# Some remarks on the maximality of Inner Models

P. D. Welch

Graduate School of Science & Technology,
Kobe University,
Rokko-dai, Nada-ku,
Kobe 657, Japan.
welch@pascal.seg.kobe-u.ac.jp

**Abstract.** We consider maximality properties of inner models, elementary embeddings between them, and survey some connections through the concept of Jónsson cardinal. In particular we give proofs of:
**Theorem** Assume there is no inner model with a strong cardinal and $K$ is the core model. If $\alpha$ is a regular Jónsson cardinal, then
(i) $\alpha^+ = \alpha^{+K}$;
(ii) $\{\beta < \alpha \mid \beta$ regular, $\beta^+ = \beta^{+K}\}$ is stationary in $\alpha$;
(iii) $\forall A \subseteq \alpha$, $A^\#$ exists.
**Theorem** Assume there is no inner model with a Woodin cardinal, there is a measurable cardinal $\Omega$, and $K$ is the Steel core model. If $\alpha < \Omega$ is a regular Jónsson cardinal, then (i) and (ii) above hold.

## 1 Introduction

The first half of this paper constitutes the lecture given at the conference meeting, and contains some general discussion and a survey of the area. The second half gives proofs of some new results that were mentioned but not given. The reader interested only in these, (the theorems of the abstract) can turn to §2.2 Theorem 12 and §2.3. We assume at those points a working familiarity with fine structural arguments, and in particular with [19] and [23].

*Themes :*
1. Maximality properties of inner models.
2. Elementary embeddings.
3. Some connections through the concept of Jónsson cardinal.

**Definition 1.** *An* Inner Model $M$ *is a transitive class of sets, a model of ZFC axioms with* $On \subseteq M$.

We abbreviate this as $IM(M)$. We regard $V$ as an inner model of itself. Gödel's universe $L$ is then a paradigm for an inner model. We write $\exists IM(\Psi)$ to mean there exists an inner model with property $\Psi$ (usually $\Psi$ is simply "a large cardinal".)

Examples of what one could mean by "Maximality" of an inner model $M \subseteq V$:

a) The cardinals of $M$ are the cardinals of $V$.
b) The reflection of various cardinal properties from $V$ down to $M$.
c) Certain cardinal computations are correctly done in $M$.
E.g. if $\lambda$ a cardinal, is $\lambda^+ = \lambda^{+M}$?
d) Other Covering properties, e,g, the Strong Covering Lemma:
$CL(M)$ : If $X \subseteq \lambda, \text{card}(X) > \omega$, then $\exists Y \in M, Y \supseteq X$, $\text{card}(Y) = \text{card}(X)$.

As cardinals can be collapsed and altered so easily by forcing, we shall have almost nothing to say about the situation in a) here, but we shall look at the other notions with particular reference to the canonical inner models known as "core models". We shall adopt the usual conventions concerning set theoretical notation. Closed and unbounded sets of ordinals will be called cub.

**Definition 2.** *A class $j$ is an* elementary embedding *between the inner models $M, N$ if $j : M \longrightarrow_{\Sigma_1} N$. (By this we mean $j$ is $\varphi$-preserving for any formula $\varphi \in \Sigma_1 \cap \mathcal{L}_{\{\dot\in,\dot=\}}$.)*

**Definition 3.** *The* critical point *of $j$,* $\quad crit(j) \approx \mu\alpha\, j(\alpha) > \alpha$.

By $ZF_{\{j\}}$, or $ZFC_{\{j\}}$, we shall mean the usual axioms of $ZF$ or $ZFC$, but in the language $\mathcal{L}_{\{\dot\in,\dot=,j\}}$, that is with an extra predicate letter $j$.

**Theorem 1** (*Kunen 1971* [13]) $ZFC_{\{j\}} \vdash j : V \longrightarrow_{\Sigma_1} V \Longrightarrow j = id$.

*Open Question 1)*: Is this true for $ZF_{\{j\}}$?

A.Suzuki has witten out an explicit proof that this must be true of any $ZF$*-definable* class term $j$ [25].

**Fact 2** $ZFC \vdash \exists j : V \longrightarrow_{\Sigma_1} M \wedge \kappa = crit(j) \longleftrightarrow \kappa$ *is a measurable cardinal.*

*Question*: (Kunen)[14] Does
$\text{Con}(ZFC_{\{j\}} + j : M \longrightarrow_{\Sigma_1} V, j \neq \text{id}) \implies \text{Con}(ZFC + \exists$ a measurable cardinal)?

If $j : M \longrightarrow_{\Sigma_1} N \quad IM(M), IM(N)$ and $j \neq \text{id}$, then $j \restriction L^M : L^M \longrightarrow L^N$. Of course by absolunteness of the construction of $L$, $L^M = L^N = L$. Hence $V \neq L$, by Kunen's result (or indeed by an earlier result of Scott proving directly that there were no measurable cardinals in $L$).

But if $j : L \longrightarrow_{\Sigma_1} L, j \neq \text{id}$, and $\kappa = \text{crit}(j)$ we may define a filter measuring all constructible subsets of $\kappa$: $X \in \mathcal{U} \iff X \in \mathcal{P}(\kappa)^L \wedge \kappa \in j(X)$. One may check that this is a normal filter, and we may form an ultrapower. The ultrapower is wellfounded as we may define a $\Sigma_0$-preserving embedding $k$ of it, back into $L$, by: $k([f]_{\mathcal{U}}) = j(f)(\kappa)$. By a result of Kunen, we may "iterate" this ultrapower of $L$

by $\mathcal{U}$, and the resulting images and critical points form a closed and unbounded in $On$ class of indiscernibles for L - known as "$O^{\#}$ exists".

But it is not necessary to start out with an elementary embedding $j$ of all of $L$ in order to achieve this.

*Example* Let $j : L_\alpha \longrightarrow L_\beta, \mathrm{crit}(j) < \alpha$, with $\alpha$ regular, let $\lambda > \alpha, \beta$ be any regular cardinal. Then we may directly perform a "lift-up" or "pseudo-ultrapower" finding $\tilde{j} : L_\lambda \longrightarrow_{\Sigma_1} L_\lambda$with $\tilde{j} \supseteq j$.

The construction is well known and has a long history. We give an outline once again here. Of course using the regularity of $\alpha$ one could just define a measure $\mathcal{U}$ as above on the critical point of $j$, and simply determine that all of $L$ has a wellfounded ultrapower by $\mathcal{U}$. (There are far stronger statements possible than that of the Example, but we illustrate with a procedure that will have other applications later.)

Let $\Gamma = \{f \in L_\lambda \mid \mathrm{dom} f = u \in L_\alpha\}$. Form a domain $D = \{\langle a, f\rangle \mid f \in \Gamma \wedge a \in j(\mathrm{dom} f)\}$ and define relations $I, e$ by:

$$\langle a, f\rangle {}^{I}_{e} \langle b, g\rangle \Longleftrightarrow \langle a, b\rangle \in j(\{\langle u, v\rangle \mid f(a) {}^{=}_{\in} g(v)\}).$$

Set $\mathcal{D} = \langle D, I, e\rangle$. We now use the following version of Łos Theorem:

$$\text{For } \varphi \in \Sigma_0 \cap \mathcal{L}_{\{\dot{\in}, \dot{=}\}} : \quad \mathcal{D} \models \varphi(\langle a_i, f_i\rangle) \Longleftrightarrow \boldsymbol{a}_i \in j(\{\boldsymbol{u} \mid \varphi[f_i((\boldsymbol{u})_i)]_{L_\lambda}\})$$

*If* $e$ is wellfounded let $[\ \ ] : \langle D, I, e\rangle \widetilde{\longleftrightarrow} \langle M, =, \in\rangle$. Then $M$ is transitive and

$$[x] {}^{=}_{\in} [y] \Longleftrightarrow x {}^{I}_{e} y (\text{ for } x, y \in D).$$

Define $\tilde{j} : L_\lambda \longrightarrow M$ by $\tilde{j}(x) = [\langle \phi, \{\langle \phi, x\}]$. One can *check* (i) $\tilde{j}$ is $\Sigma_0$ and cofinal (so $\Sigma_1$-preserving) into $M$; (ii) $\tilde{j} \supseteq j$.

*Why is e wellfounded*? Suppose there are $\langle a_i, f_i\rangle$ with $(*)$ $[\langle a_{i+1}, f_{i+1}\rangle] \in [\langle a_i, f_i\rangle]$ $(i < \omega)$. Let $\delta$ be sufficiently large $< \lambda$ so that $\{f_i\} \subseteq L_\delta$. Let $\mu < \alpha$ be sufficiently large so that $\{\mathrm{dom} f_i\} \subseteq L_\mu$.

Let $\tau : L_{\overline{\delta}} \widetilde{\longleftrightarrow} X$ where $\mu \cup \{f_i\} \subseteq X \prec L_\delta$ with $\mathrm{card}(X) = \mathrm{card}(\mu)$. Then $\overline{\delta} < \alpha$. Let $\tau(\overline{f}_i) = f_i$.

By $(*)$ $\langle a_{i+1}, a_i\rangle \in j(\{\langle u, v\rangle \mid f_{i+1}(u) \in f_i(v)\})$

$\longleftrightarrow \langle a_{i+1}, a_i\rangle \in j(\{\langle u, v\rangle \mid \overline{f}_{i+1}(u) \in \overline{f}_i(v)\})$ [Elementarity of $\tau$]

$\longleftrightarrow \langle a_{i+1}, a_i\rangle \in \{\langle u, v\rangle \mid j(\overline{f}_{i+1})(u) \in j(\overline{f}_i)(v)\}$ [Elementarity of $j$]

But then $\ldots j(\overline{f}_{i+1})(a_{i+1}) \in j(\overline{f}_i)(a_i)$!

There is no need to stop at any particular $\lambda$ in the above: one may simply lift $j$ to a $\tilde{j} : L \to L$ of all of $L$.

### 1.1 Covering and Reflection

We shall be considering covering properties between inner models and $V$. The paradigm here is Jensen's original Covering Lemma for $L$.

**Theorem 3** (*Jensen 1974*)[1] $(\neg 0^{\#})$

**(i)** *d) holds for $M = L$, that is $CL(L)$. This has the following consequences:*
**(ii)** *$\kappa$ singular $\longrightarrow$ ($\kappa$ singular)$^L$*
**(iii)** *$L$ computes correctly successors of singular, weakly compact cardinals. In fact $cf((\gamma^+)^L) \geq \mathrm{card}(\gamma)$ for $\gamma \geq \omega_2$.*

But note that, in general, no other successors need be correctly calculated. Other reflection properties:

- $\kappa$ weakly compact $\longrightarrow$ ($\kappa$ weakly compact)$^L$
- $\kappa$ weakly compact $\longrightarrow \forall A(\forall\alpha < \kappa\ A \cap \alpha \in L \longrightarrow A \in L)$. Hence if $\kappa$ is weakly compact and $V_\kappa = L_\kappa$ then $V_{\kappa+1} = (V_{\kappa+1})^L$.
- $\kappa$ singular cardinal $\wedge\, V_\kappa = L_\kappa \longrightarrow V_{\kappa+1} = (V_{\kappa+1})^L$.

Note that only the last of these is a genuine consequence of the (proof of the) Covering Lemma for $L$, whilst the first two are simply consequences of the weak compactness property, or rather the tree property. There are many other examples of "small" large cardinal properties that relativise to $L$ which we could list.

*Back to Kunen's Question:*
Let $\Phi(j)$ denote $j : M =_{df} \mathrm{dom}(j) \longrightarrow_{\Sigma_1} V$, $j \neq \mathrm{id}$.

- Note no such $j$ definable by a class term $\varphi \in \mathcal{L}_{\{\dot\in,\dot=\}}$ from parameters.
- If $j : M \longrightarrow V$, then $j : L \longrightarrow L, j : L[0^{\#}] \longrightarrow L[0^{\#}] \ldots$

It is thus natural to ask how far one can go with this.

**Theorem 4** (*Vickers-Welch*)[26] *$ZFC_{\{j\}} + \Phi(j) \Longrightarrow Con(ZFC +$ "There is a proper class of 'almost Ramsey' cardinals").*

**Definition 4.** *$\kappa$ is* almost Ramsey $\Longleftrightarrow \forall\delta < \kappa\ \ \kappa \longrightarrow (\delta)^{<w}$*;*
*$\kappa$ is* Ramsey $\Longleftrightarrow \kappa \longrightarrow (\kappa)^{<w}$.

The answer to Kunen is 'No'. We first recall the definition:

**Definition 5.** *An infinite cardinal $\kappa$ is* Jónsson, *if for any algebra $\mathcal{A} = \langle \kappa, (f_n)_{n<w} \rangle$ there is $\mathcal{B} \prec \mathcal{A}$ a proper subalgebra of $\mathcal{A}$ of cardinality $\kappa$.*

We list some facts about Jónsson cardinals; references to these may be found in [11].

**Fact 5** (*Rowbottom*) *$\kappa$ Ramsey $\Longrightarrow \kappa$ regular Jónsson. ( $\not\Longleftarrow$ Devlin);*
*$\kappa$ not Jónsson $\Longrightarrow \kappa^+$ not Jónsson.*
(*Tryba, Woodin ind.*) *$\kappa$ regular $\Longrightarrow \kappa^+$ not Jónsson.*
(*Devlin-Rowbottom*) *the least Jónsson (if it exists) is weakly inaccessible or has cofinality $\omega$.*
(*Prikry*) *A singular limit of measurables is Jónsson.*

The following is a long-standing problem.

*Open Question 2)* Can $\aleph_\omega$ be Jónsson?

The above is at one end of a scale of questions concerning possibly small or accessible Jónsson cardinals. The following result(s) showed that such cardinals would yield strong inner models.

**Theorem 6** (*Donder-Koepke, Koepke*)[4], [12] *($\kappa$ regular but not weakly $\kappa$-Mahlo) or ($\kappa = \omega_\xi > \xi$) then $\kappa$ Jónsson $\Longrightarrow$ $IM$ (Proper class of measures).*

The following outright $ZFC$ theorem of Shelah (an implication of pcf theory) clinches the matter.

