An Accelerated Domain Decomposition Procedure Based on Robin Transmission Conditions

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Abstract. A domain decomposition procedure based on Robin transmission conditions applicable to elliptic boundary problems was first introduced by P. L. Lions and later discussed by a number of authors. In all of these discussions, the weighting of the flux and the trace of the solution were independent of the iterative step number. For some model problems we introduce a cycle of weights and prove that an acceleration of the convergence rate similar to that occurring for alternating-direction iteration using a cycle of pseudo-time steps results. In some discrete cases, the cycle length can be taken to be independent of the mesh spacing.

1 Introduction

Consider the model boundary value problem

$$\begin{aligned} -\Delta u &= f, \qquad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{1}$$

where the domain $\Omega = (-\infty, \infty) \times (0,1)$ is the unit strip. Split Ω into two subdomains $\Omega_1 = (-\infty, 0) \times (0,1)$ and $\Omega_2 = (0, \infty) \times (0,1)$, and let $\Gamma = \Gamma_{12} = \Gamma_{21}$ denote the interface between Ω_1 and Ω_2. Then, under reasonable conditions, (1) is equivalent to solving the two problems

$$\begin{aligned} -\Delta u_i &= f, \qquad && x \in \Omega_i, \quad i = 1,2, \\ u_i &= 0, && x \in \partial\Omega \cap \partial\Omega_i, \end{aligned}$$

subject to the two consistency conditions

$$\begin{aligned} u_1 &= u_2, \qquad && x \in \Gamma, \\ \nabla u_1 \cdot \nu_1 + \nabla u_1 \cdot \nu_2 &= 0, && x \in \Gamma, \end{aligned} \tag{2}$$

where ν_i is the outer unit to Ω_i on Γ. Let $\beta > 0$. Then, P. L. Lions noted the simple fact that the Robin conditions

$$\beta \nabla u_i \cdot \nu_i + u_i = -\beta \nabla u_j \cdot \nu_j + u_j, \qquad x \in \Gamma_{ij}; \quad i = 1,2; \quad j \neq i, \tag{3}$$

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is equivalent to (2) and based a domain decomposition procedure on (3).

In this short paper, we shall study a version of Lions's iteration when the parameter β depends on the iteration index. We shall treat the differential problem first and then consider a discrete approximation by the standard five-point finite difference operator. The object is to indicate that a properly chosen parameter sequence can lead to a distinct speedup in the convergence rate for the iteration. We wish to emphasize that this iterative procedure, with easily chosen parameters, can suppress not just the high frequency modes but also the low frequency ones in a very stable manner.

Let us define Lions's iteration for the differential problem (1). Since the model problem we are considering can be treated by separation of variables in an elementary way, we shall omit technical details, such as worrying over the proper Sobolev spaces for the procedure. The algorithm can be given as follows:

- Let $u_1^0 \mid_{\Gamma 12}= U$ be arbitrary, so long as $u_1^0(0,0) = u_1^0(0,1) = 0$. Then, solve the Dirichlet problem

$$\begin{aligned} -\Delta u_1^0 &= f, & x \in \Omega_1, \\ u_1^0 &= U, & x \in \Gamma_{12}, \\ u_1^0 &= 0, & x \in \partial\Omega_1 \cap \partial\Omega. \end{aligned}$$

- For $n = 1, 2, \ldots$, let

$$\begin{aligned} -\Delta u_2^n &= f, & x \in \Omega_2, \\ \beta_n \nabla u_2^n \cdot \nu_2 + u_2^n &= -\beta_n \nabla u_1^{n-1} \cdot \nu_1 + u_1^{n-1}, & x \in \Gamma_{21}, \\ u_2^n &= 0, & x \in \partial\Omega_2 \cap \partial\Omega, \end{aligned}$$

$$\begin{aligned} -\Delta u_1^n &= f, & x \in \Omega_1, \\ \beta_n \nabla u_1^n \cdot \nu_1 + u_1^n &= -\beta_n \nabla u_2^n \cdot \nu_2 + u_2^n, & x \in \Gamma_{12}, \\ u_1^n &= 0, & x \in \partial\Omega_1 \cap \partial\Omega. \end{aligned}$$

Note that we have employed a simple version of red-black ordering of the subdomains; obviously, this concept generalizes to many ways of partitioning Ω in either two-or three-space. It also applies to bounded domains, instead of the strip domain being treated here, and to equations with variable coefficients. In both the differential and difference cases, the analysis given below extends for the Laplace operator, with some algebraic complication, to a rectangular domain decomposed into a union of rectangles without interior vertices; the authors are still considering the case when interior vertices occur. The extension to variable coefficients can be made rigorously by employing this iteration as a preconditioner in a two-stage iteration.

