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Zeros of equivariant vector fields: Algorithms for an invariant approach

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We present a symbolic algorithm to solve for the zeros of a polynomial vector field equivariant with respect to a finite subgroup of $O(n)$. We prove that the module of equivariant polynomial maps for a finite matrix group is Cohen-Macaulay and give an algorithm to compute a fundamental basis. Equivariant normal forms are easily computed from this basis. We use this basis to transform the problem of finding the zeros of an equivariant map to the problem of finding zeros of a set of invariant polynomials. Solving for the values of fundamental polynomial invariants at the zeros effectively reduces each group orbit of solutions to a single point. Our emphasis is on a computationally effective algorithm and we present our techniques applied to two examples.

1. Introduction

Vector fields which commute with the action of a symmetry group arise frequently from simple mathematical models. The study of generic bifurcations of such systems has developed into the field called equivariant bifurcation theory. Many researchers have contributed to this work and the survey articles by Field (1986) and Stewart (1988) and the book by Golubitsky *et al.* (1988) contain many of the results and references. We mention two significant themes of this theory. The first is that the generic bifurcation of families of equivariant vector fields can be much more complicated than generic (non-equivariant) bifurcation. For an extreme example of this, see Guckenheimer and Worfolk (1992). Second, many aspects of the behavior of symmetric systems are independent of the specific model and depend solely on the representation of the symmetry group. Consequently, equivariant bifurcation involves the study of abstract groups and their representations. Given a specific representation of a symmetry group, one basic analysis is to examine the equilibrium branching pattern of the generic equivariant bifurcation. This involves, at the very least (Field and Richardson, 1992a,b), computing an equivariant polynomial normal form and solving for its equilibrium points.

The computation of a normal form is often nontrivial. Since these calculations must be performed for each representation of an abstract group, many researchers have spent time performing these calculations for a variety of different groups (Field and Swift, 1991;

Golubitsky *et al.*, 1988; Guckenheimer and Worfolk, 1992; Melbourne, 1984). We present a symbolic computational algorithm for this task.

Solving for the zeros of a set of nonlinear polynomial equations is well known to be a difficult problem, however the equations we work with have a symmetrical structure which may be taken advantage of in order to simplify the computational task. Our techniques are directed towards those systems of equations which have closed form solutions but which are computationally too hard to solve using more naive approaches. An example of one such problem is presented in Section 8 and was the motivating example for the development of these algorithms.

This discussion is restricted to polynomial vector fields which commute with a finite group. The analysis proposed in this paper exploits the symmetry structure of the solution set. The orbit of any zero under the symmetry group of the vector field gives a set of solutions. Each orbit of points is collapsed to a single point by considering polynomials invariant under the action of the symmetry group. The goal is to derive a set of equations in the invariants whose zeros coincide with the zeros of the vector field. The number of solutions to these new equations will correspond to the number of orbits of zeros of the original equations. This approach has been used by Jaric *et al.* (1984) with crystallographic point groups. The algorithms needed naturally lend themselves to symbolic computer algebra techniques. This paper emphasizes computable results.

The linearization of a vector field at an equilibrium point determines local stability and bifurcation. Frequently the most complicated dynamical behavior of a vector field occurs when parameter values are close to those where the linearization is degenerate. Therefore we are also interested in computing relations on the parameters which result in degenerate linearizations at the equilibrium points. We show how these relations can be written in terms of the invariants.

In Section 2, we present our notation and summarize the mathematical theory of polynomial invariants. This discussion includes little more than the material we will need for our analysis and we do not give any proofs of the results which may be found easily in the literature. We have relied heavily here on the work of Stanley (1979) and Sturmfels (1993). In Section 3, we give a brief summary of the results from Gröbner basis theory on which some of our techniques rely. Gröbner bases are the computational tool developed by Buchberger (1985) which makes possible the symbolic analysis. For additional references we also refer the reader to the texts of Cox *et al.* (1992) and Becker and Weispfenning (1993).

In Section 4, we develop the equivariant theory in a fashion which parallels the development of the invariant theory. Proofs of all the theorems and propositions are given because the equivariant results are harder to find in the literature. Most of the results and proofs are direct analogues of the invariant theory results. The emphasis in these sections is on the theory which allows us to effectively compute bases for the module of equivariant polynomial maps. A knowledge of the bases will allow us to easily compute normal forms for equivariant bifurcation problems.

In Section 5, the invariant and equivariant theories are applied to the problem of solving for zeros of equivariant polynomial maps. We shall think of these maps as coming from the defining expressions of a differential equation for which we wish to compute equilibria. Next, in Section 6, we describe how to use the analysis to compute information about the linearization of the differential equation.

The analysis is only useful if it can be used to solve problems which cannot be solved efficiently using other techniques. This work was motivated by the desire to compute the

asymmetric zeros of the equivariant normal form studied in the author's Ph.D. dissertation (Worfolk, 1993) and the paper by Guckenheimer and Worfolk (1992). In order to illustrate the process, we will present the analysis of a simple example in Section 7 as well as the more complicated, motivating example in Section 8. We have implemented all the algorithms for this analysis in Maple (Char *et al.*, 1991), except for some of the larger Gröbner basis calculations which were done with Macaulay (Stillman *et al.*, 1989).

2. Review of Polynomial Invariant Theory

The invariant theory of finite groups is a well developed mathematical field which combines elements of combinatorial theory and modern commutative algebra. Only the elementary results needed to follow the forthcoming analysis are presented here, and the reader is referred to the literature for proofs and more extensive discussions. Our presentation is taken mainly from Stanley (1979) and Sturmfels (1993). Proofs of all the Theorems and Lemmas stated in this section may also be found in Stanley (1979) or Sturmfels (1993).

Let V be a finite-dimensional vector space and $\Gamma \subset GL(V)$ a finite matrix group, with its order denoted by $|\Gamma|$. Let $\mathbf{C}[V]$ denote the ring of polynomials with complex coefficients in $n = dim(V)$ variables. Denote the subring of invariant polynomials by $\mathbf{C}[V]^\Gamma$,

$$\mathbf{C}[V]^\Gamma \equiv \{p \in \mathbf{C}[V] \mid p \circ \gamma = p \text{ for all } \gamma \in \Gamma\} .$$

The goal is to understand the structure of the invariant ring with an emphasis on those characteristics which are useful for computing invariants. We begin with the following two theorems.

THEOREM 2.1. *If $dim(V) = n$ then there exist n, but not $n+1$, algebraically independent invariants over $\mathbf{C}$.*

THEOREM 2.2. (NOETHER'S DEGREE BOUND) *If $dim(V) = n$, then $\mathbf{C}[V]^\Gamma$ is generated as an algebra over $\mathbf{C}$ by at most $\begin{pmatrix} |\Gamma| + n \\ n \end{pmatrix}$ homogeneous invariants whose degree is bounded by $|\Gamma|$.*

Note that $\mathbf{C}[V]^\Gamma$ is a graded algebra. It is the direct sum of the finite-dimensional vector spaces $\mathbf{C}[V]^\Gamma_d$, where $\mathbf{C}[V]^\Gamma_d$ denotes the set of all homogeneous invariants of degree d. The following lemma gives an explicit algorithm for computing bases of the vector spaces $\mathbf{C}[V]^\Gamma_d$. This may be thought of as averaging monomials over the group.

LEMMA 2.3. *The polynomial invariants of degree d, $\mathbf{C}[V]^\Gamma_d$, are generated as a vector space by the $\begin{pmatrix} n+d-1 \\ d \end{pmatrix}$ polynomials $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} f \circ \gamma$, where f ranges over all $\begin{pmatrix} n+d-1 \\ d \end{pmatrix}$ monomials of degree d.*

From Theorem 2.2 and Lemma 2.3, we already have a direct method which may be used to find a finite generating set for $\mathbf{C}[V]^\Gamma$; however, we wish for more information

about the structure of the invariant ring. We would like to find a canonical form for the invariant polynomials which will allow a more complete description.

The Hilbert series, or Poincaré series, of the graded algebra $\mathbf{C}[V]^\Gamma$, gives the dimension of the vector space of invariants at each degree and is by definition the generating function

$$\Phi_\Gamma(z) = \sum_{d=0}^{\infty} \dim(\mathbf{C}[V]_d^\Gamma) z^d \ .$$

Molien's Theorem gives an explicit expression for the rational function $\Phi_\Gamma(z)$.

THEOREM 2.4. (MOLIEN'S THEOREM) *The Hilbert series of* $\mathbf{C}[V]^\Gamma$ *is given by*

$$\Phi_\Gamma(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{det(1_\Gamma - z\gamma)} \ .$$

This remarkable result will facilitate the generating set calculations.

As previously mentioned, Noether's degree bound and the associated lemma provide a construction for a set of homogeneous generators for the invariant ring. By Theorem 2.1 there are exactly n algebraically independent invariants from this set. Now consider the subring of $\mathbf{C}[V]^\Gamma$ generated by n algebraically independent elements $p_1, \ldots, p_n$. We will denote this subring by $R_p \equiv \mathbf{C}[p_1, \ldots, p_n]$.

LEMMA 2.5. $R_p = \mathbf{C}[p_1, \ldots, p_n]$ *has Hilbert series*

$$H(R_p, z) = \frac{1}{(1 - z^{d_1}) \cdots (1 - z^{d_n})} \ ,$$

where d_i *is the degree of* p_i.

This combinatorial result follows easily from the algebraic independence of the p_i.

A set $\{p_1, \ldots, p_n\}$ of n homogeneous polynomials with positive degree is called a *homogeneous system of parameters* (henceforth, an *h.s.o.p.*) if $\dim(\mathbf{C}[V]^\Gamma) = n$ and $\mathbf{C}[V]^\Gamma$ is a finitely generated module over $\mathbf{C}[p_1, \ldots, p_n]$. For further results in commutative algebra, see, for example, Atiyah and Macdonald (1969). The following theorem helps in the identification of an h.s.o.p.

THEOREM 2.6. *A set* $\{p_1, \ldots, p_n\}$ *of* n *homogeneous invariant polynomials is an h.s.o.p. for* $\mathbf{C}[V]^\Gamma$, $n = dim\ V$, *if and only if the polynomials in* $\{p_1, \ldots, p_n\}$ *have no common roots except* 0.

A ring is called *Cohen-Macaulay* if it is finitely generated as a *free* module over the ring of any h.s.o.p. The following theorem asserts that invariant polynomials of finite groups form Cohen-Macaulay rings.

