Teaching fractions in elementary school: A manual for teachers

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[Added, March 1999] *This is a preliminary draft of what is projected to be a monograph. Even as a draft, this electronic version is incomplete because some line drawings and annotations to drawings are absent due to the author's incompetence with computer graphics. Nevertheless, I have decided to post this incomplete version on the web because its central idea of how to approach the teaching of fractions from grade 5 and up is sound, and at present nothing resembling it seems to be available.*

It is to be emphasized that (1) the primary audience of this article is teachers of grades 5–8, not students, and (2) this approach to fractions is definitely not for grades 2–4 where fractions first make their tentative appearances. The overriding concern here is that, after children's informal encounter with fractions in the early grades, they reach the point in grade 5 where their haphazard knowledge needs consolidation and their initial foray into abstract mathematics (fractions) needs some structure for support. This article is designed to help teachers face up to the challenge of leading students to the next phase of mathematical achievement.

This manual is essentially the content of a workshop I conducted in March of 1998 for teachers of grades 5–8. The main goal is to outline a different way to teach (positive) fractions *to the 5th or 6th grade*. This approach to fractions is a geometric one, where a fraction is identified unambiguously as a point on the number line. Thus a fraction becomes a clearly defined mathematical object. Moreover, the emphasis throughout is on mathematical reasoning as we progress from topic to topic; a reason is supplied for each and every step. Special care has been given to avoid invoking the usual device of *deus ex machina* found in almost all school textbooks, whereby new properties are periodically conferred on fractions—as if by heavenly decrees—without any mathematical justification.

To someone not familiar with the mathematics of elementary school, the fact that we give a clearcut definition of a fraction must seem utterly trivial. After all, how can we ask students to add and multiply and divide fractions if we don't even tell them what fractions are? Unfortunately, it is the case that school texts usually do not define fractions. Try looking up the index of an elementary school mathematics text can be a very disconcerting experience. Under "fractions" one can find many items, such as "add", "multiply", "mixed numbers", or "estimate", but never "definition" or "meaning". This state of ambiguity not surprisingly spawns endless confusion about the concepts of "ratio", "proportion", and "percent"; see the last last section (§7) of this manual. It would be reasonable to conjecture that the Fear of Fractions (cf. [1] and [3]) can also be traced to this singular pedagogical phenomenon.

Certain mathematical and pedagogical aspects of the following exposition are worth mentioning. On the mathematical front, the difficulty with the teaching of fractions lies in the fact that the correct definition of fractions (rational numbers) as equivalence classes of ordered pairs of integers is abysmally unsuitable for teaching at the level of the 5th or 6th grade. Furthermore, the formal definitions of the arithmetic operations as those dictated by the requirements of the field axioms are also grossly improper for instruction at this level because of the lack of motivation. It has long been recognized that a compromise between what is mathematically correct and what is pedagogically feasible must be forged. For this reason, I have chosen to define a fraction as a point on the number line (while dodging the explanation of what a "number" is), and have introduced addition, multiplication, and division on fractions by extending the intuitive idea—gleaned from experiences with natural numbers—of what these operations *ought to be*. All this is part of an effort to keep the mathematics plausible as much as possible. There is a price, however. The expositon almost seems to "prove" that my definitions are the correct ones, whereas it merely motivates them. But I decided not to enter into a discussion of the subtle distinction between the two on account of my belief that, *at this level*, it is more important to build upon students' intuition and prior knowledge than to prematurely impose on them a rigid logical scaffolding. Having said this, I am obliged to point out that while this approach is by design mathematically faulty, it is nevertheless a step forward compared with the methods used by most textbooks to teach fractions because it would be difficult to find any merit in the latter.

On the pedagogical front, it should be pointed out that inasmuch as this manual is not a textbook on fractions but the content of a workshop for teachers, the level of the presentation is higher than is appropriate for a 5th grade class (say). For example, at times I make use of symbolic notation in discussions (e.g., $\frac{k}{l} + \frac{m}{n}$) in order to save time. Actual teaching in a 5th grade classroom should probably avoid that. I also make use of fairly large numbers (in both denominators and numerators) for my first examples on each topic, whereas in a classroom it would be advisable to start off with $\frac{1}{2}$, $\frac{1}{3}$ and the likes. In addition, since the workshop was designed also to increase the mathematical knowledge of the participants, I have supplied more proofs than would be advisable in a 5th grade classroom. I hope teachers reading this would make judicious choices on what to give to their students. In the same breath, let me also express my strong personal belief that such choices are best made only *after* they have mastered *everything* in this manual.

I mentioned in the preceding paragraph that I used fairly large numbers for illustration. Part of the reason is that I was inspired to give my workshop on fractions by the following passage on p. 96 of the NCTM Standards [2]:

> The proficiency in the addition, subtraction, and multiplication of fractions and mixed numbers should be limited to those with simple denominators that can be visualized concretely and pictorially and are apt to occur in real-world settings; such computation promotes conceptual understanding of the operations. This is not to suggest however that valuable time should be devoted to exercises like $\frac{17}{24} + \frac{5}{18}$ or $5\frac{3}{4} \times 4\frac{1}{4}$ which are much harder to visualize and unlikely to occur in real life situations. Division of fractions should be approached conceptually.

I could not envision what kind of school mathematics education would, as a matter of principle, favor fractions with simple denominators over those with large ones, or how the so-called conceptual understanding of fractions could be applicable only to the former kind of fractions. As to the trivial computations of $\frac{17}{24} + \frac{5}{18}$ and $5\frac{3}{4} \times 4\frac{1}{4}$, they are the kind that every student who claims to know *anything* about fractions should be able to do in his or her sleep. There is clearly room for improvement in this part of the school mathematics curriculum. The present manual is my contribution in this direction.

After I had finished writing this manual, I got to read the volume [9] on rational numbers; it is apparently somewhat difficult to find, but my colleague Leon Henkin was kind enough to show me a copy. Its primary audience is also school teachers and its Chapters 2-6 are devoted to what we call "fractions" here. There are points of contact between [9] and this manual. In particular, I was both surprised and gratified by the close similarity between the first two sections of this manual and Chapter 2 of [9] due to Henkin.

1

To begin the mathematics, the *natural numbers* 0, 1, 2, 3, 4, . . . will be represented as points on a line. So take an infinite line segment extending to the right from a fixed point. That point will be designated as 0. Select a unit of length (1 in., 1 cm, etc.) and mark it off to the right as 1. Then mark off segments of unit length successively to the right to get 2, 3, 4, etc., as on a ruler. We create thereby an "infinite ruler" where the postion of each natural number n on the line indicates the length of the segment from 0 to n. We shall refer to it as the *number line*.

```
-------------------------------------------------------
0       1       2       3       4       5      etc.
```

In this setting, each natural number is not only a point on the number line, but also a length. For example, the (line) segment from 0 to the number 4 on this infinite ruler has length exactly 4. Incidentally, we shall denote such a segment by the symbol $[0, 4]$. (By the same token, $[m, n]$ will denote the segment from m to n for any natural numbers m and n.) The double meaning attached to a number would not be a source of difficulty in students' learning because they are already used to a similar situation with the ordinary ruler.

Our immediate goal is to translate the usual four arithmetic operations on natural numbers into geometric terms. In a classroom, this would require giving students sufficient number of drills until the geometric interpretations become *instinctive* with them. Let us start with addition: adding (say) 4 to a number is the same as shifting that number to the right of the infinite ruler by 4 units. In analogy with (the ancient instrument of) a slide rule, we can visualize this as follows:

```
0    1    2    3    4    5    6    7    8    9   10   11  etc.
---------------------------------------------------------------
                                   |                   |
                    --------------------------------------------
                    0    1    2    3    4    5    6    7    8
```

Imagine sliding the infinite ruler to the right until the 0 is under 4, then the value of $4+n$ can be obtained by locating the number in the upper ruler that is directly above the number n in the lower ruler. In the figure above, we have indicated how 7 and 11 are the values of $4+3$ and $4+7$, respectively.

For us, a more useful way to look at addition is the following:

> $4+5=$ the length of two abutting segments one of length 4 and the other of length 5

Note that here as later, when we speak of *abutting segments* we mean that the segments are lined up in a straight line from end to end. For example, the segments $[0,4]$ and $[0,5]$ may be used for this purpose:

```
                0   1   2   3   4   5
-------------------------------------
0   1   2   3   4

<-----------total length = 9-------->
```

In general, for any two natural numbers l and k, we define:

$$l+k \;=\; \text{the length of two abutting segments one of length } l \text{ and the other of length } k \qquad (1)$$

Next, subtraction. One can obtain the answer to $5 - 2$, for example, by close analogy with the above "sliding rule" method we employed for addition: slide the lower infinite ruler this time to the left until 2 is right under the 0 of the upper ruler, then the number (which is naturally 3) on the upper ruler which is right above the 2 of the lower ruler provides the answer. But again, a more useful way to think of this is the following: for natural numbers k and l such that $k > l$,

$$k - l = \text{the length of the remaining segment when a segment of length } l \text{ is removed from a segment of length } k \qquad (2)$$

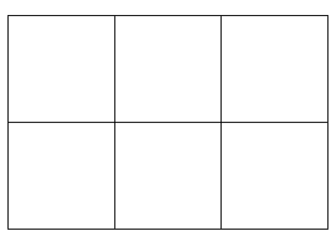
Multiplication is more complicated. 3×2 is of course the length of 3 abutting segments each of length 2, and in general for natural number k and l,

$$k \times l = \text{the length of } k \text{ abutting segments each of length } l \qquad (3)$$

It goes without saying that in the preceding statement, k and l can be interchanged. This amounts to the commutative law of multiplcation, but we shall refer to this more formal aspect of fractions only sparingly. This said, we turn to a more useful version of the multiplication of natural numbers in terms of the concept of area of a rectangle:

$$3 \times 2 = \text{the area of a rectangle with sides 3 and 2}$$

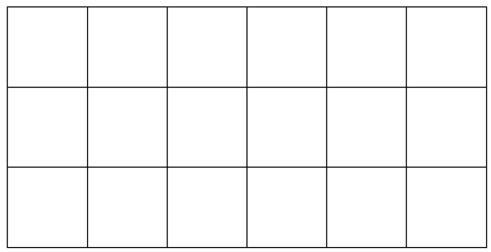
This is seen by partitioning this rectangle into *unit squares* (*i.e.,* squares with sides of unit length) and count that there are 6 of them. Since each unit square has area equal to 1, the area of the rectangle is 6.