Lions [3] gave a proof of the convergence of the iteration for fixed $\beta > 0$ under some reasonable hypotheses; however, the argument provides no estimate of the rate of convergence. He states that in the finite-dimensional version of the problem, geometric convergence results, but no estimate of the rate in terms of a discretization parameter is given. As an aside in his thesis on the application of this technique to the more difficult Helmholtz problem, Després [1] extended Lions's proof to a more general partition but again without an estimate of an explicit rate of convergence. Douglas, Paes Leme, Roberts, and Wang [2] did obtain convergence rates for fixed β for a mixed finite element approximation of (1) under a number of different hypotheses on the coefficients and the partition.

The paper is organized as follows. In §2, we consider the differential model problem in the plane and indicate how its analysis carries over trivially to three-space. A finite difference analogue of the model problem in two space variables will be treated in §3; it will be necessary to reinterpret the Robin condition in order to have the iteration compatible with the difference equation on the unpartitioned domain. The three-dimensional problem will be treated in §4.

2 The Model Differential Problem

Let $v^n = u - u^n$ represent the error in the n^{th} iteration, so that

$$-\Delta v_i^n = 0, \qquad x \in \Omega_i, \tag{4}$$

and the problem is to determine the effect of an initial error

$$v_1^0 \mid_{\Gamma_{12}} = \sum_k \alpha_k \sin \pi k y. \tag{5}$$

It suffices to treat each mode separately; thus, let

$$v_1^0 \mid_{\Gamma_{12}} = \sin \pi k y.$$

Then, it is easy to see that

$$v_1^0 = e^{\pi k x} \sin \pi k y, \quad x \in \Omega_1.$$

Evaluating the Robin transmission condition for v_2^1 on Γ_{21} leads to

$$-\beta_1 \frac{\partial v_2^1}{\partial x} + v_2^1 = -\beta_1 \frac{\partial v_1^0}{\partial x} + v_1^0 = (1 - \beta_1 \pi k) \sin \pi k y \text{ on } \Gamma_{12}.$$

Then, since $v_2^1 = A_2^1 e^{-\pi k x} \sin \pi k y$, the boundary condition at $x = 0$ implies that

$$A_2^1 = \frac{1 - \beta_1 \pi k}{1 + \beta_1 \pi k},$$

so that

$$v_2^1 \mid_{\Gamma_{21}} = \frac{1 - \beta_1 \pi k}{1 + \beta_1 \pi k} \sin \pi k y.$$

Next, evaluate the Robin condition for v_1^1 on Γ_{12} and solve for v_1^1. By symmetry, it is clear that

$$v_1^1 \mid_{\Gamma_{12}} = \left(\frac{1 - \beta_1 \pi k}{1 + \beta_1 \pi k} \right)^2 \sin \pi k y.$$

Note that the choice

$$\beta_1 = \frac{1}{\pi k}$$

gives $v_i^1 = 0$ on Ω_i, $i = 1, 2$; i.e., if the initial error consists of a single harmonic on Γ, the error can be eliminated in one iteration; moreover, for any $\beta > 0$, each nonzero coefficient of a mode in the initial condition is reduced. For those familiar with alternating-direction iteration, this observation can bring to mind the possibility of choosing a cycle $\{\beta_j\}$ of parameters, in place of repeating a single β, in order to accelerate the convergence of the iteration.