THEOREM 2.7. *For* Γ *finite, there exist* $n = dim(V)$ *homogeneous polynomial invariants* $p_1, \ldots, p_n$ *such that* $\mathbf{C}[V]^\Gamma$ *is finitely generated as a free module over the subring* $\mathbf{C}[p_1, \ldots, p_n]$. *Then there exist* $q_1, \ldots, q_m \in \mathbf{C}[V]^\Gamma$ *with each* q_i *homogeneous, such that*

$$\mathbf{C}[V]^\Gamma = \bigoplus_{i=1}^{m} q_i \mathbf{C}[p_1, \ldots, p_n] \ ,$$

i.e., $\{p_1, \ldots, p_n\}$ *is an h.s.o.p., and* $\mathbf{C}[V]^\Gamma$ *is a Cohen-Macaulay ring.*

This theorem indicates that given an h.s.o.p., $\{p_1, \ldots, p_n\}$, then there is a finite set of homogeneous invariants $\{q_1, \ldots, q_m\}$ such that any invariant may be written uniquely as a linear combination of the q_i with polynomials in the p_i as coefficients. Usually, the p_i are called *primary invariants*, and the q_i, *secondary invariants*. Together, they make up a set of *fundamental invariants* for Γ. The first goal of an algorithm which computes fundamental invariants is to produce a set of primary invariants. These invariants are not unique, nor are their degrees, and it is often interesting to look at different sets of primary invariants. However, once their degrees are fixed, the number and degrees of the secondary invariants are also fixed. This follows from the following theorem.

THEOREM 2.8. *For* Γ *finite and* $p_1, \ldots, p_n$ *a set of primary invariants for* $\mathbf{C}[V]^\Gamma$ *with degrees* $d_1, \ldots, d_n$*, the number of secondary invariants* m *is given by*

$$m = \frac{d_1 \cdots d_n}{|\Gamma|} .$$

The degrees with multiplicity of the secondary invariants are the exponents of the generating function

$$\Phi_\Gamma(z)/H(R_p, z) = z^{e_1} + \cdots + z^{e_m} ,$$

where $H(R_p, z)$ *is the Hilbert series of* $R_p = \mathbf{C}[p_1, \ldots, p_n]$ *(see Lemma 2.5).*

Thus, once primary invariants are computed, with the aid of this theorem we may search through the generating invariants given by Lemma 2.3 to find the secondary invariants. Note that any constant is an invariant, so we may always assume that $q_1 = 1$. Sturmfels (1993) and McShane (1992) present explicit symbolic computer algebra algorithms based on the theoretical results considered here. For the purposes of our forthcoming analysis, we consider the calculation of a set of fundamental invariants to be a solved problem.

We will need to be able to write an invariant polynomial in terms of the fundamental invariants. This may be done by separating an invariant into its homogeneous components. Each component may then be written in standard form by solving a system of linear equations. Because of the structure of the ring of invariants, these systems of linear equations will always possess unique solutions.

3. Gröbner Bases

The method of Gröbner bases provides an algorithmic solution to the problem of solving for the zeros of a set of algebraic equations. This is only one of a variety of problems for which Gröbner bases may be used. We do not attempt to discuss much of this theory developed by Bruno Buchberger, but simply present the result which is most useful to us. The paper by Buchberger (1985) and the books by Cox *et al.* (1992) and Becker and Weispfenning (1993) are good introductions. The Gröbner basis techniques required for the computation of fundamental invariants are presented and discussed in Sturmfels (1993) and McShane (1992).

The essential idea for applying Gröbner bases to elimination theory is as follows. Given a finite set of polynomials which generate an ideal and an ordering on the monomials,

Buchberger's algorithm gives an alternative generating set, called a Gröbner basis. Given an appropriate ordering of monomials, the Gröbner basis has some special properties.

THEOREM 3.1. *(Buchberger, 1985) Let G be a Gröbner basis for an ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ computed with respect to the purely lexicographic ordering of monomials. Assume that $x_1 < x_2 < \ldots < x_n$. Then*

$$I \cap \mathbf{C}[x_1, \ldots, x_i] = Ideal(G \cap \mathbf{C}[x_1, \ldots, x_i])$$

for $i = 1, \ldots, n$, where the ideal on the right hand side is formed in $\mathbf{C}[x_1, \ldots, x_i]$.

Therefore we can solve for the value of one variable at a time, starting with x_1, by factoring univariate polynomials and then substituting the solution into the remaining equations. Of course, the factorization of a univariate polynomial is not always possible for degrees greater than four. This is a limitation to solving algebraic equations that no algorithm can avoid. We will refer to the use of Gröbner bases to solve for the zeros of sets of multivariate polynomials as the Gröbner solve procedure.

4. Polynomial Equivariant Theory

Specific results on the equivariant theory of finite groups are difficult to find in the literature. In this section we present the equivariance results, with proofs, which are the analogues of the theorems on invariants. The reader will notice the similarities with, and heavy dependence on, the invariance results.

Let V and W be finite dimensional vector spaces over $\mathbf{C}$. Let Γ be a finite group with homomorphisms $\Gamma \to GL(V)$ and $\Gamma \to GL(W)$. We shall simply use γ to represent the corresponding element in $GL(V)$ or $GL(W)$ and occasionally γ_V or γ_W, respectively, when there is the possibility for confusion. Then we may define the action of Γ on $\mathbf{C}[V]$and $\mathbf{C}[W]$ by $\gamma(f) = f \circ \gamma^{-1}$. Identify the space of polynomial mappings from V to W with $\mathbf{C}[V] \otimes W$, which is a module over $\mathbf{C}[V]$. The polynomial mappings which commute with Γ are called equivariants and we shall denote them by $(\mathbf{C}[V] \otimes W)^\Gamma$,

$$(\mathbf{C}[V] \otimes W)^\Gamma \equiv \{g \in \mathbf{C}[V] \otimes W \mid g \circ \gamma_V = \gamma_W g \text{ for all } \gamma \in \Gamma\} .$$

A mapping $g \in \mathbf{C}[V] \otimes W$ is an equivariant if and only if g is invariant under the action of $\gamma \in \Gamma$ given by $\gamma(g) = \gamma \cdot (g \circ \gamma^{-1})$. Thus, the action of γ on $\mathbf{C}[V] \otimes W$ is linear and is defined by $\gamma : f(x) \otimes y \mapsto f \circ \gamma^{-1}(x) \otimes \gamma y$. The goal of this section will be to demonstrate that there exists a computable, finite fundamental set of equivariants that generates the $\mathbf{C}[V]^\Gamma$-module $(\mathbf{C}[V] \otimes W)^\Gamma$. In the algebraic literature, equivariants are often called covariants.

We first prove that the equivariants are a module over the invariants.

LEMMA 4.1. *(Golubitsky et al., 1988) $(\mathbf{C}[V] \otimes W)^\Gamma$ is a module over $\mathbf{C}[V]^\Gamma$.*

PROOF. Let $f \in \mathbf{C}[V]^\Gamma, f : V \to \mathbf{C}$, and let $g \in (\mathbf{C}[V] \otimes W)^\Gamma, g : V \to W$. For all $\gamma \in \Gamma$, compute

$$\begin{aligned} (fg) \circ \gamma &= (f \circ \gamma)(g \circ \gamma) \\ &= f \cdot (\gamma g) \\ &= \gamma \cdot (fg) \ . \end{aligned}$$

This shows that $fg : V \to W$ and $fg \in (\mathbf{C}[V]\otimes W)^\Gamma$, and consequently proves the assertion. □

Consider now the polynomial ring $\mathbf{C}[V\oplus W^*]$. This ring can be given the grading transported from $\mathbf{C}[W^*]$ and we define the action of Γ by

$$\gamma(h(x,y^*)) = h(\gamma^{-1}x, \gamma^T y^*) \ .$$

Let Π_d be the projection of a polynomial in $\mathbf{C}[V\oplus W^*]$to its d-th graded components. So $\Pi_1(\mathbf{C}[V\oplus W^*]) = (\mathbf{C}[V\oplus W^*])_1$ is exactly the space of polynomials which are linear in y^*. We will show that $(\mathbf{C}[V]\otimes W)^\Gamma \cong (\mathbf{C}[V\oplus W^*]^\Gamma)_1$ and then use the invariant theory to study $(\mathbf{C}[V\oplus W^*]^\Gamma)_1$. To begin this, we define the linear mapping $H : \mathbf{C}[V]\otimes W \to \mathbf{C}[V\oplus W^*]$ by

$$H(f(x)\otimes w) = f(x)\sum_{i=1}^{m} w_i y_i^* \ ,$$

where $m = \dim(W^*)$.

Theorem 4.2. *$H : \mathbf{C}[V]\otimes W \to \mathbf{C}[V\oplus W^*]$ is a Γ-compatible $\mathbf{C}[V]$-monomorphism, and, therefore, $(\mathbf{C}[V]\otimes W)^\Gamma \cong (\mathbf{C}[V\oplus W^*]^\Gamma)_1$.*

Proof. By construction, it is clear that H is a $\mathbf{C}[V]$-monomorphism so we just need to check that it is Γ-compatible. By linearity of the action of Γ it is sufficient to consider elements of $\mathbf{C}[V]\otimes W$of the form $f(x)\otimes w$. Now,

$$\begin{aligned}
H(\gamma(f(x)\otimes w)) &= H(f\circ\gamma^{-1}(x)\otimes\gamma w) \\
&= f\circ\gamma^{-1}(x)\sum_i(\gamma w)_i y_i^* \\
&= f\circ\gamma^{-1}(x)\sum_i(\sum_j \gamma_{i,j} w_j)y_i^* \\
&= f\circ\gamma^{-1}(x)\sum_j w_j(\sum_i \gamma_{i,j} y_i^*) \\
&= f\circ\gamma^{-1}(x)\sum_j w_j(\gamma^T y^*)_j \\
&= \gamma(f(x)\sum_j w_j y_j^*) \\
&= \gamma(H(f(x)\otimes w)) \ .
\end{aligned}$$

This proves that H is Γ-compatible and since $\mathbf{C}[V]\otimes W \cong (\mathbf{C}[V\oplus W^*])_1$ by H, the conclusion of the theorem follows. □

It is now straightforward to apply the invariant theory results to $\mathbf{C}[V\oplus W^*]^\Gamma$and to look for generators of $(\mathbf{C}[V\oplus W^*]^\Gamma)_1$ as a $\mathbf{C}[V]^\Gamma$-module. Through H, or rather H^{-1}, we will then have the results we want for the equivariants. In the next proposition, we demonstrate how to find a finite set of generators for the equivariants.

