Annals of Discrete Mathematics 6 (1980) 41–55

ON PERMUTATION CLIQUES

M. DEZA
C.N.R.S., Paris, France

The symmetric group S_n is a metric space with distance $d(a, b) = |E(a^{-1}b)|$ where $E(c)$ is the set of points moved by $c \in S_n$. Let L be a given subset of $\{1, 2, \ldots, n\}$, a *permutation clique* $A = A(L, n)$ is any subset $A \subseteq S_n$ with $d(a, b) \in L$ whenever $a, b \in A$, $a \neq b$. We give a framework of new and known information on some special $A = A(L, n)$: maximal, largest, largest subgroups of S_n, subscheme of Hamming metric scheme, permutation geometry and some other problems related to this metric space. Some links with classical problems of classification of permutation groups and with extremal problems on finite sets are given.

1. Introduction

We will use the following notations. Let n be a given integer, $n \geq 2$; $N = \{1, 2, \ldots, n\}$; S_n is the symmetric group of all permutations of N.

We will consider each $a \in S_n$ as an n-vector $a = (a_1, a_2, \ldots, a_n)$ over N where $a(i) = a_i$; denote $E(a) = \{i \in N: a_i \neq i\}$; $d(a, b) = |E(a^{-1}b)|$ for $a, b \in S_n$ (it is the *distance* on S_n; $d(a, b) = |\{i \in N: a_i \neq b_i\}|$, i.e. it is the Hamming distance on S_n considered as a subset of a set of all n-vectors over N; $d(ac, bc) = d(a, b) = d(ca, cb)$).

Let $L = \{l_1, l_2, \ldots, l_r\}$ be a given set of integers, $1 \leq l_1 < l_2 < \cdots < l_r \leq n$, $r \geq 1$; denote $\bar{L} = N - L$. Let $A \subseteq S_n$; denote

$$A = A(L, n) \quad \text{if } d(a, b) \in L \text{ for any } a, b \in A, \quad a \neq b;$$

we will call $A(L, n)$ an (L, n)-*clique*; below, we will study the following special (L, n)-cliques.

Let $A = A(L, n)$; we call it:

(1) $A^*(L, n)$ if A is a subgroup of S_n;

(2) $A(L, n)$-*scheme* if A is a metric association scheme where we define $a, b \in A$ is i-*associated* whenever $d(a, b) = l_i$;

(3) *maximal* $A(L, n)$ if $\{a\} \cup A \neq A(L, n)$ for any $a \in S_n - A$;

(4) *largest* $A(L, n)$ if $A' = A(L, n)$ implies $|A'| \leq |A|$; *largest* $A^*(L, n)$ if $A' = A^*(L, n)$ implies $|A'| \leq |A|$;

(5) *sharp* $A(L, n)$ if $|A(L, n)| = \prod_{i \in L} i$.

(6) $A(\leq t, n)$ if $L = \{1, 2, \ldots, t\}$; $A(\geq t, n)$ if $L = \{t, t+1, \ldots, n\}$; $A(t, n)$ if $L = \{t\}$.

Denote $\langle 1 \rangle$ as the *identity* permutation; the subset $A \subset S_n$ is *unitary* iff $\langle 1 \rangle \in A$. Of course, for any $A = A(L, n)$ and $b \in S_n$ we have $B = \{ab : a \in A\} = A(L, n)$ with

$|B| = |A|$; B has any of the properties (2)–(7) iff A has that property. In the case $b^{-1} \in A$ the set B will be unitary.

The distance $d(a, b)$ was introduced in [16]. Some other distances on S_n (for example, two in [6, p. 97, Vol. 2], four in [15] and many others in [5]) have interesting links with statistics, number theory, calculation of permanents etc., but we will consider below only $d(a, b)$.

The metric space (S_n, d) gives a good common point of view for many different objects. For example:

(a) $A(t, n)$ is an orthogonal (r, λ)-design $(r = n, \lambda = n - t)$ or E.P.A. (i.e. equidistant permutation array) in other terms;

(b) sharp $A(\geq t, n)$ is sharply $(n - t + 1)$-transitive subset of S_n or (in terms of German geometers [17]) Minkowski $(n - t - 1)$-structure of order $t + 1$;

(c) some classification theorems for permutation groups are equivalent to classification of some sharp $A^*(L, n)$;

(d) some $A(\geq t, n)$ have good properties as codes (for example, for majority type decoding [4]); the coding aspect of $A(L, n)$ is considered in [1].

My essential motivation, however, was to start a large project of extremal theory on subsets of S_n by analogy with an extremal theory on systems of finite sets. For example, we have easily $\max |A^*(\leq t, n)| = t!$, i.e. an analog to Erdos–Ko–Rado's theorem for $A^*(\leq t, n)$. (We conjecture $\max |A(\leq t, n)| = t!$ for $n \geq n_0(n - t)$; Frankl (personal communication) gave $\max |A \leq t, n)| \leq (t + \log n)!$ for any $n, t > \frac{1}{2}n$).

Imitating my teacher Paul Erdös, I tried, during the last 2 years, to push this project with my mathematician friends around the world. The survey of bounds on largest $A(L, n)$ with specified L is given in [9, 11], and on maximal $A(\leq t, n)$ is studied in [14]. Below we give information on some $A(L, n)$ with arbitrary L.

In Section 2, we will give, via $\bar{L} = N - L$, a characterization of maximal $A(L, n)$ and some bounds for its cardinality. In Section 3 we shall give (via permutation representation of S_n) a bijection between the set of all $A(L, n)$ and the set of all systems B of n-subsets of a given n^2-set such that $L = \{\frac{1}{2}|C \cup D - C \cap D| : C, D \in B\} - \{0\}$, and the system B is 2-resolvable. In Section 4, we will give an application of Q^v (i.e. the set of all v-vectors over $Q = \{0, 1, \ldots, q - 1\}$) into S_{vq}, permitting us to give examples of $A(L, n)$-schemes. We give some other examples also. Sections 3, 4 are slight developments of [10, 12], respectively.