**Fact 7** (*Shelah*)[22] *$ZFC \vdash \kappa$ regular Jónsson $\Longrightarrow \kappa$ is weakly $\kappa$-Mahlo; if $\kappa$ singular $\wedge\ \kappa^+$ Jónsson $\Longrightarrow \kappa$ is a limit of regular Jónssons. E.g. $\aleph_{\omega+1}$ not Jónsson.*

Let $\kappa$ be a strongly inaccessible Jónsson, let $\mathcal{A} = \langle V_\kappa, \in \rangle$. Let $\mathcal{B} \prec \mathcal{A}$, $\mathrm{card}(B) = \kappa, B \cap \kappa \neq \kappa$. Let $j : \langle M, \in \rangle \overset{\sim}{\longrightarrow} \mathcal{B} \prec \mathcal{A}$.

Now let $\langle V_\kappa, \in, j, M \rangle$ arise as above from the definition of $\kappa$ as a strongly inaccessible Jónsson cardinal. Then $\langle V_\kappa, \in, j, M \rangle \models \Phi(j)$. Since a Ramsey cardinal is a strongly inaccessible Jónsson we have shown:

**Theorem 8** (*Vickers-Welch*)[26] *$ZFC + \exists$ Ramsey cardinal $\Longrightarrow$ $Con(ZFC_{\{j\}} + \Phi(j))$.*

$\Phi(j)$'s consistency strength is thus sandwiched between that of a Ramsey and an almost Ramsey cardinal. It is interesting to ask further questions about the possible range of $j$.

**Theorem 9** (*Vickers-Welch*)[26] *(i) $ZFC_{\{j\}} + \Phi(j) + ran j$ contains unboundedly many regular fixed points $\Longrightarrow Con(ZFC + \exists$ proper class of Ramseys$)$.*
*(ii) $ZFC_{\{j\}} + \Phi(j) + ran j$ contains stationary many fixed ponts $\delta$ of cofinality $\omega_1 \Rightarrow$ "$O^{sword}$exists".*

(We explain this latter term below.)
*Open Question 3)* What is a good upper bound for the consistency strength of (ii)?
(The natural conjecture here is something like a measurable with Mitchell order $\omega_1$.)
**Remark:** It is interesting to compare (i) of the last theorem with the following. If $G$ is a class $\mathbb{P}$-generic for the proper class form of Woodin's full stationary tower forcing, over $V$ where $V$ contains a proper class of *completely* Jónsson cardinals (a notion ostensibly stronger, but actually equiconsistent with the definition of Jónsson above, and so with Ramsey), then in $V[G]$ one may define an embedding of $V \longrightarrow V[G]$ (my thanks to A. Kanamori for emphasising this point). But care must be taken here, since the structure $\langle V[G], V, \in \rangle$ is not a

$ZFC$-model. In one sense the model $V[G]$ is "near" $V$ here (being a forcing extension), whilst the model $M$ of Theorem 9 (i) is "thin" inside $V$. Thus the latter perhaps is an easier situation to realise, and thus requires smaller large cardinal assumptions.

## 2 The computation of successor cardinals

### 2.1 Beyond $0^{\#}$, beyond $L$

The generalizations of $L$ to other inner models, the "core models" of $V$, also have some "maximality" assuming some anti-large cardinal axiom. Usually the fullest expression of this maximality is achieved through a Covering Lemma. Such models $K$ are of the form $L[\boldsymbol{E}]$ where $\boldsymbol{E}$ is a sequence of filters, or extenders. The Dodd-Jensen core $K^{DJ}$ (see [2]) was the first step out from the constructible universe, and enjoys the full Covering Lemma, if $\neg\exists IM$(measurable cardinal). For models containing more measurable cardinals it was known by work of Mitchell that a weaker result would have to suffice.

A Weak Covering Lemma over an inner model $M$:

$$WCL(M) : \gamma \text{ singular cardinal } \longrightarrow \gamma^+ = \gamma^{+M}.$$

Other models have been built subsequently: Jensen's model for measures of Mitchell order zero, $K^{MOZ}$, previously Mitchell's model $K^{Mitchell}$ for both of which the Weak Covering Lemma held (under the appropriate assumption, that $\neg\exists j : K^{MOZ} \longrightarrow K^{MOZ}$ (known as $\neg O^{sword}$) for the former model, or for the latter, assuming there was no inner model with a measurable cardinal of maximal Mitchell order: $\kappa^{++}$; $K^{strong}$ (Jensen, assuming no # for an inner model of a strong cardinal, or "$\neg O^{¶}$"), and Steel's models, the first of which we shall write as $K^{Steel}$, which is built assuming both that there is no $IM$(Woodin), and a technical assumption to assure iterability, that there is some measurable cardinal $\Omega$ in the universe. Such a model also enjoys Weak Covering under these assumptions (Mitchell-Schimmerling-Steel, [18].)

General Moral: The larger cardinals $V$ might contain, the greater difficulty in the construction of the relevant core, and the greater difficulty in proving 'maximality'.

**Definition 6.** *A cardinal $\kappa$ is* strong *if* $\forall\alpha\exists j_\alpha(crit(j_\alpha) = \kappa \wedge j_\alpha : V \longrightarrow_e M_\alpha \wedge V_\alpha \subseteq V_\alpha^{M_\alpha})$.

**Fact 10** *(i) Jensen* [7]*(No # for an $IM$(Strong))* $WCL(K^{strong})$.

$$(*) \qquad \kappa \geq \omega_2 \wedge (\tau = \kappa^+)^{K^{MOZ}} \longrightarrow cf(\tau) \geq \text{card}(\kappa).$$

*(ii) (Mitchell-Schimmerling-Steel)*[18], [17],*(No $IM$(Woodin), but $\exists$ a measurable cardinal). $WCL(K^{Steel})$, and $(*)$ holds for $K^{Steel}$.*

Once one has larger models the door is open to prove further reflection properties from $V$ into the model, that would not have been possible for $L$.

The following reflection properties (inconsistent with $L$) hold.

- Let $cf(\alpha) \geq \omega_1$ and suppose that $\kappa$ is $\alpha$-Erdős. Then ($\kappa$ is $\alpha$-Erdős)$^{K^{DJ}}$ (Jensen)[3]
- $\kappa$ is Jónsson $\longrightarrow$ ($\kappa$ is Ramsey)$^{K^{DJ}}$ (Jensen, Mitchell)[3][15]
- $\kappa$ is Jónsson $\longrightarrow$ ($\kappa$ is Jónsson or measurable)$^{K^{MOZ}}$ (Jensen-Vickers)[9].
- $\kappa$ is $\delta$-Jónsson $\longrightarrow$ ($\kappa$ is $\delta$-Jónsson)$^{K^{Steel}}$ (Mitchell)[16].

**Definition 7.** (*Mitchell* [16]) $\kappa$ *is* $\delta$-Jónsson *(for any regular $\delta \leq \kappa$) if we require only in the definition of Jónsson that the subalgebra $\mathcal{B}$ have order type $\delta$.*

## 2.2 Jónsson cardinals and $L[A]$ models

A Ramsey cardinal is weakly compact, and this together with the fact $\kappa$ Ramsey implies $\forall a \in V_\kappa$ "$a^\#$ exists" yields:
$\kappa$ Ramsey $\Longrightarrow \forall A \subseteq \kappa A^\#$ exists (so $V \neq L[A]$).

Do regular Jónsson's enjoy this property? (Singular Jónssons need not: by Prikry's result in Fact 5, in $L[\boldsymbol{\mu}]$ where $\boldsymbol{\mu}$ is an $\omega$-sequence of measures, then in $L[\boldsymbol{\mu}]$, $\gamma =_{df} \sup \boldsymbol{\mu}$ is Jónsson, but $\exists A \subseteq \gamma(\neg A^\#$ exists).) The first observation lends plausibility to this.

**Theorem 11** (*Mitchell* [16], *Welch*) *$\kappa$ regular Jónsson, then $\neg \exists A \subseteq \kappa$ $V = L[A]$.*

**Proof:** Suppose for a contradiction $V = L[A]$ for some $A \subseteq \kappa$. Let $j : L_\kappa[\overline{A}] \longrightarrow L_\kappa[A]$ arise from the Jónsson property, with $j$ chosen so that $\mathrm{crit}(j)$ is least over all possible such $\overline{A}$. Just as in the Introduction, perform a "lift-up" replacing $L_\alpha$ there by $L_\kappa[\overline{A}]$ here, and using as $D$ now the class of pairs using $\Gamma = \{f \in L[\overline{A}] \mid \mathrm{dom} f = u \in L_\kappa[\overline{A}]\}$); thus yielding a $\tilde{\jmath} : M = L[\overline{A}] \longrightarrow V = L[A]$. The argument is identical. From all this we may construct a class term, defining a map of an inner model $M$ of the universe to $V$, which is itself impossible. **Q.E.D.**

The next result shows that it must be true (or at least there must be inner models with some large cardinals).

**Theorem 12** *$\kappa$ regular Jónsson $\wedge \exists A \subseteq \kappa \neg A^\# \Longrightarrow \exists$ a $\#$ for an IM(Strong Cardinal).*

We give some detail of the proof of the above. We assume some familiarity with the notion of premouse and mouse, and adopt the formalism of [19], with their associated good extender hierarchies. We explain some of this nomenclature below before giving a proof, which is itself a simple iteration and comparison argument, although the reader must be referred to [19] for an explanation of all terms. For the hierarchies under consideration, that is "below $O^\P$" this material is also summarised in [24] §1.

**Definition 8.** *(i) A premouse $M$ of the form $\mathcal{J}_\alpha^{E^M}$ is* below $O^\P$ *iff whenever $E$ is a $(\kappa,\beta)$ extender on the $E^M$-sequence and $\lambda < \kappa$, then* $\sup\{\gamma \mid crit(E^M_\gamma) = \lambda\} < \kappa$.
*(ii) $O^\P$ is the unique sound mouse $M$ such that $M$ is not below $O^\P$ , but every proper initial seg. of $M$ is below $O^\P$ .*

It is easy to see that $O^\P$ is active (*i.e.* its last extender predicate - in fact equivalent to a filter - is nonempty), and that if $\mu$ is the critical point of this last extender, then there is a $\kappa < \mu$ such that for cofinally many $\lambda < \mu$, there is a $(\kappa, \lambda)$ extender on the $O^\P$ sequence. It is easy to see that a premouse $M$ being below $O^\P$ is equivalent to the condition that if $crit(E^M_\beta) = \kappa$ then $\mathcal{J}^M_\kappa \models$ " there are no strong cardinals ". If we are restricting ourselves to premice below $O^\P$, then the iterations arising are of the following simple kind:

**Definition 9.** *An iteration tree $\mathcal{T}$ of length $\theta$ is* almost linear *if for any $\alpha < \beta < \theta \quad T\text{-}pred(\beta+1) = \alpha \Rightarrow \beta = \alpha + n$ for some $n < \omega \wedge \forall k \leq n(crit(E^{\mathcal{T}}_\beta) = crit(E^{\mathcal{T}}_{\alpha+k}))$.*

It is easy to argue from the initial segment condition that for premice below $O^\P$ that every iteration tree on $M$ is almost linear (cf. §8 of [23]). In what follows the reader will lose little by picturing the iterations to be completely linear: in a sense we have only the possibility of non-linear iterations due to the formal arrangement of our extender hierarchies. If we restrict ourselves to working below $O^\P$, then our notion of a "universal" inner model $L[\boldsymbol{E}]$ is somewhat simplified, and we have the fact that any such "universal weasel" is an iteration of the core model $K^{strong}$. (Fact14 below.)

Correctly computing successors of singulars characterises universal weasels.

**Fact 13** (cf. [19]§3) *$W$ is a universal weasel if and only if on a stationary class of $\beta$, $\beta^+ = \beta^{+^W}$ . Further, if assume there is no inner model for a strong cardinal, then $W$ is a universal weasel if and only if $\beta^+ = \beta^{+^W}$ for arbitrarily large cardinals $\beta$.*

**Fact 14** ([23] *8.13 for example*) *If $\neg O^\P$, then for every universal weasel $W$, there is an almost linear iteration tree $T$ on $K$ with last model $W$, and such that $T$ does not drop in model or degree. Similarly if $j : K^{strong} \longrightarrow M$, then $M$ is universal, and $j$ is (the) iteration map taking $K^{strong}$ to $M$.*

**Proof:** (of Theorem 12) Suppose for a contradiction that $\kappa$ is a regular Jónsson cardinal and that $A \subseteq \kappa$ has no #. Then $\kappa$ is (weakly) inaccessible. Without loss of generality we assume $A$ is such that $K^{strong} \restriction \kappa \subseteq (V_\kappa)^{L[A]}$. By the Jónsson property there is $\overline{A} \subseteq \kappa$ and $j$ with $j : L_\kappa[\overline{A}] \longrightarrow L_\kappa[A]$. Again, using the regularity of $\kappa$, perform a "lift-up" yielding a $\tilde{j} : L[\overline{A}] \longrightarrow L[A]$. It is easy to see that we also must have $\neg\overline{A}^\#$ (otherwise we could carry indiscernibles for $L[\overline{A}]$ *via* $j$ to $L[A]$). By the $L$-Covering Lemma relativised to the models $L[\overline{A}]$ and $L[A]$, we must have that $\kappa^+$ in both models has cofinality greater than or equal to $\kappa$, and in fact the full covering lemma holds for subsets of ordinals above $\kappa$. But now let $K_1 =_{df} K^{L[\overline{A}]}$ and $K_2 =_{df} K^{L[A]}$. We claim these are both "universal" weasels. To see this note, by the Weak Covering Lemma for $K$ applied inside $L[A]$ $(L[\overline{A}])$ we have that successors of singulars greater than $\kappa$ are correctly calculated (from the point of view of $L[A]$ (respectively $L[\overline{A}]$)) by $K_2$ (resp. $K_1$). But then unboundedly often successors are calculated correctly by the models $K_1, K_2$, and thus by the Fact 13 they are both universal, and hence simple (almost linear) iterates of the true core model $K^{strong}$. Set $K = K^{strong}$. Now run a comparison of the two models. This results in iteration trees, call them, $\mathcal{T}$ on $M_0^{\mathcal{T}} = K_1$ and $\mathcal{U}$ on $M_0^{\mathcal{U}} = K_2$, with a common final inner model, $W = M_\infty^{\mathcal{T}} = M_\infty^{\mathcal{U}}$ say. (By "universality" we cannot have any truncation of any model occurring in these trees, and the final models on $\mathcal{T}, \mathcal{U}$ must be identical.) Note if we let $\overline{\theta} \geq 0$ be least with $M_{\overline{\theta}}^{\mathcal{U}} \restriction \kappa = K_1 \restriction \kappa$ then a) $\theta \in [0, \infty)_U$ (where $U$ is the underlying almost linear tree ordering of $\mathcal{U}$); and by our assumption on $A$: b) the tree $\mathcal{T} \restriction \overline{\theta}$ is trivial (that is for $i \leq \overline{\theta}\, M_i^{\mathcal{T}} = M_0^{\mathcal{T}} = K_1$ and $\pi_{i,j}^{\mathcal{T}} = \mathrm{id}\,(i < j \leq \overline{\theta})$). This is because the same is true for the comparison of $(K_1, K)$. (This is tantamount to saying that $K_2 \restriction \kappa = K \restriction \kappa$: which it is, since we ensured that $K \restriction \kappa \subseteq L_\kappa[A]$.)