Proposition 4.3. *The image of a generating set of invariants for $\mathbf{C}[V\oplus W^*]^\Gamma$ under Π_1*

generate $(\mathbf{C}[V \oplus W^*]^\Gamma)_1$ *as a* $\mathbf{C}[V]^\Gamma$*-module. Let* $n = dim(V)$ *and* $m = dim(W)$*, then* $(\mathbf{C}[V] \otimes W)^\Gamma$ *is generated as a* $\mathbf{C}[V]^\Gamma$*-module by at most* $m \cdot \binom{|\Gamma| + n}{n}$ *homogeneous equivariants whose degree is bounded by* $|\Gamma| - 1$*.*

Proof. The first part of the proposition follows from the grading of $\mathbf{C}[V \oplus W^*]^\Gamma$. By Theorem 2.2, $\mathbf{C}[V \oplus W^*]^\Gamma$is generated by homogeneous invariants whose degrees are bounded by $|\Gamma|$. Generators of $\mathbf{C}[V \oplus W^*]^\Gamma$which lie in $(\mathbf{C}[V \oplus W^*]^\Gamma)_1$ are generated from monomials in $(\mathbf{C}[V \oplus W^*])_1$, thus a count on these monomials with degree bounded by Γ gives the maximum number of generators. Since mapping by H^{-1} lowers the degree by one, generators of the equivariants have degree bounded by $|\Gamma| - 1$. □

This proposition gives us one way to compute a set of generating equivariants. However, a better method will be presented later in this section.

We continue to study the structure of the equivariant module by observing that it is a graded module. $(\mathbf{C}[V] \otimes W)^\Gamma$ is the direct sum of the finite-dimensional vector spaces $(\mathbf{C}[V]_d \otimes W)^\Gamma$, where $(\mathbf{C}[V]_d \otimes W)^\Gamma$ denotes the set of all homogeneous equivariants of degree d. Elements of $(\mathbf{C}[V]_d \otimes W)^\Gamma$ are equivariant vectors whose components are either a homogeneous polynomial of degree d or zero.

Proposition 4.4. $(\mathbf{C}[V]_d \otimes W)^\Gamma$ *is generated by the* $m \cdot \binom{n+d-1}{d}$ *equivariants*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma(H^{-1}(g)) \ ,$$

where g *ranges over all* $m \cdot \binom{n+d-1}{d}$ *monomials in* $(\mathbf{C}[V \oplus W^*])_1$ *of degree* $d+1$*.*

Proof. The kernel of Π_1 contains all monomials of $\mathbf{C}[V \oplus W^*]$which are not linear in the second coordinate. The image under H of any equivariant of degree d is an invariant polynomial in $(\mathbf{C}[V \oplus W^*]^\Gamma)_1$ of degree $d+1$. These invariants are generated by the monomials of degree $d+1$ which are linear in the second coordinate. A count of these monomials completes the proof. □

Note that any basis for the vector space of degree $d+1$ polynomials in $(\mathbf{C}[V \oplus W^*]^\Gamma)_1$ could be substituted for the specific, simple monomial one chosen here.

We have demonstrated that the module of equivariants is finitely generated and have given a bound on the degree of the generators. Our proofs have been constructive, so we may actually compute a set of generating equivariants. It is not necessary to compute a set of fundamental invariants for $\mathbf{C}[V \oplus W^*]^\Gamma$ since we are only interested in a small subset of them. However, if this were desirable, we note that knowing sets of fundamental invariants for $\mathbf{C}[V]^\Gamma$ and $\mathbf{C}[W^*]^\Gamma$ helps considerably. We state this in a proposition whose proof is easy.

Proposition 4.5. *Let* $\{p_1(x), \ldots, p_n(x)\}$ *and* $\{q_1(x), \ldots, q_m(x)\}$ *be sets of primary and secondary invariants, respectively, for* $\mathbf{C}[V]^\Gamma$*. Also, let* $\{\tilde{p}_1(y^*), \ldots, \tilde{p}_r(y^*)\}$ *and* $\{\tilde{q}_1(y^*), \ldots, \tilde{q}_s(y^*)\}$ *be sets of primary and secondary invariants, respectively, for* $\mathbf{C}[W^*]^\Gamma$*. Then,*

$\{p_1(x), \ldots, p_n(x), \tilde{p}_1(y^*), \ldots, \tilde{p}_r(y^*)\}$ is a set of primary invariants for $\mathbf{C}[V \oplus W^*]^\Gamma$. With these primary invariants, the set $\{q_1(x), \ldots, q_m(x), \tilde{q}_1(y^*), \ldots, \tilde{q}_s(y^*)\}$ can be taken as a subset of a set of secondary invariants for $\mathbf{C}[V \oplus W^*]^\Gamma$.

Now we continue to discuss the structure of the module in order to develop the theory in a fashion parallel to our presentation of the invariant theory. The Hilbert series of the graded module $(\mathbf{C}[V] \otimes W)^\Gamma$ is by definition the generating function

$$\Psi_\Gamma(z) = \sum_{d=0}^{\infty} \dim((\mathbf{C}[V]_d \otimes W)^\Gamma) z^d \ .$$

In an equivariant version of Molien's Theorem, we give an explicit expression for the rational function $\Psi(z)$. This can aid in the enumeration of the equivariants.

THEOREM 4.6. (EQUIVARIANT MOLIEN THEOREM) *(Sattinger, 1979, Theorem 5.10) $(\mathbf{C}[V] \otimes W)^\Gamma$ has a Hilbert series given by*

$$\Psi_\Gamma(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{trace(\gamma_W^{-1})}{det(1_\Gamma - z\gamma_V)} \ .$$

Sattinger (1979) gives the equivariant Molien series for a unitary group representation. In that case, $\mathrm{trace}(\gamma^{-1}) = \overline{\mathrm{trace}(\gamma)}$, and our formula reduces to his. In the case of an orthogonal group representation on W, $\mathrm{trace}(\gamma^{-1}) = \mathrm{trace}(\gamma)$.

PROOF. For each $\gamma \in \Gamma$ there is an induced linear transformation $\gamma_{(d)}$ on the vector space $\mathbf{C}[V]_d \otimes W$. Then $\dim((\mathbf{C}[V]_d \otimes W)^\Gamma)$ will be the dimension of the invariant subspace of $\mathbf{C}[V]_d \otimes W$ with respect to the induced group $\{\gamma_{(d)} : \gamma \in \Gamma\}$. Write $\gamma_{(d)} = \gamma^{(d)} \otimes \gamma_W$, where $\gamma^{(d)}$ is the induced linear transformation of γ on $\mathbf{C}[V]_d$.

Now, $\dim\,(\mathbf{C}[V]_d \otimes W)^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{trace}(\gamma_{(d)})$, and

$$\begin{aligned} \mathrm{trace}(\gamma_{(d)}) &= \mathrm{trace}(\gamma^{(d)} \otimes \gamma_W) \\ &= \mathrm{trace}(\gamma^{(d)}) \mathrm{trace}(\gamma_W) \ . \end{aligned}$$

We need the identity, $\sum_{d=0}^{\infty} \mathrm{trace}(\gamma^{(d)}) z^d = 1/\det(1_\Gamma - z\gamma_V^{-1})$ (Sturmfels, 1993, Theorem 2.2.1). Finally, calculate

$$\begin{aligned} \Psi_\Gamma(z) &= \textstyle\sum_{d=0}^{\infty} \dim\,(\mathbf{C}[V]_d \otimes W)^\Gamma z^d \\ &= \textstyle\sum_{d=0}^{\infty} \left(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{trace}(\gamma_{(d)}) \right) z^d \\ &= \textstyle\sum_{d=0}^{\infty} \left(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{trace}(\gamma^{(d)}) \mathrm{trace}(\gamma_W) \right) z^d \\ &= \textstyle\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \mathrm{trace}(\gamma_W) \left(\sum_{d=0}^{\infty} \mathrm{trace}(\gamma^{(d)}) z^d \right) \\ &= \textstyle\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\mathrm{trace}(\gamma_W)}{\det(1_\Gamma - z\gamma_V^{-1})} \\ &= \textstyle\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\mathrm{trace}(\gamma_W^{-1})}{\det(1_\Gamma - z\gamma_V)} \ , \end{aligned}$$

giving the desired result. □

Analogous to the definition of a homogeneous system of parameters for a ring, one can define an h.s.o.p. for modules. A set $\{p_1, \ldots, p_n\}$ of n homogeneous polynomials with positive degree is called an h.s.o.p. of the $\mathbf{C}[V]^\Gamma$-module $(\mathbf{C}[V] \otimes W)^\Gamma$ if $\dim(\mathbf{C}[V]^\Gamma) = n$

and $(\mathbf{C}[V]\otimes W)^\Gamma$ is a finitely generated module over $\mathbf{C}[p_1,\ldots,p_n]$. A module is called Cohen-Macaulay if it is finitely generated as a free module over the ring of any h.s.o.p. We prove in the following proposition that the equivariant mappings of a finite group form a Cohen-Macaulay module.

THEOREM 4.7. *For Γ finite, there exist $n = dim(V)$ homogeneous polynomial invariants $p_1,\ldots,p_n$ such that $(\mathbf{C}[V]\otimes W)^\Gamma$ is finitely generated as a free module over the ring $\mathbf{C}[p_1,\ldots,p_n]$. Then there exist $g_1,\ldots,g_s \in (\mathbf{C}[V]\otimes W)^\Gamma$ with each g_i homogeneous, such that*

$$(\mathbf{C}[V]\otimes W)^\Gamma = \bigoplus_{i=1}^{s} g_i\mathbf{C}[p_1,\ldots,p_n] \ ,$$

i.e., $\{p_1,\ldots,p_n\}$ is an h.s.o.p., and $(\mathbf{C}[V]\otimes W)^\Gamma$ is a Cohen-Macaulay module.

This theorem follows directly from results presented in Jaric *et al.* (1984) or Stanley (1983). The proof we give follows Sturmfels's (1993, Theorem 2.3.5)) proof showing that the ring of invariants is Cohen-Macaulay. The differences arise from working with modules as opposed to rings. Some of the steps are given in more detail by Sturmfels (1993).

PROOF. A standard result in commutative algebra says that a module with an h.s.o.p. is Cohen-Macaulay if it is a finitely generated and free $\mathbf{C}[p_1,\ldots,p_n]$-module for some, and equivalently, every, h.s.o.p. $\{p_1,\ldots,p_n\}$ (Sturmfels, 1993, Theorem 2.3.1). Consequently, we simply need to give one h.s.o.p. from which $(\mathbf{C}[V]\otimes W)^\Gamma$ is finitely and freely generated.