In Section 5, we introduce a semilattice such that S_n is its upper fiber; it gives a permutation analog of the packing and covering problem on sets.

In Section 6, permutation analog of a matroid is introduced.

In Section 7 is given a link with group theory. The proofs are usually omitted.

2. Duality

Proposition 2.1. $|A(L, n)| \cdot |A(\bar{L}, n)| \leq n!$

Proposition 2.2 (generalization). *Let $A = A(L, n)$, such that either $A = A^*(L, n)$ or A is a $A(L, n)$-scheme. Let $L = L' \cup L''$ where $L' \cap L'' = \emptyset \neq L', L''$. Let $B', B'' \subset A$ and $B' = A(L', n)$, $B'' = A(L'', n)$. Then*

$$|B'| \cdot |B''| \leq |A|.$$

Proposition 2.1 is the case $A = S_n$ of Proposition 2.2. The case "A is an $A(L, n)$-scheme" is a special case of the inequality proved by Delsarte for any association scheme. In the case "$A = A^*(L, n)$" of Proposition 2.2 (actually, we need only $a, b \in A \Rightarrow a \cdot b \in A$, i.e. A is a semigroup), we have $a'b' \neq a''b''$ (for any $a', a'' \in B'$; $b', b'' \in B''$, $a' \neq a''$) because otherwise $b' = (a')^{-1}a''b''$, $b'(b'')^{-1} = (a')^{-1}a''$, $|E(b' \cdot (b'')^{-1})| = |E((a')^{-1} \cdot a'')|$, $d(b', b'') = d(a', a'')$, $L' \cap L'' \neq \phi$, a contradiction. So $|B'B''| = |B'| \cdot |B''|$, and Proposition 2.2 follows from $B'B'' \subseteq A$.

Proposition 2.3. *Let $A = A(L, n)$. Then A is a maximal $A(L, n)$ iff*

$$\bigcup_{a \in A} S^a_{L,n} = S_n$$

where

$$S^a_{L,n} = \{b \in S_n : d(a, b) \in \bar{L}\} \cup \{a\}.$$

In fact, from the definition of $S^a_{L,n}$, it follows that $A \cup \{c\} = A(L, n)$ iff $c \in S_n \setminus (\bigcup_{a \in A} S^a_{L,n})$.

We remark that, for any $A = A(L, n)$, $B \subset A$, $L \supset L' \neq \phi$, $B = A(L', n)$, we have that B is $A(L', n)$ *maximal in* A iff

$$A \cap \left(\bigcup_{a \in B} S^a_{L-L',n} \right) = A.$$

We remark also that (for any $a \in S_n$)

$$|S^a_{L,n}| = 1 + \sum_{i \in L} \binom{n}{i} D_i,$$

where D_i is the number of *derangements* of degree i ($b \in S_i$ is a derangement iff $|E(b)| = i$; we know that

$$D_i = i! \sum_{0 \leq j \leq i} (-1)^j (1/j!) \sim i!/e$$

and $D_i = 0, 1, 2, 9, 44, 265, \ldots$ for $i = 1, 2, 3, 4, 5, 6, \ldots$). For $L = \{1, 2, \ldots, t\}$, the number $|S^a_{L,n}|$ is just the volume of a sphere of radius t in the metric space (S_n, d); denote it $|S_{t,n}|$.

Corollary 2.4. *Let A be a maximal $A(L, n)$. Then*

$$n!/|S^a_{L,n}| \leq |A| \leq n!/|A(\bar{L}, n)|$$

for any $A(\bar{L}, n)$.

In the special case of $L=\{t+1, t+2, \ldots, n\}$ we have, for any $A=A(>t, n)$ maximal, the following inequality

$$n!/|S_{t,n}| \leqslant |A| \leqslant n!/|S_{[t/2],n}|.$$

The right side of this inequality is an analog to Rao-Hamming's (sphere-packing) upper bound for largest error correcting codes; C. Landauer studied (in [21]) the case of equality. The left side is an analog to the Gilbert lower bound, but we have no lower bound for largest $A^*(L, n)$ (i.e. an analog to Varshamov's modification of Gilbert's bound for subgroups) because S_n is not commutative. But for any *abelian* $A=A^*(L, n)$, $B \subset A$, $L \supset L' \neq \phi$, $B=A^*(L', n)$ *maximal* (as $A^*(L', n)$) *in* A we have

$$|B| \geqslant |A| / \max_{a \in B} |A \cap S^a_{L-L',n}|.$$

The equality in Proposition 2.1 corresponds to a "factorization" of S_n by 2 dual cliques. In the case of 2 dual sharp cliques, we know only that

(a) for $L=\{t\}$ a sharp $A^*(L, n)$ exists; there exists sharp $A(\bar{L}, n)$ if and only if $t=n$; for $t=n$ $(S_n)_\alpha$ is a sharp $A^*(\bar{L}, n)$;

(b) for $L=\{1, 2, \ldots, n-t\}$ a sharp $A^*(L, n)$ exists; a sharp $A(\bar{L}, n)$ (or sharp $A^*(\bar{L}, n)$) exists iff there exists a sharply t-transitive subset (or subgroup) of S_n;

(c) $\max |A(\geqslant 5, 6)| = 18$, $\max |A(<5, 6)| = 4!$

Some information on $A=A(L, n)$ or maximal $A(L, n)$ can be obtained from the set $E(A)=\{E(a): a \in A\}$ using the evident inclusion

$$E(a) \cup E(b) \supseteq E(a^{-1}b) \supseteq E(a) \Delta E(b).$$

For example, in the case $|E(a) \cap E(b)| \leqslant 1$ $\forall a, b \in A$ (i.e. $E(A)$ is finite linear space) we have

$$d(a, b) = |E(a) \cup E(b)| \quad \forall a, b \in A.$$

Proposition 2.5. *Let A be a complete A (L, n), i.e. $E(a) \in E(A) \Rightarrow a \in A$. Then $|E_1 \cup E_2| \in L$ $\forall E_1, E_2 \in E(A)$ and A is a maximal $A(L, n)$ if*

$$B \subseteq \{1, \ldots, n\},\ |B| \neq 1,\ B \notin E(A) \Rightarrow |B \cup E_1| \notin L \quad \text{for some } E_1 \in E(A).$$

In [14], the author considered *complete* maximal $A(\leqslant t, n)$; they correspond (via $a \to E(a)$) to upper fibres of hereditary families $\{E_i\}$ such that $E_i \subseteq \{1, \ldots, n\}$, $|E_i| \neq 1$, $|E_i \cup E_j| \leqslant t$ and $\{E_i\}$ is maximal (non extendable). Similar results can be given a link of $d(a, b)$ with $|E(a) \Delta E(b)|$, i.e. with Hamming distance on $E(A)$.

