On the existence of a short admissible pivot sequence for feasibility and linear optimization problems

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Abstract

Finding a pivot rule for the simplex method that is strongly polynomial is an open question. In fact, the shortest length of simplex pivots from any feasible basis to some optimal basis is not known to be polynomially bounded. An admissible pivot is a common generalization of simplex and dual simplex pivots, and there are various admissible pivot methods that are finite, including the least-index criss-cross method. No polynomial admissible algorithm is known.

The key question we address here is the existence of a short sequence of admissible pivots (where short means linear in the basis and nonbasis sizes). More precisely, we extend the existence result due to Fukuda, Lüthi and Namiki for nondegeneate LPs. For the feasibility problem, we prove the existence of a short admissible pivot sequence from an arbitrary basis to a feasible basis. Furthermore, for the general LP, the existence of a short admissible pivot sequence from an arbitrary basis to an optimal basis is proved without any nondegeneracy assumptions. The question remains: is it possible to design a strongly polynomial admissible pivot algorithm?

1 Introduction

Let us consider the primal and dual linear programming (LP) problems in canonical form:

$$\begin{array}{llll} \text{(P)} & \max c^T x & \text{subject to} & Ax \le b, \quad x \ge 0, \\ \text{(D)} & \min b^T y & \text{subject to} & A^T y \ge c, \quad y \ge 0, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ are given. The simplex method has been studied extensively since it was invented in 1947 by Dantzig [6, 5]. The primal (dual) simplex method is a family of methods which start with a (dual) feasible basis and use pivot operations

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selected with a proper sign restriction and using the ratio test in order to preserve (dual) feasibility of the basis and guarantee monotonicity of the objective value. Other pivot methods, e.g. the finite criss-cross methods [17, 10] do not preserve feasibility, they do not use ratio test, but impose the same sign restrictions in pivot element selection as the simplex method or the dual simplex method. Such pivots in general will be called *admissible pivots* (for a precise definition, see Section 1.1 and Figure 2).

Unfortunately, there is no known pivot algorithms that is polynomial. To date, two families of polynomial algorithms for LP exist. Ellipsoid methods [12] and interior-point methods [11, 15]. These algorithms are not strongly polynomial since in all variants the number of arithmetic iterations to solve an LP depends on the size of the input matrix. In fact, the termination criteria of these algorithms depends heavily on the size of input and thus indicate that there is a very little hope for these algorithms to be modified to be strongly polynomial. Therefore the only reasonable approaches to find a strongly polynomial algorithm remains to be pivot-based.

We do not know yet if such a pivot algorithm exists, however recently a basic question was raised and partially solved by Fukuda, Lüthi and Namiki [7]. They studied the following question:

> (*) Let an LP problem be given. What is the length of a shortest admissible pivot sequence *from any* (not necessarily feasible) basis to an *optimal* basis? Is it polynomially bounded in m and n?

Note that if we replace "admissible" by "simplex" above then there is no known polynomial upper bound for the length in terms of m and n. This means that the simplex method may not be polynomial even if it always selects (magically) a shortest path. Assuming that the LP problem is totally (i.e. primally and dually) nondegenerate, it was proved in [7] that the length of an admissible pivot sequence can be bounded from above by $\min\{m,n\}$. For degenerate problems the authors suggested to use a perturbation technique. In this paper the nondegeneracy assumption is removed. Our constructive proof relies on similar ideas that were developed for strongly polynomial basis identification techniques in interior point methods [14, 15].

One might suspect that our results would follow easily from the fundamental theorem of linear programming:

(a) If an LP has a feasible solution, it has a feasible basic solution.

(b) If an LP has an optimal solution, it has an optimal basic solution.

It is well known that the theorem can be proved constructively by elementary arguments with short pivot sequences, see, e.g. [13, Section 2.4]. Here, we mean by *short* linearly bounded in m and n. Such procedures, sometimes called *purification* [1], are simply not sufficient to prove our results which require each pivot to be admissible. Note that an admissible pivot must be **guided by** (the sign patterns of the simplex tableau associated with) the current basic solution. Furthermore the straightforward proof for **(b)** merely provides a pivot sequence to an optimal basic solution and not to an optimal basis. In contrast our results immediately yield an algorithm to find an optimal basis from a given dual pair of optimal solutions using at most $m+n$ admissible pivot operations. This corollary is a strengthening of Meggido's theorem [14, Theorem 0.2] which states: there is a strongly polynomial algorithm that finds an optimal basis, given a dual pair of optimal solutions. Magiddo's approach is based on building an optimal basis incrementally by elementary matrix operations, which can be implemented as (typically non-admissible) pivot operations. Consequently his procedure is not applicable for our problem. Our proof uses only one type of elementary operations, i.e. admissible pivots. Finally, and most importantly, we can now claim that there is a good reason to search for a strongly polynomial admissible pivot method for the feasibility and the LP problems. As we noted earlier, the situation is highly uncertain for the special case of simplex pivots.

In the rest of this section we fix our notations and give formal definitions. In Section 2.1 for the primal and in Section 2.2 for the dual feasibility problem it will be shown that from any basis to a feasible basis an admissible pivot sequence exists whose length is bounded by n and m, respectively. Our main result, in Section 3 shows that the answer to the question (*) is positive: there always exists an admissible pivot sequence consisting of no more than $m+n$ pivots. The paper closes by summarizing our results and giving some outlook to possible further generalizations.