Let $\{p_1,\ldots,p_n\}$ be an h.s.o.p. for $\mathbf{C}[V]^\Gamma$, in other words, a set of primary invariants. We first prove that $\{p_1,\ldots,p_n\}$ is also an h.s.o.p. for $\mathbf{C}[V]\otimes W$. Since the ring $\mathbf{C}[V]$ is integral over $\mathbf{C}[V]^\Gamma$, $\{p_1,\ldots,p_n\}$ is an h.s.o.p. for $\mathbf{C}[V]$. Also, $\mathbf{C}[V]\otimes W$ is a finitely generated free $\mathbf{C}[V]$-module (consider the trivial h.s.o.p. $\{x_1,\ldots,x_n\}$), so $\mathbf{C}[V]\otimes W$ is Cohen-Macaulay and $\{p_1,\ldots,p_n\}$ is also an h.s.o.p. of $\mathbf{C}[V]\otimes W$.

Define the operator $\mathcal{R}$ on the space of polynomial maps $\mathbf{C}[V]\otimes W$ by

$$\mathcal{R}(g) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma(g) \ .$$

It is easy to verify that $\mathcal{R}$ has the following three properties:

1 $\mathcal{R}$ is $\mathbf{C}$-linear.
2 $\mathcal{R}$ restricts to the identity on $(\mathbf{C}[V]\otimes W)^\Gamma$.
3 $\mathcal{R}$ is a $\mathbf{C}[V]^\Gamma$-module homomorphism.

Define $\mathcal{U} = \{g \in \mathbf{C}[V]\otimes W : \mathcal{R}(g) = 0\} = \{g - \mathcal{R}(g) : g \in \mathbf{C}[V]\otimes W\}$. Then the space of polynomial maps is a direct sum of two $\mathbf{C}[V]^\Gamma$-modules,

$$\mathbf{C}[V]\otimes W = (\mathbf{C}[V]\otimes W)^\Gamma \oplus \mathcal{U} \ .$$

From this module decomposition, we get a decomposition of the finite-dimensional vector space:

$$\mathbf{C}[V]\otimes W \ / \ \sum_{i=1}^{n} p_i(\mathbf{C}[V]\otimes W) \ = \ \left[(\mathbf{C}[V]\otimes W)^\Gamma \ / \ \sum_{i=1}^{n} p_i(\mathbf{C}[V]\otimes W)^\Gamma\right] \oplus \left[\mathcal{U} \ / \ \sum_{i=1}^{n} p_i\mathcal{U}\right]$$

$$g + \sum_{i=1}^{n} p_i h_i = \left[\mathcal{R}(g) + \sum_{i=1}^{n} p_i \mathcal{R}(h_i)\right] + \left[(g - \mathcal{R}(g)) + \sum_{i=1}^{n} p_i (h_i - \mathcal{R}(h_i))\right],$$

with $h_i \in \mathbf{C}[V]\otimes W$. By the Cohen-Macaulay property of $\mathbf{C}[V]\otimes W$, we can choose a homogeneous $\mathbf{C}$-basis $\{\bar{g}_1, \ldots, \bar{g}_s, \bar{g}_{s+1}, \ldots, \bar{g}_t\}$ for $\mathbf{C}[V]\otimes W \,/\, \sum_{i=1}^{n} p_i(\mathbf{C}[V]\otimes W)$ such that $\{\bar{g}_1, \ldots, \bar{g}_s\}$ is a $\mathbf{C}$-basis for $(\mathbf{C}[V]\otimes W)^\Gamma / \sum_{i=1}^{n} p_i(\mathbf{C}[V]\otimes W)^\Gamma$ and $\{\bar{g}_{s+1}, \ldots, \bar{g}_t\}$ is a $\mathbf{C}$-basis for $\mathcal{U} / \sum_{i=1}^{n} p_i \mathcal{U}$. We lift $\{\bar{g}_1, \ldots, \bar{g}_s\}$ to homogeneous elements $\{g_1, \ldots, g_s\}$ of $(\mathbf{C}[V]\otimes W)^\Gamma$and lift $\{\bar{g}_{s+1}, \ldots, \bar{g}_t\}$ to homogeneous elements of $\mathcal{U}$.

Thus,

$$\begin{aligned}
\mathbf{C}[V]\otimes W &= (\bigoplus_{i=1}^{t} g_i \mathbf{C}) \oplus (< p_1, \ldots, p_n > \otimes W) \\
&= (\bigoplus_{i=1}^{t} g_i \mathbf{C}) \oplus \left(\bigoplus_{(i_1,\ldots,i_n)\in \mathbf{N}^n \setminus \{0\}} \bigoplus_{i=1}^{t} g_i p_1^{i_1} \cdots p_n^{i_n} \mathbf{C} \right) \\
&= \bigoplus_{i=1}^{t} g_i \mathbf{C}[p_1, \ldots, p_n] \,.
\end{aligned}$$

Consequently,

$$(\mathbf{C}[V]\otimes W)^\Gamma = \bigoplus_{i=1}^{s} g_i \mathbf{C}[p_1, \ldots, p_n] \,,$$

demonstrating that $(\mathbf{C}[V]\otimes W)^\Gamma$ is a finitely and freely generated $\mathbf{C}[p_1, \ldots, p_n]$-module. □

The proof of this proposition not only shows that the module of equivariants is Cohen-Macaulay, but also shows that the set of primary invariants is an h.s.o.p. This is very convenient because we already know how to compute primary invariants. We will call the g_i a set of *fundamental equivariants* for the module $(\mathbf{C}[V]\otimes W)^\Gamma$. Just as in the invariant case, we can enumerate the number and degrees of the fundamental equivariants.

Proposition 4.8. *For Γ finite and $p_1, \ldots, p_n$ a set of primary invariants for $\mathbf{C}[V]^\Gamma$ with degrees $d_1, \ldots, d_n$, the number of fundamental equivariants s is given by*

$$s = m \frac{d_1 \cdots d_n}{|\Gamma|} \,,$$

where $m = dim(W)$. The degrees with multiplicity of the fundamental equivariants are the exponents of the generating function

$$\Psi_\Gamma(z)/H(R_p, z) = z^{e_1} + \cdots + z^{e_s} \,,$$

where $H(R_p, z)$ is the Hilbert series of $R_p = \mathbf{C}[p_1, \ldots, p_n]$ (see Lemma 2.5).

Proof. The proof of this proposition is essentially identical to that of Theorem 2.8, which may be found in Sturmfels (1993, Proposition 2.3.6). □

The formulation of a new algorithm to compute the fundamental equivariants follows from these results. The idea is to first compute the primary invariants. Then search

through a basis of equivariants at each degree (given by Proposition 4.4). It is only a linear algebra problem at each degree to see whether a new equivariant should be added to the set of fundamental equivariants. The information gleaned from applying Proposition 4.8 can be used to help with the search and determine at which degree to stop. The outline of a simple implementation is given in the following algorithm.

Fundamental_Equivariants($\Gamma, p(x)$)
INPUT
Γ is a finite group with actions on V and W.
$p(x) = (p_1(x), \ldots, p_n(x))$, where the p_i are a set of primary invariants for $\mathbf{C}[V]^\Gamma$.
OUTPUT
A set of fundamental equivariants.

$s := m \cdot (\prod_{i=1}^n \deg(p_i)) / |\Gamma|$ % the number of fundamental equivariants
$t := 0$ % the number of fundamental equivariants computed
$d := 1$ % the degree we are searching
WHILE $t < s$ DO
$G :=$ vector space basis of $(\mathbf{C}[V]_d \otimes W)^\Gamma$ % using Proposition 4.4
FOREACH $g \in G$ DO
IF $g \notin \bigoplus_{i=1}^t g_t \mathbf{C}[p_1, \ldots, p_n]$ THEN $t := t + 1,\ g_t := g$
$d := d + 1$
RETURN $\{g_1, \ldots, g_s\}$

Given a module decomposition as presented in Theorem 4.7, we may write any equivariant $f(x)$ uniquely in the form

$$f(x) = \sum_{i=1}^{s} h_i(p_1(x), \ldots, p_n(x)) g_i(x) \ ,$$

where the g_i are a fundamental equivariants and the h_i are polynomials in the primary invariants. Expanding the $h_i(p)$ formally in Taylor series and then truncating terms, one can write a normal form for a Γ-equivariant vector field to any specified degree.

5. Solution Using Invariants

Our goal is to find the equilibrium points of an equivariant polynomial vector field, $x' = f(x)$. Although this work was motivated by problems in dynamical systems, so that we continue to view our equivariant equations as the defining equations of a differential equation, this analysis can be applied to find the zeros of any system of equivariant polynomial equations. Thus we consider $f \in (\mathbf{C}[V] \otimes W)^\Gamma$, with Γ finite.

First we present three definitions from equivariant bifurcation theory.

Definition 5.1. *The Γ-orbit of a point $x \in V$ is defined by*

$$\Gamma x = \{\gamma x \ : \ \gamma \in \Gamma\}.$$

The set of Γ-orbits is called the orbit space and is denoted by V/Γ.

Definition 5.2. *The isotropy subgroup of a point $x \in V$ is defined by*

$$\Sigma_x = \{\gamma \in \Gamma \ : \ \gamma x = x\}.$$

The isotropy subgroup of a point is the subgroup of Γ which fixes that point. Points in the Γ-orbit of x have conjugate isotropy subgroups ($\Sigma_{\gamma x} = \gamma\Sigma_x\gamma^{-1}$), so we will be interested in conjugacy classes of isotropy subgroups, $[\Sigma_x]$. The set of conjugacy classes of isotropy subgroups is finite and can be given an ordering based on inclusion. This ordering is called the isotropy lattice of the group Γ acting on V.

Definition 5.3. *The fixed point subspace of an isotropy subgroup $\Sigma \subset \Gamma$ is defined by*

$$Fix(\Sigma) = \{x \in V \ : \ \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

The fixed point subspace of an isotropy subgroup Σ_x is the linear subspace of V which contains all points with the same symmetry as x. Fixed point subspaces are dynamically important because they are invariant under the flow.

We will hunt for zeros in each fixed point subspace. For all isotropy subgroup conjugacy classes except $[\{1_\Gamma\}]$, the fixed point subspace has dimension less than n. Hence, the problem of finding zeros lying in the subspace is easier: The vector field restricted to that subspace may be written and the problem reformulated. Consequently, to display the power of our technique, we shall focus on solving for the asymmetric zeros, those which lie in $\mathrm{Fix}(\{1_\Gamma\})$ but in no other fixed point subspace.