In general, if n and k are natural numbers, then

$$n \times k = \text{the area of a rectangle with sides } n \text{ and } k \qquad (4)$$

By the times students are in the 5th grade, they should be already familiar with this interpretation of multiplication. Just to be sure, one should make them do many examples, such as $3 \times 6 = 18$:



Division among natural numbers of course applies only to the situation where one number evenly divides another. When we say $k \div l = n$, we mean $k = n + n + \cdots + n$ (l times), or, "k can be divided into l equal parts, and each part is of size n".[1] In other words,

$$k \div l = n \quad \text{exactly when} \quad k = l \times n \tag{5}$$

In geometric terms, we use (5) to interpret $l \times n$ as the length of l abutting segments each of length n, so that for $k = l \times n$,

$$k \div l \;=\; \text{the length of a subdivision when a segment of length } k \text{ is divided into } l \text{ equal parts} \tag{6}$$

2

With the arithemtic of natural numbers out of the way, we are ready to tackle fractions. Our first task is to give meaning to $\frac{2}{3}$, $\frac{4}{7}$, or more generally, $\frac{m}{n}$ where m and n are natural numbers. These will be points on the number line. However, to prepare students for this definition, it is important they get used to the idea of dividing a given line segment into l equal part, where l is an arbitrary natural number. In *practical* terms, of course, this can be easily accomplished: if we are asked to divide (say) $[0, 1]$ into 7 equal parts,

[1] We have purposely ignored the alternate interpretation of $k \div l = n$ as the division of k into n parts each of which has size l.

just use grid paper and choose a unit length to consist of 7 or 14 grids (or for that matter, any multiple of 7). The main point is that students should feel entirely comfortable with the idea of equal subdivision of a segment because it will serve as the foundation of this particular approach to fractions.

To this end, we introduce a geometric construction that routinely divides a given segment into any number of equal parts. This construction, strictly speaking, is not needed in our development of fractions. However, it is a fun activity that students would enjoy and it helps to put students psychologically at ease with the idea of arbitrary equi-subdivisions of a given segment. Contrary to the prevailing trend in education, it is also a manual activity using ruler, compass, and triangles rather than an electronic one. Thus suppose we have to divide the segment AB below into 3 equal parts. We draw a ray ρ issuing from A and, using a compass, mark off three points C, D, and E in succession on ρ so that $AC = CD = DE$—the precise length of AC being irrelevant. Join BE, and through C and D draw lines parallel to BE which intersect AB at C' and D'. A basic theorem of Euclidean geometry then guarantees that C' and D' are the desired subdivisions of AB, *i.e.,* $AC' = C'D' = D'B$.

```
-------------------------
A                       B
```

It should be added that while there is a standard Euclidean construction with ruler and compass of a line through a given point parallel to a give line, the practical (and quick) way to draw such lines using a ruler and a (plastic) triangle is of interest. So position such a triangle so that one side is exactly over the points B and E. *Keeping the triangle fixed in this position*, now place a ruler snugly along another side of the triangle. Since there are two other sides of the triangle, choose one so that when *the ruler is now held fixed* and the triangle is allowed to glide along the ruler, the side of the triangle that was originally on top of B and E now passes over the points D and C as the triangle slides along the ruler in the direction of the point A. When this side is over D, hold the triangle and draw the line along this side through D; this line is then parallel to BE. Same for the point C.

There is nothing special about the number 3 in the preceding construction. We could have asked for a division of AB into 7 equal parts, in which case we would mark off 7 points $A_1, A_2, \ldots, A_7$ in succession on the ray ρ so that $AA_1 = A_1A_2 = \cdots = A_6A_7$. Join A_7B, and then draw lines through $A_1, \ldots, A_6$ parallel to A_7B as before. The intersections of these parallel lines with AB then furnish the desired subdivisions of AB.

Be sure to devote sufficient class time to this activity until students can treat the division of a segment into any number of equal parts as a routine matter before proceeding any further.

Now we can give the definition of fractions. First a simple case: $\frac{2}{3}$. Divide $[0, 1]$ into 3 equal parts, then by definition,

$\frac{1}{3}$ is the length of the any one of these subdivisions,
$\frac{2}{3}$ is the length of 2 such abutting subdivisions,
$\frac{3}{3}$ is the length of 3 such abutting subdivisions,
$\frac{4}{3}$ is the length of 4 such abutting subdivisions, etc.

In anlaogy with the way the natural numbers were marked down on our infinite ruler, we now add more markings to this ruler. The point to the right of 0 of distance $\frac{1}{3}$ from 0 will be $\frac{1}{3}$. Continuing on to the right, each point of distance $\frac{1}{3}$ will from the preceding one will give rise to $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, $\ldots$, $\frac{n}{3}$, $\ldots$ Note that $\frac{3}{3}$ coincides with 1, $\frac{6}{3}$ coincides with 2, and in general $\frac{3n}{3}$ coincides with n for any natural number n. So writing $k/3$ for $\frac{k}{3}$, we have:

```
0                 1                 2                 3
-----------------------------------------------------------------------------
|     |     |     |     |     |     |     |     |     |     |     |
0    1/3   2/3   3/3   4/3   5/3   6/3   7/3   8/3   9/3    etc
```

In general, let k, l be natural numbers with $l > 0$. Then by definition:

$$\frac{1}{l} \;=\; \text{the length of a subdivision when } [0,1] \text{ is divided into } l \text{ equal parts} \qquad (7)$$

$$\frac{k}{l} = \text{the length of } k \text{ abutting segments each of length } \frac{1}{l} \tag{8}$$

For example, with $l = 5$, and writing $k/5$ for $\frac{k}{5}$ we have:

```
0                       1                        2
--------------------------------------------------------------
    |    |    |    |    |    |    |    |    |    |    |     |
0  1/5  2/5  3/5  4/5  5/5  6/5  7/5  8/5  9/5  10/5 11/5  etc.
```

$\frac{k}{l}$ is called a *fraction* of the natural numbers k and l. Sometime we also refer informally to $\frac{k}{l}$ as a *quotient* of k and l, but the word quotient is used in many situations not necessarily connected with natural numbers. Our infinite ruler now has many more markers: in addition to the natural numbers, we also have all the fractions $\dfrac{k}{l}$, where k and l are natural numbers and $l \neq 0$. Moreover, in case k is a multiple of l, say $k = n \times l$, then

$$\frac{n \times l}{l} = n, \quad \text{for all natural numbers } n,\ l, \text{ where } l > 0 \tag{9}$$

In particular,

$$\frac{n}{1} = n$$

for any natural number n. This can be seen directly from our definition of a fraction: $\frac{n}{1}$ is the length of n abutting segments each being a subdivision of $[0, 1]$ into *one* part, and hence equals n. By its defintion, $\frac{k}{l}$ has the following interpretation:

$$\text{If we divide a segment of unit length into } l \text{ equal parts, then the total length of } k \text{ of them is } \tfrac{k}{l} \tag{10}$$

In common language, we would say that $\frac{k}{l}$ is "k-lth's of 1", e.g., $\frac{2}{7}$ is two-sevenths of 1 and $\frac{4}{5}$ is four-fifths of 1. Of course we do not always deal with lengths of segments in real life, but more likely with a bag of candy, a box of crayons, a portion of a pie, etc. Nevertheless, it should not be difficult for students to draw the parallel between two-sevenths of a gallon of water and two-sevenths of a unit segment, so that the consideration of segments is indeed relevant to the real world. More will be said on this topic later. (See the end of this section and §§4 and 6.)

> *It is likely that teachers would expose students to other pictorial representations of fractions at this point, such as a square divided into 4 equal parts, or a pie cut into 6 equal parts, etc., so a caveat is in order. There is certainly no harm in introducing these models, but I would suggest doing so only* after *students have become proficient at working with the number line and the division of line segments. One reason is that our reasoning throughout the development of fractions is done with the help of the number line. But there is another reason: the pie representation of fractions, for instance, has the drawback of being clumsy at representing fractions* > 1 *because teachers and students alike balk at drawing many pies. Reasoning done with the pie model therefore tends to accentuate the importance of small fractions. Could this be the explanation of the passage on p. 96 of the NCTM Standards* ([2]) *quoted in the introduction? On the other hand, the number line automatically puts all fractions, big or small, on an equal footing so that they can all be treated in a uniform manner. An additional advantage is the flexibility of this model in all kinds of discussions, and this advantage would become most apparent when we come to the multiplication of fractions.*

EXAMPLE *Describe roughly where* $\frac{84}{17}$ *is on the number line.* Since $68 < 84 < 85$ and $68 = 4 \times 17$ and $85 = 5 \times 17$, $\frac{84}{17}$ is between 4 and 5, close to 5.

We pause to note the convention that the symbol of a fraction $\frac{k}{l}$ automatically assumes that $l > 0$. Students probably need some explanation as to why we do not consider $\frac{k}{0}$: in definition (7), $\frac{k}{0}$ would necessitate the consideration of "dividing $[0, 1]$ into 0 equal parts, which is meaningless. Or one could wait till the second interpretation of a fraction in (13) is in place before reminding them why we do not divide by 0.

Our next assertion is of great importance in this approach to fraction. It is the *cancellation law* for fractions, and will be seen to be the linchpin of almost everything else that follows. It asserts that $\frac{k \times m}{l \times m} = \frac{k}{l}$. The special case where $k = 1$ is just (9) and, although it is not obvious, the reason why (9) is true is in fact the reason why the general case must be true. As usual, let us begin with a simple example: we shall show why

$$\frac{4 \times 3}{4 \times 2} = \frac{3}{2}$$

Recall the definition of $\frac{3}{2}$: mark off to the right of 0 on the number line $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, . . .

```
0                       1                       2
-----------------------------------------------------
            ^           ^           ^           ^
```

Call this the *first subdivision*. Then $\frac{3}{2}$ is the third division point of the first subdivision. Each part of the first subdivision has length $\frac{1}{2}$. Now subdivide each of these parts into 4 equal parts; call this the *second subdivision*. Each part of the latter has length $\frac{1}{4\times 2}$. Put another way: 2 parts of the first subdivision fill out $[0, 1]$, but it takes 4×2 parts of the second subdivision to fill out $[0, 1]$. So the length of each part of the second subdivision is $\frac{1}{4\times 2}$.

```
0                       1          3/2          2
-----------------------------------------------------
            ^           ^           ^           ^
   |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
```

Now 3 parts of the first subdivision fill out $[0, \frac{3}{2}]$, so it takes 4×3 parts of the second subdivision to fill out the same segment $[0, \frac{3}{2}]$. Thus, by (8), $\frac{3}{2} = \frac{4\times 3}{4\times 2}$.