Now extend using a "lift-up" $j$ to $\tilde{\jmath} : M_{\overline{\theta}}^{\mathcal{U}} \longrightarrow W$ with $W \supseteq K_2 \restriction \kappa = K \restriction \kappa$. Again one can do this just as in the Introduction: $M_{\overline{\theta}}^{\mathcal{U}}$ is a model of the form $L[\boldsymbol{E}]$, with $H_\alpha^{L_\alpha[\boldsymbol{E}]} = |L_\alpha[\boldsymbol{E}]|$ for $L[\boldsymbol{E}]$-cardinals $\alpha$. This enables us to form a domain $\mathcal{D}$ using functions in $M_{\overline{\theta}}^{\mathcal{U}}$ with domains in $L_\kappa[\boldsymbol{E}] = M_{\overline{\theta}}^{\mathcal{U}} \restriction \kappa$. This gives us a wellfounded class $W$. Note that we get iterability of the model $W$ virtually for free: any set-sized initial segment of $W$ is iterable by any iteration definable within $W$ (by elementarity of $\tilde{\jmath}$); but in the region of almost linear iterations any iteration whatsoever can be "embedded" back into a "universal" iteration of $W$ that can be defined *internally* to $W$. (This "universal iteration" procedure, and the construction of the "embedding back" of a general iteration is described in full detail in [24] Def. 2.8 and Lemma 2.10.) If all initial segments of $W$ are iterable, then clearly $W$ is iterable. Then $W$ is a normal iterate of $K$ (Fact 14). So let $\sigma : K \longrightarrow W$ be the (unique) iteration map. But then $\sigma = \tilde{\jmath} \circ \pi_{0,\overline{\theta}}^{\mathcal{U}}$ by the same Fact. But $W \restriction \kappa = K \restriction \kappa$, hence as $crit(\sigma) = \alpha = crit(\tilde{\jmath}) = crit(j)$ this can only happen if for some $\nu > \kappa$, $crit(E_\nu^K) = \alpha$. By the Initial Segment Condition (part of the condition for the construction of the "good extender sequence" hierarchy for such $L[E]$ models) we must have that $\{\delta \mid crit(E_\delta^K) = crit(E_\nu^K)\}$ is unbounded in any $K$-cardinal $\lambda \leq \kappa$ with $crit(E_\nu^K) < \lambda$. But as we are below $O^{\P}$, this means there are no measurable cardinals of $K$ in $(\alpha, \kappa)$. But this implies that $\overline{\theta} = 0$, *i.e.*, $K_1 \restriction \kappa = K \restriction \kappa$. But then $\tilde{\jmath} : K \longrightarrow W$ is an iteration map; hence $\alpha$ is measurable in $K$; but then so is $j(\alpha)$. A contradiction! **Q.E.D.**

Clearly this argument could be pushed a little further, but beyond linear iterations, one usually invokes the existence of a measurable cardinal $\Omega$ (or at least #'s for sets in $V_\Omega$) to construct $K^c$ and hence $K$; this somewhat obviates the need for any proof at all! A small corollary is immediate from the argument however:

**Corollary 15** ($\neg O^{\P}$) *In $K^{strong}$ there is no "strong past a Jónsson". That is if $\kappa$ is Jónsson, then for no $\lambda < \kappa$ do we have* $\sup\{\nu < \kappa \,|\, crit(E^K_\nu) = \lambda\} = \kappa$.

Let $\kappa$ be weakly compact cardinal. Suppose $\neg\exists j : L \longrightarrow L$ : then $\kappa^+ = \kappa^{+L}$. [Because $\kappa^{+L} < \kappa^+ \longrightarrow cf(\kappa^{+L}) = \text{card}(\kappa^{+L}) = \kappa$ (by $WCL(L)$)]. Weak compactness implies there is a $\kappa$-complete $L$-ultrafilter on $\mathcal{P}(\kappa)^L$. So $Ult(L,\mathcal{U})$ is wellfounded and $\exists j : L \longrightarrow Ult(L,\mathcal{U}) \cong L$. For relatively small core models similar considerations work. The following theorem however requires deeper arguments about the nature of justification concerning which extenders are added to the good extender sequence that is inductively defined to build $K$,

**Theorem 16** (*Schimmerling-Steel*)[21] *(No $IM(Woodin) + \exists$ measurable cardinal $\Omega$)*

$$\forall\beta < \Omega(\beta \text{ weakly compact } \longrightarrow \beta^+ = \beta^{+K^{steel}}).$$

Much recent work has focussed on looking at $\Box$ sequences (or weakenings thereof).

**Definition 10.** *Let $\gamma \in Card$. $\Box_\gamma$ is the following principle:*
$\exists\langle C_\alpha \mid \alpha < \gamma^+\rangle$ *such that*
*(i)* $\forall Lim(\alpha) < \gamma^+ \quad C_\alpha \subseteq \alpha$ *is c.u.b.*
*(ii)* $\forall\delta \in C^*_\alpha \; C_\alpha \cap \delta = C_\delta$
*(iii)* $otp(C_\alpha) \leq \gamma$

The definition is due to Jensen [5] where it was also shown that $\forall\beta\,\Box_\beta$ holds in $L$; as core models were developed proofs of $\Box_\gamma$ were given for $K^{DJ}$ (Welch [28]), $K^{MOZ}$ (Wylie [29]), and for $K^{Strong}$ (Jensen [8]). (In fact it is not the point of this survey to go into this matter so thoroughly, but stronger "global" $\Box$ principles are possible, and in fact these global principles were shown for the models in the works cited for $L, K^{DJ}$ and $K^{MOZ}$; for $K^{strong}$ see [10],[30].)

If one weakens the definition of $\Box$ to allow at each $\alpha < \gamma^+$ a non-empty finite family of cub-in-$\alpha$ guesses $\mathcal{C}_\alpha$ for $C \in \mathcal{C}_\alpha$ (or, weaker still, $\mu \leq \gamma$ many guesses - thus at (ii), for any limit point $\delta$ of any $C \in \mathcal{C}_\alpha$ we require that $C \cap \delta \in \mathcal{C}_\delta$), then one obtains the weaker principles $\Box^{<\omega}_\gamma$ (or $\Box^\mu_\gamma$).

**Theorem 17** (*Schimmerling*)[20] *(No $IM(Woodin) + \exists$ measurable cardinal $\Omega$)*
$\forall\beta < \Omega \quad \Box^{<\omega}_\beta$ *holds in $K^{Steel}$.*

In fact much more than this is proven in [20]: there it is shown true in $L[\boldsymbol{E}]$ models without superstrong extenders. It now appears that full $\Box_\gamma$ would hold in the above theorem for all $\gamma < \Omega$, (and in fact in a wider class of $L[\boldsymbol{E}]$ models which do not contain a measurable Woodin cardinal) by work of Zeman [31].

- Now a $\Box_\beta$ or $\Box^{<\omega}_\beta$ sequence is absolute between an IM(M), and $V$, provided $\beta^+ = \beta^{+M}$. Hence
- (No $IM$ (Woodin) $+\exists$ measurable cardinal $\Omega$) $\Rightarrow$ $\forall\kappa$ singular, or weakly compact $\Box^{<\omega}_\kappa$ holds in $V$.

*Question (Vickers)*: What about successors of Jónssons?

**Remark:** The point of the question (which could have been asked about $K^{DJ}$ (and answered) some years ago) is that a regular Jónsson cardinal can fail to be weakly compact.

**Theorem 18** (*Vickers-Welch*[27]) *Let $\kappa$ be Jónsson.*
*a)* $V_\kappa = V_\kappa^{K^{MOZ}} \longrightarrow V_{\kappa+1} = V_{\kappa+1}^{K^{MOZ}}$
*b) Assume* $\neg O^{sword}$*. Then* $\kappa^+ = \kappa^{+K^{MOZ}}$ *[Hence* $\square_\kappa$ *in* $V$*].*

### 2.3

The last could equally well have been proven assuming $\neg O^\P$ and using $K^{strong}$. We sketch the proof of:

**Theorem 19** ($\neg O^\P$) *If $\kappa$ is a regular Jónsson cardinal, then* $\kappa^+ = \kappa^{+K^{strong}}$.

**Proof:** We do not wish to repeat large segments of the proof of Theorem 13 above (or of Theorem 18), so we give a sketch outline which should suffice for this and to see the truth of Theorem 20. Let $K$ be $K^{strong}$. Suppose the theorem false; let $\kappa$ be Jónsson and suppose $\kappa' = \kappa^{+K} < \kappa^+$ and by the Covering Lemma for $K$ $\mathrm{cf}(\kappa^{+K}) = \kappa$. let $D : \kappa \longrightarrow \kappa'$ be a monotone cofinal map. Appealing to the Jónsson property of $\kappa$ in $V$, let $X \prec \langle K_{\kappa'}, E^K \restriction \kappa', D\rangle$ where $card(X) = \kappa$, $X \cap \kappa \neq \kappa$. Let $j : \langle H, \overline{E}, \overline{D}\rangle \longrightarrow_{\Sigma_\omega} \langle K_{\kappa'}, E^K \restriction \kappa', D\rangle$ come from the inverse of the transitive collapse on $X$. Then $j \neq id$, $j(\kappa) = \kappa$, $j``On\cap H$ is cofinal in $\kappa'$. Let $\overline{\kappa} = On \cap H$. Now form the coiteration of $(H, K)$ setting $M_0^{\mathcal{T}} = H, M_0^{\mathcal{U}} = K$, via almost linear trees $\mathcal{T}$,$\mathcal{U}$ of length $\theta$. Corollary 15 ensures that if $\theta > \kappa$ then for $\kappa \leq i < \theta$, $crit(E_i^{\mathcal{U}}) \geq \kappa$. Let $\overline{\theta}$ be least so that $M_{\overline{\theta}}^{\mathcal{T}} \restriction \kappa = M_{\overline{\theta}}^{\mathcal{U}} \restriction \kappa$.

*Claim 1*: $\pi_{0,\overline{\theta}}^{\mathcal{U}}``\kappa \not\subseteq \kappa$.

Remark: In the proof of Theorem 12 we had the universality of $K_1$ to use, in order to argue that the first $\kappa$-steps at least of the iteration involved no movement on the $K_1$-side. We cannot use that here; but for the moment let us assume there is no movement on the $H$ side, and that the tree $\mathcal{T}$ is trivial for $i < \kappa$.

**Proof:** (of Claim 1) If this failed then $M_{\overline{\theta}}^{\mathcal{T}} \restriction \kappa$ (here assumed to be $H \restriction \kappa$) is an initial segment of $M_{\overline{\theta}}^{\mathcal{U}}$. In fact the latter must be a proper class: any truncation at some step $i < \overline{\theta}$ would result from this point onwards in an iteration of an initial segment of the $i$'th model of cardinality $< \kappa$, but this would contradict our assumption that no ordinal in the iteration gets sent up beyond $\kappa$. But now we are in the same situation as the last part of the proof of Theorem 12 and get a contradiction *via* a lift-up argument as there. **Q.E.D.(Claim 1)**

We therefore have (+) $\exists i_0 < \kappa \exists \delta < \kappa\, \pi_{i_0,\overline{\theta}}^{\mathcal{U}}(\delta) \geq \kappa$.

*Claim 2*

(i) $\overline{\theta} = \kappa$,

(ii) There is a cub $C \subseteq \kappa$ so that $i < j \in C \Longrightarrow \pi^{\mathcal{U}}_{ij}(\kappa_i) = \kappa_j = j$ where $\kappa_i = \text{crit}(E^{\mathcal{U}}_i)$.