Consider the general initial condition (5) and m iterations employing the parameter sequence $\{\beta_1, \ldots, \beta_m\}$. Then,

$$\begin{aligned} v_1^m \mid_{\Gamma_{12}} &= \sum_k \alpha_k \cdot \prod_{n=1}^m \left(\frac{1 - \beta_n \pi k}{1 + \beta_n \pi k} \right)^2 \cdot \sin \pi k y \\ &= \sum_k \alpha_k \prod_{n=1}^m R(k, \beta_n) \sin \pi k y. \end{aligned}$$

There is no finite set $\{\beta_1, \ldots, \beta_m\}$ for which $\prod_{n=1}^m R(k, \beta_n)$ tends to zero uniformly for all k; however, in the discrete case to follow, there will be only $N = O(h^{-1})$ components of the error. Now, in order to lead our intuition properly in the discrete case, let us fix the number of components at N (hence, assume $\alpha_k = 0$ for $k > N$) and try to find a more or less optimal parameter cycle. Let

$$\mathcal{R}(k, m) = \prod_{n=1}^m \left(\frac{1 - \beta_n \pi k}{1 + \beta_n \pi k} \right)^2, \quad k = 1, \ldots, N.$$

Let $\varepsilon > 0$ and define $\beta_1 > \ldots > \beta_m$ and $\pi = \eta_0 < \eta_1 < \ldots < \eta_m$ by

$$\frac{1 - \beta_n \eta_{n-1}}{1 + \beta_n \eta_{n-1}} = \varepsilon^{\frac{1}{2}} = - \frac{1 - \beta_n \eta_n}{1 + \beta_n \eta_n}, \quad n = 1, \ldots, m,$$

where η_m is the first η_n to satisfy $\eta_m \geq N\pi$. Thus, $\{\beta_n\}$ and $\{\eta_n\}$ form geometric sequences,

$$\eta_n = \pi \left(\frac{1 + \varepsilon^{\frac{1}{2}}}{1 - \varepsilon^{\frac{1}{2}}} \right), \quad \text{and} \quad m \sim \frac{\log N}{4 \varepsilon^{\frac{1}{2}}}.$$

Moreover, for any $k \in \{1, \ldots, N\}$, $\pi k \in [\eta_{n-1}, \eta_n]$ for some $n \in \{1, m\}$, so that $0 \leq R(k, \beta_n) \leq \varepsilon$. Hence,

$$\max_{1 \leq k \leq N} \mathcal{R}(k, m) \leq \varepsilon$$

and

$$\|v_1^m\|_{L^2(\Gamma)} \leq \varepsilon \|v_1^0\|_{L^2(\Gamma)},$$

so that we have achieved an alternating-direction-like acceleration of the convergence of the iteration for the differential model problem under the assumption that the number of error components is boundable in advance.

It is clear that the three-dimensional case of the model problem on $\Omega = (-\infty, \infty) \times (0, 1)^2$ can be treated similarly. In the case that $\Omega = (-1, 0) \times (0, 1)$, the same conclusion can be reached after a slightly messier argument.

3 A Model Finite Difference Problem in Two-Space

Let us consider the discretization of (1) by the standard five-point finite difference method; i.e., let

$$\begin{gathered} h = 1/N, \quad x_k = kh, \; y_\ell = \ell h, \quad f(x_k, y_\ell) = f_{k\ell}, \\ \partial_{xx} f_{k\ell} = (f_{k+1,\ell} - 2 f_{k\ell} + f_{k-1,\ell})/h^2, \end{gathered}$$

and approximate (4) by

$$\begin{aligned} -\Delta_h u = -(\partial_{xx} + \partial_{yy})u = f, \qquad & (x_k, y_\ell) \in \Omega, \\ u = 0, \qquad & (x_k, y_\ell) \in \partial\Omega. \end{aligned}$$

We wish to carry over the procedure of the previous section to the discrete problem. So, first we solve the discrete Dirichlet problem on Ω_1. Then, the object is to solve the discrete problems with boundary values specified on $\partial\Omega_i \cap \partial\Omega$ and a discrete analogue of the Robin condition (3) on Γ_{ij} thereafter. However, a straightforward discretization of (1) fails to lead to a relation consistent with the finite difference equation for grid points on the interface Γ_{ij}, since the iteration would converge to a solution that is linear on each line $y = y_j$ between x_{-1} and x_1, a clear violation of the global difference equation. Thus, we are forced to modify the Robin condition; one modification is indicated below.