1.1 Matrix Notation and LP Dictionary

Here we present some basic notations and definitions for matrices and linear systems.

For finite sets I and J, an $I \times J$ *matrix* is an array of doubly indexed numbers or variables

$$A = (a_{ij} : i \in I, j \in J)$$

where each member of I is called a *row index,* each member of J is called a *column index* and each a_{ij} is called a *component*. For $R \subseteq I$ and $S \subseteq J$, the $R \times S$ matrix $(a_{rs} : r \in R, s \in S)$ is called a submatrix of A, and will be denoted by A_{RS}. We use simplified notations like, A_R for A_{RJ}, $A_{\cdot S}$ for A_{IS}, A_i for $A_{\{i\}J}$, and $A_{\cdot j}$ for $A_{I\{j\}}$.

Given a primal LP:

$$\text{(P)} \quad \max c^T x \quad \text{subject to} \quad Ax \le b, \quad x \ge \mathbf{0},$$

where $A \in R^{m\times n}$, $b \in R^m$ and $c \in R^n$, we define the *initial dictionary*[1] system as

$$\begin{aligned} x_f &= 0x_g + c^T x_N \\ x_B &= bx_g - Ax_N, \end{aligned} \tag{1}$$

where $N = \{1, \dots, n\}$ and $B = \{n+1, \dots, n+m\}$ are the *initial nonbasis* and *basis*, respectively. Note that A is considered as $B \times N$ matrix, x_f is a new variable representing the objective value and x_g is one to make the system homogeneous. The original variable vector in (P) is only the $B = \{1, \dots, n\}$ part of an extended vector $x = (x_B, x_N, x_f, x_g)^T$. For convenience, we set $E = B \cup N$ and $\overline{E} = E \cup \{f, g\}$. By setting

$$D = \begin{bmatrix} 0 & c^T \\ b & -A \end{bmatrix}, \quad \overline{B} = B \cup \{f\}, \quad \overline{N} = N \cup \{g\}$$

[1] The notion of dictionary was first introduced in [16] and elaborated in [2].

the dictionary system can be written as

$$x_{\overline{B}} = D x_{\overline{N}}. \tag{2}$$

The matrix $D = D(B)$ is called the *dictionary* of the LP associated with a basis B. The associated basic solution $x(B)$ is the unique solution of the system with $x(B)_g = 1$ and $x(B)_N = \mathbf{0}$. If for $i \in B$ and $j \in N$, the (i, j) component d_{ij} of D is nonzero, then one can transform the system (2) to an equivalent system with the new basis $B := B - i + j$, the new nonbasis $N := N - j + i$ and the new $\overline{B} \times \overline{N}$ dictionary matrix D. The replacement is known as a *pivot operation on* (i, j). For each dictionary system (2), the dual dictionary system is defined by

$$y_{\overline{N}} = -D^T y_{\overline{B}}. \tag{3}$$

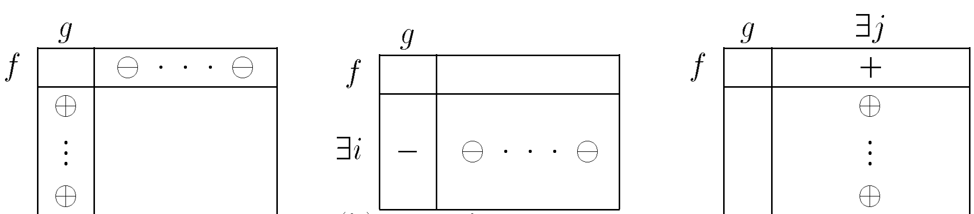
This represents a dictionary system of the dual problem:

$$\text{(D)} \quad \min b^T y \quad \text{subject to} \quad A^T y \geq c, \quad y \geq \mathbf{0}$$

in such a way that the original dual variable y is the $\{n+1, \dots, n+m\}$ part of the extended variable vector $y = (y_B, y_N, y_f, y_g)^T$ and the variable y_g expresses the negative of the original objective function $b^T y$. The *basic dual solution* $y(B)$ is the unique solution of (3) with $y(B)_f = 1$ and $y(B)_B = \mathbf{0}$.

A basis B (or a dictionary) is called *primal feasible* if $x(B)_B \geq \mathbf{0}$, *dual feasible* if $y(B)_N \geq \mathbf{0}$, and *optimal* if it is both primal and dual feasible. A basis B (or a dictionary) is called *primal inconsistent* if there exists $i \in B$ such that $d_{ig} < 0$ and $D_{iN} \leq \mathbf{0}$, and *dual inconsistent* if there exists $j \in N$ such that $d_{fj} > 0$ and $D_{Bj} \geq \mathbf{0}$.

We call a basis (a dictionary) *terminal* if it is either optimal, primal or dual inconsistent. The sign structure of optimal, primal and dual inconsistent dictionaries are illustrated in Figure 1, where we indicate the positive, nonnegative, negative, nonpositive and zero components by $+, \oplus, -, \ominus, 0$, respectively.



(a) optimal (b) primal inconsistent (c) dual inconsistent

Figure 1: Terminal dictionary structures.