We remark also that for any set $\alpha^0, a^1, \ldots, a^s \in S_n$, we have

$$\{i \in \{1, \ldots, n\}: \alpha^0_i = a^1_i = \cdots = a^s_i\} = \bigcap_{j=1}^{s} \{i \in \{1, \ldots, n\}: a^0_i = a^j_i\}$$
$$= \{1, \ldots, n\} - \bigcup_{j=1}^{s} E((a^0)^{-1} a^j).$$

3. A characterization of permutation cliques

For any permutation $a \in S_n$ we denote by $m(a)$ the following binary n^2-vector

$$m(a) = (m_{11}, m_{12}, \ldots, m_{1n}, m_{21}, \ldots, m_{2n}, \ldots, m_{n1}, \ldots, m_{nn}),$$

where $m_{ij} = 1$ if $a_i = j$ and $m_{ij} = 0$ otherwise, i.e. $m(a)$ is just a "linearized" permutation matrix $((m_{ij}))_n^n$ representing a in a *natural* (*permutation*) representation of S_n (the trace of this matrix equals $n - |E(a)|$; it is the number of fixed points of a and value $\theta(a)$ on a of permutation character θ). Replacing double indexing with i, j by indexing with t, we obtain

$$m(a) = (m_1, m_2, \ldots, m_t, \ldots, m_{n^2})$$

where

$$m_{ij} = m_t \ (t = (i-1)n + j) \quad \text{and} \quad m_t = m_{ij} (t \equiv j \pmod{n},\ i = (t-j)/n).$$

Denote

$$\tilde{a} = \{t \in \{1, 2, \ldots, n^2\} : m_t = 1\}.$$

It is easy to see that $\tilde{a}$ is an n-subset of the n^2-set $\{1, 2, \ldots, n^2\}$ and

$$\tfrac{1}{2} d(a, b) = \sum_{t=1}^{n^2} (m_t(a) \oplus m_t(b)) / \mathrm{mod}\ 2) \tag{1}$$

i.e. equal to the Hamming distance of $m(a)$, $m(b)$; $\frac{1}{2} d(a, b) = |\tilde{a} \,\Delta\, \tilde{b}|$ where $\tilde{a} \,\Delta\, \tilde{b} = \tilde{a} \cup \tilde{b} - \tilde{a} \cap \tilde{b}$ is the symmetric difference of $\tilde{a}, \tilde{b}$;

$$n - d(a, b) = (m(a) \cdot m(b)), \tag{2}$$

i.e. equal to the scalar product of $m(a)$, $m(b)$; $n - d(a, b) = |\tilde{a} \cap \tilde{b}|$.

Denote by $[n^2]^n$ the set of all n-subsets of the n^2-set $\{1, 2, \ldots, n^2\}$. Let $B \subseteq [n^2]^n$; suppose that $R = \{R_1, R_2, \ldots, R_n\}$ is a partition of $\{1, 2, \ldots, n^2\}$ such that for any i, $1 \leqslant i \leqslant n$, the sets $\{b \in B : t \in b\}$, $t \in R_i$, form a partition of B. In the case of the existence of such a partition R we will call B *resolvable* (or 1-*resolvable*), and we will call $R = \{R_1, R_2, \ldots, R_n\}$ the *resolution* of B with *classes* $R_1, R_2, \ldots, R_n$. Given an integer $f \geqslant 0$, we will call a subset $B \subset [n^2]^n$ $(f+1)$-*resolvable* iff there exist $f+1$ resolutions $R^{(0)}, R^{(1)}, \ldots, R^{(f)}$ of B such that $|R_{i'}^{(i)} \cap R_{j'}^{(j)}| \leqslant 1$ for any $0 \leqslant i < j \leqslant f$ and $1 \leqslant i', j' \leqslant n$. The referee of this paper remarked that condition $|R_i| = n$, $1 \leqslant i \leqslant n$, follows from $(f+1)$-resolvability only for $f \geqslant 1$.

Proposition 3.1. *Let* $B \subseteq [n^2]^n$. *Then*

(a) B *is resolvable with resolution* $\{R_1, R_2, \ldots, R_n\}$, $|R_i| = n$, *iff* $B = \{\tilde{a} : a \in A\}$ *for some set of* n*-vectors over* N;

(b) B *is* 2*-resolvable iff* $B = \{\tilde{a} : a \in A\}$ *for some set* $A \subseteq S_n$;

(c) B *is* $(f+1)$*-resolvable* (f *integer*, $f \geqslant 0$) *iff* $B = \{\tilde{a} : a \in A^{(1)}\}$ *for some orthogonal system* $A^{(1)}, \ldots, A^{(f)}$ *of subsets of* S_n.