Let $\partial_{\nu_i} u_i$ denote the interior, first, divided difference quotient in the exterior normal direction on Γ_{ij} of the function $u_i = u|_{\Omega_i}$; i.e., with $u_{i;k,\ell} = u_i(x_k, y_\ell)$,

$$\partial_{\nu_1} u_{1;0,\ell} = (u_{1;0,\ell} - u_{1;-1,\ell})/h.$$

As in the differential case, let there be two values defined at a mesh point on Γ, $u_{1;0,\ell}$ and $u_{2;0,\ell}$, and write the difference equation in the form

$$\partial_{\nu_i} u_i + \partial_{\nu_j} u_j - \theta h(\partial_{yy} u_i + f) - (1-\theta)h(\partial_{yy} u_j + f) = 0, \quad i \neq j, \quad x_{0,\ell} \in \Gamma,$$

where $0 \le \theta \le 1$. Then, for $i \neq j$ and $x_{0,\ell} \in \Gamma$, consistent modifications of the Robin transmission condition can be given by

$$\beta\big(\partial_{\nu_i} u_i - \theta h(\partial_{yy} u_i + f)\big) + u_i = -\beta\big(\partial_{\nu_j} u_j - (1-\theta)h(\partial_{yy} u_j + f)\big) + u_j, \quad (6)$$

again for $0 \le \theta \le 1$. The choice $\theta = 1$ weights the information at the new iterate on Ω_i the heaviest and seems to be preferable to smaller values for θ; thus, we shall require the following transmission condition:

$$\beta\big(\partial_{\nu_i} u_i - h(\partial_{yy} u_i + f)\big) + u_i = -\beta \partial_{\nu_j} u_j + u_j, \quad i \neq j, \quad x_{0,\ell} \in \Gamma.$$

The choice $\theta = \frac{1}{2}$ leads to convergence estimates quite similar to those for the differential problem; this case will be indicated later.

Let the superscript $^\circ$ on Ω_i or Γ_{ij} indicate the mesh points of these sets interior to Ω. We can summarize the iterative algorithm as follows. Let $u_i^n = 0$ on $\partial\Omega$, $n \ge 0$. Then, let

- $u_1^0 |_{\Gamma_{12}^\circ}$ arbitrary,
  $-\Delta_h u_1^0 = f, \quad x_{k\ell} \in \Omega_1^\circ,$

- For $n \ge 1$,

  $$\begin{aligned} &\beta_n[\partial_{\nu_2} u_2^n - h\partial_{yy} u_2^n - hf] + u_2^n = -\beta_n \partial_{\nu_1} u_1^{n-1} + u_1^{n-1}, \ x_{k\ell} \in \Gamma_{21}^\circ, \\ &-\Delta_h u_2^n = f, \quad x_{k\ell} \in \Omega_2^\circ, \end{aligned}$$

  $$\begin{aligned} &\beta_n[\partial_{\nu_1} u_1^n - h\partial_{yy} u_1^n - hf] + u_1^n = -\beta_n \partial_{\nu_2} u_2^n + u_2^n, \qquad x_{k\ell} \in \Gamma_{12}^\circ, \\ &-\Delta_h u_1^n = f, \quad x_{k\ell} \in \Omega_1^\circ. \end{aligned}$$

Let $v^n = u - u^n$ represent the error in the n^{th} iteration, beginning with an initial error on Γ_{12} given by (5) with the range of k given by $k \in \{1, \dots, N-1\}$, where $N = h^{-1}$. Thus,

- $$v_1^0 \mid_{\Gamma_{12}} = \sum_{k=1}^{N-1} \alpha_k \sin \pi k y, \tag{7}$$
  $$-\Delta_h v_1^0 = 0, \quad x_{k\ell} \in \Omega_1^\circ,$$