**Proof:** Due to Claim 1, if $\overline{\theta} < \kappa$ we should have to have an extender $E^{\mathcal{U}}_i$ with critical point $< \kappa$ but of length $> \kappa$; but there are no "strong past a Jónsson" such extenders. Hence (i) holds. (ii) is a standard regressive set argument. **Q.E.D.(Claim 2)**

Now consider the $\kappa$'th model in $\mathcal{U}$: $M^{\mathcal{U}}_\kappa$. For $i \in C$ we have that $\pi^{\mathcal{U}}_{i,\kappa}(\kappa_i) = \kappa$, and moreover sup $\pi^{\mathcal{U}}_{i,\kappa}$"$(\kappa_i^+)^{M^{\mathcal{U}}_i} = (\kappa^+)^{M^{\mathcal{U}}_\kappa}$. (As always for a transitive model $N$ with $\lambda$ the largest $N$-cardinal, we adopt the convention that $(\lambda^+)^N$ is $On \cap N$.) Call the latter $\delta$. Then $cf(\delta) < \kappa$ whilst $cf(\overline{\kappa}) = \kappa$. But $\delta > \overline{\kappa}$ (otherwise the next step of the coiteration would require a truncation on the $\mathcal{T}$-side, contradicting the universality of $K$. (Note as no extenders overlap $\kappa$ this would really require a truncation rather than simply using an extender $E^{M^{\mathcal{T}}_\kappa}_\delta$ with critical point below $\kappa$.) But in that case there is $N$ a proper initial segment of $M^{\mathcal{U}}_\kappa$ and some $n < \omega$ with $\rho^{n+1}_N = \kappa < \overline{\kappa} \leq \rho^n_N$. We now invoke the procedure of constructing a "fine structure" preserving lift-up, (a considerable refinement of the previous lift-up procedure for providing wellfounded and iterable models which we shall not go into here). This method provides an extension $k$ of the embedding $j$ and an iterable model $P$ of the form $L_\gamma[F]$ with $\rho^{n+1}_P = \kappa < \kappa' \leq \rho^n_P$, where $k$ is a fine-structure preserving embedding $k : N \longrightarrow P$ with $k \supseteq j$. The remark to make is that $P$ is sufficiently sound and is coiterable with $K$ in order to argue that in fact there is a code for $P$ in $K$. The two crucial points for the proof to go through here are: (i) $j$ is a cofinal map of $\overline{\kappa}$ into $\kappa'$; (ii) $H \models$ "$\kappa$ is the largest cardinal" and $cf(\overline{\kappa}) > \omega$. These ensure that the $P$ constructed has $On \cap P \geq \kappa'$, and is iterable. But definably over $P$ there is a map of $\kappa$ onto $\kappa'$! But this contradicts the assumption that $\kappa' = (\kappa^+)^K$. (The reader may find the further detail of this argument in [27] Thm. 1.)

Now consider what happens to this argument if $\mathcal{T}$ is nontrivial below $\kappa$.

*Claim 3* $\pi^{\mathcal{T}}_{0,\overline{\theta}}$"$\kappa \subseteq \kappa$.

**Proof:** Suppose otherwise; if $\pi^{\mathcal{U}}_{0,\overline{\theta}}$"$\kappa \subseteq \kappa$, the embedding $j$ would yield a contradiction to the Dodd-Jensen Lemma (*cf.* [19] 5.3 - or in other words it would imply that $K \restriction \kappa <_* H \restriction \kappa$ where $<_*$ is the mouse ordering). Hence (+) holds. But a standard regressive set argument would then show that on a cub $D \subseteq \kappa$ we should have agreement between the extenders used on the two sides, contradicting the definition of comparison. **Q.E.D.(3)**

We thus have again $M^{\mathcal{T}}_{\overline{\theta}} \restriction \kappa$ a proper initial segment of $M^{\mathcal{U}}_{\overline{\theta}}$. We also have that

$$(1) \quad \tilde{\kappa} =_{df} (\kappa^+)^{M^{\mathcal{T}}_{\overline{\theta}}} (= On \cap M^{\mathcal{T}}_{\overline{\theta}}) = \pi^{\mathcal{T}}_{0,\overline{\theta}}(\overline{\kappa}) = \sup \pi^{\mathcal{T}}_{0,\overline{\theta}}\text{“}\overline{\kappa}$$

and hence has cofinality $\kappa$. We copy the iteration $\mathcal{T} \restriction \overline{\theta}$ *via* $j$ to an iteration $j\mathcal{T}$ on $K \restriction \kappa'$ (again *cf.* [27] or [19] 5.2) with resulting maps $j_i : M^{\mathcal{T}}_i \longrightarrow M^{j\mathcal{T}}_i$

so that we have a final map $\overline{\jmath} =_{df} j_{\overline{\theta}}$, $\overline{\jmath} : M^{\mathcal{T}}_{\overline{\theta}} \longrightarrow \widetilde{K}$ where $\widetilde{K} = M^{j\mathcal{T}}_{\overline{\theta}}$. We use this as the base for our lift-ups. In our proof of Claim 1 we embedded the proper class $M^{\mathcal{U}}_{\theta} \longrightarrow W$ *via* a lift-up map $\tilde{\jmath}$ extending $j \restriction (H \restriction \kappa)$. Now we should lift-up $\overline{\jmath} \restriction (M^{\mathcal{U}}_{\overline{\theta}} \restriction \kappa)$ to a $\tilde{\jmath} : M^{\mathcal{U}}_{\kappa} \longrightarrow W$ with $W \supseteq \widetilde{K} \restriction \kappa$. With this variation Claim 1 can be concluded in the same way. For the final argument (of the paragraph following Claim 2) we should have some initial segment $N$ of $M^{\mathcal{U}}_{\kappa}$ whose $n+1$'st projectum drops to $\kappa$ but whose height is $\geq \tilde{\kappa}$. We then lift-up (fine-structurally) $\overline{\jmath} : M^{\mathcal{T}}_{\overline{\theta}} \longrightarrow \widetilde{K}$ to a fine-structure preserving map $k : N \longrightarrow P$. Note that $\overline{\jmath}$ is cofinal into $On \cap \widetilde{K}$, and both $\widetilde{K}$, $M^{\mathcal{T}}_{\overline{\theta}} \models$ "$\kappa$ is the largest cardinal" and $cf(\tilde{\kappa}) = \kappa > \omega$. The crucial points mentioned above thus still hold true. **Q.E.D.(Theorem 19)**

In fact it simply reflects :

**Theorem 20** ($\neg O^{\P}$) *If $\kappa$ regular Jónsson, then $\{\beta < \kappa \mid \beta$ regular, $\beta^+ = \beta^{+K}\}$ is stationary in $\kappa$.*

**Proof:** Let $D \subseteq \kappa$ be cub. Let $\eta > \kappa$ be sufficiently large, and by the Jónsson property, let $j : \langle \overline{V}, \in, \overline{K}, \overline{D} \rangle \longrightarrow \langle V_\eta, \in, K \restriction \eta, D \rangle$ (where $K = K^{strong}$) be an elementary embedding with card(ran $j \cap \kappa) = \kappa$, $j \restriction \kappa \neq$ id. The point is that the Claims of the last proof hold here (they were really only based on the universality of $K$). Thus, if we consider the coiteration of $M^{\mathcal{T}}_0 = \overline{K}$ and $M^{\mathcal{U}}_0 = K \restriction \eta$, some point will be sent up past $\kappa$ on the $\mathcal{U}$-side of the coiteration, and there will be a cub in $\kappa$ set $C$ with the properties of Claim 2. But note that for $i < j \in C$, $\pi^{\mathcal{U}}_{i,j}((\kappa_i)^+)^{M^{\mathcal{U}}_i}) = (\kappa_j)^+)^{M^{\mathcal{U}}_j}$ and in fact $\pi^{\mathcal{U}}_{i,j}$ is continuous between these two ordinals. Hence all such successors have the same fixed cofinality $\delta$ for some $\delta < \kappa$. By the Mahloness of $\kappa$ (*cf.* [22]), there is a regular $\gamma \in \overline{D} \cap C$, with $\delta < \gamma$ and with the additional property that $\pi^{\mathcal{T}}_{0,\gamma}$"$\gamma \subseteq \gamma$ (and hence $\pi^{\mathcal{T}}_{0,\gamma}(\gamma) = \gamma$). Then we shall have that $\pi^{\mathcal{T}}_{0,\gamma}$ takes $(\gamma^+)^{\overline{K}}$ cofinally to $(\gamma^+)^{M^{\mathcal{T}}_\gamma}$. Observe now that we simply cannot have $(\gamma^+)^{\overline{V}} > (\gamma^+)^{\overline{K}}$: by the Weak Covering Lemma applied inside $\overline{V}$, we should have $\overline{V} \models cf((\gamma^+)^{\overline{K}} \geq \gamma$. But by the previous comment $cf((\gamma^+)^{\overline{K}}) = cf((\gamma^+)^{M^{\mathcal{T}}_\gamma}) = cf((\gamma^+)^{M^{\mathcal{U}}_\gamma}) = \delta$. Therefore we must have that $\overline{V} \models \gamma^+ = (\gamma^+)^{\overline{K}}$. But then $j(\gamma)^+ = (j(\gamma)^+)^K$. And $j(\gamma) \in D$. **Q.E.D.**

Extensions of these arguments show:

**Theorem 21** *Assume there is no IM(Woodin) $\wedge \exists \Omega(\Omega$ a measurable cardinal). Let $K = K^{Steel}$, and suppose that $\kappa$ is Jónsson . Then*
*a) $\kappa^+ = \kappa^{+K}$;*
*b) $\{\alpha < \kappa \mid \alpha$ regular $, \alpha^+ = \alpha^{+K}\}$ is stationary below $\kappa$.*

**Proof:** We work towards a proof of a) first. We shall need some notation for kind of phalanx that we shall use. We adopt that of Definition 9.6 of [23]. This runs as follows:

**Definition 11.** *A* phalanx $\Phi$ *of premice of length* $\theta+1$ *is a pair*

$$(\langle(\mathcal{M}_\gamma, k_\gamma) \mid \gamma \leq \theta\rangle, \langle\rho_\gamma \mid \gamma < \theta\rangle)$$

*such that for all* $\gamma \leq \theta$

*(1)* $\mathcal{M}_\gamma$ *is a premouse,* $k_\gamma \leq \omega$*, and if* $k_\gamma \neq 0$ *then* $\mathcal{M}_\gamma$ *is a* $k_\gamma$*-sound premouse;*

*(2) if* $\gamma < \delta < \theta$*, and* $\rho_\delta \neq 0$ *then* $\rho_\gamma < \rho_\delta$*;*

*(3) if* $\gamma < \delta \leq \theta$*,* $\rho_{k_\delta}(\mathcal{M}_\delta) > \rho_\gamma$*;* $\mathcal{M}_\gamma$ *agrees with* $\mathcal{M}_\delta$ *below* $\rho_\gamma$*.*

Actually the definition cited allows for a more exotic choice of creatures than just premice, but we shall not need them. Nor do we need to consider the protomice of [18]. [23] also uses the notation $deg^{\Phi}(\gamma)$, and $\nu(\gamma,\Phi)$ for $k_\gamma, \rho_\gamma$. There is a further coordinate $\lambda_\gamma$, that we have suppressed which is used for handling Type III premice. We allow for models with $\rho_\delta = 0$. The point of this is only to cater for some "dummy" models that have arisen from deriving a phalanx from a comparison tree with padding steps. They play no dynamic role in the constructions and we could simply cut them out, but this would only involve another step at some later point. We make the further simplifications:
(i) if any $k_\gamma = 0$, it is omitted (for example if $\mathcal{M}_\gamma$ is a weasel);
(ii) if the phalanx is of length 2 and the first model is a weasel, we shall thus just write $((\mathcal{M}_0, (\mathcal{M}_1, k_1)), \rho_0)$ and even omit $k_1$ if it is understood. This will be especially true when $k_1$ is just $n(\mathcal{M}_1, \rho_0) =_{df} \max\{n \mid \rho^n_{\mathcal{M}_1} > \rho_0\}$ (if the latter exists, otherwise set it to $\omega$). We note that for the structures arising in our proofs the $\mathcal{M}_\gamma$ for $\gamma < \theta$ will all be weasels, with only the last "starting model" $\mathcal{M}_\theta$ possibly being a less than $\Omega$ sized premouse (although this is a fact we should need to demonstrate).
(iii) If $\Phi$ is a phalanx of length $\theta$ and $((Q,n),\gamma)$ a suitable pair to enlarge the phalanx one point, we write $\Phi^\frown((Q,n),\gamma)$ for this extension of length $\theta+1$.

**Definition 12.** *An* iteration tree of length $\tau$ on the phalanx $\Phi$ *of length* $\theta+1$ *is an* $\omega$*-maximal padded tree on* $\Phi$*:*
*for* $\beta \leq \theta, deg^{\mathcal{T}}(\beta) = k_\beta$*, and for* $\theta < \beta \leq \tau, deg^{\mathcal{T}}(\beta)$ *is the unique* $k \leq \omega$ *such that* $M^{\mathcal{T}}_\beta = Ult_k((M^{\mathcal{T}}_\beta)^*, E^{\mathcal{T}}_\beta)$ *if* $\beta = \delta+1$ *, and* $k$ *and* $(M^{\mathcal{T}}_\beta)^*$ *are chosen according to the rules of [19] Def. 6.1.2 for forming* $\omega$*-maximal trees, and* $E^{\mathcal{T}}_\beta \neq \emptyset$*; it is set to* $deg^{\mathcal{T}}(\delta)$ *otherwise. If* $\beta$ *is a limit, then* $deg^{\mathcal{T}}(\beta)$ *is the eventual value of* $deg^{\mathcal{T}}(\gamma)$ *for sufficiently large* $\gamma$ *with* $\gamma T \beta$*.*

Given an iteration tree on a model we can derive a phalanx from it as follows:

**Definition 13.** *Let* $\mathcal{T}$ *be an* $\omega$*-maximal padded iteration tree of length* $\theta+1$*. Then* $\Phi(\mathcal{T})$ *is the unique phalanx of length* $\theta+1$ *so that:*
*(i)* $M^{\Phi(\mathcal{T})}_\beta = M^{\mathcal{T}}_\beta$ *and* $k^{\Phi(\mathcal{T})}_\beta = deg^{\mathcal{T}}(\beta)$ *for all* $\beta \leq \theta$*;*
*(ii)* $\rho^{\Phi(\mathcal{T})}_\beta = \rho^{\mathcal{T}}_\beta$*.*

(Recall that in a padded iteration tree arising from the comparison process for example, there are ordinals $\rho^{\mathcal{T}}_{\beta}$ defined as the natural length of $E^{\mathcal{T}}_{\beta}$ if the latter is non-empty, but can possible be 0 otherwise; as mentioned above this is how models with $\rho^{\Phi(\mathcal{T})}_{\beta} = 0$ arise in the derived phalanx).

