak{I}_s = (\zeta_{m_s} - t)$, where $t$ is an integer whose reduction modulo $l$ generates $(\mathbb{Z}/l\mathbb{Z})^*$. From [15, Theorem 4.20], we have $h_s = \frac{N(\mathfrak{I}_s)}{N(a_s)} \leq O(l^{\phi(d(s)-4})$.

Recall that the class number $h_n$ of the maximal real subfield of $\mathbb{Q}(\zeta_n)$ divides $h_n$. Much is known about the quotient $h_n/h_n^+ := h_n^-$, while comparatively little is known about $h_n^+$ (these quantities are discussed in detail in [15], for example). From [15, Theorem 4.20], we have

$$h_n^- \sim n^{\frac{1}{2} \phi(n)}.$$  

Taking $n = l^{u-1}(l-1)$ and using the bound (10) in place of $h$ in (9) (which is cheating, of course, since (10) does not account for $h^+$) already gives a much larger bound for $\min_{\mathfrak{I}_s \text{ principal}, \mathfrak{I}_s \subseteq a_s} |a_s : \mathfrak{I}_s|$ than we obtain via (8).

To the author’s knowledge, there is no known asymptotic formula (or even non-trivial lower bound) for $h_n^+$.

Thus we find that $\prod_{s \in \Omega'} \min |a_s : \mathfrak{I}_s|$ can be bounded above by $O(l^C)$, where, by (8), we may take $C$ to be

$$\frac{1}{4} \left( \sum_{i=1}^b v_i (l^{u_i-2}(l-1)\phi(l-1) - 4) \right).$$  

We can also obtain an effective upper bound for $\min_{\mathfrak{I}_s \text{ principal}, \mathfrak{I}_s \subseteq a_s} |a_s : \mathfrak{I}_s|$ for each $s \in \Omega'$, and thus for $\text{Irr}(\mathbb{Q}, A)$, by means of (2) from Section 3. We obtain

$$\min_{\mathfrak{I}_s \text{ principal}, \mathfrak{I}_s \subseteq a_s} |a_s : \mathfrak{I}_s| \leq \frac{1}{l} \left( \frac{1}{2} l^{1/2} + 1 \right)^{\phi(d(s))}.$$  

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Remark 10. One checks that the bound for $\text{Irr}(\mathbb{Q}, A)$ coming from (12) (and thus from (11)) beats the bound one gets from the rational field $\mathbb{Q}(x_1, \ldots, x_{|A|})^{S|A|}$, of which $\mathbb{Q}(A)$ is a degree $(|A| - 1)!$ extension.

4 Rational Subfields of $K(A)$

Let $A$ be an abelian group and $K$ a field, and suppose there is a rational field $K'$ inside $K(A)$. To simplify the presentation, we will assume that $\text{char} K$ is prime to $|A|$; if $\gcd(\text{char} K, |A|) \neq 1$, the results of this section are easily modified since $K(A)$ is rational over $L(I_\pi)$ (notation as in Section 3.2). Ideally, we would like a lower bound for $[K(A) : K']$ in terms of $K$ and $A$, which would provide a lower bound for $\text{Irr}(K, A)$. In this section, we investigate certain rational subfields of $K(A)$, and discuss how various hypotheses lead to conditional lower bounds for $\text{Irr}(K, A)$.

We begin by setting notation. Let $\Omega' = \{s_1, \ldots, s_n\}$ be the odd elementary divisors of $|A|$, and let $\bar{\Omega}$ be a maximal subset among all subsets of $\Omega'$ whose elements are distinct. Let $I = \bigoplus_{s \in \Omega} J_s$, as defined in Section 3.2. For $s \in \Omega'$, set $\pi_s = G(K(\zeta_s)/K)$, with order $m_s$. Let $e$ be the exponent of $|A|$, let $l = K(\zeta_e)$, and let $\pi = G(l/K)$. Define the following sets:

$$U = \{x_{1,0}, \ldots, x_{1,s_1-1}, \ldots, x_{n,1}, \ldots, x_{n,s_n-1}\}, \quad |U| = \sum s_i$$

$$S = \{x_{1,0}, \ldots, x_{1,m_{s_1}-1}, \ldots, x_{n,0}, \ldots, x_{n,m_{s_n}-1}\}, \quad |S| = \sum m_{s_i}$$

$$T = U \setminus S, \quad |T| = |U| - |S|.$$ 

In order for our methods to work, we need to make the following assumptions, which we do for the remainder of Section 4:

For all $s \in \Omega'$, either $m_s = [K(\zeta_s) : K]$ is even or $\zeta_s \in K$. \hfill (13)

For all $s \in \Omega$, $K(\zeta_s)/K$ is cyclic. \hfill (14)

Figure 2:

We take the subset $S_i := x_{i,0}, \ldots, x_{i,m_{s_i}-1}$ of $S$ to be a maximal transcendence basis for $l(J_{s_i})^\pi$ over $K$. We have $d := [l(I)^\pi : K(S)] \leq \text{Irr}(l(I)^\pi)$. By the generalization in [7]...
of our Proposition 2, \( K(A) \) is a rational extension of a subfield isomorphic to \( l(I)^\pi \), which we identify with \( l(I)^\pi \). We may take \( l(I)^\pi(T) = K(A) \). In [7], Lenstra establishes that \( l(I)^\pi \) is rational if and only \( K(A) \) is rational (one direction being obvious). In other words, \( \text{Irr}(K, A) = \text{Irr}(l(I)^\pi) \) in the case that \( \text{Irr}(K, \mathbb{Z}/p\mathbb{Z}) = 1 \), and we might speculate that \( \text{Irr}(K, A) = \text{Irr}(l(I)^\pi) \) in all cases.

Let \( m_e = [l : K] \). For each \( k, 1 \leq k \leq n \), define

\[
z_{k;i} = \sum_{j=0}^{s_k-1} \zeta_{s_k}^{ij} x_{k;j}.
\]

This is the discrete Fourier transform of vector spaces: \( \bigoplus_{0 \leq i \leq s_k-1} lx_{k;i} \rightarrow \bigoplus_{0 \leq i \leq s_k-1} lz_{k;i} \), so for each \( k, l(x_{k;0}, \ldots, x_{k;s_k-1}) = l(z_{k;0}, \ldots, z_{k;s_k-1}) \). The group \( \pi \) acts, via its quotient \( \pi_{s_k} \), on \( l(z_{k;0}, \ldots, z_{k;s_k-1}) \). Let

\[
I' = \text{Free abelian group on all the } z_{k;i}, \text{ and } I'_k = \text{Free abelian group on the set } \{z_{k;i}\}_{0 \leq i \leq s_k-1},
\]

so that \( I' = \bigoplus_{1 \leq k \leq n} I'_k \).

The group \( \pi_{s_k} \) is isomorphic to a subgroup of \( \text{G}(\mathbb{Q}(\zeta_{s_k})/\mathbb{Q}) \), which cyclically permutes the set \( \{z_{k;i}\}_{\text{gcd}(i,s_k)=1} \). Therefore, as a \( \mathbb{Z}[\pi_{s_k}] \)-module, \( I'_k \) contains at least one copy of \( \mathbb{Z}[\pi_{s_k}] \). Since \( \pi \) permutes the \( z_{k;i} \), \( I' \) is a \( \mathbb{Z}[\pi] \)-permutation module and \( I' \) decomposes into a direct sum of \( \mathbb{Z}[\pi] \)-permutation modules with each summand having a \( \mathbb{Z} \)-basis on which \( \pi \) acts transitively. Therefore, recalling that \( F_{m_s}(\mathbb{Z}[\pi_s]) \cong \mathbb{Z}[\zeta_{m_s}] \) and using [7, Corollary 2.5], we have

\[
l(I')^\pi \cong l(\bigoplus_{s \in O'} m'_s F_{m'_s}(I'))^\pi = l(\bigoplus_{s \in \Omega} \mathbb{Z}[\zeta_{m_s}] \oplus R)^\pi,
\]

where \( R \) just denotes the direct sum of the remaining summands.