- For $n \geq 1$,

  $$\beta_n[\partial_{\nu_2} v_2^n - h\partial_{yy} v_2^n] + v_2^n = -\beta_n \partial_{\nu_1} v_1^{n-1} + v_1^{n-1}, \quad x_{k\ell} \in \Gamma_{21}^\circ,$$
  $$-\Delta_h v_2^n = 0, \quad x_{k\ell} \in \Omega_2^\circ,$$

  $$\beta_n[\partial_{\nu_1} v_1^n - h\partial_{yy} v_1^n] + v_1^n = -\beta_n \partial_{\nu_2} v_2^n + v_2^n, \qquad x_{k\ell} \in \Gamma_{12}^\circ,$$
  $$-\Delta_h v_1^n = 0, \quad x_{k\ell} \in \Omega_1^\circ.$$

Since $\partial_{yy} \sin \pi k y = -4h^{-2} \sin^2(\pi k h/2) \cdot \sin \pi k y$, it suffices to consider the modes one at a time and seek a solution in the form

$$\begin{aligned} v_1^n(x_i) &= A_1^n z_{k,1}^i \sin \pi k y, \qquad i = 0, -1, -2, \dots, \\ v_2^n(x_i) &= A_2^n z_{k,2}^i \sin \pi k y, \qquad i = 0, 1, 2, \dots, \end{aligned}$$

where $A_1^0 = 1$,

$$\lambda_k = 2 \sin^2(\pi k h/2),$$

and $z_{k,1}$ and $z_{k,2}$ are the roots

$$z_{k,j} = 1 + \lambda_k - (-1)^j \sqrt{(1 + \lambda_k)^2 - 1}, \quad j = 1, 2,$$

of the characteristic equation

$$z^2 - 2(2 - \cos \pi k h) z + 1 = 0$$

associated with separating variables in the Laplace difference equation. Note that $z_{k,1} z_{k,2} = 1$ and $0 < z_{k,2} < 1$. (Note that the particularly simple form of the solution above results from considering the infinite strip; if the domain were a rectangle, the solution would be a linear combination of the two independent solutions in each subdomain.)

Let

$$\gamma_n = \beta_n / h.$$

Then, a simple calculation shows that

$$A_2^1 = \frac{1 - (1 - z_{k,2})\gamma_1}{1 + (1 - z_{k,2} + 2\lambda_k)\gamma_1}, \quad A_1^1 = (A_2^1)^2.$$

More generally, for the initial condition (7),

$$v_1^n(x_i, y_j) = \sum_{k=1}^{N-1} \alpha_k \prod_{\ell=1}^{n} \left[\frac{1 - (1 - z_{k,2})\gamma_\ell}{1 + (1 - z_{k,2} + 2\lambda_k)\gamma_\ell} \right]^2 z_{k,1}^i \sin \pi k y_j.$$

N \ np	1	2	3	4	5
10	.356e-1	.227e-3	.152e-5	.719e-8	.851e-10
20	.100e0	.324e-2	.372e-4	.279e-5	.291e-6
50	.238e0	.291e-1	.621e-3	.175e-3	.705e-4
100	.365e0	.839e-1	.286e-2	.144e-2	.786e-3

Table 1: $\max_k \prod_{\ell=1}^{np} \left[\frac{1-(1-z_{k,2})\gamma_\ell}{1+(1-z_{k,2}+2\lambda_k)\gamma_\ell} \right]^2, \qquad \theta = 1$

Note that, as $h \to 0$,

$$\lambda_1 \sim \frac{1}{2}\pi^2 h^2, \qquad \lambda_{N-1} \sim 2,$$

so that

$$1 - z_{1,2} \sim \pi h, \quad 1 - z_{N-1,2} \sim 3 - 2\sqrt{2} \approx .17.$$

The choice $\gamma_1 = (1 - z_{1,2})^{-1} \sim 1/\pi h$ suppresses the fundamental mode; to see what this choice does to the other modes, consider (with $\gamma_1 = 1/\pi h$)

$$f(\lambda, \gamma_1) = \frac{1 - (1 - z_2(\lambda))\gamma_1}{1 + (1 - z_2(\lambda) + 2\lambda)\gamma_1} = \frac{\lambda - \sqrt{(1+\lambda)^2 - 1} + \pi h}{\lambda + \sqrt{(1+\lambda)^2 - 1} + \pi h}$$