The problem may be expressed as finding the zeros of the components of the equivariant $f = (f_1, \ldots, f_n)$,

$$\begin{aligned} \mathcal{Z} &\equiv Z(\{f_1, \ldots, f_n\}) \\ &= \{x \in V | f_i(x) = 0, 1 \leq i \leq n\} \ . \end{aligned}$$

This problem can be reduced to the problem of finding the zeros of a univariate polynomial by using Gröbner bases, as introduced in Section 3. Unfortunately, this method is often not practical for polynomials of much complexity. Additionally, this technique does not take advantage of any special structure which the equations may have, namely symmetry.

Assume that we are working with a polynomial equivariant in n variables. If $x \in \mathcal{Z}$ then each point in the Γ-orbit of x is in $\mathcal{Z}$. This is why it is natural to work in the orbit space. This also motivates the introduction of polynomial invariants of the symmetry group since these are constant on each orbit. We wish to find a change of coordinates to a new set of equations whose zeros give us information on the zero set of the original equations. Each solution of the new equations should correspond to an orbit of solutions for the original equations.

We now require that Γ act orthogonally or unitarily on W with respect to an innter produce $< \cdot, \cdot >_W$ depending on whether the action on W is real linear or complex linear. The following lemma shows how to convert to sets of equations in the invariants.

Lemma 5.4. *(Jaric et al., 1984, p. 7) Let $f = (f_1, \ldots, f_n) \in (\mathbf{C}[V] \otimes W)^\Gamma$ and $\Gamma_W \subset O(W)$ (or $\Gamma_W \subset U(W)$), and let the set $\{g_1, \ldots, g_l\}$ generate $(\mathbf{C}[V] \otimes W)^\Gamma$. Then*

$$Z(\{f_1, \ldots, f_n\}) = Z(\{< g_1, f, >_W, \ldots, < g_l, f >_W\}) \ .$$

Proof. It is clear that $Z(\{f_1, \ldots, f_n\}) \subset Z(\{< g_1, f >_W, \ldots, < g_l, f >_W\})$. The equivariant f may be expanded in terms of the g_i, $f = \sum_{i=1}^{l} h_i g_i$, where $\{h_i\}$ are invariants. So $< f, f >_W = \sum_{i=1}^{l} h_i < g_i, f >_W$. If $x \in Z(\{< g_1, f >_W, \ldots, < g_l, f >_W\})$, then $< f(x), f(x) >_W = 0$. We conclude that $f_i(x) = 0$ and $x \in Z(\{f_1, \ldots, f_n\})$. $\square$

Each $< g_i, f >_W$ from the preceding lemma is an invariant polynomial and can be written in terms of the fundamental invariants of Γ. Writing an invariant in standard form is a linear algebra problem as described at the end of Section 2. Denote by ϕ_i the polynomials in standard form in the invariants such that

$$\phi_i(p_1(x), \ldots, p_n(x), q_1(x), \ldots, q_m(x)) =< g_i(x), f(x) >_W \quad .$$

We compute $Z(\{\phi_1, \ldots, \phi_l\})$, where we consider ϕ_i to be a function in the variables p_i and q_i, in order to give expressions for the invariants in terms of the parameters of the problem. This consists of a Gröbner basis calculation which includes the relationships, or syzygies, between the fundamental invariants. We will assume that we already know the values of the invariants at the symmetric zeros, so that we can factor out these solutions. The final computation requires the factorization of a univariate polynomial with degree equal to the number of group orbits of asymmetric solutions. This will result in expressions for the values of the invariants at zeros of the original problem in terms of the parameters of the original problem. These ideas are outlined in the following algorithm.

Invariant_Zeros($f(x), p(x), q(x), g(x)$)
INPUT
  $f(x)$ is the equivariant vector field
  $p(x) = (p_1(x), \ldots, p_n(x))$, where $p_i(x)$ are primary invariants.
  $q(x) = (q_1(x), \ldots, q_m(x))$, where $q_i(x)$ are secondary invariants.
  $g(x) = (g_1(x), \ldots, g_l(x))$, where $g_i(x)$ are fundamental equivariants.
OUTPUT
  The values of the invariants at zeros of $f(x)$.

FOR $i := 1$ TO l
  $\phi_i(p, q)$:=Standard_Invariant_Form($< g_i(x), f(x) >_W, p(x), q(x)$)
FOR $i := 1$ TO m
  FOR $j := 1$ TO i
    $Q_{i,j}(p, q)$:=Standard_Invariant_Form($q_i(x)q_j(x), p(x), q(x)$)
$S := \{\phi_i(p, q)\}_i \cup \{Q_{i,j}(p, q) - q_i q_j\}_{i \geq j}$
RETURN Gröbner_Solve($S, [p, q]$)

SUBROUTINE Standard_Invariant_Form($f(x), p(x), q(x)$)
INPUT:
  $f(x)$ is an invariant polynomial
  $p(x) = (p_1(x), \ldots, p_n(x))$, where $p_i(x)$ are primary invariants.
  $q(x) = (q_1(x), \ldots, q_m(x))$, where $q_i(x)$ are secondary invariants.
OUPTPUT
  The polynomial $f(x)$ written in terms of the p_i and q_i and linear in q.

SUBROUTINE Gröbner_Solve($\mathcal{I}$, vars)
INPUT:
  $\mathcal{I}$ is a set of polynomials.
  vars is a list of the variables in $\mathcal{I}$.
OUTPUT:

The zero set of the ideal found by computing elimination ideals using Gröbner bases.

To understand what has been computed, it is useful to consider the map $\Omega : V \to \mathbf{C}^{n+m}$ defined by

$$\Omega : (x) \mapsto (p_1(x), \ldots, p_n(x), q_1(x), \ldots, q_m(x))$$

and usually called the Hilbert map. The image $\Omega(V)$ is an embedding of V/Γ as an algebraic variety into $\mathbf{C}^{n+m}$. The relations (syzygies) between the fundamental invariants are the equations defining the variety. The previous algorithm produces the points on this variety which correspond to Γ-orbits of zeros of the equivariant equation. The actual zero may be computed by applying the Gröbner solve procedure to the set of equations

$$\{p_1 - p_1(x), \ldots, p_n - p_n(x), q_1 - q_1(x), \ldots, q_m - q_m(x)\} \ .$$

The computed values of the p_i and q_i in terms of the parameters for the orbit of interest must be substituted into the resulting expressions for x.

The main idea of this section, may be summarized as follows: Taking advantage of the structure of a hard problem, we break the problem into two smaller problems, each one easier to solve than the original problem. We apply the same Gröbner basis computational techniques twice, just as we could have applied it to solving for the zeros directly. However, solely due to the complexity of the problems we would like to solve, the solution of the two easier problems is sometimes possible when solution of the harder problem appears not to be. This is exhibited in the example in Section 8.

6. Linearization at Zeros

In dynamical systems theory, we are not interested in just the equilibrium points, but we would like to know information about the linearized vector field at these points in order to understand the local flow. Frequently we are interested in parameter values for which the linearization is in some sense degenerate. Degeneracy conditions are typically formulated in terms of conditions on the eigenvalues. One of the simplest is the existence of a zero eigenvalue, and this may indicate a change of stability for the equilibrium point. For families of vector fields, the set of parameter values which result in degenerate linearizations are part of the bifurcation set and are frequently of interest. We will assume that the conditions for degeneracy can be written in terms of the coefficients of the characteristic polynomial for the linearized vector field. The case of a zero eigenvalue corresponds to a zero determinant.

Lemma 6.1. *Let $f \in (\mathbf{C}[V] \otimes V)^\Gamma$, then the coefficients of the characteristic polynomial of Df are invariant polynomials.*

Proof. The characteristic polynomial is given by $\det(Df|_x - \lambda 1)$ and is a polynomial in λ whose coefficients are polynomials in x. Since $D(f \circ \gamma)|_x = Df|_{\gamma x}\gamma$, we compute

$$\begin{aligned} Df|_{\gamma x} &= D(f \circ \gamma)|_x \gamma^{-1} \\ &= D(\gamma f)|_x \gamma^{-1} \\ &= \gamma Df|_x \gamma^{-1} \ . \end{aligned}$$

The characteristic equation of a matrix is preserved under similarity transformations of the matrix, so the coefficients of the characteristic equation are invariant polynomials. □

We may use the expressions for the coefficients of the characteristic polynomial to write the degeneracy conditions in terms of the invariants. Once the values of the invariants at the equilibrium points are known, the degeneracy conditions may be expressed in terms of the parameters of the problem. We will have a set of constraints for each orbit of solutions, each of which defines a component of the bifurcation set. It is a natural consequence of the symmetry that all the equilibria in the same group orbit will bifurcate simultaneously. These calculations have been performed on the examples in Sections 7 and 8.

7. The Simple Example

A simple example will illustrate the theory and algorithms discussed in this paper. Let $\Gamma = Z_4$ acting on $\mathbf{R}^2$ with the representation defined by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma \ .$$

Theorem 2.4 gives the Hilbert series for the invariant ring:

$$\begin{aligned} \Phi_\Gamma(z) &= \tfrac{1}{4}\left(\tfrac{1}{(1-z)^2} + \tfrac{2}{1+z^2} + \tfrac{1}{(1+z)^2}\right) \\ &= 1 + z^2 + 3z^4 + 3z^6 + 5z^8 + 5z^{10} + O(z^{12}) \ . \end{aligned}$$

Since $|\Gamma| = 4$, sets of homogeneous generating invariants have elements with degree at most equal to 4. Averaging monomials according to Lemma 2.3 gives basis elements for the vector spaces which grade $\mathbf{C}[V]^\Gamma$:

$$\begin{aligned} \mathbf{C}[V]^\Gamma_0 &= \{1\} \\ \mathbf{C}[V]^\Gamma_2 &= \{x^2 + y^2\} \\ \mathbf{C}[V]^\Gamma_4 &= \{x^4 + y^4, x^2y^2, x^3y - xy^3\} \ . \end{aligned}$$

Even without looking at the Hilbert series, we know there are no invariants with odd degree since $-1_\Gamma \in \Gamma$.