Now define

\[
I'' = I \oplus \mathbb{Z}(T),
\]

where \( \mathbb{Z}(T) \) denotes the (multiplicative) free abelian group on the set \( T \). The group \( \pi \) acts trivially on \( T \) and \( \mathbb{Z} \) acts by taking powers. Note that if \( \zeta_{s_k} \in K \), then \( \pi \) acts trivially on \( J_{s_k} \), so by construction, \( l(J_{s_k}) = l(J_{s_k})^\pi = l(x_{k;0}) \). We have the diagram of fields in Figure 3, which follows from Figure 2.

By [7, Corollary 2.5], we have

\[
l(I'')^\pi \cong l(\bigoplus_{s \in O'} m'_s F_{m'_s}(I \oplus \mathbb{Z}(T)))^\pi.
\]

It follows from Proposition 4 that \( F_m(\mathbb{Z}(T)) = 0 \) unless \( m = 1 \), in which case \( F_m(\mathbb{Z}(T)) = \mathbb{Z}[T] \). Recalling that \( I = \bigoplus_{s \in \Omega'} J_s \) and using line (6) from Section 3.2, we have

\[
\bigoplus_{s \in O', m'_s | m_s} F_{m'_s}(I \oplus \mathbb{Z}(T)) = \left( \bigoplus_{s \in O'} \left( \bigoplus_{m'_s \neq m_s} \mathbb{Z}[\zeta_{m'_s}] \right) \oplus a_s \right) \oplus \mathbb{Z}[T].
\]

We would like to say something about \( d \), which we will do by working with \([l(I'') : l(I')]\). We have \( l(I'_k) \subseteq l(J_{s_k} \oplus T_k) \), where \( T_k \) is the free abelian group on \( \{x_{k;m_{s_k}}, \ldots, x_{k;s_k-1}\} \). Since
From Corollary 2.5, we have

\[ K_1' l(I)^\pi = l(I')^\pi(T) = l(I')^\pi \]

where \( R \)

4.1 Conditional Lower Bounds for \( d \)

Proposition 4. Set \( \text{first isomorphism in each by [7, Corollary 2.5], and the second by [7, Proposition 3.6] and (13).} \)

Recall that the cyclic group \( \pi \) acts through its quotient \( \pi_{sk} \) on \( l(N) \), acting on each direct summand separately, and within each summand, acting by permuting monomials. Set \( w = m_{sk} \), and let \( M \) be a copy of \( \mathbb{Z}[\zeta_w] \) in \( N' \). Let \( \sigma \) be a generator of \( \pi \). As \( M \) is a free \( \mathbb{Z}[\zeta_w] \)-module of rank 1, there exists an element \( f \in l(N) \) so that

\[ l(M) = l(f, f^\sigma, \ldots, f^{\sigma^{r-1}}), \]

\( I_i' \) and \( I_j' \) are algebraically independent (viewed as subsets of \( l(I') \)) for \( i \neq j \), as are \( J_{s_i} \oplus T_i \) and \( J_{s_j} \oplus T_j \) (viewed as subsets of \( l(I') \)), we have

\[ [l(I') : l(I''')] = \prod_{1 \leq k \leq n} [l(J_{s_k} \oplus T_k) : l(I_k')] \]