By WCL(K) we may assume that $\kappa$ is regular and, using the notation of Theorem 19, that $\kappa'$ has cofinality $\kappa$. Again let $D : \kappa \to \kappa'$ be monotone and cofinal. Let $W$ be the "very soundness" witness for $J^K_\kappa$ obtained by iterating each order zero measure $\lambda > \kappa'$ (*cf.* [23] Ch. 8). Let $X \prec \langle V_\eta, E^W, D, \in \rangle$ where $\eta$ is some sufficiently large cardinal, $\mathrm{card}(X) = \kappa$, but $X \cap \kappa \neq \kappa$. $X$ exists by the Jónsson property at $\kappa$. Let $j : \langle H, \overline{E}, \overline{D}, \in \rangle \overset{\sim}{\longleftrightarrow} \langle X, E^W \cap X, D \cap X, \in \rangle$ be the inverse of the transitive collapse. Let $j(\overline{\kappa}) = \kappa'$, and thus $j$ is continuous at $\overline{\kappa}$ by virtue of $\overline{D}$. Then $j : \overline{W} \to_{\Sigma_\omega} W$, where $\overline{W} = J^{\overline{E}}_{\mu}$, $\mu = On \cap H$. Following Theorem 19 we should like to compare $(\overline{W}, W)$ at least up to agreement below $\kappa$, *via* (padded) trees $\mathcal{T},\mathcal{U}$, and prove a version of Claim 1, where some $\alpha < \kappa$ is sent up to $\kappa$ by $\pi^{\mathcal{U}}_{0,\kappa}$.

We first suppose for simplicity that $\mathcal{T}$ is trivial, and there is no movement on the $\overline{W}$ side, and that the statement of Claim 1 of Theorem 19 fails. Then again there can be no truncation on the branch $[0,\kappa]_{\mathcal{U}}$ (as otherwise we'd have a structure of size $< \kappa$ iterating up to one of size $\kappa$).Thus $M^{\mathcal{U}}_\kappa$ is a proper class, *i.e.* a weasel. We could perform a "lift-up" of $j$ to $\tilde{\jmath} : M^{\mathcal{U}}_\kappa \to W'$ with $W' \supseteq W \restriction \kappa$. Wellfoundedness of $W'$ is again not an issue, but sufficient co-iterability of $W'$ with $W$ is: in particular we should want the phalanx $((W, W'), \kappa)$ to be coiterable with $W$.

We adopt Mitchell's solution to this iterability problem (in a slightly more general situation) from [16], which essentially is to look ahead as the iteration tree $\mathcal{U}$ grows, and to check if we were to perform a similar lift-up of $\tilde{\jmath} : M^{\mathcal{U}}_i \to W'_i$ that $((W, W'_i), \kappa(i))$ has no "badly behaved" iteration trees on it. (This means, no trees with an ill-founded last model, or, if of limit length, that the only cofinal branches through the tree are ill-founded.) If there are such bad trees, then there are witnesses to this in a small substructure $Y \prec M^{\mathcal{U}}_i$ of size $< \kappa$. We set $M^{\mathcal{U}}_{i+1}$ to be the collapse of this $Y$, and we continue the comparison by dropping to this set-sized premouse. The tree $\mathcal{U}$ built in this way using this kind of "special drop" Mitchell calls a quasi-iteration tree.

The point of this manœuvre is that finally one is guaranteed either a fully iterable phalanx $((W, M^{\mathcal{U}}_\kappa), \kappa)$, or a small structure of size $< \kappa$ iterating up to the cardinal $\kappa$. In the former case [16] can still show some ordinal is sent up to $\kappa$, and in the latter the quasi-iteration tree guarantees this, and in both cases the critical points (on a tail) form a cub in $\kappa$ set of indiscernibles for $\mathcal{J}^{\overline{E}}_\kappa$ (which is what [16] is looking for). We shall need somewhat more than this.

A comparison of the arguments that follow with [16] will reveal the extent of our indebtedness to Mitchell's proof where this iterability problem is overcome. Indeed, the reader may view much of the current proof as an account of his argument, but within a changed overall framework of a proof such as that of Theorem 19. We should like to thank him for letting us use these ideas in this article.

One query that may arise is that of the continuability properties of the quasi-iteration tree $\mathcal{U}$ if there is such a special drop. We need a guarantee of unique wellfounded branches at limit stages. [16] constructs an auxiliary regular tree $\overline{\mathcal{U}}$ and an embedding of $\mathcal{U}$ into $\overline{\mathcal{U}}$: the unique branches of $\overline{\mathcal{U}}$ can be used to guarantee a good branch in $\mathcal{U}$. There is thus a "derived iteration strategy" for the tree $\mathcal{U}$ (*cf.* the kind of argument on p.75 of [19]). Our ultimate aim is, as in Theorem 19's proof, to have a set-mouse $N$, an initial segment of $M^{\mathcal{U}}_{\kappa}$, to send *via* a lift-up $k : N \to P$, where $P$ is some premouse which may be compared with $W$. To ensure the sufficient iterability of $((W, P), \kappa)$ now, we add a second argument to Mitchell's recipe of special drops, and thus look for initial segments of our current structure $M^{\mathcal{U}}_i$ which could lift to badly behaved phalanges.

We define the comparison trees $\mathcal{T}, \mathcal{U}$ arising from coiterating $\overline{W}, W$ as follows. $\mathcal{T}$ will be an entirely standard tree, arising just as in the standard comparison process, but $\mathcal{U}$ will be augmented by the non-standard special drop outlined above. To ensure the continuability of the process we embed the tree $\mathcal{U}$ into an iteration tree $\overline{\mathcal{U}}$ *via* maps $\pi_\alpha$. The existence of the "uniqueness strategy" for $\overline{\mathcal{U}}$ which picks out for us the unique cofinal wellfounded branch, can be used to give us a strategy for forming branches in $\mathcal{U}$. We use the comparison notation of [19] Sect. 5. We define by induction $\mathcal{T} \restriction \alpha+1, \mathcal{U} \restriction \alpha+1, \overline{\mathcal{U}} \restriction \alpha+1$ and a $deg^{\mathcal{U}}(\alpha)$ embedding $\pi_\alpha : M^{\mathcal{U}}_\alpha \to M^{\overline{\mathcal{U}}}_\alpha$ as follows:

$\underline{\alpha = 0}$. $M^{\mathcal{T}}_0 = \overline{W}$; $\quad M^{\mathcal{U}}_0 = W = M^{\overline{\mathcal{U}}}_0$; $\quad \pi_0 = id; \rho^{\mathcal{U}}_0 = 0 = \rho^{\mathcal{T}}_0$.

$\underline{\alpha = \lambda \text{ a limit}}$. $\overline{\mathcal{U}} \restriction \lambda$ is then an $\omega$-maximal iteration tree, and the "uniqueness strategy" picks for us the unique cofinal branch $\overline{b} \subseteq [0, \lambda)_{\overline{\mathcal{U}}}$ with a wellfounded direct limit model. Let $\gamma < \lambda$ be such that no drops of any kind have occurred in $\overline{b} \backslash \gamma \subseteq [0, \lambda)_{\overline{\mathcal{U}}}$ (there will only be finitely many of these along any such putative branch), and so that no $\nu + 1 \in [\overline{\gamma}, \lambda)$ is from a special pair (see below in Case IIa) for this latter notion). Set $M^{\overline{\mathcal{U}}}_\lambda = M^{\overline{\mathcal{U}}}_{\overline{b}}$ - the direct limit model. Then set $\langle M^{\mathcal{U}}_\lambda, \langle \pi^{\mathcal{U}}_{\tau,\lambda} \rangle \rangle =_{df} \underrightarrow{Lim} \langle \langle M^{\mathcal{U}}_\tau \rangle, \langle \pi^{\mathcal{U}}_{\tau,\tau'} \rangle_{\gamma < \tau <_{\mathcal{U}} \tau' \in \overline{b}} \rangle$. Set $deg^{\mathcal{U}}(\lambda) = deg^{\overline{\mathcal{U}}}(\lambda)$ to be the eventual value of $deg^{\overline{\mathcal{U}}}(\tau)$ as $\tau \to \lambda, \tau \in \overline{b}$. $\pi_\lambda$ is defined to be the limit of the embeddings $\pi_\tau$ for $\tau \in \overline{b}$, defined to ensure commutativity.

$\underline{\alpha = \beta + 1}$. We are given $\mathcal{U} \restriction \beta + 1, \overline{\mathcal{U}} \restriction \beta + 1, \mathcal{T} \restriction \beta + 1$ and so have models $M^{\mathcal{U}}_\beta, M^{\overline{\mathcal{U}}}_\beta, M^{\mathcal{T}}_\beta$. For most $\beta$ we shall extend $\mathcal{T}$ and $\mathcal{U}$ using the normal comparison process. We shall ensure that the comparison process will come to a clean halt with models $M^{\mathcal{U}}_\delta, M^{\mathcal{T}}_\delta$ at some inaccessible cardinal stage $\delta \leq \kappa$ with the two models agreeing up to $\kappa$ (in fact $\delta$ will be $\kappa$). (If it would "normally" halt before, we simply put in some "padding steps" to make it nominally halt at the next inaccessible stage - as can be surmised, this is an inessential part of the proof. This is somewhat unnecessarily to ensure we only look for special drops at notationally convenient stages $\beta \in \Sigma$ below.)

**Definition 14.** $\Sigma = \{\delta < \kappa \mid \delta \textit{ is inaccessible } \wedge j``\delta \subseteq \delta\}$.

*Case I* $\beta \notin \Sigma$ or $\beta < \sup_{\gamma<\beta} \rho^{\mathcal{U}}_{\gamma}$.

If $M^{\mathcal{T}}_{\beta}$ is an initial segment of $M^{\mathcal{U}}_{\beta}$ or *vice versa*, then as indicated in the last parenthetical remark above, we pad out one more step everywhere setting $M^{\mathcal{W}}_{\beta+1} = M^{\mathcal{W}}_{\beta}$, $\pi^{\mathcal{W}}_{\beta,\alpha} = id$, $\rho^{\mathcal{W}}_{\beta} = 0$, for $\mathcal{W} \in \{\mathcal{T}, \mathcal{U}, \overline{\mathcal{U}}\}$, and $\pi_{\alpha} = \pi_{\beta}$. Otherwise there is some least ordinal $\eta$ witnessing some difference in the hierarchies $E^{M^{\mathcal{T}}_{\beta}}$, $E^{M^{\mathcal{U}}_{\beta}}$. Suppose $E^{\mathcal{U}}_{\beta} \neq \emptyset$ in this definition. Then we set $E^{\overline{\mathcal{U}}}_{\beta} = E^{M^{\overline{\mathcal{U}}}_{\beta}}_{\pi_{\beta}(\eta)}$. If $M^{\mathcal{U}*}_{\alpha}$ is (an initial segment of) $M^{\mathcal{U}}_{\gamma}$ then $M^{\overline{\mathcal{U}}*}_{\alpha}$ is (an initial segment of) $M^{\overline{\mathcal{U}}}_{\gamma}$. $\pi_{\alpha}$ is again defined to commute through the ultrapowers: $\pi_{\alpha}([\langle a, f\rangle]^{M^{\mathcal{U}}_{\gamma}}_{E^{\mathcal{U}}_{\beta}} = [\langle \pi_{\beta}(a), \pi_{\gamma}(f)\rangle]^{M^{\overline{\mathcal{U}}}_{\gamma}}_{E^{\overline{\mathcal{U}}}_{\beta}}$ (stating only the "$n = 0$" case). Then $\pi_{\alpha}$ is a weak $deg^{\mathcal{U}}$-embedding. (Note that actually $\pi_{\alpha}$ is fully elementary until there is a "normal" drop $\alpha \in D^{\mathcal{U}}$.) If $E^{\mathcal{U}}_{\beta} = \emptyset$ then set $\pi_{\alpha} = \pi_{\beta}$. $\rho^{\mathcal{U}}_{\beta}$ and $\rho^{\mathcal{T}}_{\beta}$ are defined as for a normal comparison process step.

*Case II* Otherwise.

*Case IIa) $M^{\mathcal{U}}_{\beta}$ is a weasel and $\widetilde{M} =_{df} Ult(M^{\mathcal{U}}_{\beta}, E_j \restriction \overline{\beta})$ yields a badly behaved tree on the phalanx $((W, \widetilde{M}), \overline{\beta})$ for some $\overline{\beta} \in \Sigma, \overline{\beta} \geq \beta$, where $M^{\mathcal{U}}_{\beta}, M^{\mathcal{T}}_{\beta}$ agree up to $\overline{\beta}$.*

Note that if $\mathcal{V}$ is such a bad tree for some least $\overline{\beta} \geq \beta$, then there are initial segments $W', M'$ of $W, \widetilde{M}$ respectively, which are *properly small* in the sense of [23] Def.6.12, on which we can construe $\mathcal{V}$ as a bad tree (see the arguments particularly of 6.14 *op.cit.*). As our phalanx is properly small and we are assuming there is no IM(Woodin), $\mathcal{V}$ is simple.

Let $X \prec Y \prec V_{\Omega+1}$ be chosen with $X$ countable, $j, \mathcal{V}, \mathcal{T} \restriction \beta+1, \mathcal{U} \restriction \beta+1, \overline{\beta} \in X$. Let $\pi : R \widetilde{\longleftrightarrow} X$ with $\pi(\overline{\mathcal{V}}, \overline{W}, \overline{M}) = \mathcal{V}, W', M'$ etc. The absoluteness argument of 6.14 cited shows that in $V$ $\overline{\mathcal{V}}$ is truly a bad tree on $((\overline{W}, \overline{M}), \overline{\gamma})$ where $\pi(\gamma, \overline{\gamma}) = \beta, \overline{\beta}$. We assume $Y$ chosen so that $\overline{\beta} + 1 \subseteq Y$, card$(Y) \leq \overline{\beta}$. Let $\sigma : S \widetilde{\longleftrightarrow} Y$, and $\psi : R \to S$ be given by $\psi = \sigma^{-1} \circ \pi$. If $\psi(\overline{M}) = \widetilde{Q}$, then $\widetilde{Q} = Ult(Q, E_j \restriction \overline{\beta})$, where $Q = \sigma^{-1}(M^{\mathcal{U}}_{\beta})$. The bad tree $\overline{\mathcal{V}}$ can be copied *via* $\psi$ to yield a bad tree on $((\psi(\overline{W}), \psi(\overline{M})), \psi(\overline{\gamma})) = ((\psi(\overline{W}), \widetilde{Q}), \overline{\beta})$, and using the embedding $\sigma \restriction \psi(\overline{W}) : \psi(\overline{W}) \to W'$ we see:

(1) There is a bad tree on $((W, \widetilde{Q}), \overline{\beta})$.

