

Mathematical investigations of functional interpretation of a constructive arithmetic system with the bar induction (of type 0) and definitions by transfinite recursion

1 Historical remark of an axiom *TRD*

The relationship between the *comprehension axiom* and the *inductive definition* has been investigated since 1950's. For example, Kleene in [?] proved that every Π_1^1 -relation on natural numbers is inductive. From his article, it follows that every Π_1^1 -set can be formalized by ID_2 (twice iterated inductive definition). (See [?] or P24 in [?].) In 1970, Feferman in [?] also established a relationship between ID_ω (ω -iterated inductive definition), Π_1^1 -comprehension axiom and the bar induction. On the other hand, Takeuti in [?] established a consistency proof of Π_1^1 -CA (the second order arithmetic with Π_1^1 -comprehension axiom). The consistency proof is carried out in the form of Gentzen-style consistency proof of first order arithmetic by applying the system of ordinal diagrams.

Yasugi in (1) (Section 4 below) established the well-foundedness of ordinal diagrams, which is called an "accessibility proof (of ordinal diagrams)". She formalized the accessibility proof in a formal system of constructive arithmetic, called **ASOD**, which contains a certain inductive definition called *TRD*. In this work, she formalized (the essential part of) the consistency proof of Π_1^1 -CA by an weakened version of inductive definition instead of directly constructing Π_1^1 -sets by traditional inductive definition.

As a reference, let us state a traditional form of inductive definition as well as TRD (**D**efinition by **T**ransfinite **R**ecursion).

1. *ID*: Let $F(x, X)$ be a formula with a set parameter X , where we assume that X occur positively in F . Then, the axiom $ID(F)$ ($= ID_1(F)$) claims:

$$(i) F(x, \{y | H(y)\}) \Rightarrow H(x);$$

$$(ii) \forall x[F(x, \{y | B(y)\}) \Rightarrow B(x)] \Rightarrow \forall x[H(x) \Rightarrow B(x)] \text{ for each formula } B(x).$$

where we fix a predicate constant H , which represents the least fix point of F .

2. *TRD*: Preceding the definition of *TRD*, we need to fix a primitive recursive well-order $<_I$ (In practice, we needed only orders below ε_0 .) and a formula $G(x, X)$ which satisfies that no scope of an existential quantifier in G contains X . (Such a formula is called *admissible*.) Then, the axiom $TRD(<_I, G)$ claims:

$$\forall x[G(x, \{y | y <_I x \ \& \ H(y)\}) \Leftrightarrow H(x)].$$

(The definition of *TRD* in our papers is slightly different from the form above for technical reasons, but it is not an essential matter.)

2 Functional interpretations of systems with TRD

Functional interpretations for constructive arithmetic systems have been studied by many logicians for a number of reasons. For example, Gödel in [?] proposed an interpretation of \mathbf{HA} -formulas in terms of quantifier-free formulas of a functional theory called \mathbf{T} . This interpretation is often called *Dialectica interpretation*. Gödel showed that \mathbf{HA} is Dialectica interpretable in \mathbf{T} . By this theorem, the consistency of \mathbf{PA} can be reduced to the consistency of \mathbf{T} .

At the end of [?], he suggested the possibility of extending his functional interpretation to stronger theories in two ways. The first suggestion is to define functional systems stronger than \mathbf{T} by adding certain inferences useful for constructive mathematics. Upon this suggestion, Spector introduced the bar-induction (of finite types) and he established a functional interpretation of $\mathbf{HA+BI}$ (bar induction) to a functional system with bar-recursion. (See [?] or Section 6 in [?].)

The second suggestion is to define functional systems stronger than \mathbf{T} by admitting transfinite types. Upon this suggestion, several functional interpretations for systems with the axiom called inductive definition have been studied. For example, Avigad in [?]; Avigad1998 established a Dialectica interpretation of a weakened version of finitely iterated inductive definitions into systems called predicative functional systems.

Spector's interpretation provides a functional interpretation of the full second-order arithmetic, while Avigad's interpretation provides a characterization of functions which are provably total in \mathbf{ATR}_0 . Therefore, Gödel's suggestions are very important.

On the other hand, Yasugi in (1) independently have developed a functional interpretation for \mathbf{ASOD} . In (1), she obtained the functional interpretation by extending the mr(modified realizability)-translation for \mathbf{HA} , which had been defined by Kreisel, to definition by transfinite recursion. Using this mr-translation, she showed that the functional structure of ordinal diagrams can be represented by a term-system called \mathbf{HP} which was defined in (1).

3 \mathbf{TRDB} and \mathbf{TRM}

In (2) (Section 4 below), Yasugi and Hayashi formulated a constructive arithmetic called \mathbf{TRDB} , which is a streamlined version of \mathbf{ASOD} . Roughly speaking, \mathbf{TRDB} is a constructive system of (essentially) first order arithmetic consisting of \mathbf{HA} (with function variables), TRD and the bar induction of type 0 applied to *admissible formulas*. \mathbf{ASOD} is a special case of \mathbf{TRDB} .

It is interesting to develop a nice functional interpretation of a system with TRD so that the interpretation reflects an algorithm inherent in a proof of the

system, which involves properties of the well-order in *TRD*. So, the authors expect that **TRDB** is a mathematically interesting system on its own right.