For $r \in B - f$ and $s \in N - g$ with $d_{rs} \neq 0$, a pivot on (r, s) is said to be *admissible* if either (I) $d_{rg} < 0$ and $d_{rs} > 0$ or (II) $d_{fs} > 0$ and $d_{rs} < 0$. See Figure 2. Admissible pivots are natural for pivot algorithms because it exists whenever a basis is not terminal. All variants of the primal and dual simplex methods [4], Dantzig's parametric self-dual dual simplex algorithm [4] and variants of finite criss-cross methods [17, 10] use only admissible pivots.

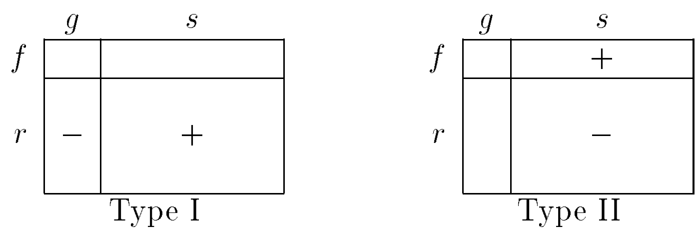


Figure 2: Two types of admissible pivots

2 Feasibility problems

In this section both the primal and dual feasibility problems are considered. Although by dualization one result can be derived from the other, we prefer to present both the primal and the dual case, because both are needed for the general LP case.

2.1 The primal feasibility problem

Let us consider the linear feasibility problem:

$$Ax \leq b, \quad x \geq 0 \tag{4}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

We make use of the notation introduced in the Section 1.1. We use the dictionary notation defined there, except we ignore the f-row representing the objective function that does not exists for the feasibility problem.

Our first problem is as follows:

Problem 2.1 *Let an arbitrary basis B and a feasible solution x^* be given. Find a short admissible pivot sequence from B to a feasible basis.*

We prove constructively by presenting an algorithm which terminates in at most n admissible pivots. We will see that the given feasible solution x^* will serve as a sort thread to a feasible basis.

The algorithm is as follows.

Admissible Pivot Algorithm

(Primal feasibility problem)

Initialization
Given: basis B; feasible solution $\overline{x}^*$.
Let $E = \{1, \cdots, m+n\}$, $I_* = \{i \in E \,|\, \overline{x}^*_i > 0\}$.
Set-up
Set up a subproblem by defining $F = I_* \cup B$. Fix the variables not in F to nonbasis permanently. The subproblem $\overline{A}_F \overline{x}_F = 0$, $\overline{x}_F \geq 0$, $\overline{x}_g = 1$ is feasible. The dictionary at this stage is as given at Figure 3.

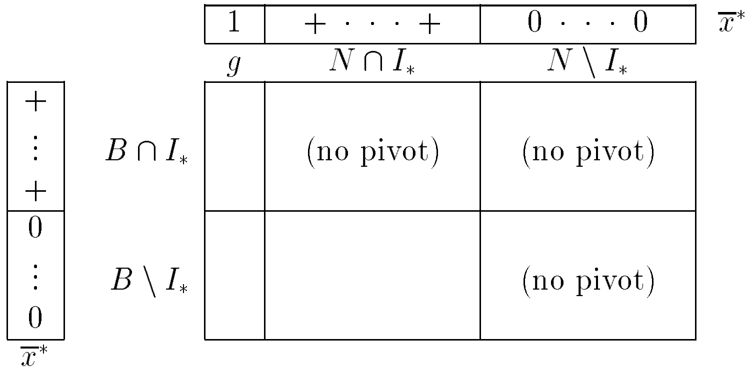


Figure 3: Initial dictionary, primal case.

Set $\overline{x} = x(B)$. We have the following cases:

If $\overline{x}_B \geq 0$, then a feasible basis is found, **Stop**.
If $\overline{x}_r < 0$, for some $r \in B \backslash I_*$, then **Reduce-F**;
else **Modify-$\overline{x}^*$**.

Reduce-F
Now there is an $r \in B \backslash I_*$ such that $\overline{x}_r < 0$. Then there is an $s \in I_* \backslash B = N \cap I_*$ such that pivot (r, s) is admissible (see Figure 4). This is true, because the problem $\overline{A}_F \overline{x}_F = 0$, $\overline{x}_F \geq 0$, $\overline{x}_g = 1$ is feasible.

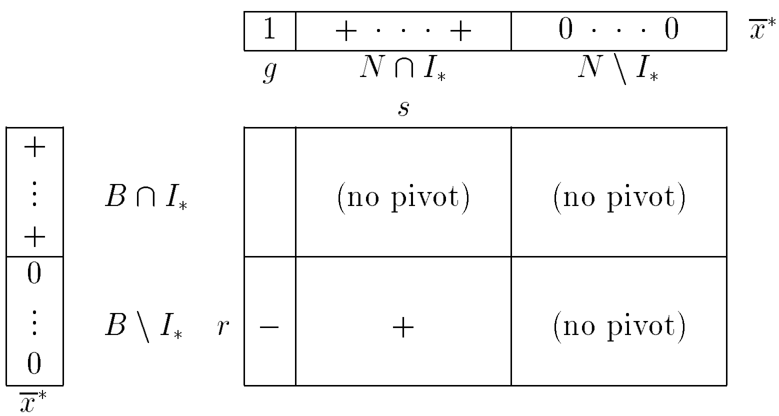


Figure 4: The situation when F is reduced.