The two invariants $x^2 + y^2$ and $x^4 + y^4$ are algebraically independent and have no nontrivial common zeros, so they form an h.s.o.p. for the invariant ring. Let $p_1 = x^2 + y^2$ and $p_2 = x^4 + y^4$. Lemma 2.5 and Theorem 2.8 give the degrees of the secondary invariants by computing,

$$\begin{aligned} \Phi_\Gamma(z)/H(R_p, z) &= \Phi_\Gamma(z)/\left(\tfrac{1}{(1-z^2)(1-z^4)}\right) \\ &= 1 + z^4 \ . \end{aligned}$$

This reveals that there are two secondary invariants, one with degree 0 and the other with degree 4. The degree 0 secondary invariant is $q_1 = 1$ and, since $x^2y^2 = (p_1^2 - p_2)/2 \in \mathbf{C}[p_1, p_2]$, let $q_2 = x^3y - xy^3$. Consequently, any element of $\mathbf{C}[V]^\Gamma$ may be expressed uniquely in the following form:

$$1 \cdot P_1(x^2 + y^2, x^4 + y^4) + (x^3y - xy^3) \cdot P_2(x^2 + y^2, x^4 + y^4),$$

where P_1 and P_2 are polynomials.

We now turn our attention to computing generating and fundamental equivariants

for $(\mathbf{C}[V]\otimes V)^\Gamma$. Proposition 4.6 gives the Hilbert series for the equivariants,

$$\begin{aligned}\Psi_\Gamma(z) &= \tfrac{1}{4}\left(\tfrac{2}{(1-z)^2}+\tfrac{-2}{(1+z)^2}\right)\\ &= 2z+4z^3+6z^5+8z^7+10z^9+O(z^{11})\ .\end{aligned}$$

Proposition 4.3 indicates that the generating equivariants will have degree at most equal to $|\Gamma|-1=3$, and Proposition 4.4 gives a method to generate a vector space basis of equivariants at each degree.

$$\begin{aligned}(\mathbf{C}[V]_1\otimes V)^\Gamma &= \left\{\begin{pmatrix} x\\ y\end{pmatrix},\begin{pmatrix} y\\ -x\end{pmatrix}\right\}\\ (\mathbf{C}[V]_3\otimes V)^\Gamma &= \left\{\begin{pmatrix} x^3\\ y^3\end{pmatrix},\begin{pmatrix} x^2y\\ -xy^2\end{pmatrix},\begin{pmatrix} xy^2\\ x^2y\end{pmatrix},\begin{pmatrix} y^3\\ -x^3\end{pmatrix}\right\}\ .\end{aligned}$$

We add both degree 1 equivariants to the set of generating equivariants, and then consider the degree 3 equivariants. If we add $(x^3,y^3)^T$ and $(y^3,-x^3)^T$, then the remaining two equivariants in $(\mathbf{C}[V]_3\otimes V)^\Gamma$ can be generated:

$$\begin{aligned}\begin{pmatrix} x^2y\\ -xy^2\end{pmatrix} &= (x^2+y^2)\begin{pmatrix} y\\ -x\end{pmatrix}-\begin{pmatrix} y^3\\ -x^3\end{pmatrix}\\ \begin{pmatrix} xy^2\\ x^2y\end{pmatrix} &= (x^2+y^2)\begin{pmatrix} x\\ y\end{pmatrix}-\begin{pmatrix} x^3\\ y^3\end{pmatrix}\ .\end{aligned}\tag{7.1}$$

Therefore a set of generating equivariants is

$$\left\{\begin{pmatrix} x\\ y\end{pmatrix},\begin{pmatrix} y\\ -x\end{pmatrix},\begin{pmatrix} x^3\\ y^3\end{pmatrix},\begin{pmatrix} y^3\\ -x^3\end{pmatrix}\right\}\ .$$

This is also a Cohen-Macaulay basis since the relations given in Equation 7.1 do not use the secondary invariants. We verify that no more equivariants are necessary for this basis by computing the degrees of the fundamental equivariants according to Proposition 4.8:

$$\begin{aligned}\Psi_\Gamma(z)/H(R_p,z) &= \Psi_\Gamma(z)/\left(\tfrac{1}{(1-z^2)(1-z^4)}\right)\\ &= 2z+2z^3\ .\end{aligned}$$

This indicates that a fundamental set will contain 2 degree one equivariants and 2 degree three equivariants which is what our set contains. Consequently, any equivariant may be written uniquely in the following form:

$$H_1(x^2+y^2,x^4+y^4)\cdot\begin{pmatrix} x\\ y\end{pmatrix}+H_2(x^2+y^2,x^4+y^4)\cdot\begin{pmatrix} y\\ -x\end{pmatrix}+$$

$$H_3(x^2+y^2,x^4+y^4)\cdot\begin{pmatrix} x^3\\ y^3\end{pmatrix}+H_4(x^2+y^2,x^4+y^4)\cdot\begin{pmatrix} y^3\\ -x^3\end{pmatrix}\ ,$$

where the H_i are arbitrary polynomials in two variables.

To write a normal form for a Γ-equivariant map to any specific degree we simply take a Taylor series of the above expression using arbitrary parameters and truncate at the desired degree. The degree three normal form may be written as

$$(a+b(x^2+y^2))\begin{pmatrix} x\\ y\end{pmatrix}+(c+d(x^2+y^2))\begin{pmatrix} y\\ -x\end{pmatrix}+e\begin{pmatrix} x^3\\ y^3\end{pmatrix}+f\begin{pmatrix} y^3\\ -x^3\end{pmatrix}\ .\tag{7.2}$$

If we consider these equations to be the defining equations of a Γ-equivariant vector field

on the plane, then zeros of these equations correspond to equilibrium points. We will apply our analysis to Equation 7.2 to examine its zero set.

Note that the action of Γ on R^2 has only the trivial fixed point subspaces, the origin and $V = R^2$ itself. The origin is the trivial equilibrium. Other equilibria will belong to an orbit of $|\Gamma| = 4$ points. By Bezout's theorem (Shafarevich, 1977, p. 198), we expect a total of nine isolated zeros — since nine is the product of the degrees of the equations — so there will be two nontrivial orbits of zeros. Each orbit of solutions will have a single solution in the invariant problem, so we will be reducing to a problem with three solutions, one of which is trivial. This contrasts with solving the unreduced problem which, if we could reduce to solving a univariate polynomial would have nine roots. Admittedly, one of the roots will be zero, and the others would occur in pairs with opposite signs, so we would need to solve only a quartic. But this is only the simple example!

Let $F(x, y)$ denote the cubic normal form given in Equation 7.2. We write $< F, g_i >$ for each i in terms of the fundamental invariants to find:

$$\begin{array}{rcl} < F, g_1 > & = & ap_1 + bp_1^2 + ep_2 + fq_2 \\ < F, g_2 > & = & cp_1 + dp_1^2 + fp_2 - eq_2 \\ < F, g_3 > & = & ap_2 + (b + 3/2e)p_1p_2 - 1/2ep_1^3 - (c + dp_1)q_2 \\ < F, g_4 > & = & cp_2 + (d + 3/2f)p_1p_2 - 1/2fp_1^3 + (a + bp_1)q_2 \ . \end{array} \tag{7.3}$$

We also compute the relations between the products of the q_i. For this problem there is a single nontrivial secondary invariant, so we just need to write q_2^2 in standard form:

$$q_2^2 = -(p_1^2 - p_2)(1/2p_1^2 - p_2) \tag{7.4}$$

Using an elimination order, compute a Gröbner basis of the ideal defined by Equations 7.3 and the relation from Equation 7.4 using an elimination order. With Macaulay, we find the following basis of 10 elements:

$$\begin{array}{l} p_1a^2 + 2p_1^2ab + p_1^3b^2 + p_1c^2 + 2p_1^2cd + p_1^3d^2 + 3/2p_1^2ae + 3/2p_1^3be + 1/2p_1^3e^2 + \\ \quad 3/2p_1^2cf + 3/2p_1^3df + 1/2p_1^3f^2 \\ p_2e^2 + p_2f^2 + p_1ae + p_1^2be + p_1cf + p_1^2df \\ p_2ce + p_2p_1de + p_2af + p_2p_1bf + 3/2p_2p_1ef + p_1ac + p_1^2bc + p_1^2ad + p_1^3bd - \\ \quad 1/2p_1^3ef \\ p_2ae + p_2p_1be - p_2cf - p_2p_1df - 3/2p_2p_1f^2 - p_1c^2 - 2p_1^2cd - p_1^3d^2 - 3/2p_1^2ae - \\ \quad 3/2p_1^3be - 1/2p_1^3e^2 - 3/2p_1^2cf - 3/2p_1^3df \\ p_2a^2 + 2p_2p_1ab + p_2p_1^2b^2 + p_2c^2 + 2p_2p_1cd + p_2p_1^2d^2 + 3p_2p_1cf + 3p_2p_1^2df + \\ \quad 9/4p_2p_1^2f^2 + 3/2p_1^2c^2 + 3p_1^3cd + 3/2p_1^4d^2 + 7/4p_1^3ae + 7/4p_1^4be + \\ \quad 3/4p_1^4e^2 + 7/4p_1^3cf + 7/4p_1^4df \\ q_2f + p_2e + p_1a + p_1^2b \\ q_2e - p_2f - p_1c - p_1^2d \\ q_2c + q_2p_1d - p_2a - p_2p_1b - 3/2p_2p_1e + 1/2p_1^3e \\ q_2a + q_2p_1b + p_2c + p_2p_1d + 3/2p_2p_1f - 1/2p_1^3f \\ q_2^2 + p_2^2 - 3/2p_2p_1^2 + 1/2p_1^4 \ . \end{array}$$

And now we see the beauty of the Gröbner basis calculations. The first equation is a cubic polynomial in p_1. Factor this equation to solve for all possible values for p_1. Then substitute these values for p_1 into the second equation to solve for p_2. Note that the second equation is linear in p_2. Finally, take one of the equations which is linear in q_2, the sixth for example, and solve it. These calculations are messy, but they are easy to do

with computer algebra. We get the solutions:

$$p_1 = 0 \ , \ \ p_2 = 0 \ , \ \ q_2 = 0 \ ,$$

and

$$p_1 = \frac{-4(ab+cd) - 3(ae+cf) \pm \sqrt{T1}}{2((2b+e)(b+e) + (2d+f)(d+f))} \ ,$$

$$p_2 = -p_1 \frac{(be+df)p_1 + (ae+cf)}{e^2+f^2} \ ,$$

$$q_2 = \frac{1}{e}(cp_1 + dp_1^2 + fp_2) \ ,$$

with

$$T1 = (ae+cf)^2 - 8(2(ad-bc) + (af-ec))(ad-bc+af-ec) \ .$$

To solve for x and y, take $p_1 = x^2+y^2$, $p_2 = x^4+y^4$ and compute:

$$x = s_1 \sqrt{(p_1 + s_3\sqrt{2p_2 - p_1^2})/2} \ ,$$

$$y = s_2 \sqrt{(p_1 - s_3\sqrt{2p_2 - p_1^2})/2} \ ,$$

where $s_1, s_2, s_3 \in \{-1, 1\}$. Note that there are eight solutions and we only expect an orbit of four. This is because we have not yet considered q_2. If we substitute the expressions for x and y into $q_2 = x^3y - xy^3$, then we find a constraint on the s_i which reduces the number of solutions to four.