The authors also formulated a term-system **TRM** (a reformulation of the term-system **HP** defined in (1)). **TRM** is a functional system in which the algorithms inherent in proofs of **TRDB** can be interpreted. Each type of **TRM** is of transfinite nature, that is, a kind of parametric type, which is called a “type-form”. Among the constructors of type-forms are the transfinite recursor (along some well-ordered structure), parametric abstraction, projection to parametric objects and the conditional definition. Terms (called term-forms) belonging to such type-forms are defined in terms of abstraction, conditional definition, primitive recursion and bar recursion (of type 0).

Note. Let us consider the set $H (= \{x|H(x)\})$ defined in the axiom *ID* above. H is taken to be the least fix point of F which is fixed in *ID* in advance, that is, H is defined to be the least set satisfying $(*) \forall x(F(x, X) \Rightarrow x \in X)$. In the set theoretical way, we can formalize such an H by the Π_1^1 -formula below:

$$a \in H \Leftrightarrow \forall X[\forall x(F(x, X) \Rightarrow x \in X) \Rightarrow a \in X].$$

In such a case, H is defined by considering all sets satisfying $(*)$. (That is, we assume a principle strong enough to construct power sets.)

On the other hand, in *TRD*, the set H is regarded to be defined “recursively” along a well-order $(I, <_I)$ (defined in advance). That is, for each element $i \in I$, $H(i)$ is defined by the properties of G and $\{H(j) \mid j <_I i\}$ “in a linear way”. In this sense, we can regard a set defined by *TRD* as being of “first order”.

Similarly, we do not define the (transfinite) types of **TRM** as those of the second order lambda calculus F . In fact, each type of **TRM** is defined along the well-order $<_I$ recursively, that is, in a way of constructing “first order” objects

4 Papers for TRDB

Since (1) was published, Yasugi and others have investigated properties of **TRDB** and its functional interpretation. We list several papers devoted to such investigations below.

(1) **M. Yasugi**,
Hyper-principle and the functional structure of ordinal diagrams,
Comment. Math. Univ. st. Pauli, **34** (1985) 227-263 (the opening part); **35** (1986) 1-37 (the concluding part).

In the opening part of this article, the author carries out a formal accessibility proof of the system of ordinal diagrams. There, she employs a constructive system of arithmetic called **ASOD**.

The first half of the concluding part is dedicated to the theory of “methods”, which was called the “hyper-principle” (**HP**). It serves as the basis of the functional interpretation of ordinal diagrams, and is an entirely new theory. The theory (term-system) **HP** consists of generalized notion of types, functionals and formulas. The author also develops semantics of **HP**.

In the latter half, the mr-translation is applied to the formulas of **ASOD**. The author proves that **ASOD** is mr-interpreted in **HP**. It follows that the functional structure of ordinal diagrams is represented by **HP**.

(2) **M. Yasugi and S. Hayashi,**
A functional system with transfinitely defined types,
Springer-Verlag, Lecture Notes in Computer Science, 792 (1994) 31-60.

In this paper, the authors present a revised version of the system of term-forms, named **TRM** and consider **ASOD** in a more general setting, renaming it as **TRDB**.

The authors prove the normalization theorem in **TRM**, that is, normalizability of type-forms and term-forms. They also present a direct translation of **TRDB** to **TRM**, associating formulas with type-forms and proofs with term-forms. It follows that the computational mechanism of **TRDB** is distilled via the translation.

M. Yasugi and S. Hayashi,
Interpretations of transfinite recursion and parametric abstraction in types,
Words, Languages and Combinatorics II, World Scientific Publishing Co., (1994) 452-464.

In this paper, a set model of type-forms is presented, and the term-forms are interpreted as functions in the image sets of type-forms. Since a map from **TRDB** to **TRM** is defined in [3] above, this means that **TRDB** also has a set model. Another theme in this article is the reverse map, that is, translation of terms of **TRM** into proofs of **TRDB**. In other words, one can induce proofs from programs, a reversal of the usual study.

(4) **O. Takaki,**
Normalization of natural deductions of a constructive arithmetic for transfinite recursion and bar induction,
Notre Dame Journal of Formal Logic, 38:3 (1997) 350-373.

In this paper, the author proves the strong normalization theorem for the natural deduction system of **TRDB**. Applying the strong normalization theorem, the author establishes a syntactic consistency of this system and prove the existence-property and the disjunction-property.

(5) **O. Takaki and M. Yasugi,**
Modified realizability for TRDB,
Algebraic Engineering, World Scientific Publishing Co., to appear.

The main purpose of this paper is to introduce an extension of a constructive arithmetic **TRDB** so that the extension is complete with respect to the modified realizability interpretation for **TRDB**. The authors first introduce an extension of **TRDB**, called **S**, and then define a modified realizability interpretation on **S**. As the main theorem, they show the completeness of **S** with respect to the modified realizability interpretation.

(6) **O. Takaki and M. Yasugi,**
An MR-complete extension of TRDB and its functional interpretation,
Submitted to the Proceedings of the 7th Asian Logic Conference

This paper is a sequel to the previous work in (5) above. In this paper, the authors investigate the semantics of **S** in terms of **TRM**, stating that the algorithm which is claimed by *MR*-interpretation of a theorem of **S** is realized by a functional in **TRM**. As corollaries, we can claim that all provably total functions in **S** can be realized in **TRM** and that **S** is syntactically consistent.

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