Here we call *orthogonal* any system $A^{(1)}, \ldots, A^{(f)}$ of equicardinal (with $|A^{(i)}| = m$) ordered (with $A^{(i)} = \{a^{(i)}_{(1)}, \ldots, a^{(i)}_{(m)}\}$) subsets of S_n such that

$$d(a^{(i)}_{(i')}, a^{(i)}_{(j')}) = d(a^{(j)}_{(i')}, a^{(j)}_{(j')}) \tag{a}$$

for any $1 \leq i, j \leq f$ and $1 \leq i' < j' \leq m$ (i.e. any two $A^{(i)}, A^{(j)}$ are isometric),

$$(a^{(g)}_{(i)j}, a^{(g')}_{(i)j}) = (a^{(g)}_{(i')j'}, a^{(g')}_{(i')j'}) \tag{b}$$

iff $j = j'$ and

$$a^{(1)}_{(i)j} = a^{(1)}_{(i')j}, \qquad a^{(2)}_{(i)j} = a^{(2)}_{(i')j}, \ldots, a^{(f)}_{(i)j} = a^{(f)}_{(i')j}.$$

So, $(f+1)$-resolvable subsets of $[n^2]^n$, consisting of m *disjoint* n-subsets of $\{1, 2, \ldots, n^2\}$, correspond (via c) to a system of f pairwise orthogonal Latin rectangles $(A^{(i)})$ with m rows and n columns; in particular, for $m = f+1 = n$, it corresponds to a complete set of Latin squares.

Proposition 3.2. *Let* $B \subseteq [n^2]^n$, $|B| > 1$, B *is* $(f+1)$*-resolvable. Then*

$$2(f+1) \leq \max_{a,b \in B} |a \Delta b| \leq 2n$$

with both equalities if B *consists of* n *disjoint sets and there exists* PG(2, n), *i.e. a projective plane of order* n.

The orthogonal system $A^{(1)}, \ldots, A^{(f)}$ of ordered subsets of S_n can be visualized as a 3-dimensional matrix of size $n \times m \times f$ over N; we can consider ordered subsets of S_n (following the terminology of Cheema and Motzkin) as *multipermutations.*

From the mapping $a \longrightarrow m(a)$, and the corresponding bound for n-subsets of a given n^2-set follows

Proposition 3.3. *Given* $A = A(L, n)$, *we have* g.c.d.$\{i \in L\} \nmid n \Rightarrow |A| \leq n^2$.

4. Examples of an $A(L, n)$-scheme

The set of all binary n^2-vectors with n ones (and, in general, the set Q^v of all v-vectors over $Q = \{0, 1, \ldots, n-1\}$) is a *metric association scheme* if one defines $x = (x_1, \ldots, x_v)$, $y = (y_1, \ldots, y_v)$ to be i-associates whenever their Hamming distance

$$d(x, y) = |\{j \in \{1, 2, \ldots, v\} : x_j \neq y_j\}| = i$$

(see e.g. [7]). It is called the *Hamming scheme.* Some subsets of a Hamming scheme are also schemes with the same definitions of distance and i-association (for example, the set of all binary v-vectors, each with the same number of ones). Now we will give an example of such subschemes of a Hamming scheme consisting only of permutations of $\{0, 1, \ldots, q-1\}$.

Let $n = n_1 n_2$ where n_1, n_2 are integers >1, i.e. n is not a prime. For any n_2-vector $x = (x_1, \ldots, x_{n_2})$ over $\{0, 1, \ldots, n_1 - 1\}$, we define the following permutation of

$$\{1, \ldots, n\} = \{i + jn_2 : i \in \{1, \ldots, n_2\},\ j \in \{0, 1, \ldots, n_i - 1\}:$$

$$l(x) = \prod_{i=1}^{n_2} (c_0^i, \ldots, c_{n_1-1}^i) \quad \text{where } c_j^i = i + n_2((x_i \oplus j)/\text{mod } n_1)$$

i.e. $l(x)$ is a product of some disjoint cycles of length n_2 whenever $x_i \neq 0$. For $n_1 = 2$ (i.e. for n even) $l(x)$ is some involution because it is a product of disjoint transpositions.

Proposition 4.1. *Let x, y be two n_2-vectors over $\{0, 1, \ldots, n_1 - 1\}$. Then*

(a) $l((x \oplus y)/\text{mod } n_1) = l(x)l(y)$;

(b) $n_1 d(x, y) = d(l(x), l(y))$.

Corollary 4.2. *Let B be a set of n_2-vectors over $\{0, 1, \ldots, n_1 - 1\}$. Then*

(a) *B is a subgroup (whose addition is defined componentwise and modulo n_1) of the group of all n_2-vectors over $\{0, 1, \ldots, n_1 - 1\}$ iff $\{l(x) : x \in B\}$ is a subgroup of S_n (the symmetric group under usual composition of permutations);*

(b) *B is a subscheme of the Hamming scheme of all n_2-vectors over $\{0, 1, \ldots, n_1 - 1\}$ iff $\{l(x) : x \in B\}$ is a subscheme of the Hamming scheme of all n-vectors over $\{1, 2, \ldots, n\}$.*

So, for example, the set of all involutions of S_n of even degrees n contains a subgroup of order $2^{n/2}$, which is a metric association scheme (with $1 + \frac{1}{2}n$ classes) with respect to our Hamming distance on permutations.

Corollary 4.3. *Let $R = ((r_{ij}))_n^m$ be the distance matrix which is isometric to some ordered set of m binary n-vectors with Hamming distance. Then $2R$ is isometric to some ordered set of m involutions of S_{2n} (with Hamming distance).*

Another example of an $A(L, n)$-scheme is provided for the case $L = \{l_1, l_2\}$ by an $A = A(L, n)$ which is such that the *graph* defined on A (for $a, b \in A$ the edge (a, b) exists iff $d(a, b) = l_2$) is strongly regular. In this case, $A(L, n)$ will be a scheme with $|L| + 1 = 3$ classes. Any complete multipartite graph is strongly regular. For example, a sharp $A(\{n, n-1\}, n)$ corresponds to a complete n-partite graph on $n(n-1)$ vertices; a sharp $A(\{n, n-1\}, n)$ exists iff there exists PG(2, n).

5. Subpermutations; packing, covering

Let us fix some element $\alpha = -\infty$ ("joker") *exterior to* $N = \{1, 2, \ldots, n\}$. Denote by $(N \cup \{\alpha\})^v$ the set of all v-vectors $a = (a_1, \ldots, a_v)$ over $N \cup \{\alpha\}$. This set is a partially ordered set with order $a \leqslant b$ iff $a_i \neq b_i$, $1 \leqslant i \leqslant v \Rightarrow a_i = \alpha$. Define the

height $\|a\|$ of an element a as $|\{i \in [1, v]: a_i \neq \alpha\})$. The smallest element is $0 = (\alpha, \alpha, \ldots, \alpha)$ having height 0. The largest height is v, but the largest element does not exist for $|N| > 1$.

