# Constructive Closed Range and Open Mapping Theorems

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**Abstract**

We prove a version of the closed range theorem within Bishop's constructive mathematics. This is applied to show that if an operator $T$ on a Hilbert space has an adjoint and a complete range, then both $T$ and $T^*$ are sequentially open.

## 1 Introduction

In this paper we continue our constructive exploration of the theory of operators, in particular operators on a Hilbert space ([4], [5], [6]). We work entirely within Bishop's constructive mathematics, which we regard as mathematics with intuitionistic logic. For discussions of the merits of this approach to mathematics—in particular, the multiplicity of its models—see [3] and [10]. The technical background needed in our paper is found in [1] and [9].

Our main aim is to prove the following result, the constructive **Closed Range Theorem** for operators on a Hilbert space (cf. [16], pages 99-103):

**Theorem 1** *Let $H$ be a Hilbert space, and $T$ a linear operator on $H$ such that $T^*$ exists and* $\mathrm{ran}(T)$ *is closed. Then* $\mathrm{ran}(T)$ *and* $\ker(T)$ *are both located, and* $\mathrm{ran}(T^*)$ *is closed.*

(Recall that a subset $S$ of a normed space $X$ is **located** if

$$\rho(x, S) = \inf \{\|x - s\| : s \in S\}$$

exists for each $x \in S$.) This theorem will enable us to produce one good constructive substitute for the classical open mapping theorem of functional analysis.

To set the scene for our main results, consider how the closed range theorem is proved classically. The first move is to observe that $\mathrm{ran}(T^*T)$ is dense in $\mathrm{ran}(T^*)$ : this is shown by noting the trivial identity $\ker(T^*T) = \ker(T^*)$ and then taking orthogonal complements. (Unfortunately, this simple proof does not work constructively—see [4]; but the final result does hold, as we shall see shortly.) So it will suffice to prove that if $\mathrm{ran}(T)$ is closed, then so is $\mathrm{ran}(T^*T)$.

To this end, let $(x_n)$ be a sequence such that $(T^*Tx_n)$ converges to a limit $\xi \in H$. Applying the Uniform Boundedness Theorem to the bounded linear functionals $f_n$ defined on the Hilbert space $\mathrm{ran}(T)$ by

$$f_n(Tx) = \langle Tx, Tx_n \rangle = \langle x, T^*Tx_n \rangle ,$$

we obtain $M > 0$ such that $\|f_n\| \leq M$ for each $n$. Hence the linear functional $Tx \mapsto \langle x, \xi \rangle$ on $\mathrm{ran}(T)$ is bounded by $M$. By the Riesz Representation Theorem, there exists $Tx_\infty \in \mathrm{ran}(T)$ such that

$$\langle x, \xi \rangle = \langle Tx, Tx_\infty \rangle = \langle x, T^*Tx_\infty \rangle$$

for all $x$. It follows that $\xi = T^*Tx_\infty \in \mathrm{ran}(T^*T)$.

This proof fails constructively in two places: first, in its use of the Uniform Boundedness Theorem, which, in its standard classical form, does not look susceptible of constructive proof (although a contrapositive version is provable and is used below); and secondly, in its application of the Riesz Representation Theorem, which requires the linear functional to be not just bounded but be **normable**—that is, have a norm—a property that, in view of the extra hypotheses needed for a constructive version of the least–upper–bound principle ([1], page 37, (4.3)), cannot be guaranteed. Fortunately, all these difficulties can be overcome, as we will show in the rest of this paper.

## 2 The Main Results

We first extend a lemma which appeared, with an incorrect proof, as Lemma 2 of [4]. For the proof of this lemma we need to observe

- ▷ that a subset $S$ of the Hilbert space $H$ is **weakly totally bounded** if for each $y \in H$, $\{\langle x, y \rangle : x \in S\}$ is a totally bounded subset of $\mathbf{C}$; and
- ▷ that nonempty, weakly totally bounded, convex subset of a Hilbert space is located [14].

**Lemma 2** *If $T$ is an operator with an adjoint on a Hilbert space $H$, then* $\mathrm{ran}(TT^*)$ *is dense in* $\mathrm{ran}(T)$.

**Proof.** For each positive integer $n$ the closed ball $B_n = \overline{B}(0, n)$ in $H$ is weakly totally bounded. So for each $y \in H$ the set

$$\{\langle TT^*x, y \rangle : x \in B_n\} = \{\langle x, TT^*y \rangle : x \in B_n\}$$

is totally bounded in $\mathbf{C}$. In other words, $TT^*(B_n)$ is weakly totally bounded; since it is convex and contains 0, it is located. Given $x \in H$ and $\varepsilon > 0$, we need only show that $\rho(Tx, TT^*(B_n)) < \varepsilon$. for some $n$. To this end, construct an increasing binary sequence $(\lambda_n)$ such that

$$\begin{aligned} \lambda_n = 0 \quad &\Rightarrow \quad \rho(Tx, TT^*(B_n)) > \tfrac{\varepsilon}{2}, \\ \lambda_n = 1 \quad &\Rightarrow \quad \rho(Tx, TT^*(B_n)) < \varepsilon. \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, then, applying the Separation Theorem ([1], page 336, (4.3), and [11], Theorem 2) and the Riesz Representation Theorem ([1], page 419, (2.3)), choose a unit vector $y_n \in H$ such that

$$\langle TT^*z, y_n\rangle + \tfrac{\varepsilon}{2} < \langle Tx, y_n\rangle \quad (z \in B_n).$$