Make a pivot at (r, s); let $B := B \cup \{s\} \backslash \{r\}$; $F := F \backslash \{r\}$.
Go back to Set-up.
Observe: By a pivot $|F|$ decreases by one; $\overline{x}^*$ does not change, only the basis.
Modify-$\overline{x}^*$
Here $\overline{x}_i \geq 0$ for all $i \in B \backslash I_*$, but there is an $i \in B \cap I_*$ such that $\overline{x}_i < 0$ (see Figure 5).

By taking an appropriate convex combination of the current basic solution and the feasible solution $\overline{x}^*$ we eliminate a positive coordinate of $\overline{x}^*$.

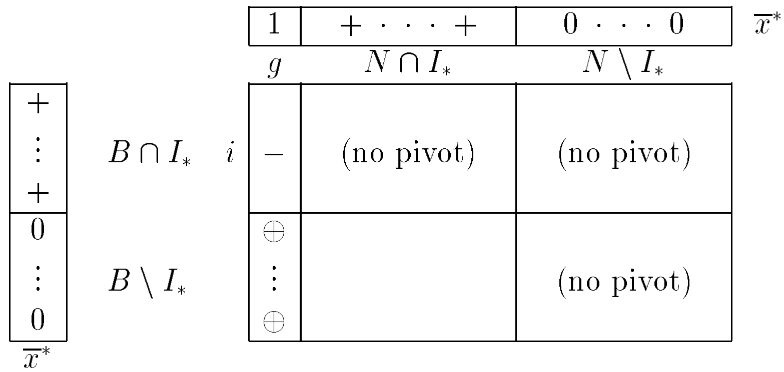


Figure 5: The situation when $\overline{x}^*$ is modified.

Let

$$\lambda := \min \left\{ \frac{\overline{x}_i^*}{\overline{x}_i^* - \overline{x}_i} \,|\, \overline{x}_i < 0 \right\} = \frac{\overline{x}_r^*}{\overline{x}_r^* - \overline{x}_r}.$$

Then by defining

$$\overline{x}^* := \lambda \overline{x} + (1 - \lambda) \overline{x}^* \geq 0$$

we have $A_I \overline{x}_I^* = b$ and so a new feasible solution with fewer nonzero coordinates is obtained. Let $I_* = \{i \in E \,|\, \overline{x}_i^* > 0\}$.
Go back to Set-up.
Observe: The set I_* changes. In the next iteration, because $\overline{x}_r < 0$ and $r \notin I_*$, the step Reduce-F will be applied and the set F will be reduced.

Now we are ready to prove strongly polynomial complexity of the algorithm.

Theorem 2.2 *For any given feasible solution $\overline{x}^*$ and any given basis B, there exists an algorithm to generate a sequence of at most n admissible pivots from B to some feasible basis.*

Proof. The presented admissible pivot algorithm produces an admissible pivot path which is initiated by the basis B and stops only if a feasible basis is found. At the step Modify-$\overline{x}^*$, no pivots is performed and the step Reduce-F will be applied in the next iteration. At the step Reduce-F, the cardinality of F is reduced by one. At initialization the cardinality of F is at most $m + n$, while at termination F contains at least m elements, thus the algorithm needs at most n pivots to find a feasible basis. ∎

Remark 2.3 *The information that a feasible solution is known was heavily used. This reminds the reader of basis identification techniques in interior point methods. However, there are two major differences: here only admissible pivots are allowed; convex combination of two solutions with ratio test had to be taken.*

Remark 2.4 *We do more than just purification of a feasible solution to a basic solution. Purification procedures allow arbitrary pivots while we use only admissible ones.*

Remark 2.5 *The algorithm does not give a strongly polynomial algorithm to solve the feasibility problem. The strongly polynomial complexity holds only, if a feasible, not necessarily basic, solution is known. However, this result makes clear that a short admissible pivot path always exist.*

2.2 The dual feasibility problem

Let us consider the dual linear feasibility problem:

$$A^T y \geq c, \qquad y \geq 0, \tag{5}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

We use the dictionary notation defined in Section 1.1, except we ignore the g-column representing the dual objective function that does not exists for the dual feasibility problem.

Our second problem is as follows:

Problem 2.6 *Let an arbitrary basis B and a feasible solution y^* be given. Find a short admissible pivot sequence from B to a dual feasible basis.*

Since the dual feasibility is equivalent to the primal feasibility, the results of the previous section apply directly and yield a (dual) algorithm which terminates in at most m admissible pivots.

We shall describe the dual of the feasibility algorithm below to prepare for the presentation of an LP algorithm in the next section that combines both.

Admissible Pivot Algorithm

(Dual feasibility problem)

Initialization
Given: basis B; dual feasible solution $\overline{y}^*$.
Let $E = \{1, \cdots, m+n\}$, $J_* = \{i \in E \,|\, \overline{y}_i^* > 0\}$.
Set-up
Set up a subproblem by defining $G = J_* \cup N$. Fix the variables not in G to be basis variables permanently. The dictionary at this stage is as given in Figure 6.

Set $\overline{y} = y(B)$. We have the following cases:
If $\overline{y} \geq 0$, then a dual feasible basis solution is found, **Stop**.
If $\overline{y}_s < 0$ for some $s \in N \backslash J_*$, then **Reduce-G**;
else **Modify-$\overline{y}^*$**.