If \( \zeta_{sk} \notin K \), then by assumption (13) at the beginning of Section 4, we know that \( m_{sk} = [K(\zeta_s) : K] \) is even. Let \( d_k = [l(J_{s_k} \oplus T_k) : l(I'_k)] \), and note that \( d_k = 1 \) if \( \zeta_{sk} \in K \).
where \( r = \phi(w) \). The action of \( \sigma \) on \( f \) is given by viewing \( f \) as an element of \( l(N) \), on which \( \sigma \) acts as a field automorphism. Note that although \( \sigma \) is an automorphism of \( l(N) \) that restricts naturally to \( l(N') \), when viewed as \( \mathbb{Z}[\zeta_w] \)-modules, the action of \( \mathbb{Z} \) on \( N \) (taking powers of elements of \( N \)) is not compatible with the action of \( \mathbb{Z} \) on \( N' \) (taking powers of elements of \( N' \)). The element \( \sigma^w \) acts on \( f \) as inversion since \( \zeta_w^{w/2} = -1 \) (recall that \( w \) is assumed to be even). We can identify \( l(N) \) with \( l(y_0, \ldots, y_{s_k-1}) \) for indeterminates \( y_i \), so that we may write \( f = \frac{g}{h} \) with \( g, h \in B := l[y_0^{\pm 1}, \ldots, y_{s_k-1}^{\pm 1}] \), where \( g \) and \( h \) have no common factors (recall that \( B \) is a unique factorization domain with unit group equal to the group of monomials). Setting \( a = \frac{g}{h} \), we have

\[
\frac{h}{g} = f^{-1} = f^{\sigma^a} = \frac{g^{\sigma^a}}{h^{\sigma^a}},
\]

so, up to units, \( h = g^{\sigma^a} \).

If \( f \in B^* \), we have the following conditional results.

### 4.1.1 The Case \( f \in B^* \)

All results in this section are only valid under the assumption that \( f \in B^* \).

For the moment we additionally assume:

If \( s \in \Omega' \) and \( \zeta_{m_s} \notin K \), then the element \( s \) appears in \( \Omega' \) exactly once.

(20)

Here, \( f \) must be a monomial in \( l[y_0^{\pm 1}, \ldots, y_{s_k-1}^{\pm 1}] \). If \( f \) has coefficient \( \alpha \in l \), then taking \( f' := \frac{1}{\alpha} f \), we may replace \( M \) by \( M' \), where \( M' \) is the \( \mathbb{Z}[\pi] \)-module generated by \( f' \). So \( M' \) is contained in \( N \).

Write \( f' = \prod_{i=0}^{s_k-1} y_i^{a_i} \), where we may assume that \( y_0, \ldots, y_{r-1} \) correspond to \( a_{s_k} \) (recall \( r = \phi(w) \)). We claim that \( a_i = 0 \) for all \( r \leq i \leq s_k - 1 \), and thus \( M' \subseteq a_{s_k} \). To see this, note that if \( \tau \) is the \( w \)-cyclotomic polynomial in \( \mathbb{Z}[^{\sigma}] \), then \( f^{\tau} = 1 \). The element \( \tau \) acts on monomials in the variables \( y_{b_0}, \ldots, y_{b_{r-1}} \), where these groupings correspond to separate summands of \( N \) (so, for example, \( b_0 = 0, b_1 = r \)). Therefore \( \tau \) acts trivially on each subproduct \( y_{b_0}^{a_{b_0}} \cdots y_{b_{r-1}}^{a_{b_{r-1}}} \). But this only holds if \( i = 0 \) or if each exponent in the subproduct is zero since \( y_{b_j} \), for \( b_j \geq r \), satisfies \( y_{b_j}^{\Phi_w(\sigma)} = 1 \), for some \( w' \neq w \), and all cyclotomic polynomials are irreducible. We conclude that \( M' \) is a principal ideal contained in \( a_{s_k} \).

Remark 11. If \( s_k \) occurred with multiplicity \( n_k \) in \( \Omega' \), then we would only be assured that \( M' \) is contained in \( a_{s_k}^{n_k} \) (direct sum).