Finally, we will use the invariants to find constraints on the parameters for degenerate linearizations. The determinant and trace of the Jacobian matrix of the vector field are computed and written in terms of the invariants as:

$$\begin{aligned} \det(Df) &= a^2 + c^2 + (2(ab+cd) + c(ae+cf))p_1 + \\ &\quad (b^2 + d^2 + 3(be+df) + 9/2(e^2+f^2))p_1^2 - 9/2(e^2+f^2)p_2 \ , \\ \mathrm{trace}(Df) &= -(2b+3e)p_1 - 2a \ . \end{aligned}$$

This implies that (using the expression for p_2 at a zero)

$$p_1 = \frac{-(15/2(cf+ae) + 2(ab+cd)) \pm \sqrt{T2}}{2(b^2+d^2+15/2(ae+df) + 9/2(e^2+f^2))}$$

with

$$T2 = (15/2(cf+ae) + 2(ab+cd))^2 - 4(a^2+c^2)(b^2+d^2+15/2(be+df) + 9/2(e^2+f^2))$$

in order for the determinant to equal zero, and

$$p_1 = -2a/(2b+3e)$$

in order for the trace to equal zero. Equating one or both of these expressions for p_1 with the expression that we already have gives additional constraints on the parameters which result in degenerate linearizations at the equilibrium points.

8. The Motivating Example

The example presented in this section is much more complicated than the previous one and was the motivating problem for this work. Let $\Gamma = (Z_4 \cdot Z_2^4)' \subset W(B_4)$ act on $\mathbf{R}^4$ with the group action generated by

$$p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ and } r = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Applying Molien's Theorem 2.4 gives the Hilbert series:

$$\begin{aligned} \Phi_\Gamma(z) &= \tfrac{1}{32}\left(\tfrac{16}{1+z^4} + \tfrac{1}{(1-z)^4} + \tfrac{1}{(1+z)^4} + \tfrac{10}{(1-z^2)^2} + \tfrac{4}{(1+z^2)^2}\right) \\ &= 1 + z^2 + 3z^4 + 6z^6 + 13z^8 + 19z^{10} + 31z^{12} + 44z^{14} + 65z^{16} + \\ &\quad 85z^{18} + 115z^{20} + 146z^{22} + 189z^{24} + 231z^{26} + 287z^{28} + 344z^{30} + \\ &\quad 417z^{32} + O(z^{34}) . \end{aligned}$$

Using Noether's Lemma 2.3 we compute the basis elements of the following vector spaces:

$$\begin{aligned} \mathbf{C}[V]_0^\Gamma &= \{1\} \\ \mathbf{C}[V]_2^\Gamma &= \{x^2 + y^2 + z^2 + w^2\} \\ \mathbf{C}[V]_4^\Gamma &= \{x^4 + y^4 + z^4 + w^4,\ x^2z^2 + y^2w^2,\ x^2y^2 + y^2z^2 + z^2w^2 + w^2x^2\} \\ \mathbf{C}[V]_6^\Gamma &= \{x^6 + y^6 + z^6 + w^6,\ x^4y^2 + y^4z^2 + z^4w^2 + w^4x^2, \\ &\quad x^3yzw - y^3zwx + z^3wxy - w^3xyz,\ x^2y^4 + y^2z^4 + z^2w^4 + w^2x^4, \\ &\quad x^4z^2 + y^4w^2 +^4 x^2 + w^4y^2,\ x^2y^2z^2 + y^2z^2w^2 + z^2w^2x^2 + w^2x^2y^2\} \\ \mathbf{C}[V]_8^\Gamma &= \{x^8 + y^8 + z^8 + w^8,\ z^4x^2w^2 + z^2x^4y^2 + z^2y^4w^2 + x^2y^2w^4, \\ &\quad z^4y^2w^2 + z^2x^2y^4 + z^2x^2w^4 + x^4y^2w^2,\ z^3x^3yw - zxy^3w^3, \\ &\quad z^4x^2y^2 + z^2x^4w^2 + z^2y^2w^4 + x^2y^4w^2,\ z^2x^2y^2w^2, \\ &\quad z^4x^4 + y^4w^4,\ z^5xyw + zx^5yw - zxy^5w - zxyw^5, \\ &\quad z^4y^4 + z^4w^4 + x^4y^4 + x^4w^4,\ z^6x^2 + z^2x^6 + y^6w^2 + y^2w^6, \\ &\quad z^6w^2 + z^2y^6 + x^6y^2 + x^2w^6,\ z^6y^2 + z^2w^6 + x^6w^2 + x^2y^6, \\ &\quad z^3xy^3w - z^3xyw^3 - zx^3y^3w + zx^3yw^3\} . \end{aligned}$$

From the above, select four algebraically independent invariants with no nontrivial common zeros to be the primary invariants: Let $p_1 = x^2 + y^2 + z^2 + w^2$, $p_2 = x^4 + y^4 + z^4 + w^4$, $p_3 = x^6 + y^6 + z^6 + w^6$, and $p_4 = x^8 + y^8 + z^8 + w^8$. We note that the choice of primary invariants is quite arbitrary. Sets with lower degrees will produce a smaller number of secondary invariants and fundamental equivariants.

Computing $H(R_p, z)$ according to Lemma 2.5, we find that

$$\begin{aligned} \Phi_\Gamma(z)/H(R_p, z) &= \Phi_\Gamma(z)/\tfrac{1}{(1-z^2)(1-z^4)(1-z^6)(1-z^8)} \\ &= 1 + z^4 + 2z^6 + 4z^8 + 2z^{10} + z^{12} + z^{16} . \end{aligned}$$

There are $1 + 1 + 2 + 4 + 2 + 1 + 1 = 12$ secondary invariants. We solve a linear algebra problem at each degree in order to determine which invariants to add to the set of

secondary invariants, resulting in:

$$\begin{aligned}
q_1 &= 1\\
q_2 &= x^2z^2 + y^2w^2\\
q_3 &= x^4y^2 + y^4z^2 + z^4w^2 + w^4x^2\\
q_4 &= x^3yzw - xy^3zw + xyz^3w - xyzw^3\\
q_5 &= xyz^3w^3 - x^3yzw^3 + x^3y^3zw - xy^3z^3w\\
q_6 &= x^6z^2 + y^6w^2 + x^2z^6 + y^2w^6\\
q_7 &= xy^5zw - xyz^5w + xyzw^5 - x^5yzw\\
q_8 &= x^2y^2w^4 + x^4y^2z^2 + y^4z^2w^2 + x^2z^4w^2\\
q_9 &= q_2q_3\\
q_{10} &= q_2q_4\\
q_{11} &= q_3q_4\\
q_{12} &= q_5q_6\ .
\end{aligned}$$

Note that with these four primary invariants, $\mathbf{C}[V]^\Gamma$ is generated by a total of eleven homogeneous invariants (the four primary ones plus seven nontrivial secondary invariants), despite the fact that the Cohen-Macaulay basis needs more elements. We may write any Γ-invariant polynomial uniquely in standard form:

$$\sum_{i=1}^{12} P_i(p_1,p_2,p_3,p_4)q_i\ ,$$

where the P_i are arbitrary polynomials in four variables.

Proposition 4.6 gives the Hilbert series for the equivariants $(\mathbf{C}[V]\otimes V)^\Gamma$,

$$\begin{aligned}
\Psi_\Gamma(z) &= \tfrac{1}{32}\left(\tfrac{4}{(1-z)^4} + \tfrac{-4}{(1+z)^4}\right)\\
&= z + 5z^3 + 14z^5 + 30z^7 + 55z^9 + 91z^{11} + 140z^{13} +\\
&\quad 204z^{15} + 285z^{17} + 385z^{19} + 506z^{21} + 650z^{23} +\\
&\quad 819z^{25} + 1015z^{27} + 1240z^{29} + 1496z^{31} + O(z^{33})\ .
\end{aligned}$$

Proposition 4.3 indicates that the generating equivariants will have degree at most equal to $|\Gamma| - 1 = 31$. Proposition 4.4 gives a method to generate a basis of equivariants at each degree. The first two non-empty vector space bases of the graded components are:

$$\begin{aligned}
(\mathbf{C}[V]_1 \otimes V)^\Gamma &= \left\{ \begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} \right\}\\
(\mathbf{C}[V]_3 \otimes V)^\Gamma &= \left\{ \begin{pmatrix} x^3\\ y^3\\ z^3\\ w^3 \end{pmatrix}, \begin{pmatrix} xy^2\\ yz^2\\ zw^2\\ wx^2 \end{pmatrix}, \begin{pmatrix} xz^2\\ yw^2\\ zx^2\\ wy^2 \end{pmatrix}, \begin{pmatrix} xw^2\\ yx^2\\ zy^2\\ wz^2 \end{pmatrix}, \begin{pmatrix} yzw\\ -xzw\\ xyw\\ -xyz \end{pmatrix} \right\}\ .
\end{aligned}$$

Once again, it is solely a linear algebra problem to determine at each new degree whether each basis element needs to be added to a generating set or not. Computing

$$\begin{aligned}
\Psi_\Gamma(z)/H(R_p,z) &= \Psi_\Gamma(z)/\left(\tfrac{1}{(1-z^2)(1-z^4)(1-z^6)(1-z^8)}\right)\\
&= z + 4z^3 + 8z^5 + 11z^7 + 11z^9 + 8z^{11} + 4z^{13} + z^{15}\ .
\end{aligned} \tag{8.1}$$

we see that the fundamental set has 48 elements. It is not necessary to have a fundamental

set to solve for zeros of equivariant mappings, so we will just give a generating set here. Recall that a fundamental set is a set which generates the equivariants freely as a $\mathbf{C}[p_1, p_2, p_3, p_4]$-module, while a generating set simply generates the equivariants as a $\mathbf{C}[V]^\Gamma$-module. A generating set may be selected as a subset of a fundamental set and is typically smaller. The generating set we compute is:

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \begin{pmatrix} xy^2 \\ yz^2 \\ zw^2 \\ wx^2 \end{pmatrix}, \begin{pmatrix} xz^2 \\ yw^2 \\ zx^2 \\ wy^2 \end{pmatrix}, \begin{pmatrix} xw^2 \\ yx^2 \\ zy^2 \\ wz^2 \end{pmatrix}, \begin{pmatrix} yzw \\ -xzw \\ xyw \\ -xyz \end{pmatrix}, \begin{pmatrix} x^5 \\ y^5 \\ z^5 \\ w^5 \end{pmatrix}, \right.$$

$$\begin{pmatrix} xz^4 \\ yw^4 \\ x^4z \\ y^4w \end{pmatrix}, \begin{pmatrix} x^3w^2 \\ x^2y^3 \\ y^2z^3 \\ z^2w^3 \end{pmatrix}, \begin{pmatrix} xw^4 \\ x^4y \\ y^4z \\ z^4w \end{pmatrix}, \begin{pmatrix} -y^3zw \\ xz^3w \\ -xyw^3 \\ x^3yz \end{pmatrix}, \begin{pmatrix} -yzw^3 \\ x^3zw \\ -xy^3w \\ xyz^3 \end{pmatrix}, \begin{pmatrix} yz^3w \\ -xzw^3 \\ x^3yw \\ -xy^3z \end{pmatrix},$$

$$\left. \begin{pmatrix} x^7 \\ y^7 \\ z^7 \\ w^7 \end{pmatrix}, \begin{pmatrix} xw^6 \\ x^6y \\ y^6z \\ z^6w \end{pmatrix}, \begin{pmatrix} -yzw^5 \\ x^5zw \\ -xy^5w \\ xyz^5 \end{pmatrix}, \begin{pmatrix} yz^5w \\ -xzw^5 \\ x^5yw \\ -xy^5z \end{pmatrix}, \begin{pmatrix} -yz^3w^3 \\ x^3zw^3 \\ -x^3y^3w \\ xy^3z^3 \end{pmatrix} \right\} .$$

The fact that this set generates the equivariants may be verified by checking that it generates all equivariants up to degree 15, the highest degree of a fundamental equivariant as given when computing $\Psi_\Gamma(z)/H(R_p, z)$.

As in the previous example we will solve for the zeros of the cubic Γ-equivariant normal form. To do this we need the subset of degree at most 3 of the fundamental set of equivariants. This is given to us by the degree 3 equivariants in the generating set. We see from the computation in Equation 8.1 that no more basis elements than the ones provided are required. The degree three Γ-equivariant normal form is

$$(l + ap_1) \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} + b \begin{pmatrix} xy^2 \\ yz^2 \\ zw^2 \\ wx^2 \end{pmatrix} + c \begin{pmatrix} xz^2 \\ yw^2 \\ zx^2 \\ wy^2 \end{pmatrix} + d \begin{pmatrix} xw^2 \\ yx^2 \\ zy^2 \\ wz^2 \end{pmatrix} + e \begin{pmatrix} yzw \\ -xzw \\ xyw \\ -xyz \end{pmatrix} , \quad (8.2)$$

with

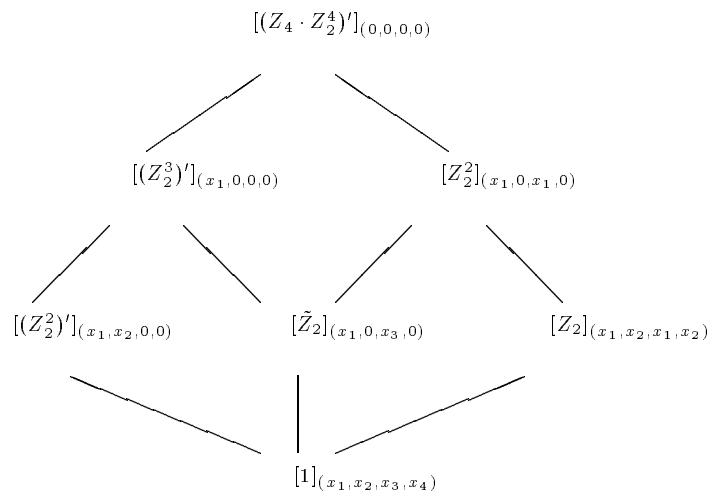
$$p_1 = x^2 + y^2 + z^2 + w^2 .$$

We shall apply our analysis to solve for the zeros of this map in the remainder of this section.

The isotropy lattice for the action of Γ on R^4 is given in Table 1. Representative zeros for the invariant subspaces are also given in Table 2. All the zeros with nontrivial symmetry are easily computable by hand. Alternatively, the problem can be reformulated in each invariant subspace and our techniques may be applied to the reduced problem. By solving the full problem, all the zeros will be solved for simultaneously, but knowledge of some of the zeros will help the calculations.

We now proceed as in the simple example and compute $F^T g_i$ for each i where F is the right hand side of Equation 8.2. The relations between the products of the secondary invariants are also computed. The Gröbner basis of the resulting set of equations can be computed using Macaulay and takes approximately twelve hours on a Sun 4/360. The basis is quite lengthy so we shall not include it here, however the first polynomial is

Table 1. Isotropy lattice and representative fixed point subspaces.



Group	Conjugacy class	Fixed point subspace	Dim
$[(Z_4 \cdot Z_2^4)']$	$[G]$	$(0,0,0,0)$	0
$[(Z_2^3)']$	$[\{1,\ rp^2,\ rp^{-1}rp,\ rp^{-1}rp^{-1}r\}]$	$(x_1,0,0,0)$	1
$[Z_2^2]$	$[\{1,\ r,\ p^{-1}rp,\ rp^{-1}rp\}]$	$(x_1,0,x_1,0)$	1
$[(Z_2^2)']$	$[\{1,\ rp^{-1}rp^{-1}r\}]$	$(x_1,x_2,0,0)$	2
$[\tilde{Z}_2]$	$[\{1,\ rp^{-1}rp\}]$	$(x_1,0,x_3,0)$	2
$[Z_2]$	$[\{1,\ r\}]$	(x_1,x_2,x_1,x_2)	2
$[1]$	$[\{1\}]$	(x_1,x_2,x_3,x_4)	4

Table 2. Zeros for isotropy conjugacy classes.

Isotropy group	Representative equilibrium points	n
$[(Z_4 \cdot Z_2^4)']$	$x_1 = x_2 = x_3 = x_4 = 0$	1
$[(Z_2^3)']$	$x_1^2 = \frac{-\lambda}{a},\ x_2 = x_3 = x_4 = 0$	8
$[Z_2^2]$	$x_1^2 = x_3^2 = \frac{-\lambda}{2a+c},\ x_2 = x_4 = 0$	8
$[(Z_2^2)']$	$\frac{x_1^2}{b} = \frac{x_2^2}{d} = \frac{-\lambda}{a(b+d)+bd},\ x_3 = x_4 = 0$	16
$[\tilde{Z}_2]$	—	0
$[Z_2]$	$\frac{x_1^2}{b-c+d+e} = \frac{x_2^2}{b-c+d-e} = \frac{x_3^2}{b-c+d+e} = \frac{x_4^2}{b-c+d-e} = \frac{-\lambda}{(b-c+d)(4a+b+c+d)-e^2}$	16
$[1]$	?	32

degree 6 in p_1 and contains no other invariants. This tells us that p_1 can take on six different values. Each value corresponds to an orbit of solutions, but we already know the value of p_1 on five of the six orbits. Factoring the above polynomial using the known solutions will give the value of p_1 on the orbit of asymmetric zeros. Substituting this value into the basis equations, we can also solve for the other invariants, to find

$$p_1 = -\frac{(\kappa - e^2)l}{a\kappa + d\kappa_0 + b\kappa_2 - ae^2}$$

$$p_2 = \frac{(\kappa_0^2 + c^2\kappa_1^2 + \kappa_2^2 - 2e^2(\kappa + T1) + e^4)l^2}{(a\kappa + d\kappa_0 + b\kappa_2 - ae^2)^2}$$

$$p_3 = -\frac{(\kappa_0^3 + c^3\kappa_1^3 + \kappa_2^3 - 3e^2(\kappa_0^2 + c^2\kappa_1^2 + \kappa_2^2 + T2) + 3e^4(\kappa + T1) - e^6)l^3}{(a\kappa + d\kappa_0 + b\kappa_2 - ae^2)^3}$$

$$p_4 = -\frac{s(e)\ \ l^4}{((\kappa_0 - c\kappa_1 + \kappa_2 + e^2)^2 - (2e(b-d))^2)(a\kappa + d\kappa_0 + b\kappa_2 - ae^2)^4}\ ,$$

where

$$\begin{aligned}
s(e) &= (\kappa_0^4 + c^4\kappa_1^4 + \kappa_2^4)(\kappa_0 - c\kappa_1 + \kappa_2)^2 + \ldots \\
&\quad -2(3\kappa + 2T1 - 2c(b+d))e^{10} + e^{12} \\
T1 &= bd - c(b+d) \\
T2 &= b^3c - 2b^2c^2 + bc^2d - 2c^2d^2 + cd^3 \\
\kappa_0 &= b^2 + d(c-b) \\
\kappa_1 &= b - c + d \\
\kappa_2 &= d^2 + b(c-d) \\
\kappa &= \kappa_0 + c\kappa_1 + \kappa_2\ .
\end{aligned} \tag{8.3}$$

The full expression for $s(e)$ is lengthy and may be obtained from the author.

These solutions are clearly unwieldy to work with by hand, but using computer algebra it is easy to evaluate the expression at any given set of parameters. The next step is to solve for the zeros in terms of the invariants. This problem reduces to solving a quartic polynomial in the square of one of the coordinate variables. This polynomial may be found using Gröbner basis techniques just as we solve the larger sets of equations. The resulting polynomial is :

$$\begin{aligned}
&w^8 - w^6p_1 + 1/2w^4p_1^2 - 1/6w^2p_1^3 + 1/24p_1^4 - 1/2w^4p_2 + \\
&1/2w^2p_1p_2 - 1/4p_1^2p_2 + 1/8p_2^2 - 1/3w^2p_3 + 1/3p_1p_3 - 1/4p_4
\end{aligned}$$

This may be factored and then we may solve for the other coordinate variables.

We may similarly continue the analysis and study the linearization at these fixed points in terms of the invariants. This becomes less useful as the problem becomes more complicated since very little may be learned from the lengthy expressions. However, as soon as the number of parameters is decreased or their values are fixed, then the calculations may be simplified and the expressions become more interesting. The conditions for degenerate linearization are complicated and not very illuminating. They also may be obtained from the author on request.

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