Reduce-G
Now there is an $s \in N \backslash J_*$ such that $\overline{y}_s < 0$, and there is an $r \in B \cap J_*$ such that pivot (r, s) is admissible. (True, because the subproblem defined by G is feasible.) For the structure of the dictionary see Figure 7.

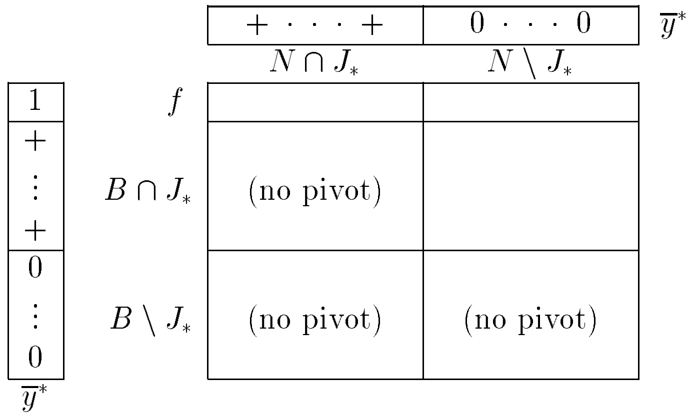


Figure 6: Initial dictionary, dual case.

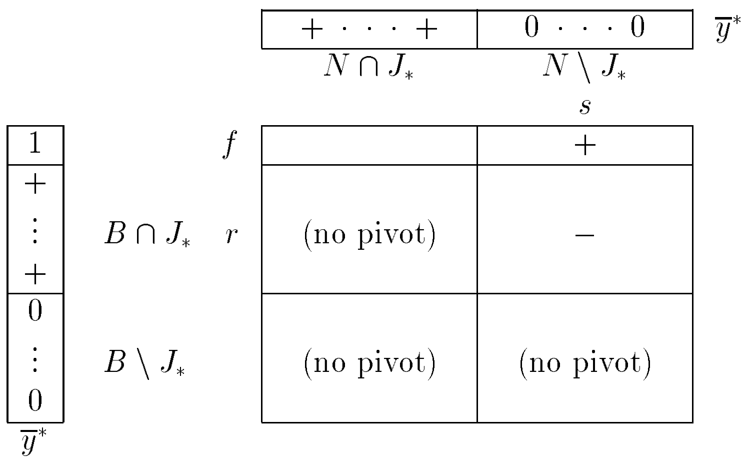


Figure 7: The situation when G is reduced.

Make a pivot at (r, s); let $B := B \cup \{s\} \setminus \{r\}$; $G := G \setminus \{s\}$.

Go back to Set-up.

Observe: Here $|G|$ decreases by one; $\overline{y}^*$ does not change, only the basis changes.

Modify-$\overline{y}^*$.

Here $\overline{y}_j \geq 0$ for all $j \in N \setminus J_*$, but there is a $j \in N \cap J_*$ such that $\overline{y}_j < 0$ (see Figure 8). By taking an appropriate convex combination of the current basic solution and the feasible solution $\overline{y}^*$ we eliminate a positive coordinate of $\overline{y}^*$.

Let

$$\lambda := \min \left\{ \frac{\overline{y}^*_j}{\overline{y}^*_j - \overline{y}_j} \,|\, \overline{y}_j < 0 \right\} = \frac{\overline{y}^*_s}{\overline{y}^*_s - y_s}.$$

Then by defining $\overline{y}^* := \lambda \overline{y} + (1 - \lambda) \overline{y}^*$ a new feasible solution with fewer nonzero coordinates is obtained. Let $J_* = \{ j \in E \,|\, \overline{y}^*_j > 0 \}$.

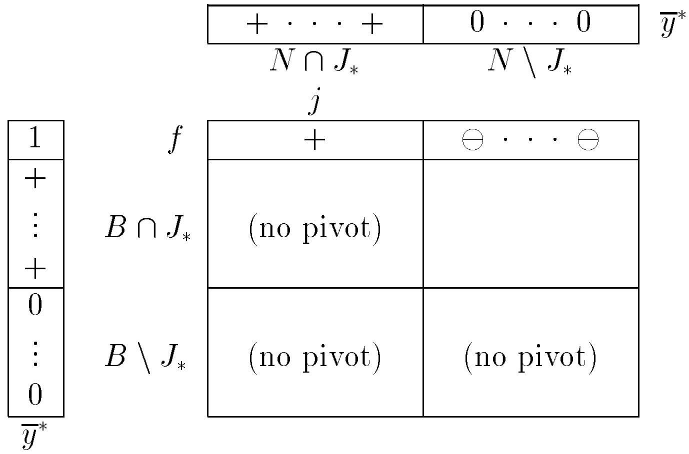


Figure 8: The situation when $\overline{y}^*$ is modified.

Go back to Set-up.
Observe: The set J_* changes. In the next iteration, because $\overline{y}_s < 0$ and $s \notin J_*$, the step Reduce-G will be applied and the set G will be reduced.

Now we are ready to prove strongly polynomial complexity of the algorithm, namely that at most m pivots are needed.