Let $v = n$; denote by $\mathcal{P}$ the subset of $(N \cup \{\alpha\})^v$ which consists only of all n-vectors $a = (a_1, \ldots, a_n)$ such that $a_i = a_j$, $1 \leq i \neq j \leq n \Rightarrow a_i = a_j = \alpha$. Any $a \in \mathcal{P}$ we will call *subpermutation* (of height $\|a\|$); the set of all subpermutations of largest height n is just our symmetric group S_n.

Let $A \subseteq S_n$, $1 \leq t \leq n-1$; we call it

t-packing $P(t, n)$ if for any $a \in \mathcal{P}$ with $\|a\| = t$ there exists *at most* one $b \in A$ with $a < b$;

t-covering $C(t, n)$ if for any $a \in \mathcal{P}$ with $\|a\| = t$ there exists *at least* one $b \in A$ with $a < b$;

sharply t-transitive set $T(t, n)$ if for any $a \in \mathcal{P}$ with $\|a\| = t$ there exists *exactly* one $b \in A$ with $a < b$.

Proposition 5.1. *Let* $A \subseteq S_n$, $1 \leq t \leq n-1$. *Then:*

(a) $A = P(t, n)$ *iff* $A = A(>n-t, n)$;

(b) $A = T(t, n)$ *iff* A *is a sharp* $A(>n-t, n)$;

(c) *in the case* $\langle 1 \rangle \in A$

$$A = C(t, n) \quad \text{iff } A \text{ is } t\text{-transitive subset of } S_n;$$

(d) $|P(t, n)| \leq n!/(n-t)! \leq |C(t, n)|$ *and each of these inequalities become equality iff there exists a sharp* A $(>n-t, n)$.

So we have a direct analogy with the situation for finite sets and, in particular, $T(t, n)$ is a permutation analog to the Steiner system $S(t, k, v)$ (i.e. to t-design). We remark that both of them are special cases of q-ary *T-designs* introduced by Delsarte [7]. C. Landauer remarked that $42 \leq \min |C(2, 6)| \leq 60$.

Proposition 5.2. (a) $A = P(t, n)$ *iff* $\{\tilde{a}: a \in A\}$ *is a 2-resolvable packing of t-subsets of an* n^2*-set by its n-subsets;*

(b) *Let B be a 1-resolvable covering of t-subsets,* $1 \leq t \leq n-1$, *of an* n^2*-set by its n-subsets; then it is the smallest covering of 1-subsets, consisting of n disjoint n-subsets and* $\{a \in S_n: \tilde{a} \in B\} = T(1, n)$.

So a unique $A = C(t, n)$ such that $\{\tilde{a}: a \in A\}$ is a *t-covering of t*-subsets, is a Latin square $T(1, n)$.

Denote $A = \bar{P}(t, n)$ (*maximal t*-packing) if $A \cup \{a\} \neq P(t, n)$ $\forall a \in S_n - A$; denote $A = \underline{C}(t, n)$ (*minimal t*-covering) if $A - \{a\} \neq C(t, n)$ $\forall a \in A$. We have $\min |\bar{P}(1, n)| = n$ (and also smallest maximal packing of 1-subsets of an n^2-set consists of n disjoint subsets). Largest minimal covering of 1-subsets of an n^2-set consists of $n^2 - n + 1$ n-subsets containing a given $(n-1)$-set. We have $\max |\underline{C}(1, n)| = n(n-2)$ (see [3]).

We have $\min|\bar{P}(2,3)|=6$ and, of course, $\min|\bar{P}(n-1,n)|=n!$. But $\min|\bar{P}(2,n)|\leqslant n$ for any even n, because any cyclic Latin square has no transversals. For $t>2$ we can use the lower bound of Proposition 2.4. $\bar{P}(t,n)$ is just maximal $A(n-t,n)$; so

$$\min|\bar{P}(t,n)|\geqslant n!/|S_{n-t,n}|.$$

We remind that

$$\exists T(2,n)\Leftrightarrow\exists S(2,n,n^2),$$

because $S(2,n,n^2)$ is just a set of lines of $AG(2,n)$, and

$$\exists T(2,n)\Leftrightarrow\exists\text{ sharp }A(\{n-1,n\},n)\Leftrightarrow\exists PG(2,n)\Leftrightarrow\exists AG(2,n).$$

For a given subset $B\subset S_n$ we call *B-sorting* any $A\subset S_n$ such that $b\in B$, $a\in A$ implies $b\not<a$. A. Nozaki communicated to me that the problem of finding the largest B-sorting (for the case of B consisting of all subpermutations a of height 4 such that

$$a_{i_1},a_{i_2},a_{i_3},a_{i_4}\neq\alpha,\quad i_1<i_2<i_3<i_4\Rightarrow a_{i_1}>a_{i_2}>a_{i_3}>a_{i_4})$$

is equivalent to some problem in computer science (best sorting of permutations).

We call a given t-covering $A\subset S_n$ *c-uniform* if for any $a\in\mathcal{P}$ with $\|a\|=t$, there exist exactly c elements $b_1,\ldots,b_c\in A$ with $a<b_1,\ldots,a<b_c$. It is easy to see that any c-uniform t-covering A has $|A|=c\cdot n!/(n-t)!$ and that any t-transitive group G is a c-uniform t-covering with $c=|G|\cdot(n-t)!/n!$. K. Lieberherr (private communication) raised the following problem: to find an infinite sequence $\{A(n)\}$, $n\to\infty$, of c-uniform t-coverings $A(n)$, such that c is bounded by a polynom of n. Of course, either sharply 1-transitive (or 2, or 3) groups are c-uniform with $c=1$ for either any n (or $n=p^a$ or $n=p^a+1$); so we consider only $t>3$ and also we prefer c as small as possible. This problem comes from computer science also (design of fast algorithms for construction of interpretations of conjunctive normal forms).