For a second example, let us look at why

$$\frac{2 \times 3}{5 \times 3} = \frac{2}{5}$$

$\frac{2}{5}$ is by definition the second marker when multiples of $\frac{1}{5}$ are marked off on the number line. Call these the *first sub-division*.

```
0                             1
------------------------------------
      ^     ^     ^     ^     ^
```

Now subdivide each segment of the first subdivision into 3 equal parts, and call this the *second subdivision*.

```
0                             1
------------------------------------
      ^     ^     ^     ^     ^
  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
```

Each segment of the second subdivision has length $\frac{1}{5\times 3}$, because $[0,1]$ has been divided into 5×3 equal parts by the second subdivision. Since $\frac{2}{5}$ is the 2nd marker of the first subdivision, there will be 2×3 parts of the second subdivision in $[0,\frac{2}{5}]$. Hence $\frac{2}{5}=\frac{2\times 3}{5\times 3}$, by (8).

In general, we have: if k, l, m are natural numbers with $m\neq 0$,,

$$\frac{k\times m}{l\times m}=\frac{k}{l} \tag{11}$$

The general argument is briefly the following: Mark off multiples of $\frac{1}{l}$ on the number line. Call this the first subdivision*. The k-th marker is then $\frac{k}{l}$, by definition. $[0,1]$ therefore contains exactly l segments of the first subdvision. Divide each segment of the first subdivision into m equal parts; call this the* second subdivision. *$[0,1]$ now contains $l\times m$ segments of the second subdivision; the length of each segment of the latter is therefore $\frac{1}{l\times m}$. Since the segment $[0,\frac{k}{l}]$ contains k abutting segments of the first subdvision, it contains $k\times m$ abutting segments of the second subdivision and therefore has length $\frac{k\times m}{l\times m}$, by (8). Thus $\frac{k\times m}{l\times m}=\frac{k}{l}$.*

We now turn to the second interpretation of $\frac{k}{l}$ mentioned earlier. First look at a special case: we claim

$$\frac{2}{3} \;=\; \text{the length of any subdivision when a segment of length 2 is subdivided into 3 equal parts.}$$

Here is the reasoning: Divide $[0, 2]$ into 2 equal parts; call this the *first subdivision*. Note that each part of the first subdivision has unit length.

```
0           1          2
------------------------
```

Now further divide each of these 2 parts into 3 equal parts; call this the *second subdivision*. Each part of the second subdivision now has length $\frac{1}{3}$.

```
0           1           2
-------------------------
^   ^   ^   ^   ^   ^   ^
```

$[0, 2]$ has now been divided into $2 \times 3 = 6$ equal parts. If we take 2 parts at a time, we get additional markers which indicate where $[0, 2]$ is divided into 3 equal parts:

```
0           1           2
-------------------------
^   ^   ^   ^   ^   ^   ^

|       |       |       |
```

We are being asked to show that the first marker `|` after 0 is exactly at the $\frac{2}{3}$ position. But this is so because this marker is the second division point of the second subdivision, and each part of the latter has length $\frac{1}{3}$.

Let us run over the same argument for a different fraction: $\frac{3}{4}$. We want to show that

$\frac{3}{4}$ = the length of any subdivision when $[0, 3]$ is divided into 4 equal parts.

As before, we begin by dividing $[0, 3]$ into 3 equal parts; call this the *first subdivision*. Each part of the first subdvision is thus of unit length.

```
0       1       2       3
-------------------------
```

Next divide each of these 3 parts into 4 equal parts; call this the *second subdivision*. Each part of the second subdivision therefore has length $\frac{1}{4}$.

```
0       1       2       3
-------------------------
^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^
```

$[0, 3]$ has now been divided into $3 \times 4 = 12$ equal parts. If we take 3 parts at a time, we obtain additional markers which indicate where $[0, 3]$ is divided into 4 equal parts:

```
0       1       2       3
-------------------------
^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^ ^
|     |     |     |     |
```

We must show that the first marker `|` after 0 is exactly at the $\frac{3}{4}$ position, but this is so because each part of the second subdivision has length $\frac{1}{4}$, and the marker under inspection is at the third division point of the second subdivision and is therefore exactly $\frac{3}{4}$. This completes the argument.

In a classroom, more examples of this kind should be discussed, preferably some by students themselves. Use, for example, $\frac{5}{3}$, $\frac{4}{7}$, etc. Be sure to use for this purpose some fractions with a larger numerator than denominator (improper fractions). The reasoning given above is perfectly general, and we have therefore shown why

$$\frac{k}{l} = \text{the length of any subdvision when a segment of length } k \text{ is divided into } l \text{ equal parts, } l > 0 \qquad (12)$$

Here is the brief argument for the general case. First divide $[0, k]$ *into* k *equal parts so that each part has unit length. Then subdivide each of these parts into* l *equal parts, so that the length of any part of the second subdivision is* $\frac{1}{l}$*. The segment* $[0, k]$ *has now been divided into* $k \times l$ *equal parts; now take these* $k \times l$ *parts* k *at a time, thereby obtaining a division of* $[0, k]$ *into* l *equal parts. The first division point of the latter must be shown to be at the* $\frac{k}{l}$ *postion, and this is so because it is the* k*-th of the* $k \times l$ *division points of* $[0, k]$*, and the length of each of these subdvisions is* $\frac{1}{l}$*.*

The second interpretation of $\frac{k}{l}$ then allows us to conclude, in view of (6) on the division of natural numbers, that if $k = l \times m$,

$$\frac{k}{l} = k \div l \quad (= m) \qquad (13)$$

From now on, we write $\frac{k}{l}$ in place of $k \div l$ between natural number k and l, $l > 0$. We emphasize that the content of (12) is really quite nontrivial because our original definition of $\frac{k}{l}$ is that it is the length of the a segment consisting of k abutting segments each of length $\frac{1}{l}$.

In the real classroom, it is recommended that many more concrete examples be done by the students, e.g., $\frac{3\times 5}{4\times 5} = \frac{3}{4}$, $\frac{2\times 7}{2\times 4} = \frac{7}{4}$, etc. It is very important that they are exposed to logical arguments of this sort early. They would then absorb the reasoning behind the canellation law by by osmosis instead of just memorize this fact by rote. Whether or not the general proof should actually be given can only be decided on a case-by-case basis.

We pauses to relate our definition of fractions to an everyday situation. Let us start with only natural numbers but no fractions. If one box of crayons has 6 pieces and another has 12 pieces, how many crayons are there in both boxes together? To compute the answer, we can forget about crayons and think of just natural numbers, in which case we can use the segment model and put together two segments, one of length 6 and another of length 12. Putting them together produces a segment of length 18:

```
                    0         6
 ------------------------------
0         6         12        18
```

Then we go back to crayons and conclude that these two boxes have 18 crayons. The moral of this is that, whether we like it or not, we have to resort to abstractions even when doing a simple everyday problem.

Now consider the following: how many pieces of candy are there in a fifth of a bag of 35 pieces? The first thing to note is that the everyday expression "a fifth of a bag of 35 pieces" means exactly "the number the pieces in a portion when the 35 pieces are divided into 5 equal portions." This kind of everyday expression is a social convention that is not unreasonable, and in any case is one that children can easily learn. Now, this is the same as looking for the length of a subdivision when a segment of length 35 is divided into 5 equal parts. Thus the answer we are looking for is the fraction $\frac{35}{5}$, by virtue of (12). The cancellation law (11) then gives the expected answer, which is 7. Here is then an example of how a real life problem is solved by our precise concept of a fraction. We emphasize that the reasoning is the important thing. For example, if the bag (for some curious reason) happens to have 2709 pieces of candy and you are allowed to take $\frac{1}{21}$ of it, then the number of pieces you take would be $\dfrac{2709}{21}$, by exactly the same reasoning regardless what the actual value of this fraction may be. (This value is actually 129.) The large number 2709 is purposely used here to stress the irrelevance of the size of the numbers one encounters in working with fractions, in contrast with the common advocacy among educators for the use of small numbers in teaching fractions.

Exercise Assume that you can cut a pie into any number of equal parts, what is the most efficient way to cut 11 pies in order to give equal portions to 7 kids? 7 pies to 11 kids?

3

Before we approach the addition of fractions, we first consider the more elementary concept of comparing two fractions. Which is the bigger of the two: $\frac{4}{7}$ or $\frac{3}{5}$? In terms of segments, this should be rephrased as: which of $[0, 4/7]$ and $[0, 3/5]$ is longer? Now by definition:

$\frac{4}{7}$ is the length of 4 abutting segments, each of length $\frac{1}{7}$
$\frac{3}{5}$ is the length of 3 abutting segments, each of length $\frac{1}{5}$

This comparison is difficult because the two fractions are expressed in terms of different "units": $\frac{1}{7}$ and $\frac{3}{5}$. Imagine, for example, *if* the preceding were the following:

$\frac{4}{7}$ is the length of 4 abutting segments, each of length $\dfrac{1}{6}$

$\frac{3}{5}$ is the length of 3 abutting segments, each of length $\dfrac{1}{6}$

Then we would be able to immediately conclude that $\frac{4}{7}$ is the bigger of the two because it includes one more segment (of the same length) than $\frac{3}{5}$. This suggests that the way to achieve the desired comparison is reduce both $\frac{1}{7}$ and $\frac{1}{5}$ to a common "unit". Students of this level would probably understand this idea better by first considering a more common problem: which is longer, 3500 yards or 3.2 km? In the latter case, everybody knows that we need to reduce both *yard* and *km* to a common unit, say, meter. One finds that 1 yard = 0.9144 meter and 1 km = 1000 m., so that

3500 yards = 3200.4 m
3.2 km = 3200 m

Conclusion: 3500 yards > 3.2 km.