We shall then take $M^{\mathcal{U}}_{\beta+1}$ as $Q$, $\rho^{\mathcal{U}}_{\beta} = 0$. This is essentially Mitchell's tactic to ensure a sufficiently iterable phalanx in the case his final model $M^{\mathcal{U}}_{\kappa}$ is a weasel.

**Remark:** 1 Suppose in the above $\beta = \overline{\beta}$ were such that $\Sigma$ was Mahlo in $\beta$, and the case hypotheses held, then there is a $\delta \in \Sigma$, $\delta < \beta$, $\delta \in [0, \beta)_{\mathcal{U}}$, already with the property that there is a bad tree on the phalanx $((W, \widetilde{M}), \delta)$, where $\widetilde{M} = Ult(M^{\mathcal{U}}_{\delta}, E_j \restriction \delta)$.

To see this we just have to observe in the above proof that we could use the Mahloness of $\Sigma$ below $\beta$, to find an inaccessible $\delta < \beta$ with $\delta \in \Sigma$ so that we could have taken a $Y$ with $X \prec Y \prec V_{\Omega+1}$, with $\mathrm{card}(Y) = \delta, \beta \cap Y = \delta$. Note that such a $Y$, if $\sigma : S \overset{\sim}{\longleftrightarrow} Y$ as above, with $Q = \sigma^{-1}(M^{\mathcal{U}}_\beta)$, then there is a map $\tau : Q \to M^{\mathcal{U}}_\delta$ given by $\tau = (\pi^{\mathcal{U}}_{\delta,\beta})^{-1} \circ \sigma$. That this makes sense is because $\delta$ is on the branch $[0,\beta]_{\mathcal{U}}$ (as this is cub below $\delta$ by elementarity) and $\langle M^{\mathcal{U}}_\delta, \langle \pi^{\mathcal{U}}_{\eta,\delta}\rangle\rangle =_{df} \underrightarrow{Lim}\, \langle\langle M^{\mathcal{U}}_\eta\rangle_{\eta <_{\mathcal{U}} \delta}, \langle \pi^{\mathcal{U}}_{\eta,\eta'}\rangle_{\eta <_{\mathcal{U}} \eta' <_{\mathcal{U}} \delta}\rangle$. Thus if $x \in M^{\mathcal{U}}_\delta \cap S \Rightarrow x = \pi^{\mathcal{U}}_{\eta,\delta}(\overline{x})$ for some $\eta <_{\mathcal{U}} \delta, \overline{x} \in M^{\mathcal{U}}_\eta$. So $\sigma(x) = \pi^{\mathcal{U}}_{\eta,\beta}(\overline{x}) \in M^{\mathcal{U}}_\beta$. But then the truly bad tree $\psi\overline{\mathcal{V}}$ on $((W,\widetilde{Q}),\delta)$ (in the notation of the case) can be copied *via* the induced map $\tilde{\tau}$ derived from $\tau$ between the lift-up ultrapowers $\widetilde{Q} = Ult(Q, E_j \restriction \delta)$ and $\widetilde{M} =_{df} Ult(M^{\mathcal{U}}_\delta, E_j \restriction \delta)$, to give a bad tree on $((W,\widetilde{M}),\delta)$.

We shall follow [16] and call $\{\beta, \beta+1\}$ a "special pair" and say that $\beta+1$ is a "special drop". To complete the definition of the case we put $D^{\mathcal{U}+2} = D^{\mathcal{U}+1} \cup \{\beta+1\}$. We intend now to continue the comparison using $M^{\mathcal{U}}_{\beta+1}$ rather than $M^{\mathcal{U}}_\beta$. We put $\rho^{\mathcal{U}}_\beta = 0$ and $deg^{\mathcal{U}}(\beta+1) = \omega$, set $\beta <_{\mathcal{U}} \beta+1, \beta <_{\overline{\mathcal{U}}} \beta+1$. Set $\pi_{\beta+1} : M^{\mathcal{U}}_{\beta+1} \to M^{\overline{\mathcal{U}}}_{\beta+1}$ with $M^{\overline{\mathcal{U}}}_{\beta+1} = M^{\overline{\mathcal{U}}}_\beta$. Note $\pi_{\beta+1}$ is fully elementary, and that $M^{\mathcal{U}}_{\beta+1}$ continues to agree with $M^{\mathcal{T}}_\beta$ up to $\sup_{\gamma<\beta}\{\rho^{\mathcal{T}}_\gamma, \rho^{\mathcal{U}}_\gamma\} \leq \beta$ (and the latter is only officially $< \beta$ if for some $\gamma < \beta$ $M^{\mathcal{U}}_\gamma$ is an initial segment of $M^{\mathcal{T}}_\gamma$ - or *vice versa* - which we shall later remark cannot happen). We thus pad one more step in $\mathcal{T}$ to keep the indices correct: $M^{\mathcal{T}}_{\beta+1} = M^{\mathcal{T}}_\beta$; $\rho^{\mathcal{T}}_\beta = 0$; $\pi^{\mathcal{T}}_{\beta,\beta+1} = id$.

*Case IIb) There are no such badly behaved "lift-up" models.*

If one of $M^{\mathcal{U}}_\beta$, $M^{\mathcal{T}}_\beta$ is an initial segment of the other, or if $\beta = \kappa$, we halt the comparison. Otherwise we look for the least $\eta$ witnessing a difference in their hierarchies and proceed just as in *Case I.*

This completes the definition of $\mathcal{T}, \mathcal{U}, \overline{\mathcal{U}}$. We note that since $\Sigma$ is Mahlo in $\kappa$ there are plenty of opportunities to look for badly behaved trees in *Case II.*

Let $\theta \leq \kappa$ be the length of the comparison process.

*Claim 0 a) If $M^{\mathcal{U}}_\theta$ is a weasel, and $\widetilde{W} =_{df} Ult(M^{\mathcal{U}}_\theta, E_j \restriction \kappa)$, then $((W,\widetilde{W}),\kappa)$ is co-iterable with $W$.*

*b) If $M^{\mathcal{U}}_\theta$ is a set premouse, $Q$ is a proper initial segment of $M^{\mathcal{U}}_\theta$, and $\widetilde{Q} = Ult_n(Q, E_j \restriction \theta)$ where $n = n(Q,\theta) < \omega$ exists, then $((W,(\widetilde{Q},n)),\theta)$ is co-iterable with $W$.*

**Proof:** In a) note that clearly $D^{\mathcal{U}} \cap [0,\theta]_{\mathcal{U}} = \emptyset$ (as $M^{\mathcal{U}}_\theta$ must be a weasel). Then $\theta$ must be $\kappa$ (otherwise we have not finished our coiteration!). If the Claim was false, then as $\Sigma$ is Mahlo in $\kappa$, the remark above shows there is an earlier stage $\delta < \kappa$ at which we should be compelled to take a special drop to a set-sized premouse!

For b) note again that if the conclusion failed we must have $\theta = \kappa$. If we consider the branch $b = [0,\kappa]_{\mathcal{U}}$, then $D^{\mathcal{U}} \cap b$ is finite (even if it contains a special drop). So there must be a $\overline{\gamma} < \kappa$ so that $D^{\mathcal{U}} \cap b$ is contained in $\overline{\gamma}$. We shall need the following:

Sublemma *Suppose $M^{\mathcal{U}}_\beta$ has size $< \kappa$ but there is $Q$, a proper initial segment of $M^{\mathcal{U}}_\beta$ with a badly behaved tree on $((W,(\widetilde{Q},n)),\beta)$ where $\widetilde{Q} = Ult_n(Q, E_j \restriction \beta)$ and $n = n(Q,\beta) =_{df} \min\{m \mid \rho^Q_{m+1} \leq \beta\}$ exists. Suppose further $\beta \in [0,\theta]_{\mathcal{U}}$ with $\beta$ the $\mathcal{U}$-predecessor of $\gamma+1$ with $\beta <_{\mathcal{U}} \gamma+1 <_{\mathcal{U}} \theta$. Then $(M^{\mathcal{U}}_{\gamma+1})^*$ is a proper initial segment of $Q$. Hence $\gamma+1 \in D^{\mathcal{U}}$.*

**Proof:** Let $G = E^{\mathcal{U}}_\gamma$ be the first extender used on $[\beta,\theta]_{\mathcal{U}}$. Then we claim that $lh(G) \leq On \cap Q$. For, if it were greater we should know that $Q$ was an initial segment of $M^{\mathcal{T}}_\gamma$ and hence so of $M^{\mathcal{T}}_\theta$. But then $Ult_n(Q, E_j \restriction \beta)$ can be embedded into $\widetilde{M} =_{df} Ult(M^{\mathcal{T}}_\theta \restriction \kappa+1, E_j \restriction \kappa)$; and then we should conclude $((W,\widetilde{M}),\kappa)$ had a badly behaved tree on it. But this is absurd since, (under our assumption on the triviality of $\mathcal{T}$) $\widetilde{M}$ is an initial segment of $W$!

This means that $\gamma+1 \in D^{\mathcal{U}}$: by specification $\rho^\omega_Q \leq \beta$ and hence $\beta < (crit(G)^+)^{M^{\mathcal{U}}_{\gamma+1}}$ with the latter a cardinal of $M^{\mathcal{U}}_{\gamma+1}$, but not of $M^{\mathcal{U}}_\beta$.

**Q.E.D.(Sublemma)**

So suppose $\mathcal{V}$ witnesses the falsity of part b) of the Claim and is a badly behaved tree on $\Phi = ((W,\widetilde{Q}),\theta)$ (dropping the $n$ here). Suppose $lh(\mathcal{V}) = \mu$ and $Lim(\mu)$, (we leave the successor case to the reader) and that there are no cofinal wellfounded branches $b$.

Let $X \prec Y \prec V_{\Omega+1}$ be chosen with $X$ countable, $j,\mathcal{V},Q,\mathcal{T} \restriction \kappa+1, \mathcal{U} \restriction \kappa+1 \in X$, and $Y \cap \kappa = \delta \in \Sigma$ with card$(Y) = \delta > \overline{\gamma}$. Let $\pi : R \overset{\sim}{\longleftrightarrow} X$ with $\pi(\overline{\mathcal{V}},\overline{W},\overline{M}) = \mathcal{V},W,M$ etc. Let $\sigma : S \overset{\sim}{\longleftrightarrow} Y$, and $\psi : R \to S$ be given by $\psi = \sigma^{-1} \circ \pi$. By the arguments of Section 2 of [23] $\overline{\mathcal{V}}$ is a countable, simple, ill-behaved tree on the countable phalanx $\overline{\Phi} = ((\overline{W},Q'),\overline{\kappa})$ where $\pi(\overline{\kappa}) = \kappa, Q' = Ult(\overline{Q}, E_{\overline{j}} \restriction \overline{\kappa})$, where $\overline{Q}$ is some initial segment of $M^{\overline{\mathcal{U}}}_{\overline{\kappa}}$ as, by elementarity of $\pi$, $\pi(\overline{Q}) = Q$. Note $n(Q,\kappa) = n(\overline{Q},\overline{\kappa})$. Consider the pair of elementary maps

$$\phi_0 : \overline{W} \to \psi(\overline{W}) = \sigma^{-1}(W); \quad \phi_1 : Q' \to \psi(Q') = \sigma^{-1}(\widetilde{Q})$$

given by $\psi \restriction \overline{W}, \psi \restriction Q'$ respectively. As $\mathcal{V}$ is bad on $\overline{\phi}$, if we use the elementary preserving maps $\sigma \circ \phi_0, \phi_1, \overline{\mathcal{V}}$ can be copied to a bad tree on $((W,\psi(Q'),\delta)$ (with $\delta = \sigma^{-1}(\kappa) = \psi(\overline{\kappa})$. Further $\psi(Q') = Ult_n(Q_0, \psi(E_{\overline{j}} \restriction \overline{\kappa})) = Ult_n(Q_0, E_j \restriction \delta)$ where $Q_0$ is an initial segment of $\pi(M^{\overline{\mathcal{U}}}_{\overline{\kappa}} = \sigma^{-1}(M^{\mathcal{U}}_\kappa)$. As $\delta \in [\overline{\gamma},\kappa]_{\mathcal{U}}$ we know that $\langle M^{\mathcal{U}}_\delta, \langle \pi^{\mathcal{U}}_{\eta,\delta}\rangle\rangle = \underrightarrow{Lim}\, \langle\langle M^{\mathcal{U}}_\eta\rangle_{\eta <_{\mathcal{U}} \delta}, \langle \pi^{\mathcal{U}}_{\eta,\eta'}\rangle_{\eta <_{\mathcal{U}} \eta' <_{\mathcal{U}} \delta}\rangle$. So $M^{\mathcal{U}}_\delta = \sigma^{-1}(M^{\mathcal{U}}_\kappa)$. Thus $Q_0$ is an initial segment of $M^{\mathcal{U}}_\delta$ with a badly behaved tree on $\Phi' = (W, Ult_n(Q, E_j \restriction \delta), \delta)$.

As $\delta \in \Sigma$ and is on the main branch to $\theta$ we have by the sublemma that $\gamma+1 \in D^{\mathcal{U}}$, where $\delta <_{\mathcal{U}} \gamma+1 <_{\mathcal{U}} \theta$ and $\delta$ is the immediate $\mathcal{U}$-predecessor of $\gamma+1$. This contradicts our definition of $\overline{\gamma}$. **Q.E.D.(Claim 0)**

*Claim 1* $\pi^{\mathcal{U}}_{0,\theta}$"$\kappa \not\subseteq \kappa$.