6. Sharp $A(L,n)$'s: permutation geometries $PG(L,n)$

We come back to the set (considered in the beginning of Section 5) $(\{\alpha\}\cup N)^v$ of all v-vectors over $N\cup\{\alpha\}$. Let Q be any given subset of $(\{\alpha\}\cup N)^v$. Let $a,b\in Q$ and $a\leqslant b$; we denote by $[a,b]$ the set $\{c\in Q:a\leqslant c\leqslant b\}$ and call it a *segment*. For any $c\in Q$ define $\hat{c}=\{i\in[1,v]:c_i\neq\alpha\}$; so $\|c\|=|\hat{c}|$ and $c\leqslant d\Leftrightarrow\hat{c}\subseteq\hat{d}$ (for any $c,d\in Q$). Denote by $c\wedge d$ (and call *meet* of c,d) the vector $(l_1,\ldots,l_v)$ where $l_i=c_i$ if $c_i=d_i$ and $l_i=\alpha$ otherwise. Now we specify c,d to be elements of some segment $[a,b]$. Denote by $c\vee d$ (and call *join* of c,d) the vector $(l_1,\ldots,l_v)$ where $l_i=\alpha$ whenever $c_i=d_i=\alpha$ and $l_i=b_i$ otherwise. It is possible because $c\leqslant b$, $d\leqslant b$ implies that all $c_i=\alpha,b_i$ and all $d_i=\alpha,b_i$. In some cases (and, in particular, for $Q=(N\cup\{\alpha\})^v$), Q is a lower semilattice under the operation $c\wedge d$

and $\|c \wedge d\| = |\hat{c} \cap \hat{d}|$. In some cases (and, in particular, for $Q = (N \cup \{\alpha\})^v$) the segment $[a, b]$ is an upper semilattice under the operation $c \vee d$ and $\|c \vee d\| = |\hat{c} \cup \hat{d}|$.

We can consider the number $\frac{1}{2}(\|c\| + \|d\| - 2\|c \wedge d\|)$ as a modified Hamming "distance". It is not a metric; it was introduced for the case $|N| = 2$ (actually, for $N = \{0, 1\}$) by Graham and Pollak for addressing of loop switching in some data communication system. For the case $\|c\| = \|d\| = v$ it is just the usual Hamming distance $v - \|c \wedge d\|$ on N^v. It will be zero iff the join $c \wedge d$ exists in $(N \cup \{\alpha\})^v$, i.e. iff $c \leqslant b$, $d \leqslant b$ for some $b \in (N \cup \{\alpha\})^v$.

We come back to the set L given at the beginning of Section 1 but for the case $v \neq n$ (so, only, in this Section 6) we will consider L as a subset of $\{1, 2, \ldots, v\}$. Let $l'_i = v l_{r+1-i}$, $L' = \{l'_i : l_i \in L\}$. Suppose $0, 1 \in L'$. Denote by $D(L, n, v)$ any Q such that

(a) Q contains all elements of height $\leqslant 1$ and at least one element of height v;

(b) $a \in Q$, $\|a\| \neq v \Rightarrow \|a\| \in L'$;

(c) for any $a \in Q$ with $\|a\| = l'_i \in L'$ and any element b of height 1 there exists exactly one $c \in Q$ with $\|c\| = l'_{i+1}$ (we denote here and below $l'_{r+1} = v$), $a < c$, $b \leqslant c$ whenever the following condition holds:

$$\{i \in [1, v] : b_i \neq \alpha\} \not\subseteq \{i \in [1, v] : a_i \neq \alpha\}. \tag{i}$$

Suppose now that $v = n$, $Q = Q \bigcap \mathscr{P}$, i.e. Q contains only subpermutations from $(N \cup \{a\})^n$. Suppose that Q has the properties (a), (b), (c), but condition (i) in (c) is replaced by the stronger condition

$$\begin{aligned} &\{i \in [1, v] : b_i \neq \alpha\} \not\subseteq \{i \in [1, v] : a_i \neq \alpha\}, \\ &\{b_i : i \in [1, v],\ b_i \neq \alpha\} \not\subseteq \{a_i : i \in [1, v],\ a_i \neq \alpha\}. \end{aligned} \tag{ii}$$

Then we call Q a *permutation geometry* or for short PG(L, n). It will be considered in detail in [2].

In the case $N = \{1\}$ the condition (i) takes the form

$$\{i \in [1, v] : b_i = 1\} \not\subset \{i \in [1, v] : a_i = 1\},$$

and Q will be the $D(L, n, v)$ iff $\{\hat{c} : c \in Q\}$ is the lattice of all flats of *simple perfect matroid-designs* on $\{1, \ldots, v\}$; we will call it PMD for short (a survey on PMD's is given in [13]).

Proposition 6.1. *Let $|N| > 1$ and let Q be either a $D(L, n, v)$ or a* PG(L, n). *Then*

(a) *for any segment $[a, b]$ of Q the set $\{\hat{c} : c \in Q,\ a \leqslant c \leqslant b\}$ is the set of all flats of simple* PMD's on $\hat{b}$.

(b) $(l_r - l_{r-1})\,|(l_{r-1} - l_{r-2})|\cdots|(l_2 - l_1)|\,l_1$.

In fact, (b) follows from (a) and the necessary conditions

$$(l'_2 - l'_1)\,|(l'_3 - l'_2)|\cdots|l'_r - l'_{r-1})|\,(l'_{r+1} - l_r)$$

(given by Edmonds–Murty–Young) for the existence of PMD.