In order to imitate this procedure, we have to decide on a common unit for $\frac{1}{7}$ and $\frac{1}{5}$. The cancellation law (11) suggests the use of $\frac{1}{35}$ because

$$\frac{1}{7} = \frac{5 \times 1}{5 \times 7} = \frac{5}{35} = \text{the length of 5 abutting segments each of length } \frac{1}{35}$$

$$\frac{1}{5} = \frac{7 \times 1}{7 \times 5} = \frac{7}{35} = \text{the length of 7 abutting segments each of length } \frac{1}{35}$$

More generally then,

$$\frac{4}{7} = \frac{5 \times 4}{5 \times 7} = \frac{20}{35} = \text{the length of 20 abutting segments each of length } \frac{1}{35}$$

$$\frac{3}{5} = \frac{7 \times 3}{7 \times 5} = \frac{21}{35} = \text{the length of 21 abutting segments each of length } \frac{1}{35}$$

Conclusion: $\frac{4}{7} < \frac{3}{5}$. A closer look of the preceding also reveals that this conclusion is based on the inequality $5 \times 4 < 7 \times 3$ between the numerators and denominators of the two fractions. We are thus witnessing the so-called cross-multiplication algorithm in a special case. In picture:

```
  0/7  1/7  2/7  3/7  4/7  5/7  6/7  7/7
   |    |    |    |    |    |    |    |
0  ____________________________________  1
   ^      ^      ^      ^      ^      ^
  0/5    1/5    2/5    3/5    5/5    5/5
```

Students should be made to work out for themselves the comparisons of $\frac{5}{6}$ and $\frac{4}{5}$, $\frac{5}{6}$ and $\frac{3}{4}$, $\frac{4}{9}$ and $\frac{3}{7}$, $\frac{9}{29}$ and $\frac{4}{13}$, $\frac{13}{17}$ and $\frac{19}{25}$, $\frac{12}{23}$ and $\frac{53}{102}$, etc. Make sure that in each case, it is not clear ahead of time which fraction is bigger.

In general, suppose we are to compare $\frac{k}{l}$ and $\frac{m}{n}$. The reasoning of the preceding example tells us that we should use the cancellation law (11) to rewrite them as:

$$\frac{k}{l} = \frac{kn}{ln}$$
$$\frac{m}{n} = \frac{lm}{ln}$$

where for simplicity we shall henceforth use the algebraic notation of writing kn for $k \times n$, etc. It follows that

$$\frac{k}{l} < \frac{m}{n} \quad \text{exactly when} \quad kn < lm \tag{14}$$

and for the same reason,

$$\frac{k}{l} = \frac{m}{n} \quad \text{exactly when} \quad kn = lm \tag{15}$$

Either of the two is referred to as the *cross-multiplication algorithm*.

EXAMPLE *Which is greater:* $\frac{23}{24}$ *or* $\frac{24}{25}$*? Do it both with and without computation.* Using the cross-multiplication algorithm, we see that $\frac{23}{24} < \frac{24}{25}$ because $23 \times 25 = 575 < 576 = 24 \times 24$. However, this result could also be done by inspection: Both fractions are point on the unit segment $[0, 1]$:

```
0__________________________________________1
                                       |
                                     23/24

0__________________________________________1
                                        |
                                      24/25
```

It suffices therefore to decide which is the longer segment: the one between $\frac{23}{24}$ and 1, and the one between $\frac{24}{25}$ and 1. But the first segment has length $\frac{1}{24}$ while the latter has length $\frac{1}{25}$; clearly the latter is shorter. So again $\frac{23}{24} < \frac{24}{25}$.

EXAMPLE *Do the same with the fractions* $\frac{94}{95}$ *and* $\frac{311}{314}$.
The cross-multiplication algorithm again shows that $\frac{94}{95} < \frac{311}{314}$. But we can reason without computation as follows. As in the preceding example, we need only compare which of the two is greater: $\frac{1}{95}$ or $\frac{3}{314}$. However, we see that

$$\frac{3}{314} < \frac{3}{300} = \frac{1}{100} < \frac{1}{95}.$$

So $\frac{311}{314} > \frac{94}{95}$.

We are now in a position to deal with the addition of fractions. Imitating the geometric definition of the addition of two natural numbers in (1), we define:

$$\frac{k}{l} + \frac{m}{n} \;=\; \text{the length of two abutting segments, one of length } \frac{k}{l} \text{ and the other of length } \frac{m}{n} \tag{16}$$

It follows directly from this definition that

$$\frac{k}{l} = \frac{1}{l} + \frac{1}{l} + \cdots + \frac{1}{l} \quad (k \text{ times}) \tag{17}$$

Strictly speaking, one cannot do repeated addition until one proves the associative law of addition, i.e., one must first prove

$$\left(\frac{k}{l} + \frac{m}{n}\right) + \frac{p}{q} = \frac{k}{l} + \left(\frac{m}{n} + \frac{p}{q}\right)$$

and this is true because both sides are equal to the length of three abutting segments of respective lengths $\frac{k}{l}$, $\frac{m}{n}$, *and* $\frac{p}{q}$. *This then allows us to write*

$$\frac{k}{l} + \frac{m}{n} + \frac{p}{q}$$

to denote either sum. The same argument extends to a sum of any number of terms so that repeated addition makes sense without specifying the order the additions must be carried out. While such coniderations would not be suitable for presentation in a 5th grade class, the teacher must be aware of them in case a precocious youngster raises this issue.

Also immediate from the definition is the fact that

$$\frac{m}{l} + \frac{k}{l} = \frac{m+k}{l} \tag{18}$$

because both sides are equal to the length of $m + k$ abutting segments each of length $\frac{1}{l}$.

We would like to obtain a formula for the sum in (16). The previous consideration of comparing fractions prepares us well for this task. In a concrete case such as $\frac{1}{5} + \frac{1}{3}$, we see that we cannot add them as is, in the same way that we cannot add the sum (1 m. + 1 ft.) until we reduce both to a common unit. For the latter, we have in fact (1 m. + 1 ft.) = 100 cm + (12 × 2.54) cm = 130.48 cm. For the fractions themselves, we follow (14) to arrive at

$$\frac{1}{5} + \frac{1}{3} = \frac{3}{3 \times 5} + \frac{5}{5 \times 3} = \frac{3}{15} + \frac{5}{15} = \frac{8}{15},$$

by virtue of (11). The general case is just more of the same: given $\frac{k}{l}$ and $\frac{m}{n}$, we use the cancellation law to rewrite

$$\frac{k}{l} + \frac{m}{n} = \frac{kn}{ln} + \frac{lm}{ln} = \frac{kn + lm}{ln},$$

where the last step uses (18). Thus we have the general formula:

$$\frac{k}{l} + \frac{m}{n} = \frac{kn + lm}{ln} \tag{19}$$

This formula is different from the usual formula involving the *lcm* (least common multiple) of the denominators l and n, and we shall comment on the difference below. Note however that this formula (19) has been obtained by a deductive process that is entirely natural, which should give teachers a solid foundation on which to answer students' question about why addition doesn't take the form of

$$\frac{k}{l} + \frac{m}{n} = \frac{k + m}{l + n}.$$

The special case of (19) when $l = 1$ should be singled out: since $\frac{k}{1} = k$,

$$k + \frac{m}{n} = \frac{kn + m}{n} \tag{20}$$

Thus, $5 + \frac{3}{2} = \frac{13}{2}$ and $7 + \frac{5}{6} = \frac{47}{6}$. A common *abbreviation* is to write

$$k\frac{m}{n} = k + \frac{m}{n} \quad \text{in case} \quad m < n. \tag{21}$$

So $7\frac{5}{6} = \frac{47}{6}$ and $4\frac{5}{7} = \frac{33}{7}$. The entity on the left side of (21) is usually referred to as a *mixed fraction* or *mixed number*. Operations with mixed fractions are regarded by many teachers and students with fear. The purpose of being so meticulous with the definition in (21) is to allay such fears once and for all: a mixed fraction is merely an shorthand notation to denote the sum of a natural number and a fraction, so that once we explain the distributive law for fractions ((26) below), mixed fractions can be handled in an entirely routine fashion.

EXAMPLE *Compute* $2\frac{5}{9} + \frac{7}{8}$ *and* $15\frac{4}{17} + 16\frac{12}{13}$.

$$2\frac{5}{9} + \frac{7}{8} = 2 + (\frac{5}{9} + \frac{7}{8}) = 2 + \frac{103}{72} = 2 + 1 + \frac{31}{72} = 3\frac{31}{72}.$$

$$15\frac{4}{17} + 16\frac{12}{13} = (15+16) + \frac{256}{221} = 31 + 1 + \frac{35}{221} = 32\frac{35}{221}.$$

Example *Compute* $\frac{3}{4} + \frac{5}{6}$.

$$\frac{3}{4} + \frac{1}{6} = \frac{18+20}{24} = \frac{38}{24} = \frac{19}{12}$$

which could also be written as $1\frac{7}{12}$. However, in this case one sees that it is not necessary to go to $\frac{1}{24}$ as common unit of measurement of the fractions $\frac{1}{4}$ and $\frac{1}{6}$, because we could use $\frac{1}{12}$ instead: $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{6} = \frac{2}{12}$. Hence we could have computed this way:

$$\frac{3}{4} + \frac{5}{6} = \frac{3\times 3}{12} + \frac{2\times 5}{12} = \frac{19}{12} = 1\frac{7}{12},$$

as before.