**Proof:** Suppose this failed. Then there can be no truncation at all on the branch $b = [0,\theta]_{\mathcal{U}}$. This again is obvious by the Dodd-Jensen Lemma (*cf.* [19] 5.3) if all such drops are standard ones, as $M^{\mathcal{T}}_0 = \overline{W}$ has cardinality $\kappa$. But suppose $\{\gamma, \gamma+1\}$ is the special drop on $b$, where $\gamma < \theta \leq \kappa$. Without loss of generality we can assume that $\{\gamma+1\} = D^{\mathcal{U}} \cap b$. Then $M^{\mathcal{U}}_{\gamma+1}$ is a structure of size $< \kappa$. If the Claim fails then we must have that $\theta < \kappa$; that is, $M^{\mathcal{U}}_\theta$ is a proper initial segment of $M^{\mathcal{T}}_\theta$ (here assumed to be just $\overline{W}$). Also if $\widetilde{Q} = Ult_n(M^{\mathcal{U}}_{\gamma+1}, E_j \restriction \gamma)$, by specification we must have that $((W,\widetilde{Q}),\gamma)$ has a badly behaved tree on it. But as $\pi^{\mathcal{U}}_{\gamma+1,\theta}$ is defined on all of $M^{\mathcal{U}}_{\gamma+1}$, and as we are able to embed $\widetilde{Q}$ into $W' = Ult_n(M^{\mathcal{T}}_\theta, E_j \restriction \kappa)$, we should conclude there is a badly behaved tree on $((W,W'),\kappa)$. But as above this is absurd as $W' = W$ in our assumed situation of $\mathcal{T}$'s triviality! Hence $M^{\mathcal{U}}_\theta$ is a weasel.

By *Claim 0a)* we may form the coiteration of $W$ with $((W,\widetilde{W}),\kappa)$ where $\widetilde{W} = Ult(M^{\mathcal{U}}_\theta, E_j \restriction \kappa)$, *via* trees $\mathcal{V}$ on $W$, and $\mathcal{W}$ on the phalanx $((W,\widetilde{W}),\kappa)$, with corresponding maps $\pi^{\mathcal{V}}_{i,j}, \pi^{\mathcal{W}}_{i,j}$ with final models $M^{\mathcal{V}}_\infty = P, M^{\mathcal{W}}_\infty = Q$. Let $k$ be the ultrapower map above $k : M^{\mathcal{U}}_\theta \to \widetilde{W}$. (By our temporary assumption on the triviality of $\mathcal{T}$, here $k \supseteq j \restriction \kappa$.) As $k \circ \pi^{\mathcal{U}}_{0,\theta} : W \to \widetilde{W}$ is an elementary map of the universal weasel $W$ into $\widetilde{W}$, irrespective of whether the final model $Q$ of $\mathcal{W}$ is above $W$, or $\widetilde{W}$, by Dodd-Jensen we must have that $P = Q$, and there has been no dropping of any kind on either side. But then:

(2) The main branch $b$ of $\mathcal{W}$ is $b = [1,\infty]_{\mathcal{W}}$; that is, $Q$ is above $\widetilde{W}$ not $W$.

**Proof:** If $Q$ were above $W$, we should have $\pi^{\mathcal{W}}_{0,\infty} : W \to Q$, $\pi^{\mathcal{V}}_{0,\infty} : W \to Q$, and $\pi^{\mathcal{W}}_{0,\infty}$"$Def(W,A_0) = Def(Q,A_0) = \pi^{\mathcal{V}}_{0,\infty}$"$Def(W,A_0)$ (in the notation of [23] Section 5). As $\mathcal{J}^K_\kappa \subseteq Def(W,A_0)$ this would imply $\gamma =_{df} crit(\pi^{\mathcal{V}}_{0,\infty}) = crit(\pi^{\mathcal{W}}_{0,\infty}) < \kappa$, and if $E, F$ are the first extenders used on the branches $[0,\infty]_{\mathcal{W}}$, $[0,\infty]_{\mathcal{V}}$ respectively, then $\gamma = crit(E) = crit(F)$ and if $\delta = \min\{\nu(E),\nu(F)\}$ then $E \restriction \delta = F \restriction \delta$ (*cf.* the argument of 5.1 of [23]). But this can never happen in such a comparison process. So $Q$ is above $\widetilde{W}$. **Q.E.D.(2)**

Set $\gamma = crit(\pi^{\mathcal{V}}_{0,\infty})$. Set $l = \pi^{\mathcal{W}}_{0,\infty} \circ k \circ \pi^{\mathcal{U}}_{0,\theta}$. Then $crit(l) \leq crit(k) = \alpha =_{df} crit(j)$. Then $l$"$Def(W,A_0) = Def(Q,A_0) = \pi^{\mathcal{V}}_{0,\infty}$"$Def(W,A_0)$. By the definition of $W$, $\mathcal{J}^K_\kappa \subseteq Def(W,A_0)$; we thus conclude $l \restriction \kappa = \pi^{\mathcal{V}}_{0,\infty} \restriction \kappa$. But $crit(l) \leq crit(k) = \alpha < \kappa$. So $\gamma = crit(\pi^{\mathcal{V}}_{0,\infty}) = crit(l) \leq \alpha < \kappa$. We complete Claim 1 by showing that $\pi^{\mathcal{U}}_{0,\theta}(\gamma) \geq \kappa$. As $crit(\pi^{\mathcal{W}}_{1,\infty}) \geq \kappa$ (as now $P = Q$ is above $\widetilde{W}$), and $k$"$\kappa \subseteq \kappa$, it suffices to show that $\pi^{\mathcal{V}}_{0,\infty}(\gamma) \geq \kappa$. But $\pi^{\mathcal{V}}_{0,\infty}(\gamma) < \kappa$ is absurd, since if $E$ were the first extender used on the main branch, we should have $\gamma = crit(E) < \nu(E) < \pi^{\mathcal{V}}_{0,\infty}(\gamma) < \kappa$, whilst $\widetilde{W}$ and $W$ agree up to $\kappa$!
**Q.E.D.(Claim 1)**

Just as in Theorem 19 we have (+) $\exists i_0, \gamma < \theta \quad \pi^{\mathcal{U}}_{i_0,\theta}(\gamma) \geq \theta$.

*Claim 2 (i) & (ii) of Claim 2 of Theorem 19 hold.*

**Proof:** (i). Suppose $\theta < \kappa$. Just as in Theorem 19, for this to happen an extender $E^{\mathcal{U}}_i$ would have to have been used with length larger than $\kappa$; but $\mathcal{U}$ only uses extenders of length smaller than $\kappa$. Again (ii) is standard. **Q.E.D.(Claim 2)**

We then have the same phenomenon occurring as in Theorem 19: by the differing cofinalities argument there is some least initial segment $N$ of the $\kappa$'th model of $\mathcal{U}$, $M^{\mathcal{U}}_\kappa$, over which we can definably collapse $\overline{\kappa}$. By *Claim 0 b)* - with this $N$ in place of $Q$ there, $n(N, \kappa)$, the least $n$ with $\rho^{n+1}_N = \kappa < \overline{\kappa} \leq \rho^n_N$ exists, so we shall be able to perform the comparison of $((W, P), \kappa)$ with $W$, *via* $n$-maximal trees $\mathcal{R}, \mathcal{S}$, where $P = Ult_n(N, E_j \restriction \kappa)$, $P$ the fine structural lift-up ultrapower of $N$ as before. $j$ is cofinal from $\overline{\kappa}$ into $\kappa' = j(\overline{\kappa}) = \kappa^{+K}$ by assumption. The $n+1$'st mastercode $A^{n+1}_P$ is definable over the final model $M^{\mathcal{R}}_\infty$, if the latter is above the model $P$ rather than above $W$. If this is so, $\pi^{\mathcal{R}}_{1,\infty} : P \to M^{\mathcal{R}}_\infty$, and the map is without any drops and is $\Sigma_{n+1}$-preserving. There is thus a code for the structure $P$ in $W$; but then $\kappa'$ is definably collapsed in $W$, whilst $\kappa' = \kappa^{+W}$! To show that $M^{\mathcal{R}}_\infty$ is above $P$ is a standard argument; as otherwise we must have $M^{\mathcal{R}}_\infty = M^{\mathcal{S}}_\infty$, by Dodd-Jensen and properties of $W$. Then $\pi^{\mathcal{R}}_{0,\infty} = \pi^{\mathcal{S}}_{0,\infty}$ and we should argue that we have compatibility of the first extenders used on either side - as usual a contradiction. This suffices.

We now consider the complications involved in removing the assumption that $\mathcal{T}$ was trivial. Essentially no new ideas are needed. Just as in the conclusion of Theorem 19, if $M^{\mathcal{U}}_\theta$ were a weasel we should want to compare a lift-up of $M^{\mathcal{U}}_\theta \to \overline{W}$ where now instead of comparing the phalanx $((W, \widetilde{W}), \kappa)$ with $W$ (where $\widetilde{W}$ was $Ult(M^{\mathcal{U}}_\theta, E_j \restriction \kappa)$) we have to take into consideration the ultrapowers that have been taken in $\mathcal{T}$, as $M^{\mathcal{U}}_\theta \supseteq M^{\mathcal{T}}_\theta$. We should thus want to copy $\mathcal{T}$ to $j\mathcal{T}$ with intermediate copy maps $j_\alpha : M^{\mathcal{T}}_\alpha \to M^{j\mathcal{T}}_\alpha$ for $\alpha < \theta$. We shall call $j_\theta$ $\tilde{\jmath}$ and now shall lift-up $M^{\mathcal{U}}_\theta$ using $E_{\tilde{\jmath}} \restriction \kappa$. We set $\widetilde{W} = Ult(M^{\mathcal{U}}_\theta, E_{\tilde{\jmath}})$ and compare $\Phi(j\mathcal{T})^\frown\langle \widetilde{W}, \kappa\rangle$ with $\Phi(j\mathcal{T})$. This will also require some adjustments to the comparison process, since we shall now be looking ahead to see if there are badly behaved trees on $\Phi(j\mathcal{T} \restriction \beta+1)^\frown\langle Ult(M^{\mathcal{U}}_\beta, E_{j_\beta} \restriction \beta), \beta\rangle$ for those $\beta \in \Sigma$ (that are additionally now closed under $j_\beta$). We add these features to the comparison definition: *Case I* also holds if $j_\beta$"$\beta \not\subseteq \beta$.

*Case IIa)* $M^{\mathcal{U}}_\beta$ *is a weasel, and* $\widetilde{M} = Ult(M^{\mathcal{U}}_\beta, E_{j_\beta} \restriction \overline{\beta})$ *yields a badly behaved tree on* $\Phi(j\mathcal{T} \restriction \beta+1)^\frown\langle \widetilde{M}, \overline{\beta}\rangle$ *where* $M^{\mathcal{U}}_\beta, M^{\mathcal{T}}_\beta$ *agree up to* $\overline{\beta} \geq \beta, j_\beta$"$\overline{\beta} \subseteq \overline{\beta}$, $\overline{\beta} \in \Sigma$.

The arguments given, but now replacing the phalanx of length 1, $W$, with $\Phi(j\mathcal{T} \restriction \beta+1)$ shows there is a countable $\overline{\Phi} \in R$, $\pi(\overline{\Phi}) = \Phi(j\mathcal{T} \restriction \beta+1)$, with $\overline{\mathcal{V}}$ a bad tree on $\overline{\Phi}^\frown\langle \overline{M}, \overline{\gamma}\rangle$ and so on. We then have:

(3) There is a bad tree on $\Phi(j\mathcal{T}\restriction\beta+1)^\frown\langle\widetilde{Q},\overline{\beta}\rangle$ where $\widetilde{Q}=Ult(Q,E_{j_\beta}\restriction\overline{\beta})$.

The remark in this case also holds.

**Remark:** 2 If $\overline{\beta}=\beta$ in the above, and is such that *Case IIa)* hypotheses hold, and that $T=_{df}\{\delta<\beta\,|\,\delta\in\Sigma\wedge j_\delta``\delta\subseteq\delta\}$ is Mahlo in $\beta$, then there is a $\delta<\beta,\delta\in T$, $\delta\in[0,\beta)_{\mathcal{U}}$ with the property that there is a bad tree on the phalanx $\Phi(j\mathcal{T}\restriction\delta+1)^\frown\langle\widetilde{M},\delta\rangle$ where $\widetilde{M}=Ult(M^{\mathcal{U}}_\delta,E_{j_\delta}\restriction\delta)$.

The revamped *Claim 0* reads:

*Claim 3a) If $\widetilde{W}=_{df}Ult(M^{\mathcal{U}}_\theta,E\restriction\tilde{\jmath}\restriction\kappa)$ is a weasel, then $\Phi(j\mathcal{T})^\frown\langle\widetilde{W},\kappa\rangle$ is coiterable with $\Phi(j\mathcal{T})$.*
*b) If $Q$ is a proper initial segment of $M^{\mathcal{U}}_\theta$, $n=n(Q,\theta)<\omega$, and $\widetilde{Q}=Ult_n(Q,E_{\tilde{\jmath}}\restriction\kappa)$ then $\Phi(j\mathcal{T})^\frown\langle(\widetilde{Q},n),\theta\rangle$ is coiterable with $\Phi(j\mathcal{T})$.*

The proof of Claim 3a) goes through using remark 2. For part b), the Sublemma now reads:

Sublemma *Suppose $M^{\mathcal{U}}_\beta$ has size $<\kappa$ but there is $Q$, a proper initial segment of $M^{\mathcal{U}}_\beta$ with a badly behaved tree on $\Phi(j\mathcal{T}\restriction\beta+1)^\frown\langle(\widetilde{Q},n)),\beta\rangle$ where $\widetilde{Q}=Ult_n(Q,E_{j_\beta}\restriction\beta)$ and $n=n(Q,\beta)=_{df}\min\{m\mid\rho^Q_{m+1}\le\beta\}$ exists. Suppose further $\beta\in[0,\theta]_{\mathcal{U}}$ with $\beta$ the $\mathcal{U}$-predecessor of $\gamma+1$ with $\beta<_{\mathcal{U}}\gamma+1<_{\mathcal{U}}\theta$. Then $(M^{\mathcal{U}}_{\gamma+1})^*$ is a proper initial segment of $Q$. Hence $\gamma+1\in D^{\mathcal{U}}$.*

In the proof of this, using $j_\kappa\restriction\beta=j_\beta\restriction\beta$ the conclusion there now says that $\Phi(j\mathcal{T})\restriction\kappa^\frown\langle\widetilde{M},\kappa\rangle=\Phi(j\mathcal{T})$ has a badly behaved tree on it (where $\widetilde{M}=Ult(M^{\mathcal{T}}_\theta\restriction\kappa+1,E_j\restriction\kappa)$). But this can be construed as simply the derived phalanx from a tree on the iterable $W$, and so there can be no badly behaved trees on this! The conclusion that $\gamma+1\in D^{\mathcal{U}}$ follows as before.