Suppose now that Q is either a $D(L, n, v)$ or a $\mathrm{PG}(L, n)$. Let $t(i, j, k) = |\{c \in Q : a \leqslant c \leqslant b, \|c\| = l'_j\}|$, where $a, b \in Q$ are given and $a \leqslant b$, $\|a\| = l'_i$, $\|b\| = l'_k$, $1 \leqslant i \leqslant j < k \leqslant r+1$ (remind that $l'_{r+1} = v$). From (a) of Proposition 6.1, it follows that

$$t(i, j, k) = \prod_{s=i}^{j-1} \left[\frac{l'_k - l'_s}{l'_j - l'_s} \right].$$

Let $t(i, j) = |\{c \in Q : a \leqslant c, \|c\| = l'_j\}|$, where $a \in Q$ is given and $\|a\| = l'_i$, $1 \leqslant i \leqslant j \leqslant r+1$.

Proposition 6.2. (a) *Let Q be a $D(L, n, v)$, then*

$$t(i, j) = |N|^{j-i} \prod_{s=i}^{j-1} \left[\frac{v - l'_s}{l'_j - l'_s} \right]$$

and Q contains $t(1, r+1) = |N|^r$ elements of height v;

(b) *Let $v = n$ and Q be a $\mathrm{PG}(L, n)$, then*

$$t(i, j) = \prod_{s=i}^{j-1} \left[\frac{(n - l'_s)^2}{(l'_j - l'_s)} \right],$$

and Q contains

$$t(1, r+1) = \prod_{s=1}^{r} (n - l'_s) = \prod_{i \in L} i$$

elements of height n.

In fact, in both cases (a), (b), we have

$$t(i, j) = \prod_{s=1}^{j-1} \frac{t(s, s+1)}{t(i, s, s+1)}.$$

It is easy to check that

$$\prod_{s=1}^{j-1} t(i, s, s+1) = \prod_{s=1}^{j-1} \left[\frac{l'_j - l'_s}{l'_{s+1} - l'_s} \right].$$

We have

$$t(s, s+1) = \frac{(v - l'_s)}{(l'_{s+1} - l'_s)} |N|$$

for the case (a), and

$$t(s, s+1) = \frac{(v - l'_s)(v - l'_s)}{(l'_{s+1} - l'_s)} = \frac{(n - l_s)^2}{l'_{s+1} - l'_s}$$

for the case (b). Proposition 6.2 follows.

Corollary 6.3. *Let $Q \subseteq \{N \cup \alpha\}^n$, and suppose that Q is a $\mathrm{PG}(L, n)$. Then the set of*

all its elements of height n is a sharp $A(L, n)$ *and*

$$(l_r - l_{r-1})\,|(l_{r-1} - l_{r-2})| \cdots |(l_2 - l_1)|\, l_1.$$

Proposition 6.4. *Any sharp* $A = A(>n-t, n)$ *is a set of all elements of height n of a* PG(L, n), Q *with* $L = \{n-t+1, \ldots, n\}$.

In fact we can take

$$Q = \{c \in \mathcal{P} : \text{either } c \in A \text{ or } |\hat{c}| \leqslant t-1\}.$$

For any $c \in A$ the set $\{\hat{d} \in Q : d \leqslant c\}$ will be PMD (actually, a truncation of a boolean algebra, i.e. trivoid in terminology of PMD's). So we have to prove the condition (c) in the definition of PG(L, n) only for a with $\|a\| = t-1 = n-l_r = l'_r$ But this condition for our case just consists of saying that for any subpermutation a of height t there exists exactly one $c \in A$ with $a < c$. In other words, A is a sharply t-transitive subset of S_n. From Section 5, we know that A is a sharp $A(>n-t, n)$ iff A is a sharply t-transitive subset of S_n.

Actually, we have to call a PG(L, n) *simple* because of the restriction $l'_1 = 0$, $l'_2 = 1$ (i.e. $l_{r-1} = n-1$, $l_r = n$). But it is easy to extend the above definition of a PG(L, n) to more general L. From Proposition 6.4 follows

Corollary 6.5. *Let* $L = \{t_1, t_1+1, \ldots, t_2\}$, $1 \leqslant t_1 \leqslant t_2 \leqslant n$. *Then*

(a) *any sharp* $A(L, n)$ *having* $n - t_2$ *trivial orbits (i.e. fixing some* $n - t_2$ *points of N) is the set of all elements of height n of some* PG(L, n);

(b) *any sharp* $A^*(L, n)$ *has the same property if* $t_1 < t_2$.

In fact, (b) is a special case of (a) as described in Section 7. Any sharp $A^*(t, n)$ corresponds to PG(L, n) $(L = \{t\})$ for $t = n, n-1$ or for $t = n-2, n$ odd, but for any even n there exists a sharp $A^*(n-2, n)$ which does not correspond to a PG(L, n).

We remark that conditions (i) and (ii) in the definitions of $D(L, n, v)$ and PG(L, n), correspond to 1- and 2-resolvability considered in Section 3. In both cases we replace in the axiom for flats of matroid ("for any i-flat and a point exterior to it, there exists exactly one $(i+1)$-flat containing this i-flat and point") the word "exterior" by "strongly exterior" specified by either (i) or (ii). It suggests the construction of other such things (for example, for $(f+1)$-resolvability).

Perhaps also, it will be interesting to study the sets Q, such that the sets $\{\hat{c} : c \in Q, a \leqslant c \leqslant b\}$ are not PMD's but either matroid-designs or matroids.

In order to get more simularity with matroids one can use, for example, following concepts of orthogonality and parallelism on subpermutations $(p, q, t, \ldots) - p \perp q$ iff $p \wedge q = 0$, $p \vee q \in S_n$; $p \| q$ iff $p \perp t$, $q \perp t$ for some t so, $p \| q$ is an equivalence and $p \| q$ iff $\hat{p} = \hat{q}$ and $p = sq$ for some permutation s of $\hat{p}$).

Perhaps (because of (a) of Proposition 6.1) it will be useful to consider the set

of all elements of height n of a PG(L, n) as a "good" set of automorphisms of corresponding PMD's. The existence of a PG(L, n) (especially of subgroups) and the existence of a PMD which is an extension of this PMD can be related.

$A(L, n)$-schemes coming from PG(L, n) can be obtained via "regularity" of its semilattice considered in [7].