Theorem 2.7 *For any given dual feasible solution $\overline{y}^*$ and any given basis B, there exists an algorithm to generate a sequence of at most m admissible pivots from B to some dual feasible basis.*

Proof. The presented admissible pivot algorithm produces an admissible pivot path which is initiated by the basis B and stops only if a feasible basis solution is found. At the step Modify-$\overline{y}^*$ no pivot is performed and the step Reduce-G will be applied in the next iteration. At the step Reduce-G the cardinality of G is reduced by one. At initialization the cardinality of G is at most $m+n$, while at termination G contains at least n elements, thus the algorithm needs at most m pivots to find a dual feasible basis. ∎

Remark 2.8 *The information that a feasible solution is known was heavily used. This is similar to the dual side of basis identification techniques in interior point methods. There are two major differences: here only admissible pivots are allowed; a convex combination of two solutions had to be taken.*

3 The LP problem

Now we consider the general LP problem

$$\begin{array}{ll} \max & \{\, c^T x \mid Ax \le b,\ x \ge 0 \,\}, \\ \min & \{\, b^T y \mid A^T y \ge c,\ y \ge 0 \,\}, \end{array} \tag{6}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

We use the dictionary notation defined in Section 1.1.

Our main problem to be solved without any nondegeneracy assumption is as follows:

Problem 3.1 *Let an arbitrary basis B and an optimal solution pair x^*, y^* be given. Find a short admissible pivot sequence from B to an optimal basis.*

We prove constructively, without any nondegeneracy assumption, that the length of a shortest admissible pivot path is bounded from above by $m+n$.

The algorithm is as follows.

Admissible Pivot Algorithm

(LP problem)

Initialization
Given: basis B; $\overline{x}^*$ and $\overline{y}^*$ optimal solutions.
Let $E = \{1, \ldots, m+n\}$, $I_* = \{i \in E \,|\, \overline{x}^*_i > 0\}$, $J_* = \{j \in E \,|\, \overline{y}^*_j > 0\}$.
Set-up
Let $F = I_* \cup B$ and $G = J_* \cup N$. The dictionary at this stage is determined by basis B as it is given on Figure 9.

			1	$0 \cdots 0$	$+ \cdots +$	$0 \cdots 0$	$\overline{x}^*$
			$\cdot$	$+ \cdots +$	$0 \cdots 0$	$0 \cdots 0$	$\overline{y}^*$
			g	$N \cap J_*$	$N \cap I_*$	$N \setminus (I_* \cup J_*)$	
$\cdot$	1	f	$\cdot$				
$+ \; \vdots \; +$	$0 \; \vdots \; 0$	$B \cap I_*$		(no pivot)	(no pivot)	(no pivot)	
$0 \; \vdots \; 0$	$+ \; \vdots \; +$	$B \cap J_*$		(no pivot)			
$0 \; \vdots \; 0$	$0 \; \vdots \; 0$	$B \setminus (I_* \cup J_*)$		(no pivot)		(no pivot)	
$\overline{x}^*$	$\overline{y}^*$						

Figure 9: Initial dictionary.

Let $\overline{x}$ and $\overline{y}$ be the current primal and dual basic solutions.
We have the following cases:

If $\overline{x} \geq 0$, $\overline{y} \geq 0$, then an optimal basis is found, **Stop**;
else if there is an $r \in B \setminus I_*$ with $\overline{x}_r < 0$,

or there is an $s \in N \setminus J_*$ with $\overline{y}_s < 0$,
then **Reduce-F or -G**;
else **Reduce solutions**.

Reduce-F or -G
The same arguments as those that were used in the previous section for the feasibility problems prove that if there is an $r \in B \setminus I_*$ with $\overline{x}_r < 0$ or if there is an $s \in N \setminus J_*$ with $\overline{y}_s < 0$ then a proper admissible pivot exist. The possible admissible pivot situations are indicated in Figure 10. Specifically,

- if there is an $r \in B \setminus I_*$ with $\overline{x}_r < 0$ then there is a $s \in N \cap I_*$ such that pivot (r, s) is admissible;
- if there is an $s \in N \setminus J_*$ with $\overline{y}_s < 0$ then there is an $r \in B \cap J_*$ such that pivot (r, s) is admissible.

$\overline{x}^*$	$\overline{y}^*$			g	$N \cap J_*$	$N \cap I_*$	$N \setminus (I_* \cup J_*)$
		$\overline{x}^*$		1	$0 \cdots 0$	$+ \cdots +$	$0 \cdots 0$
		$\overline{y}^*$		$\cdot$	$+ \cdots +$	$0 \cdots 0$	$0 \cdots 0$
						s	s
$\cdot$	1	f		$\cdot$		$+$	$+$
$+ \cdots +$	$0 \cdots 0$	$B \cap I_*$			(no pivot)	(no pivot)	(no pivot)
$0 \cdots 0$	$+ \cdots +$	$B \cap J_*$	r	$-$	(no pivot)	$-$; $+$ (row r)	$-$
$0 \cdots 0$	$0 \cdots 0$	$B \setminus (I_* \cup J_*)$	r	$-$	(no pivot)	$+$	(no pivot)

Figure 10: Possible admissible pivots to reduce either F or G.