Theorem 6. Let \( A \) be an abelian group with elementary divisor decomposition \( A = \bigoplus_{k \in \mathbb{Z}/s_k \mathbb{Z}, K} \), \( K \) a field, and suppose assumptions (13), (14), and (20) hold. For each \( s_k \), let \( c_k \) denote the minimum over all principal ideals \( \mathfrak{I}_k \subseteq a_{s_k} \) of \( |a_{s_k} : \mathfrak{I}_k| \). Then, using the notation above, \( c_k \leq d_k \).
Proof. We have
\[ c_k \leq [l(a_{s_k}) : l(M')] \leq [l(N) : l(N')] = d_k. \]

Suppose we remove assumption (20), so for a given \( s_k \), \( J_{s_k} \) occurs in \( I \) with multiplicity \( n_k \), which may be greater than 1. It follows from Remark 8 that
\[ c_k^{n_k} \geq \text{Irr}(l(J_{s_k}^{n_k} \oplus \mathbb{Z}(T_k)^{n_k})^\pi), \]
and from Theorem 6 that \( c_k^{n_k} \leq d_k^{n_k} \). Informally speaking, the likelihood of \( c_k^{n_k} \) (and thus \( d_k^{n_k} \)) being close to \( \text{Irr}(l(J_{s_k}^{n_k} \oplus \mathbb{Z}(T_k)^{n_k})^\pi) \) decreases as \( n_k \) gets larger. This is because the rationality of \( l(J_{s_k}^{n_k} \oplus \mathbb{Z}(T)^{n_k})^\pi \) is equivalent to that of \( l(\oplus m_{s_k}[m_{s_k}, m \neq m_{s_k}] \mathbb{Z}[\zeta_{m_{s_k}}]^{n_k} \oplus a_{s_k}^{n_k} \oplus \mathbb{Z}^{n_k(T)})^\pi \), and the latter field is rational if \( a_{s_k}^{n_k} \) is a free \( \mathbb{Z}[\pi_s] \)-module, or, equivalently [6], if \( a_{s_k}^{n_k} \) (ideal product inside \( \mathbb{Z}[\zeta_{m_{s_k}}] \)) is principal. Moreover, by Section 3, if \( a_{s_k}^{n_k} \) (direct sum) contains a free \( \mathbb{Z}[\zeta_{m_{s_k}}] \)-module \( X \) of index \( d_X \), then \( l(J_{s_k}^{n_k} \oplus \mathbb{Z}(T_k)^{n_k})^\pi \) will have degree of irrationality at most \( d_X \); we are guaranteed such an \( X \) with \( d_X \leq c_k^{n_k} \) since we can always take \( X = \mathbb{T}_k^{n_k} \) (direct sum) for a principal \( \mathcal{J}_k \subseteq \mathfrak{a}_s \), a free \( \mathbb{Z}[\zeta_{m_{s_k}}] \)-module inside \( \mathfrak{a}_{s_k}^{n_k} \).

Fix a field \( K \), and suppose the following holds:
\[ A = \mathbb{Z}/s\mathbb{Z} \text{ (so } d_1 = d) \text{ is cyclic for an odd prime power } s, \text{ and } \zeta_s \notin K. \] (21)

We then have the following conditional Theorem:

**Theorem 7.** Suppose (13), (14) and (20) hold. Then \( \text{Irr}(l(J_s)^\pi) = c_1 \). That is, \( \text{Irr}(l(J_s)^\pi) \) is equal to the minimum over all principal ideals \( \mathcal{I} \subseteq \mathfrak{a}_s \) of \( |\mathfrak{a}_s : \mathcal{I}| \).

**Proof.** By construction, \( \text{Irr}(l(J_s)^\pi) = d_1 \). From Theorem 6, \( c_1 \leq d_1 \). On the other hand, by Remark 8, \( \text{Irr}(l(J_s)^\pi) \leq c_1 \).

As corollary, to Theorem 7, we have

**Corollary 1.** Let \( K \) be field that is finitely generated over its prime subfield, and suppose that (13), (14), and (20) hold. Then for almost all (Dirichlet density 1) primes \( p \),
\[ \text{Irr}(J_p) \geq m_p + 1 = [K(\zeta_p) : K] + 1. \]

**Proof.** Lenstra [7, Corollary 7.6] shows that the set of primes for which \( K(\mathbb{Z}/p\mathbb{Z}) \) is not rational has density one. The result now follows by Theorem 7 and the fact that the minimal index of any ideal \( \mathcal{J} \) properly contained in \( \mathfrak{a}_p \) is \( m_p + 1 \) (recall \( \mathfrak{a}_p \subset \mathbb{Z}[\zeta_{m_p}] \)).