The last example brings us to the consideration of the usual formula for adding fractions. So suppose $\frac{k}{l}$ and $\frac{m}{n}$ are given. Suppose we know that a natural number A is a multiple of both n and l, then we have $A = nN = lL$ for some natural numbers L and N. One such example of A is of course $A = nl$, in which case $N = l$ and $L = n$. Another example is $A =$ the *lcm* of n and l (which was the case for 4 and 6 above, whose *lcm* is 12). Then $\frac{k}{l} = \frac{kL}{lL} = \frac{kL}{A}$ and $\frac{m}{n} = \frac{mN}{nN} = \frac{mN}{A}$, so that

$$\frac{k}{l} + \frac{m}{n} = \frac{kL + mN}{A} \quad \text{where} \quad A = nN = lL \tag{22}$$

As remarked earlier, if A is taken to be the *lcm* of n and l, then this formula is the one found in almost all the textbooks. On the rare occasion that *lcm* does not make an appearance, it is usually because no formula for the addition of two fractions is given at all, on the ground that there should be a decreased emphasis on fractions (echoes of [2]). From a mathematical vantage point, however, knowing how to add two fractions is important, partly because the underlying mathematical reasoning (as we have seen) is instructive and partly because it is an essential mathematical technique. So the addition of fractions should always be taught, and taught *correctly*, which then requires using formula (19) rather than (22).[2] The reason (22) should be avoided as

[2] The last thing I want is to make myself sound like a crusader! The truth is that any mathematician who is at all competent would advocate exactly the same thing. Such

the basic formula for addition is not only that it is less simple than (19)—and simplicity is a very important criterion in mathematics—but also that (22) says knowing how to get the *lcm* of two natural numbers is a prerequisite to knowing how to add fractions. Formula (19) lays bare the fact that such is not the case. On the pedagogical level, formula (22) is the less desirable of the two because the consideration of *lcm* distracts students from the main idea (such as the explanation given preceding (19)) of how two fractions are added. I hope no reader of this manual will ever again teach his or her students how to add fractions using (22).

Finally, a few words about the subtraction of fractions. Suppose as usual that $\frac{k}{l}$ and $\frac{m}{n}$ are given and that $\frac{k}{l} > \frac{m}{n}$. Imitating the case of natural numbers in (2), we define the difference $\frac{k}{l} - \frac{m}{n}$ as

$$\frac{k}{l} - \frac{m}{n} = \text{the length of the remaining segment when a segment of length } \frac{m}{n} \text{ is removed from a segment of length } \frac{k}{l}$$

Now $\frac{k}{l} > \frac{m}{n}$ means, by virtue of (14), that $kn > lm$. Hence the following makes sense:

$$\begin{aligned} \frac{k}{l} - \frac{m}{n} &= \frac{kn}{ln} - \frac{lm}{ln} = \text{the length of the remaining segment when} \\ &\quad lm \text{ abutting segments each of length } \frac{1}{ln} \text{ are removed} \\ &\quad \text{from } kn \text{ abutting segments each also of length } \frac{1}{ln} \\ &= \text{the length of } kn - lm \text{ abutting segments each of length } \frac{1}{ln} \\ &= \frac{kn - lm}{ln}, \end{aligned}$$

where the last equally is by definition of the fraction $\frac{kn-lm}{ln}$. This yields the formula:

$$\frac{k}{l} - \frac{m}{n} = \frac{kn - lm}{ln} \tag{23}$$

being the case, the fact that almost all the textbooks use formula (22) for the addition of fractions bespeaks long years of neglect of school mathematics education by professional mathematicians.

when $\frac{k}{l} > \frac{m}{n}$.

Exercise Large numbers are used in (a) and (b) below on purpose. You may use calculator for them provided the answers are in fractions and not decimals. (a) $81\frac{25}{311} + 145\frac{11}{102} = ?$ (b) $310\frac{22}{117} - 167\frac{3}{181} = ?$ (c) $78\frac{3}{54} - \frac{67}{14} = ?$ (d) Which is bigger: $\frac{23}{26}$ or $\frac{32}{35}$? $\frac{311}{314}$ or $\frac{94}{95}$? Do each of these with *and* without computations.

4

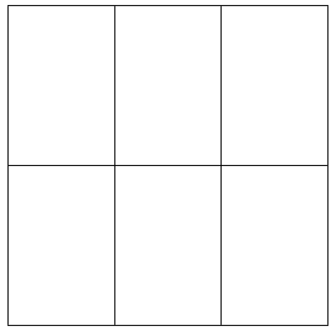
We now tackle the multiplication of fractions. The first question is: *what could* $\frac{k}{l} \times \frac{m}{n}$ *mean?* If $l = 1$, then $\frac{k}{1} \times \frac{m}{n} = k \times \frac{m}{n}$ which, in view of (3), *should* certainly mean $\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}$ (k times). When $l > 1$, the meaning of $\frac{k}{l} \times \frac{m}{n}$ in an algebraic sense is less clear. However, we recall that multiplication between natural numbers also has a geometric interpretation in terms of the area of a rectangle. See (4). This geometric interpretation makes sense even when the sides of the rectangle are no longer natural numbers. Here then is a springboard to get at the multiplication of fractions. Thus define:

$$\frac{k}{l} \times \frac{m}{n} = \text{the area of a rectangle with sides } \frac{k}{l} \text{ and } \frac{m}{n}$$

If $l = n = 1$, then this coincides with the ordinary product of the natural numbers k and m, by (4). This at least gives us some confidence that such a definition is on the right track. In order to arrive at a formula for the product, we first prove:

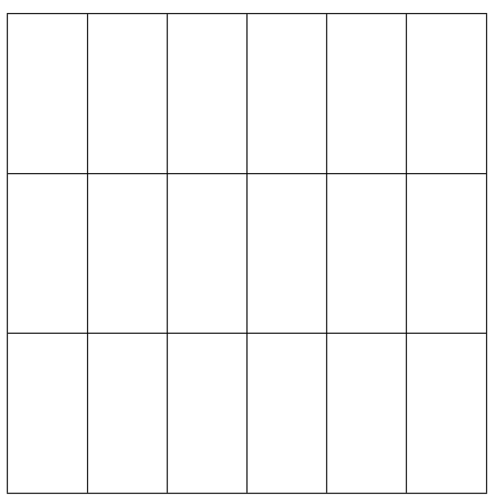
$$\frac{1}{l} \times \frac{1}{n} = \frac{1}{ln} \tag{24}$$

First consider a simple case: why is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$? Take a unit square and divide one side into 2 equal parts and the other into 3 equal parts. Joining corresponding points of the subdivision then partitions the square into 6 identical rectangles:



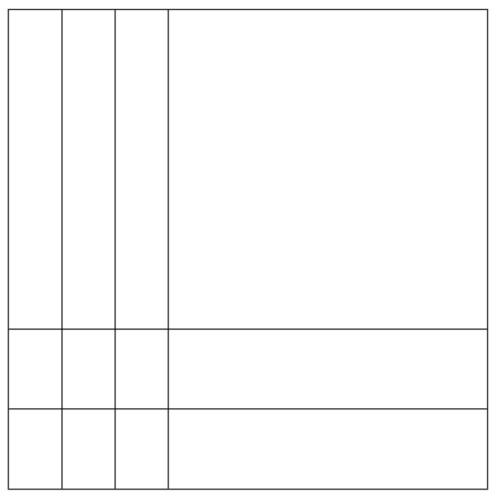
By construction, each of the 6 rectangles has sides $\frac{1}{2}$ and $\frac{1}{3}$ an its area is therefore $\frac{1}{2} \times \frac{1}{3}$. However, the total area of these 6 rectangle is the area of the unit square, which is 1, so each rectangle has area $\frac{1}{6}$, as to be shown. (In greater detail, computing the area of such a rectangle is the same as asking for the length of a subdivision when a unit segment is divided into 6 equal parts. The answer is $\frac{1}{6}$, by (7).)

Next, try showing $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$. Again we divide one side of a unit square into 3 equal parts and the other side 6 equal parts. The obvious connections of corresponding points lead to a partition of the unit square into 18 identical rectangles:



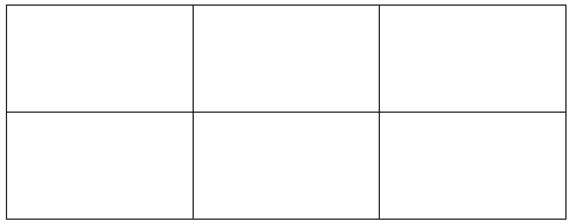
By construction, each rectangle has sides $\frac{1}{3}$ and $\frac{1}{6}$, so its area is $\frac{1}{3} \times \frac{1}{6}$. By since these 18 rectangles are identical and they partition the unit square which has area equal to 1, the area of each rectangles is $\frac{1}{18}$.

The general case in (24) can be handled in a similar way. Divide the two sides of a unit square into l equal parts and n equal parts, respectively. Joining the corresponding division points creates a partition of the unit square into ln identical rectangles.



Since each of these rectangles has sides $\frac{1}{l}$ and $\frac{1}{n}$ by construction, its area is $\frac{1}{l} \times \frac{1}{n}$ by definition. But since these ln identical rectangles partition a square of area equal to 1, each of them has area $\frac{1}{ln}$, in the same way that the length of a subdivision when a unit segment is divided into nl equal parts is $\frac{1}{ln}$, by (7). Thus $\frac{1}{l} \times \frac{1}{n} = \frac{1}{nl}$, which proves (24).

Before attacking the general case of $\frac{k}{l} \times \frac{m}{n}$, let us consider $\frac{2}{7} \times \frac{3}{4}$. This is, by definition, the area of the rectangle with the width $\frac{2}{7}$ and length $\frac{3}{4}$. Again by definition, the width consists of two abutting segments each of length $\frac{1}{7}$ while the length consists of three abutting segments each of length $\frac{1}{4}$. Joining the obvious corresponding points on opposite sides yields a partition of the original rectangle into 2×3 identical small rectangles.

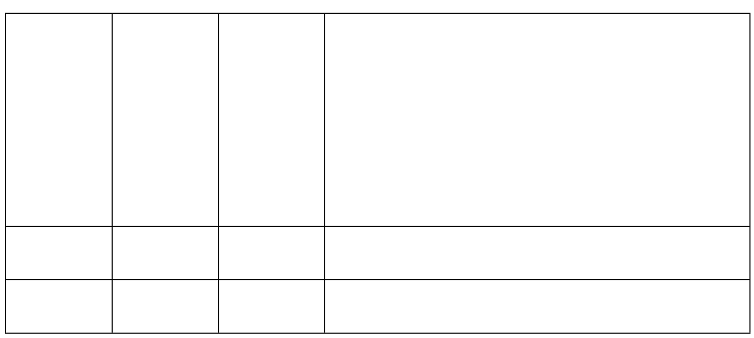


Now each of the small rectangles has sides $\frac{1}{7}$ and $\frac{1}{4}$ and therefore, by (24), has area $\frac{1}{7\times 4}$. Since the big rectangle is subdivided into 2×3 such identical rectangles, it has area equal to $\frac{2\times 3}{7\times 4}$. This shows $\frac{2}{7} \times \frac{3}{4} = \frac{2\times 3}{7\times 4}$.