Make a pivot at (r, s); update the data structure.
Go back to Set-up.
Observe: Here neither $\overline{x}^*$ nor $\overline{y}^*$ change, only the basis. One of the cardinalities of F or G decreases.
Modify-solutions
Here $\overline{x}_i \geq 0$ for all $B \setminus I_*$ and $\overline{y}_j \geq 0$ for all $N \setminus J_*$, but
- either there is an $i \in B \cap I_*$ such that $\overline{x}_i < 0$, then **Modify-$\overline{x}^*$**;
- or there is a $j \in N \cap J_*$ such that $\overline{y}_j < 0$, then **Modify-$\overline{y}^*$**.

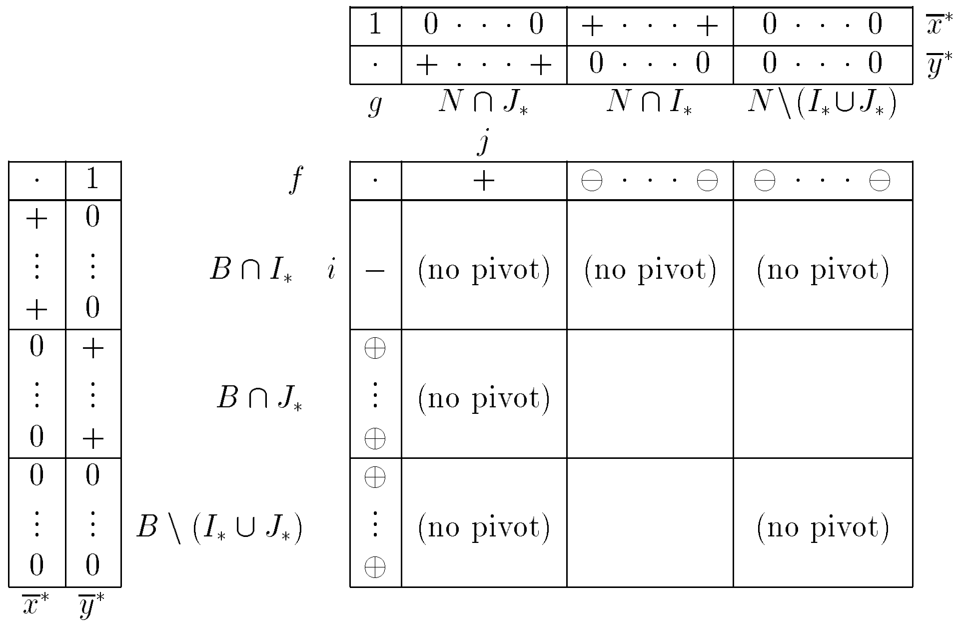


Figure 11: The situation when $\overline{x}^*$ or $\overline{y}^*$ is modified.

This situation is illustrated by Figure 11. Now we are going to show that a positive coordinate of $\overline{x}^*$ or $\overline{y}^*$ can be eliminated without losing optimality. First we make some observations that clarify the structure presented in Figure 11.

Observations.

1. Because no desired admissible pivot exists, so we have
   - $-\overline{y}_j \leq 0$ for $j \in N \backslash J_*$;
   - $\overline{x}_i \geq 0$ for $i \in B \backslash I_*$.

2. Although the current basis might be both primal and dual infeasible, the current objective value equals to the optimal one.

   Proof. Let us assume to the contrary that the current objective value is not equal to the optimal one. If it has a larger value, then we can take an appropriate convex combination $\tilde{x} = \lambda \overline{x} + (1-\lambda)\overline{x}^*$ of $\overline{x}$ and $\overline{x}^*$. For

   $$0 < \lambda \leq \min \left\{ \frac{\overline{x}_i^*}{\overline{x}_i^* - \overline{x}_i} \,|\, \overline{x}_i < 0 \right\}$$

   $\tilde{x}$ is a primal feasible solution with better than the optimal objective value. This is a contradiction.

   On the other hand, if the current objective value would be lower than the optimal value, then we could take an appropriate convex combination $\tilde{y} = \lambda \overline{y} + (1-\lambda)\overline{y}^*$ of $\overline{y}$ and y^*. For

   $$0 < \lambda \leq \min \left\{ \frac{\overline{y}_i^*}{\overline{y}_i^* - \overline{y}_i} \,|\, \overline{y}_i < 0 \right\}$$

$\tilde{y}$ is a dual feasible solution with better than the optimal objective value. This is also a contradiction, thus the current objective must be equal to the optimal one. ∎

3. We have $\overline{x}_i = 0$ for $i \in B \cap J_*$.

   Proof. If there is an $i \in B \cap J_*$ such that $\overline{x}_i > 0$ then by taking an appropriate convex combination $\tilde{x} = \lambda \overline{x} + (1-\lambda)\overline{x}^*$ of the current solution $\overline{x}$ and the primal optimal solution $\overline{x}^*$ we get that $\tilde{x}$ is optimal with a positive coordinate in J_*, i.e. it is not complementary w.r.t. the dual optimal $\overline{y}^*$. This is a contradiction. ∎

4. Similarly, we have $\overline{y}_j = 0$ for $j \in N \cap I_*$.

   Proof. If there is a $j \in N \cap I_*$ such that $\overline{y}_j > 0$ then by taking an appropriate convex combination $\tilde{y} = \lambda \overline{y} + (1-\lambda)\overline{y}^*$ of the current dual solution $\overline{y}$ and the dual optimal solution $\overline{y}^*$ we get that $\tilde{y}$ is optimal with a positive coordinate in I_*, i.e. it is not complementary w.r.t. the optimal $\overline{x}^*$. This is a contradiction. ∎

Now the sign structure presented in Figure 11 is justified. Using this sign structure the promised elimination can be made.