4.1.2 Remarks in the Case that \( f \notin B^* \)

We close by listing a few properties that \( f \) must satisfy in the case that \( f \notin B^* \).

Suppose that \( f \) is not a monomial in \( B \). If we continue to assume (13) and (14), then, as previously noted, we can write \( f = \frac{g}{h} \), with \( h = u^{-1}g^\sigma, u \in B^* \). In this case, \( f \) must have the following properties:
Proposition 5. The element $g \in B$ cannot be written as a sum of two or fewer terms in $B$.

Proof. Since $\sigma$ acts additively on monomials and $f = \frac{ug}{g^{\sigma}}$ is assumed to not be a monomial, we may assume that $g$ can be written as $A + B$ is a sum of two monomials. If $|A| = \prod y_i^{a_i}$, define $|A| = \prod y_i^{a_i}$. Furthermore, define $A^+ = \prod y_i^{a_i}$, and $A^- = \prod y_i^{a_i}$, so that $|A| = A^+ A^-$. Likewise, define $|B|, B^+, B^-$. Then

$$f = \frac{u}{h} = \frac{|A||B|}{|A||B|}, \quad \frac{A + B}{A^{-1} + B^{-1}} = \frac{u}{(A^+)^2|B| + (B^+)^2|A|} = \frac{u}{A^+ B^+(A^+ B^- + B^+ A^-)} \frac{A^{-1} B^{-1} - (A^- B^+ + A^+ B^-)}$$

which is a monomial, a contradiction.

Proposition 6. For every $b, 1 \leq b \leq w - 1$, $\frac{g}{g^{\sigma b}} \notin B^*$.

First, we have a lemma.

Lemma 2. Let $M$ be the $\mathbb{Z}[\zeta_w]$-module generated by $f$. Then $M \cap B = 1$.

Proof. Suppose $M$ contained a non-trivial element $v \in B$. Then $v$ generates a $\mathbb{Z}[\zeta_w]$-module $<v>$ inside $M$, and $M/<v>$ is finite, meaning that there exist $\delta_1, \ldots, \delta_n \in M$ such that every element of $M$ is of the form $\delta_i v'$, where $v' \in <v>$. This is impossible, however, since powers of $f$ give elements of $M$ whose numerator and denominator are both a product of arbitrarily many irreducible elements of $B$.

Proof. Suppose we had $g^{\sigma b} = vg$, for some $v \in B^*, 1 \leq b \leq w - 1$. We have $f = \frac{ug}{g^{\sigma b}}$, so

$$\frac{f}{g^{\sigma b}} = \frac{v^{\sigma a} u}{vv^{\sigma a}} := \Lambda.$$

We cannot have $\Lambda = 1$ since the lowest positive power of $\sigma$ fixing $f$ is $w$. But $\Lambda \neq 1$ contradicts Lemma 2.

Corollary 2. The element $g$ cannot be an irreducible element of $B$.

Proof. Suppose $g \in B$ were irreducible. As mentioned in Section 4.1.1, if $\tau$ is the $w$th cyclotomic polynomial in $\mathbb{Z}[\sigma]$, then $f^\tau = 1$. Since $B$ is a UFD, this would imply that up to units, $g = g^{\sigma b}$, for some $b, 1 \leq b \leq w - 1$, contradicting Proposition 6.

Corollary 3. Every non-trivial element of $m \in M$ is of the form $\frac{c}{e}$, where $e$ and $e'$ are coprime elements of $B$, each a product of $a_m$ irreducible elements of $B$, with $a_m \geq 2$.

Proof. Any non-trivial element $m \in M$ generates a free $\mathbb{Z}[\zeta_w]$-module, so as shown in the case $m = f$, $m$ must be of the form $\frac{ug}{g^{\sigma b}}$ with $u \in B^*$ and $g$ a product of at least two irreducible elements in $B$. 

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