for $\frac{1}{2}\pi^2 h^2 \le \lambda \le 2$. For λ close to $\frac{1}{2}\pi^2 h$, $f(\lambda)$ is small and negative. A calculus exercise shows that $f(\lambda, \gamma_1)$ has a minimum for

$$\lambda = \pi h/(1 - \pi h) \sim \pi h,$$

and then

$$f(\pi h, \gamma_1) \sim -(1 - \sqrt{2\pi h})/(1 + \sqrt{2\pi h}).$$

For $\lambda > \pi h$, $f(\lambda, \gamma_1)$ is increasing and $f(2, \gamma_1) \sim -.04$. For $h = 1/100$, the minimum value of $f(\lambda, \gamma_1)$ is approximately $-.7$, so that every mode has its coefficient reduced by a factor of at least two for $N \le 100$ if just this single parameter is used.

Let us consider choosing γ_j, $j = 1, \ldots, 5$, as follows:

$$\gamma_j = (1 - z_2(\lambda_j))^{-1}, \quad j = 1, 2, 4, 5; \qquad \gamma_3 = 1/\sqrt{2\pi h},$$

where

$$\lambda_j = 2\sin^2(\pi j h/2), \quad j = 1, 2; \qquad \lambda_4 = 1, \ \lambda_5 = 2.$$

This set of iteration parameters first (γ_1 and γ_2) suppresses the fundamental and second modes of the error on the interface, then (γ_3) severely reduces the coefficients of the modes with frequencies near values at which the first two parameters are least effective, and finally (γ_4 and γ_5) severely reduces the coefficients of the higher frequency modes. The minimum reduction factors for the coefficients α_k in the initial error (7) for one cycle of iteration using the parameters γ_j, $j = 1, \ldots, np$, can be computed; see Table 1.

If, instead, we had made the choice $\theta = \frac{1}{2}$ in (6), then the function $f(\lambda, \gamma)$ corresponding to this choice would have been

$$f(\lambda, \gamma) = \frac{1 - (1 - z_2(\lambda) + \lambda)\gamma}{1 + (1 - z_2(\lambda) + \lambda)\gamma},$$

N \ np	1	2	3	4	5
10	.177e-1	.350e-4	.805e-6	.158e-7	.669e-9
20	.632e-1	.977e-3	.112e-4	.163e-5	.329e-6
50	.181e0	.145e-1	.153e-3	.352e-4	.185e-4
100	.301e0	.519e-1	.127e-2	.338e-2	.167e-3

Table 2: $\max_k \prod_{\ell=1}^{np} \left[\frac{1-(1-z_{k,2})\gamma_\ell}{1+(1-z_{k,2}+2\lambda_k)\gamma_\ell} \right]^2, \qquad \theta = 1$

and, as in the differential case, we would have been led to a parameter sequence of length $\mathcal{O}(\log N \cdot \epsilon^{-\frac{1}{2}})$ to obtain a reduction of the error by a factor ϵ. It is clear from the form of the error reduction functions that the choice $\theta = 1$ is superior to $\theta = 0$; the same conclusion can be reached for $\frac{1}{2} < \theta < 1$.

4 A Model Problem in Three-Space

Let Ω be the infinite cylinder $(-\infty, \infty) \times (0,1)^2$ and consider the analogous iterative procedure for the seven-point finite difference equation, with Ω_1 being the left half of the domain and Ω_2 the right half. The entire analysis above applies to this problem if we reinterpret the index k to be $\{k_1, k_2\}$ and v_m^n to be

$$v_m^n(x_i) = A_m^n z_{k,m}^i \sin \pi k_1 y \sin \pi k_2 z, \quad m = 1, 2, \quad i^m \geq 0.$$

Then, change λ_k to be

$$\lambda_k = 2(\sin^2(\pi k_1 h/2) + \sin^2(\pi k_2 h/2)).$$

Table 2 is the three-space analogue of Table 1, with the five parameters determined as above, except that the λ-values are as follows:

$$\lambda_j = 4\sin^2(\pi j h/2), \quad j = 1, 2; \qquad \lambda_4 = 1, \ \lambda_5 = 3.$$

Again, the modified Robin transmission condition is very effective.

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