We take appropriate substructures $X\prec Y\prec V_{\Omega+1}$ as before with now $\mathcal{V}$ a badly behaved tree on $\Phi(j\mathcal{T})^\frown\langle\widetilde{Q},\theta\rangle$, and both the latter in $X$. If $\pi(\overline{\Phi})=\Phi(j\mathcal{T})$, instead of a single $\phi_0$, we have for our copy construction $\phi^0_i=_{df}\psi\restriction\overline{W}_i,\phi^0_i:\overline{W}_i\to\psi(\overline{W}_i)$ for $i<lh(\overline{\Phi})$. Thus:

$$\phi^0_i:\overline{\Phi}\rightrightarrows\sigma^{-1}(\Phi(j\mathcal{T}))\,;\quad\phi_1:Q'\to\sigma^{-1}(\widetilde{Q}).$$

**Q.E.D.(Claim 3)**

*Claim 4 $\pi^{\mathcal{U}}_{0,\theta}``\kappa\not\subseteq\kappa$;* i.e. *Claim 1 above still holds.*

**Proof:** of *Claim 4*. We argued that if the Claim failed there could be no dropping on $b=[0,\theta]_{\mathcal{U}}$: even if the latter had special drops, as $M^{\mathcal{U}}_\theta$ would be a proper initial segment of $M^{\mathcal{T}}_\theta$, we could embed $\widetilde{Q}$ into $Ult(M^{\mathcal{T}}_\theta,E_j\restriction\kappa)$. In our current situation we should say that we could embed $\widetilde{Q}=Ult_n(M^{\mathcal{U}}_{\gamma+1},E_{j_\gamma}\restriction\gamma)$ into $W'=_{df}Ult(M^{\mathcal{T}}_\theta,E_{\tilde{\jmath}}\restriction\kappa)$ and conclude there is a badly behaved tree on

$\Phi(j\mathcal{T})^\frown\langle W', \kappa\rangle$. This is again an absurdity as such a tree can be construed as one on the first member of $\Phi(j\mathcal{T})$ which is the iterable $W$ itself! Continuing by *Claim 3a)* we may form the coiteration of $\Phi(j\mathcal{T})^\frown\langle\widetilde{W}, \kappa\rangle$ with $\Phi(j\mathcal{T})$ *via* trees $\mathcal{W}$ and $\mathcal{V}$ respectively. With the notation above $k \supseteq \tilde{j} \restriction \kappa$ (rather than just $j \restriction \kappa$), the argument that the final models $P = M^{\mathcal{W}}_\infty$ and $Q = M^{\mathcal{V}}_\infty$ are equal is as before. We now have:

(4) The main branch $b$ of $\mathcal{W}$ is $b = [\theta+1, \infty]_{\mathcal{W}}$; that is, $\mathrm{root}(P) = \theta + 1$ and $P$ is above $\widetilde{W}$.

**Proof:** (Note here, $lh(\Phi(j\mathcal{T})) = \theta + 1$, and $\widetilde{W}$ is then the $\theta+1$'st model of the starting phalanx $\Phi(j\mathcal{T})^\frown\langle\widetilde{W}, \kappa\rangle$.) Suppose $\mathrm{root}_b(P) = \nu, \mathrm{root}_c(P) = \nu'$ with $\nu, \nu' \leq \theta$. Then if $E, F$ are the first extenders used on $[\nu, \theta]_{\mathcal{W}}$ and $[\nu', \theta]_{\mathcal{V}}$, both $lh(E), lh(F) > \kappa$, that is, $E, F$ do not come from any extender used in building $j\mathcal{T}$.

*Subclaim* $\nu = \nu'$.

**Proof:** If not, without loss of generality, say $\nu < \nu'$, and let $\overline{\nu}$ be the greatest stage with $\overline{\nu} \leq_{j\mathcal{T}} \nu', \nu$. Then $\overline{\nu} <_{j\mathcal{T}} \nu'$. A similar argument over the definability hulls shows: $(\pi^{\mathcal{W}}_{\nu,\infty} \circ \pi^{j\mathcal{T}}_{0,\nu})$"$Def(W, A_0) = Def(P, A_0) = (\pi^{\mathcal{V}}_{\nu',\infty} \circ \pi^{j\mathcal{T}}_{0,\nu'})$"$Def(W, A_0)$. Hence $\pi^{\mathcal{W}}_{\nu,\infty} \circ \pi^{j\mathcal{T}}_{\overline{\nu},\nu} \circ \pi^{j\mathcal{T}}_{0,\overline{\nu}} = \pi^{\mathcal{V}}_{\nu',\infty} \circ \pi^{j\mathcal{T}}_{\overline{\nu},\nu'} \circ \pi^{j\mathcal{T}}_{0,\overline{\nu}}$. If $G$ is the first extender used on the path $[\overline{\nu}, \nu']_{j\mathcal{T}}$ then $crit(G) = crit(\pi^{\mathcal{W}}_{\nu,\infty} \circ \pi^{j\mathcal{T}}_{\overline{\nu},\nu})$. If $\overline{\nu} = \nu$ then $crit(G) = crit(\pi^{\mathcal{W}}_{\nu,\infty})$, which contradicts $E \in \mathcal{W}$ as the first extender used with $crit(E) = crit(\pi^{\mathcal{W}}_{\nu,\infty})$, since then $G$ is compatible with $E$. But if $\overline{\nu} < \nu$ and $H$ is the first extender used on $[\overline{\nu}, \nu]_{j\mathcal{T}}$, then again $crit(G) = crit(H)$, $G \restriction \rho_h \neq H \restriction \rho_H$ and then they must be incompatible (*cf.* [19] p.49 at a)), and the overall maps into $W$ must differ on the two sides. **Q.E.D.(Subclaim)**

But now $\pi^{\mathcal{V}}_{\nu,\infty} = \pi^{\mathcal{W}}_{\nu,\infty}$, and so $E, F$ are compatible. But this does not happen in comparisons! Hence $\mathrm{root}_b(P) = \theta + 1$. **Q.E.D.(4)**

Set $h = \pi^{\mathcal{V}}_{\nu,\infty} \circ \pi^{j\mathcal{T}}_{0,\nu}$ (where $\nu = root(P) \leq \theta$). Set $\gamma = crit(h)$. As before set $l = \pi^{\mathcal{W}}_{\theta+1,\infty} \circ k \circ \pi^{\mathcal{U}}_{0,\theta}$. Now $k \restriction \kappa \supseteq \tilde{j} \restriction \kappa$, but we conclude as before that $l \restriction \kappa = h \restriction \kappa$ and $crit(l) \leq crit(j) < \kappa$.

*Subclaim* $h(\gamma) \geq \kappa$.

**Proof:** Suppose not. Notice that this implies that $\nu = root(P) = \theta$. For, if $\nu < \theta$ then the first extender used in $[\nu, \infty]_{\mathcal{V}}$, would have length $\geq \kappa$, whilst it would have critical point $< \kappa$, and the Subclaim would be proven. We thus have $h \restriction \kappa = \pi^{j\mathcal{T}}_{0,\theta} \restriction \kappa = l \restriction \kappa = k \circ \pi^{\mathcal{U}}_{0,\theta} \restriction \kappa$.

As $crit(h) = crit(l) = \gamma$, $crit(h) = crit(E)$ where $E = E^{j\mathcal{T}}_\tau$ where $\tau + 1 \in [0, \theta]_{\mathcal{T}} \wedge \mathcal{T}\text{-pred}(\tau+1) = 0$, and thus $E$ is the first extender used on the copied iteration on $[0, \theta]_{j\mathcal{T}}$. Let $F = E^{\mathcal{T}}_\tau$. Let $G = E^{\mathcal{U}}_\mu$ be the corresponding first extender used on the iteration $\mathcal{U}$ with $\mathcal{U}$-pred$(\mu+1) = 0 \wedge \mu+1 \in [0,\theta]_{\mathcal{U}}$. Then $crit(G) = \gamma$ also. Let $\overline{\rho} = \inf\{\rho(F), \rho(G)\}$. Then both

(5) $crit(\pi^{\mathcal{T}}_{\tau+1,\theta}), crit(\pi^{\mathcal{U}}_{\mu+1,\theta}) > \overline{\rho}$

(6) $j_\tau \restriction \overline{\rho} = j_{\tau+1} \restriction \overline{\rho} = \tilde{\jmath} \restriction \overline{\rho} = k \restriction \overline{\rho}$.

Also $\gamma = crit(E) < \alpha$ (as $E = j_\tau(F)$ whilst $\alpha \notin \mathrm{ran}(j_\tau)$ for any $\tau < \theta$). Hence for $X \subseteq \gamma \in \mathrm{dom}(j)$, $j(X) = k(X) = X$. Since $G \restriction \overline{\rho}, F \restriction \overline{\rho}$ are incompatible on $\mathcal{P}(\gamma) \cap M^{\mathcal{T}}_0 \cap M^{\mathcal{U}}_0$, we can find $a \in [\overline{\rho}]^{<\omega}, X \subseteq \gamma$ so that

$$a \in \pi^{\mathcal{T}}_{0,\tau+1}(X) \Longleftrightarrow a \notin \pi^{\mathcal{U}}_{0,\mu+1}(X). \tag{7}$$

But by (6), the left hand side here holds if and only if:

$$j_\tau(a) = j_{\tau+1}(a) \in j_\tau(\pi^{\mathcal{T}}_{0,\tau+1}(X) \cap \overline{\rho}) = j_{\tau+1}(\pi^{\mathcal{T}}_{0,\tau+1}(X) \cap \overline{\rho})$$

$$= \pi^{j\mathcal{T}}_{0,\tau+1}(j(X) \cap j_{\tau+1}(\overline{\rho})) \qquad \text{(By the copy map construction )}$$

$$= h(X) \cap j_{\tau+1}(\overline{\rho})) = l(X \cap \overline{\rho}) \qquad \text{(By (5))}$$

$$= j_\tau(\pi^{\mathcal{U}}_{0,\mu+1}(X) \cap j_\tau(\overline{\rho})). \qquad \text{(By (5), (6))}$$

But then by (7) we have deduced a contradiction:

$$j_\tau(a) \in j_\tau(\pi^{\mathcal{U}}_{0,\mu+1}(X) \cap \overline{\rho}) \Longleftrightarrow a \notin \pi^{\mathcal{U}}_{0,\mu+1}(X) \qquad \textbf{Q.E.D.(Subclaim)}$$

But this now completes *Claim 4*: if $h(\gamma) \geq \kappa$, as $h(\gamma) = l(\gamma)$, this can only arise if $\pi^{\mathcal{U}}_{0,\theta}(\gamma) \geq \kappa$. **Q.E.D.(Claim 4)**

Since $\mathcal{U}$ only uses extenders of length $< \kappa$ we have that *Claim 2* above still holds, and we can finish the argument using the differing cofinalities of the "$\kappa^+$'s" in the different models as outlined above.

For the second part of the theorem argue just as in Theorem 20, as again we know some point is sent up to $\kappa$. **Q.E.D.(Theorem 21)**

We remark that by the argument above of the second part, which is really about the failure of $K$-condensation, there will be weaker cardinals than Jónsson (such as the "$\delta$-Jónsson " of Def. 8 above) below which many successors of regular cardinals will also be correctly computed in $K$.

We conclude with some further open problems.

*Open Questions*

4) Does $ZFC \vdash \gamma$ regular Jónsson $\longrightarrow \forall A \subseteq \gamma\ A^{\#}$ exists? [True if a) $\gamma$ Ramsey, or, b) No # for $IM$(Strong) see Theorem 12 above. False if $\gamma$ singular.]
5) Does $ZFC \vdash \gamma$ strongly inaccessible Jónsson $\Longrightarrow \gamma$ weakly compact? [Conjecture : No.]
6) Is there a forcing notion that kills off the Ramsey property whilst preserving the Jónsson property and strong inaccessibility? [ccc forcing preserves Jónssonness so the emphasis here is on strong inaccessibility.]
7) Let $\gamma$ be regular Jónsson, $K = K^{Steel}$. Does $V_\gamma = V_\gamma^K \longrightarrow V_{\gamma+1} = V_{\gamma+1}^K$? [True for $K^{strong}$.]
8) Does $\gamma$ regular Jónsson $+\neg\Box_\gamma^\gamma \Longrightarrow IM$(Woodin)? [NB $\gamma^{<\gamma} = \gamma \Longrightarrow \Box_\gamma^\gamma$.]
9) Suppose no $IM$(Woodin). Let $\kappa$ be Jónsson. Show that there is a regular cardinal $\gamma < \kappa$ such that $\Box_\gamma$ holds. [The results here do not quite give this: it is another question about removing the measurable $\Omega$ (or larger cardinal) needed for the $K$ construction.]
10) Is it consistent, relative to large cardinals, that the first regular $\gamma$ for which $\Box_\gamma$ holds be greater than (or equal) to the first Jónsson ?

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