We now prove in general:

$$\frac{k}{l} \times \frac{m}{n} = \frac{km}{ln} \tag{25}$$

We construct a rectangle with width $\frac{k}{l}$ and length $\frac{m}{n}$, and our task is to show that its area is equal to $\frac{km}{ln}$. By definition, its width consists of k abutting segments each of length $\frac{1}{l}$ and its length m abutting segments each of length $\frac{1}{n}$. Joining corresponding division points on opposite sides leads to a partition of the big rectangle into km small rectangles.



Since each of these small rectangles has sides equal to $\frac{1}{l}$ and $\frac{1}{n}$, its area is $\frac{1}{ln}$ by virtue of (24). But the original rectangle has been divided into km of such small rectangles, so its area is $\frac{km}{ln}$, as desired.

Observe that if we let $l = 1$ in (25), we would have

$$k \times \frac{m}{n} = \frac{k}{1} \times \frac{m}{n} = \frac{km}{n} = (\frac{m}{n} + \frac{m}{n} + \cdots + \frac{m}{n}) \quad (k \text{ times})$$

where the last is because of (18). We therfore see that our definition of the multiplication of fractions is consistent with the usual intuitive understanding of what "multiplication by a natural number" means.

It would be a good idea to caution students at this point that, whereas in the context of natural numbers "multiply by a number" always results in magnification, in the context of fractions this is no longer true. For example, if we start with 10, then mutiplying 10 by $\frac{1}{100}$ gets $\frac{1}{10}$, which is far smaller than 10.

Up to this point we have not stressed the more formal aspects of arithmetic, such as the *commutative law* for addition and multiplication or the *associative law* for both. All these facts are straightforward consequences of (19) and (25). However, we should single out the *distributive law* because it has nontrivial computational consequences:

$$\frac{k}{l} \times \left(\frac{m_1}{n_1} \pm \frac{m_2}{n_2} \right) = \left(\frac{k}{l} \times \frac{m_1}{n_1} \right) \pm \left(\frac{k}{l} \times \frac{m_2}{n_2} \right) \tag{26}$$

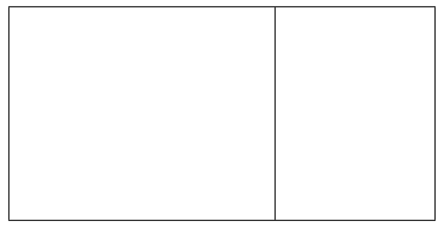
This can be proved in two ways: algebraically and geometrically. First the algebraic proof. By (19) and (25), the left side of (26) equals

$$\frac{k}{l} \times \frac{m_1 n_2 \pm m_2 n_1}{n_1 n_2} = \frac{k(m_1 n_2 \pm m_2 n_1)}{l(n_1 n_2)} = \frac{k m_1 n_2 \pm k m_2 n_1}{l n_1 n_2}$$

But by (22), we also see that the right side of (26) equals

$$\frac{k m_1}{l n_1} \pm \frac{k m_2}{l n_2} = \frac{k m_1 n_2 \pm k m_2 n_1}{l n_1 n_2}$$

So (26) holds. For the geometric proof, take the case of "+" in (26); the "−" case can be handled similarly. Consider a rectangle with one side equal to $\frac{k}{l}$, and such that the other side consists of two abutting segments of lengths $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$, respectively. This gives rise to a partition of the rectangle into two smaller rectangles as shown.



The distributive law (26) is merely the statement that the area of the big rectangle (left side of (26)) is equal to the sum of the areas of the two smaller rectangles (right side of (26)).

As an application of the distributive law, consider $181\frac{1}{6} \times \frac{3}{7}$. It is equal to

$$(181 + \frac{1}{6}) \times \frac{3}{7} = \frac{543}{7} + \frac{1}{14} = \frac{1086+1}{14} = 77\frac{9}{14}$$

Or more directly,

$$181\frac{1}{6} \times \frac{3}{7} = \frac{1087}{6} \times \frac{3}{7} = \frac{1087}{14} = 77\frac{9}{14}$$

We are now in a position to explain two everyday expressions. What does it mean when someone says "two-fifths of the people in a room" ? The precise meaning is: "the total number of people in 2 of the parts when the people in the room are divided into 5 equal parts". This is the universally accepted interpretation of this phrase, and there is no reason to lose sleep over *why* this is so, any more than to do the same over why *red* was chosen as the color of stop lights (cf. [5]). What we should do instead is to translate the latter phrase into precise mathematics. Let us say there are n people in the room. Then the number of people in each part when n is divided into 5 equal parts is the same as the length of a subdivision when a segment of length n is divided into 5 equal parts. By (12), the answer is $\frac{n}{5}$. So 2 parts would be $\frac{n}{5} + \frac{n}{5} = \frac{2n}{5} = \frac{2}{5} \times n$. Thus we see that

$$\text{"two-fifths of } n \text{ people"} \quad \text{means} \quad \text{"}\left(\frac{2}{5} \times n\right) \text{ people"} \qquad (27)$$

Similarly, "two-thirds of k cars" means "$(\frac{2}{3} \times k)$ cars", etc.

A related common expression is "65% of the students are men" or "this product contains 13% fat". Let us look at the first one. Again, we must accept that there is universal agreement on its meaning: "If we divide the students into 100 equal parts, the the men constitute 65 of those parts". In terms of our understanding of fractions, suppose there are n students altogether and we divide them into n equal parts, then each part has $\frac{n}{100}$ students. That is, if a segment has length n, then the length of each subdivision when it is divided into 100 parts is $\frac{n}{100}$, by (12). Thus the total number of students in 65 of these parts is $\frac{n}{100} + \frac{n}{100} + \cdots + \frac{n}{100}$ (65 times) $= \frac{65n}{100} = \frac{65}{100} \times n$. In short,

$$\text{“65\% of } n \text{ students”} \quad \text{means} \quad \text{“}\left(\frac{65}{100} \times n\right) \text{ students”} \qquad (28)$$

A similar statement holds for every expression involving percents.

Exercise You may use calculator for (d) and (e), but again be sure that the answers are in fractions rather than decimals. (a) $3\frac{1}{4} \times 7\frac{2}{5} = ?$ (b) $1\frac{2}{7}(3\frac{1}{8} - 2\frac{2}{5}) = ?$ (c) $12\frac{2}{7} \times \frac{7}{55} = ?$ (d) $201\frac{3}{17} \times 118\frac{6}{13} = ?$ (e) $78\frac{14}{65} \times 205\frac{211}{304} = ?$

5

Before considering the division of fractions, let us look at a seemingly unrelated problem of constructing a rectangle when one side and its area are prescribed. For example, is there a rectangle with area equal to 1 and with one side equal to $\frac{9}{4}$? If such a rectangle exists and the the other side has length s, then $\frac{9}{4}s = 1$. If s is a fraction $\frac{k}{l}$, then we want $\frac{9k}{4l} = 1$, by (25). In view of the cancellation law (11), we see by visual inspection that $k = 4$ and $l = 9$ would do. Thus the solution of our problem is a rectangle of dimensions $\frac{4}{9}$ and $\frac{9}{4}$. The same reasoning yields the fact that the rectangle with area equal to 1 and one side equal to $\frac{k}{l}$ is one with dimensions $\frac{k}{l}$ and $\frac{l}{k}$.

Consider a more general problem: suppose we want a rectangle with one side prescribed (say $\frac{9}{4}$) and its area also presecribed (say $\frac{11}{7}$), how to find the other side? Again, suppose the other side is $\frac{k}{l}$, then we saw that $k = 4$ and $l = 9$ would solve the problem if the area were equal to 1. Although the area is now $\frac{11}{7}$ and not 1, nevertheless, we see that we have the basis of a solution because: if a rectangle has area equal to A and we multiply one side by r, then the resulting rectangle will have area equal to rA. Since we already have a rectangle of area equal to 1 (that with dimensions $\frac{4}{9}$ and $\frac{9}{4}$), to get a rectangle with area $\frac{11}{7}$ all we need to do is to multiply one side by $\frac{11}{7}$. Multiplying the side of length $\frac{4}{9}$ by $\frac{11}{7}$, we get $\frac{4\times 11}{9\times 7}$. So the desired rectangle has dimensions $\frac{9}{4}$ and $\frac{4\times 11}{9\times 7}$.

In the same way, the rectangle with prescribed area $\frac{m}{n}$ and a prescribed side $\frac{k}{l}$ has second side equal to $\frac{lm}{kn}$, as $\frac{k}{l}\times\frac{lm}{kn}=\frac{m}{n}$, which is also readily confirmed by (25) and (11). We formalize the purely algebraic part of this statement as follows:

$$\text{given fractions } A \text{ and } C, \text{ with } A \neq 0, \\ \text{then there is a fraction } B \text{ so that } A\times B = C \qquad (29)$$

This is so because if we let $A=\frac{k}{l}$ and $C=\frac{m}{n}$, then we have seen that $B=\frac{lm}{kn}$ would do. Put another way,

$$\text{given } \frac{k}{l} \text{ and } \frac{m}{n}, \text{ with } \frac{k}{l}\neq 0, \text{ there is a fraction } B \text{ so that} \\ \frac{k}{l}\times B=\frac{m}{n}, \quad \text{namely,} \quad B=\frac{lm}{kn} \qquad (30)$$

This may also be the right moment to comment on the provision that $A\neq 0$, i.e., $\frac{k}{l}\neq 0$. The latter implies $k\neq 0$, which is needed because otherwise the fraction $\frac{lm}{kn}$ would have 0 as denominator. Another way to look at it is that if $A=0$, then $A\times B$ would always be 0 no matter what B is and our problem would have no solution.