7. Sharp $A^*(L, n)$

Let $G = A^*(L, n)$, i.e. G is a subgroup of S_n. Denote $f(a) = n - |E(a)|$, the number of fixed points of $a \in G$; it is the number of 1-cycles in a. The set $\{n - i : i \in L\} = \{f(a) : a \in G,\ a \neq \langle 1 \rangle\}$ is called the *type* of the group G. Sometimes one is interested only in $\{f(a) : a \in G,\ a^2 = \langle 1 \rangle\}$, i.e. only in involutions.

Proposition 7.1 (Bannai–Deza's conjecture proved by Kiyota [22]).

$$|G| \text{ divides } \prod_{i \in L} i.$$

In particular, $|G| \leq \prod_{i \in L} i$, i.e. any sharp $A^*(L, n)$ is a largest $A^*(L, n)$.

Proposition 7.2 ([23]). *Let X_t be the Bell number, i.e. $X_0 = 1 = X_1$, $X_{j+1} = \sum_{i=1}^{j} \binom{j}{i} X_{j-i}$. Then*

$$|G| \leq \sum_{a \in G} (f(a))^t / X_t = \left(n^t + \sum_{i \in L} (n-i)^t c_i \right) / X_t$$

(here $c_i = |\{a \in G : f(a) = n - i\}|$) with equality iff G is t-transitive.

It is well-known also that G has $r = (\sum_{a \in G} f(a))/|G|$ orbits. It proves directly that $|A^*(t, n)| \leq n - t$ $(r = (r + (|G| - 1)t)/|G|$, $|G| = (n-t)/(r-t) \leq n - t)$ and $A^*(\leq n - t, n) \leq (n-t)!$ $(r \geq (n + (|G| - 1)t)/|G| > t$, $|G| = n_1!\, n_2! \cdots n_r!$ where $n = n_1 + \cdots + n_r$ is a partition of n by lengths of orbits; so the maximum of G corresponds to $n = 1 + \cdots + 1 + (n - t)$, i.e. to $|G| \leq (n-t)!$) but the bound for an arbitrary L was proved via theory of characters.

In the problem of upper bound for $|G|$ (in the absence of sharp $A^*(L, n)$) we can use, for $|A^*(\geq t, n)|$ some characterization theorems. The open questions for small $n - t$ are to find largest:

(1) $A^*(\geq n - 1, n)$, i.e. Frobenius group, for $n \neq p^a$;

(2) $A^*(\geq n - 2, n)$ which is not 2-transitive, etc.

From now on we suppose G to be a sharp $A^*(L, n)$. We have a non-extremal (but typical for group theory) classification problem (for some L, for special G, either for n sufficiently small for counting or for a sufficiently large n). The problem of classification of sharp $A^*(L, n)$ (which I proposed to Bannai in Tokyo, 1977) was finally solved by his students [20] for the following cases:

(a) $L = \{t_1 + 1, t_1 + 2, \ldots, t_1 + t_2\}$, $1 \leq t_2 \leq n - t_1$; here G has $n - t_1 - t_2$ fixed

points and sharply t_2-transitive on the remaining elements of N (so we can apply Jordan's theorem that only sharply t-transitive groups with $t>3$ are $M_{11}, M_{12}, A_{t+2}, S_{t+1}, S_t$);

(b) $L=\{t, t+2\}$, here $t=4, 6, 8, 14$;

(c) $L=\{t, t+3\}$, here $t=6, 9, 15, 24, 27$. The case $L=\{n-2, n\}$ done in [25] is a case of a group of rank 3 with orbits 1, 1, $|G_a|$ of G_a. As a transitive extension of sharp $A^*(\{n-2, n\}, n)$, [25] gives $n=7, 9, 15$ only. (In general, the transitive extension of G is a sharp $(A^*(L\cup\{n+1\}, n+1)$ and we can use it for induction.)

So the degree of a sharp group of type $\{0, 1, 3\}$ can be only 7, 9, 15. We remark that a perfect matroid-design with flat sizes $\{0, 1, 3\}$ exists, for hyperplane-sizes 7, 9 and $n=13, 15, 19, 21, \ldots$ are next candidates [13].

Also in [19] a sharp $A^*(t, n)=G$ was classified for either $n-t\leqslant 3$ or G abelian.

As a final remark, we say that it will be interesting to see the relations between possibilities on the structure of $A=A(L, n)$:

(a) A is the set of all elements of height n of some PG(L, n); denote it by $A=\bar{A}(L, n)$,

(b) $A=A^*(L, n)$, i.e. A is a subgroup of S_n,

and the possibilities on its cardinality:

(1) A is sharp, write $A=\hat{A}(L, n)$,

(2) A is largest, write $A=\bar{\bar{A}}(L, n)$,

(3) A is maximal, write $A=\bar{A}(L, n)$.

For $L=\{n-t+1, \ldots, n\}$, (1)$\Rightarrow$(2), (1)$\Rightarrow$(a) ("sharp" is just sharply t-transitive set), but (1)$\not\Rightarrow$(b) in general (for $t=1$ it is a Latin square which is not a group, for $t=2$ it comes from PG$(2, n)$ which is not over near-field, for $t=3$ C. Pedrini (1966) constructed it). (But for all these counterexamples, sharp $A^*(>n-t, n)$ exists as well.) On the other hand, for $L=\{t\}$, $t\neq n$, (1) and (b)$\not\Rightarrow$(a) (at least for $t=n-2$, any even n), and (1) and (b)$\not\Rightarrow$(2). A pair of nonisomorphic sharp $A^*(\{n-q, n\}, n)$ (and, moreover, exactly one of them is a permutation geometry) exists, for example, for $n=q^2$, $q=p^\alpha$ (see [2]) and for $n=6$, $q=2$ (see [25, Theorem 1]).

In [2] the concept of permutation geometry PG(L, n) will be done in detail; in particular, a characterization of sharp $A^*(L, n)$'s which are simple PG(L, n) will be given via some Jordan groups considered by W. Kantor. Also sharp $A^*(L, n)$ is t-transitive iff $n-t'+1\in L$ for all $t'\leqslant t$.

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