Modify-$\overline{x}^*$

If $\overline{x}_i < 0$ for some $i \in B \cap I_*$, then let

$$\lambda := \min\left\{ \frac{\overline{x}_i^*}{\overline{x}_i^* - \overline{x}_i} \,|\, \overline{x}_i < 0 \right\} = \frac{\overline{x}_r^*}{\overline{x}_r^* - \overline{x}_r}.$$

Then by defining the new optimal solution as $\overline{x}^* := \lambda \overline{x} + (1-\lambda)\overline{x}^*$ we have $\overline{x}_r^* = 0$, i.e. the new primal optimal solution $\overline{x}^*$ has fewer nonzero coordinates. Let $I_* = \{i \in E \,|\, \overline{x}_i^* > 0\}$.

Go back to Set-up.

Observe: The set I_* changes. In the next iteration, because $\overline{x}_r < 0$ and $r \notin I_*$, the step Reduce-F will be applied and the set F will be reduced.

Modify-$\overline{y}^*$

If $\overline{y}_j < 0$ for some $j \in N \cap J_*$, then let

$$\lambda := \min\left\{ \frac{\overline{y}_i^*}{\overline{y}_i^* - \overline{y}_i} \,|\, \overline{y}_i < 0 \right\} = \frac{\overline{y}_s^*}{\overline{y}_s^* - \overline{y}_s}.$$

Then by defining the new dual optimal solution as $\overline{y}^* := \lambda \overline{y} + (1-\lambda)\overline{y}^*$ we have $\overline{y}_s^* = 0$, i.e. the new dual optimal solution y^* has fewer nonzero coordinates. Let $J_* = \{i \in E \,|\, \overline{y}_i^* > 0\}$.

Go back to Set-up.

Observe: The set J_* changes. In the next iteration, because $\overline{y}_s < 0$ and $s \notin J_*$, the step Reduce-G will be applied and the set G will be reduced.

Now we are ready to prove our main result.

Theorem 3.2 *For any given optimal pair $(\overline{x}^*, \overline{y}^*)$ and any given basis B, there exists an algorithm to generate a sequence of at most $m+n$ admissible pivots from B to some optimal basis.*

Proof. The presented admissible pivot algorithm produces an admissible pivot path which is initiated by the basis B and it stops only if an optimal basis is found. At the step **Reduce-F or -G** either $|G|$ reduces by one, which can happen at most $m = |B|$ times, or $|F|$ reduces by one, which can happen at most $n = |N|$ times. At the step **Modify-solutions** no pivot is performed, and in the following iteration, one of the steps Reduce-F or Reduce-G will be applied.

Consequently, we need at most $m + n$ pivots to find an optimal basis. ∎

Remark 3.3 *In the step* **Modify-solutions** *both the primal and dual optimal solutions can be reduced if there exist both* $\overline{x}_i < 0$ *for some* $i \in B \cap I_*$ *and* $\overline{y}_j < 0$ *for some* $j \in N \cap J_*$.

Remark 3.4 *The information that an optimal solution pair is known was heavily used. The algorithm contains elements similar to that used in basis identification techniques in interior point methods [14, 15]. The algorithm does not give a strongly polynomial algorithm to solve the LP problem.*

4 Summary

We have shown that with no degeneracy assumption a short sequence of admissible pivots from any basis to an optimal basis exists. The length of this admissible pivot path is bounded by $m+n$. This result is in contrast with the result proved by Fukuda, Lüthi and Namiki [7]. They proved that in the fully nondegenerate case the length of such a path can be bounded by $\min\{m, n\}$.

As an introduction to the general LP case and explore some of our techniques in degenerate situations, we have proved that from any basis a short admissible pivot path exists which leads to a feasible basis both for the primal and dual feasibility problems. The length of the respective paths are bounded by n and m, the size of the dual and primal bases, respectively.

Our LP result put the case of "simplex" pivots opposed to "admissible" pivots in sharp contract, since to date there is no easy way to analyze the length of a shortest sequence of simplex pivots from any basis to an optimal basis.

Of course, we have proved merely the existence of a short sequence of admissible pivots for the general LP, we have not designed a polynomial admissible pivot algorithm, which is an ultimate goal. We hope that our existence results provide many researchers with a good incentive and some guidance to look for such a jewel.

Finally let we make some remarks on possible extensions.

It appears to be hard to extend our main result to the case of sufficient LCP's [3, 9, 8], although it is possible for the nondegenerate case [7]. So the following problem remains unsolved:

Open Problem 4.1 *Let a sufficient* $n \times n$ *LCP be given. Without any nondegeneracy assumption prove that the length of the shortest admissible pivot sequences* from any *(not necessarily feasible) complementary basis to* a feasible complementary basis *can be bounded by* n.

It is important to note that the optimal value (not only its sign) was used in justifying our algorithm in the LP case. This fact indicates that the LP results of the present paper cannot naturally be extended to the more general setting of oriented matroids. The results for the feasibility problems generalize straightforwardly, just like in [9] where the LCP duality theorems are proved in this setting.

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