The property about fractions stated in (29) is of critical importance. It is the one property that distinguishes fractions from natural numbers, in the sense that whereas in terms of arithmetic operations one can hardly tell the difference between natural numbers and fractions, the property (29) is not shared by natural numbers. In other words, the statement that "given natural numbers A and C with $A\neq 0$, we can always find a natural number B so that $A\times B=C$" is clearly false. For example, let $A=2$ and $C=1$. In fact, property (29) is the gateway to the division of fractions, as we now explain.

As in the case of multiplication, we begin by asking, for fractions A and C, what could it mean to *divide C by A* ? Following the remark after (13), we shall henceforth use the notation

$$\frac{C}{A}$$

to stand for "C divided by A". Let us retrace our steps and re-examine what it means, *if A and C are natural numbers*, to write $\dfrac{C}{A} = B$. For example, $\frac{12}{3} = 4$ because we can divide 12 into 3 equal parts (namely, $12 = 4+4+4$), and the size of each part is 4. Similarly, $\frac{14}{7} = 2$ because we can divide 14 into 7 equal parts (namely $14 = 2+2+2+2+2+2+2$) and the size of each part is 2. In other words, $\frac{12}{3} = 4$ because $12 = 3 \times 4$, and $\frac{14}{7} = 2$ because $14 = 7 \times 2$. In the same way then, for *natural numbers* A. B, and C, $\frac{C}{A} = B$ means $C = B + B + \cdots + B$ (A times) $= A \times B$. This viewpoint is entirely consistent with the meaning given to the quotient of natural numbers in (12).

This then prompts us to give meaning to "$\dfrac{C}{A} = B$" for fractions A, B and C with $A \neq 0$ by *defining*:

$$\frac{C}{A} = B \quad \text{if } B \text{ is the fraction satisfying} \quad C = A \times B \tag{31}$$

The importance of (29) is now evident: without it, this definition (31) may not make sense because *a priori* there may not be such a B for the given A and C. In fact, using (30), we can make (31) more explicit by deriving the explicit formula for the division of fractions:

$$\frac{m/n}{k/l} = \frac{lm}{kn} \quad \text{for } \frac{k}{l} \neq 0 \tag{32}$$

because, visibly, the fraction $\frac{lm}{kn}$ satisfies $\frac{m}{n} = \frac{k}{l} \times \frac{lm}{kn}$.

The special case of (32) where $m = n = k = 1$ is of particular interest:

$$\frac{1}{1/l} = l$$

It can be interpreted as follows: Suppose we have a water tank with capacity T gallons and a bucket with capacity b gallons. Then naturally the number of buckets of water needed to fill the tank is T/b. Now suppose the tank capacity is 1 gallon, *i.e.,* $T = 1$, and the bucket capacity is $\frac{1}{l}$ of a gallon, *i.e.,* $b = \frac{1}{l}$. Then the number of buckets of water needed to fill the tank is $\dfrac{1}{\frac{1}{l}}$. However, this number can be computed directly: if $l = 2$ (bucket holds half a gallon of water), it takes 2 buckets to fill the tank, if $l = 3$ (bucket holds a thrid of a gallon of water), it takes 3 buckets to fill the tank, etc,. and for a general l the same reasoning shows that it takes l buckets to fill the tank.

Thus, $\dfrac{1}{\frac{1}{l}} = l$, exactly as predicted by our elaborate definition of the division of fractions.

Remark: As in the case of multiplications, it is good to remind students that dividing one fraction by another does not necessarily make the first fraction smaller, e.g., $2/\frac{1}{5} = 10$.

Here are some standards algebraic properties of *quotients* of fractions (in the sense of one fraction divided by another). Let A, B, $\ldots F$ be fractions (which will be assumed to be nonzero in the event any of them appears in the denominator). Then the following are valid:

(a) *Cancellation law*: if $C \neq 0$, then

$$\frac{A \times C}{B \times C} = \frac{A}{B}$$

(b) $\dfrac{A}{B} > \dfrac{C}{D}$ (resp.,=) exactly when $A \times D > B \times C$ (resp., =).
(c) $\dfrac{A}{B} \pm \dfrac{C}{D} = \dfrac{(A \times D) \pm (B \times C)}{B \times D}$
(d) $\dfrac{A}{B} \times \dfrac{C}{D} = \dfrac{A \times C}{B \times D}$
(e) *Distributive law*:

$$\frac{A}{B} \times \left(\frac{C}{D} \pm \frac{E}{F} \right) = \left(\frac{A}{B} \times \frac{C}{D} \right) \pm \left(\frac{A}{B} \times \frac{E}{F} \right)$$

Two remarks are relevant here. One is that it is important to keep in mind that each of these $\frac{A}{B}$, $\frac{C}{D}$, etc. is nothing but a fraction. See (31) and (32). Furthermore, comparing (a)–(e) with (11), (14), (15), (19), (23), (25) and (26), we see that a quotient of fractions can be treated in exactly the same way as a quotient of natural numbers. Thus formally at least, (a)–(e) should present no real difficulties to students even if the proofs are somewhat tedious. In the classroom, I would suggest giving only one or two of them.

In the following, we shall appeal liberally to (11), (14), (15), (19), (23), (25) and (26) without further comments.

Proof of (a). Let $A = \frac{k}{l}$, $B = \frac{m}{n}$, and $C = \frac{p}{q}$. Then,

$$\frac{A \times C}{B \times C} = \frac{(km)/(ln)}{(mp)/(nq)} = \frac{(km)(nq)}{(ln)(mp)} = \frac{kq}{lp} = \frac{k/l}{p/q} = \frac{A}{B}\,.$$

Proof of (b). Let $A = \frac{k}{l}$, $B = \frac{m}{n}$, $C = \frac{p}{q}$, and $D = \frac{r}{s}$. We shall use "$\Longleftrightarrow$" to denote "is exactly the same as". Then,

$$\frac{A}{B} > \frac{C}{D} \Longleftrightarrow \frac{k/l}{m/n} > \frac{p/q}{r/s} \Longleftrightarrow \frac{kn}{lm} > \frac{ps}{qr} \Longleftrightarrow knqr > lmps\,,$$

while

$$A \times D > B \times C \Longleftrightarrow \frac{kr}{ls} > \frac{mp}{nq} \Longleftrightarrow krnq > lsmp\,.$$

Hence (b) is true for the case of "$>$". The case of "$=$" is similar.

Proof of (c). Let A, B, C, and D be as in (b). Then

$$\frac{A}{B} \pm \frac{C}{D} = \frac{k/l}{m/n} \pm \frac{p/q}{r/s} = \frac{kn}{lm} \pm \frac{ps}{qr} = \frac{knqr \pm lmps}{lmqr}\,.$$

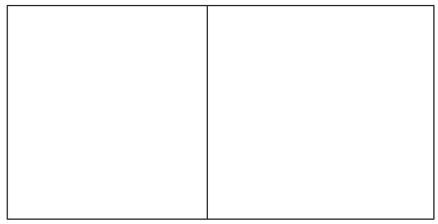
$$\begin{aligned}\frac{(A \times D) \pm (B \times C)}{B \times D} \quad &= \quad \frac{\frac{kr}{ls} \pm \frac{mp}{nq}}{\frac{mr}{ns}} = \frac{\frac{(krnq) \pm (lsmp)}{lsnq}}{\frac{mr}{ns}} \\ &= \quad \frac{ns[(krnq) \pm (lsmp)]}{mr(lsnq)} = \frac{krnq \pm lsmp}{mrlq} = \frac{A}{B} \pm \frac{C}{D}\,.\end{aligned}$$

Proof of (d). With A, B, C, D, as before,

$$\frac{A}{B} \times \frac{C}{D} = \frac{kn}{lm} = \frac{knps}{lmqr}.$$

$$\frac{A \times C}{B \times D} = \frac{(kp)/(lq)}{(mr)/(ns)} = \frac{kpns}{lqmr} = \frac{A}{B} \times \frac{C}{D}.$$

Proof of (d). An algebraic proof can be given in a routine fashion as with (26). However, a geometric proof is far more enlightening. For a change, we will handle the "$-$" case of (e) and leave the "$+$" case to the reader. Let $\mathcal{R}$ be a rectangle with one side equal to $\frac{A}{B}$ and the other side $\frac{C}{D} - \frac{E}{F}$. From the definition of $\frac{C}{D} - \frac{E}{F}$ (above (23)), the other side of $\mathcal{R}$ is the remaining segment when when a segment of length $\frac{E}{F}$ is removed from one of length $\frac{C}{D}$. See picture.



Let $\mathcal{R}_1$ be the rectangle with one side $\frac{A}{B}$ and the other side $\frac{C}{D}$ (the big rectangle), and $\mathcal{R}'$ the rectangle with one side $\frac{A}{B}$ and the other side $\frac{E}{F}$ (the smaller rectangle on the right). Then

$$\begin{aligned} \frac{A}{B} \times \left(\frac{C}{D} - \frac{E}{F}\right) &= \text{area of } \mathfrak{R} = \text{area of } \mathfrak{R}_1 - \text{ area of } \mathfrak{R}' \\ &= \left(\frac{A}{B} \times \frac{C}{D}\right) - \left(\frac{A}{B} \times \frac{E}{F}\right) \ . \end{aligned}$$

Exercise If $A = \frac{11}{5}$, $B = \frac{2}{7}$, $C = \frac{22}{21}$, $D = 2\frac{4}{5}$, $E = \frac{11}{7}$, and $F = \frac{5}{2}$, directly verify (a)–(e) above.

6

We discuss three problems to illustrate the applications of fractions.

Example 1[3] *High school math classes use 60% of class time for teaching but 8th grade math classes use only 40%. Assuming that both math classes cover the same number of pages of a 360-page Algebra I textbook per hour, and that the high school math class covers the whole book in a year, how many pages of the same book will be covered by the 8th grade math class in a year.*

Suppose there are a total of H hours in each math class per year, then the high school math class has $(\frac{60}{100} \times H)$ hours for teaching (see (28)) so that the number of pages of the algebra textbook covered per hour is $[360/(\frac{60}{100} \times H)]$

[3] I got this problem from Jerome Dancis.

pages. By hypothesis, this is also the number of pages which the 8th grade math class can cover per hour. However, the latter class has only $(\frac{40}{100} \times H)$ hours for teaching per year, so that the total number of pages covered in one year in the 8th grade math class is:

$$\begin{aligned} (\frac{40}{100} \times H) \times \frac{360}{(\frac{60}{100} \times H)} &= \frac{\frac{40}{100} \times H}{\frac{60}{100} \times H} \times 360 \\ &= \frac{40/100}{60/100} \times 360 \\ &= 240 \ \ \text{pages.} \end{aligned} \tag{33}$$

Discussion. The usual approach taught in schools is to *use proportional thinking*: if the total number of pages covered in the 8th grade math class is x, then

$$\frac{60/100}{40/100} = \frac{360}{x} \tag{34}$$

so that

$$x = \frac{40/100}{60/100} \times 360\,,$$

which is the same as (33). However, the setting up of (34) is actually quite mysterious to beginners unless the underlying reasoning of (33) has been carefully explained to them.

The next two examples are from Russia.

Example 2 *Fresh cucumbers contain 99% water by weight. 300 lbs. of cucumbers are placed in storage, but by the time they are brought to market, it is found that they contain only 98% of water by weight. How much do these cucumbers weigh?*

Since 99% of 300 lbs. is just water, there are $\frac{99}{100} \times 300 = 297$ lbs. of water ((28)) and hence only $300 - 297 = 3$ lbs. of solid. By the time the cucumbers are brought to market, some water has evaporated but the 3 lbs. of solid remain unchanged, of course. Since 98% is water, the solid is now 2% of the total weight. Hence if the total weight at market time is w lbs., we see that $3 = \frac{2}{100} \times w$, by (28) again. (30) then implies that

$$w = \frac{100 \times 3}{2} = 150 \ \ \text{lbs.}$$

Discussion. A mindless application of proportional thinking would have produced the following: Let w be the weight of the cucumbers when they are brought to market. Then,

$$\frac{99/100}{300} = \frac{98/100}{w}.$$

This is one reason why proportional thinking should be taught only after its underlying reasoning has been clearly explained.

EXAMPLE 3 *There is a bottle of wine and a kettle of tea. A spoon of tea is taken from the kettle and poured into the bottle of wine. The mixture is thoroughly stirred and a spoonful of the mixture is taken from the bottle and poured into the kettle. Is there more tea in the bottle or more wine in the kettle? Do the same problem again, but* without *assuming that the mixture has been stirred.*

Let us do the stirred version first. Let the amount of wine in the bottle, the amount of tea in the kettle, and the capacity of the spoon be b cc, k cc, and s cc, respectively. After a spoonful of tea has been added to the bottle of wine, the amount of liquid in the bottle is $(b+s)$ cc. The fraction of tea in the mixture is $\frac{s}{b+s}$, and the fraction of wine in the mixture is $\frac{b}{b+s}$. A spoonful of of the thoroughly stirred stirred mixture would therefore contain $[(\frac{s}{b+s})s]$ cc of tea and $[(\frac{b}{b+s})s]$ cc of wine (see (27)). When this spoonful is poured into the kettle of tea, there will be

$$(\frac{b}{b+s})s = \frac{bs}{b+s} \quad \text{cc of wine in the bottle.}$$

On the other hand, the mixture in the bottle originally had s cc (1 spoonful) of tea, but since $[(\frac{s}{b+s})s]$ cc has been taken away, the amount of tea left in the bottle is

$$s - (\frac{s}{b+s})s = s - \frac{s^2}{b+s} = \frac{s(b+s)}{b+s} - \frac{s^2}{b+s} = \frac{sb+s^2-s^2}{b+s} = \frac{sb}{b+s} \text{ cc}$$

This shows the amount of tea in the bottle is the same as the amount of wine in the kettle.

Now the "unstirred" case. Suppose the spoonful of mixture contains α cc of tea and β cc of wine. The $\alpha + \beta = s$, where as before s denotes the capacity of the spoon. Therefore when the spoonful of mixture is poured into the kettle, the amount of wine in the bottle is β cc. On the other hand, the bottle of mixture originally had s cc of tea. But with α cc of the tea taken away by the spoon, only $(s-\alpha)$ cc of tea is left in the bottle. Since $(s-\alpha)=\beta$, the amount of tea in the bottle is equal to the amount of wine in the kettle, as before.

The surprising aspect of the second solution is that, since it does not depend on any assumption about whether or not the mixture has been stirred, it supersedes the first solution. Thus the precise calculations of the first solution were completely unnecessary! Nevertheless, the first solution is a valuable exercise in thinking about fractions and should not be thought of as a waste of time.

7

Here are a collection of miscellaneous comments to tie up the loose ends. In the mathematical literature, a quotient of two integers $\frac{m}{n}$, $n \neq 0$, is called a *rational number*, and the collection of rational numbers is denoted by $\mathbb{Q}$. What we have been calling "fractions" are exactly the nonegative rational numbers. It is well-known that there are real numbers which are not rational, for example $\sqrt{2}$ and π, and these are called *irrational* numbers.. The assertion about the irrationality of $\sqrt{2}$ is elementary, see for example [4, p. 36]. The irrationality of π is far more difficult, see [8, p. 282]. The exact nature of the real numbers $\mathbb{R}$ as compared with $\mathbb{Q}$ is discussed in college textbooks, for example, [7, Chapter I].

We have so far only discussed the quotient of fractions, *i.e.,* the division of one fraction by another, but not the qotients of arbitrary real numbers. More precisely, what we have done is getting to know $\mathbb{Q}$ a bit, but the real numbers $\mathbb{R}$ are never seriously discussed in the curriculum of K-12. Logic would therefore dictate that when we do mathematics in K-12, we stay within $\mathbb{Q}$, *i.e.,* only use rational numbers. While this is correct, such a restriction is entirely unrealistic because it is impossible to avoid irrational numbers such as $\sqrt{2}$ or π. For example, each time you look at the diagonal of a unit square (each side has length 1) you face $\sqrt{2}$, and each time you consider the circumference of a circle, π comes up. So the **tacit assumption** in school mathematics is to try to understand fractions thoroughly and then *extrapolate all the information about fractions to all the real numbers, rational or* *irrational*. For example, school students are supposed to treat quotients such as $\frac{\pi}{\sqrt{2}}$ as if they are quotients of fractions and apply to them the identities in (a)–(e) of §5, regardless of the fact that at this point (a)–(e) make sense only for fractions and students have no idea what $\frac{\pi}{\sqrt{2}}$ is supposed to mean. As another example, in Example 3 of §6 (on wine and tea), we dealt with numbers such as $\frac{b}{s+b}$ and $\frac{s^2}{s+b}$. Nothing was said at the time about the numerators and denominators being fractions, and the computations proceeded on the implicit assumption that they were. But of course there is no reason to expect them to be rational. Thus we have already cheated without saying so.

From a pedagogical point of view, one cannot fault this *modus operandi* because if students must wait for the correct definition of $\mathbb{R}$ before they are allowed to approach such a simple problem as Example 3 of §6, they would not get to learn much mathematics in K-12. However, there *is* ample ground to object to the fact that

> (1) the **tacit assumption** seems never to be made explicit all through K-12, especially not in secondary school, and
> (2) fractions are almost never done well enough to allow for their extrapolation to real numbers.

This manual tries to make amends for both.

In any case, now that the **tacit assumption** has been made explicit, I will phrase the succeeding discussion on ratio, proportion, and percent solely in terms of fractions for the sake of clarity. You the readers will be asked to extrapolate all the statements about fractions to real numbers accordingly. Two additional comments may help to clarify the situation. One is the reminder that there is at present a tremendous confusion in terminlogy. At least in the early grades, a "fraction" always means a quotient of two natural numbers, and this is the terminology we have adopted in this manual. However, in general discussions this term is often used for the quotient of two *arbitrary real numbers*. Thus $\frac{\sqrt{2}}{3}$ would be a fraction according to the latter, but certainly not according to the former. This confusion is partly due to people's reliance on dictionaries for technical mathematical terms without realizing that, while dictionaries can serve the purpose of explaining words to the general public, they are as a rule not precise enough for the purpose of doing mathematics. In this manual, *fractions will always mean quotients of natural numbers*, but readers should be extra careful about this term in other contexts. A second comment is that although we have not discussed real numbers in general, the reader can be assured that the discussion of products and quotients of fractions in §§4-5 is *robust* in the sense that, in spirit if not in full details, the same discussion would be valid also for products and quotients of arbitrary real numbers. There is thus no need to fear that the knowledge you pick up from this manual would lead you far astray.

Here are explanations of the three terms that cause so much anxiety among students and teachers (keep in mind the **tacit assumption**):

> *Percent.* Given a fraction $\frac{k}{l}$, which typically expresses part of a whole, we can write it as $\frac{k}{l} = A \times \frac{1}{100}$ for some fraction A, by (29). Then "k-lth of something" is sometimes also expressed as "A-percent of something" (cf. (28)).
>
> *Ratio.* The ratio of two fractions A and B is just $\frac{A}{B}$ in the sense of (31). By tradition, this ratio is also written as $A : B$, and this strange notation is probably responsible for a lot of the misunderstanding.
>
> *Proportion.* The equality of two or more ratios.

It is possible to make this discussion appear more profound by more verbiage,—and the public agonizing over these terms by teachers and educators alike almost invites this kind of verbal extravagance,—but the precision with which fractions have been explained in the preceding pages renders such verbiage unnecessary.

One can perhaps understand the confusion better if one looks at the following random collection of statements by teachers and educators on the subject:

> "A fraction is a comparison of two numbers. Another word for the comparison of two numbers is ratio". [What is "comparison"? If this is not clear, how can "ratio" be explained in terms of "comparison"?]
>
> "Ratios are devices designed to communicate a relationship between two things." [What is a relationship?]
>
> "A formal definition of ratio is not wanted. Ratio should be given given a functional definition, which is best done in the context of a proposition". [What is a "proportion"? How does "functional definition" differ from a plain "mathematical definition"?]

> "According to the Mathematical Dictionary by James and James, a ratio is a fraction." [This says very little if teachers and students do not know what a fraction is. Moreover, this uses "fraction" in the sense of a quotient of two real numbers, and is likely therefore to add to the general confusion.]

At this point, perhaps the great care in giving precise definitions of all the concepts in this manual is beginning to be seen as an advantage.

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