If $\lambda_n = 1$, set $y_n = 0$. The sequence $(T^*y_n)$ converges to $0$ : for if $\lambda_n = 0$, then

$$n\,\|T^*y_n\|^2 < \langle TT^*(ny_n), y_n\rangle + \tfrac{\varepsilon}{2} < \langle Tx, y_n\rangle \le \|Tx\|$$

and so $\|T^*y_n\|^2 < n^{-1}\,\|Tx\|$. Choose a positive integer $N$ such that $\langle x, T^*y_N\rangle < \varepsilon/2$. If $\lambda_N = 0$, then

$$\tfrac{\varepsilon}{2} \le \langle Tx, y_N\rangle = \langle x, T^*y_N\rangle < \tfrac{\varepsilon}{2},$$

a contradiction. Hence $\lambda_N = 1$. Q.E.D.

Our next two lemmas will enable us to prove that if the range of our operator $T$ is closed, then so is that of $T^*T$. Their proofs employ techniques similar to those used in [13].

**Lemma 3** *Let $H$ be a Hilbert space, let $T$ be a linear operator on $H$ such that $T^*$ exists and* $\mathrm{ran}(T)$ *is complete, and let $(x_n)$ be a sequence in $H$ such that $(T^*Tx_n)$ converges to $0$. Then for all positive numbers $a, b$ with $a < b$, either $\|Tx_n\| > a$ for some $n$ or else $\|Tx_n\| < b$ for all $n$.*

**Proof.** Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\|T^*Tx_j\| < \frac{1}{(k+1)^2} \quad (j \ge n_k),$$

and, taking $n_0 = 0$, set

$$s_k = \max\{\|Tx_j\| : n_{k-1} < j \le n_k\}.$$

Construct an increasing binary sequence $(\lambda_k)$ such that

$$\begin{aligned} \lambda_k = 0 \quad &\Rightarrow \quad s_k < b, \\ \lambda_k = 1 \quad &\Rightarrow \quad s_k > a. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define a sequence $(y_k)$ in $\mathrm{ran}(T)$ as follows: if $\lambda_k = 0$, set $y_k = 0$; if $\lambda_k = 1 - \lambda_{k-1}$, choose $j$ with $n_{k-1} < j \le n_k$ and $\|Tx_j\| > a$, and set $y_i = Tx_j/(k\|Tx_j\|)$ for all $i \ge k$. Then $(y_k)$ is a Cauchy sequence: in fact, $\|y_i - y_k\| \le 1/k$ whenever $i \ge k$. So $(y_k)$ converges to a limit $Tz$ in $\mathrm{ran}(T)$. Choosing a positive integer $N$ such that $\|z\| < Na$, consider any integer $k \ge N$. If $\lambda_k = 1 - \lambda_{k-1}$, then $Tz = Tx_j/(k\|Tx_j\|)$ for some $j$ with $\|Tx_j\| > a$, so

$$a/k < \|Tx_j\|/k = \langle Tx_j, Tz\rangle = \langle T^*Tx_j, z\rangle \le \|z\|/k^2 < (Na)/k^2 < a/k,$$

a contradiction. Hence $\lambda_k = \lambda_{k-1}$ for all $k \ge N$. It follows that either $\lambda_k = 1$ for some $k$ or else $\lambda_k = 0$ for all $k$. Q.E.D.

**Lemma 4** *Let $H$ be a Hilbert space, let $T$ be a linear operator on $H$ such that $T^*$ exists and* $\mathrm{ran}(T)$ *is complete, and let $(x_n)$ be a sequence in $H$ such that $(T^*Tx_n)$ converges to* 0. *Then for all positive numbers $a, b$ with $a < b$, either $\|Tx_n\| > a$ for infinitely many $n$ or else $\|Tx_n\| < b$ for all sufficiently large $n$.*

**Proof.** Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\|T^*Tx_j\| < \frac{1}{k^2} \quad (j \geq n_k).$$

By successively applying Lemma 3 to the sequences $(x_{n_k}, x_{n_k+1}, \ldots)$, construct an increasing binary sequence $(\lambda_k)$ such that

$$\begin{array}{lcl} \lambda_k = 0 & \Rightarrow & \exists j \geq n_k (a < \|Tx_j\|), \\ \lambda_k = 1 & \Rightarrow & \forall j \geq n_k (\|Tx_j\| < b). \end{array}$$

We may assume that $\lambda_1 = 0$. Define a sequence $(y_k)$ in $\mathrm{ran}(T)$ as follows: if $\lambda_k = 0$, choose $j \geq n_k$ such that $\|Tx_j\| > a$ and set $y_k = Tx_j/(k\|Tx_j\|)$; if $\lambda_k = 1 - \lambda_{k-1}$, set $y_i = y_{k-1}$ for all $i \geq k$. Then $(y_k)$ is a Cauchy sequence and so converges to a limit $Tz \in \mathrm{ran}(T)$. Choosing a positive integer $N$ such that $\|z\| < Na$, consider any integer $k > N$. If $\lambda_k = 1 - \lambda_{k-1}$, then $Tz = Tx_j/\left((k-1)\|Tx_j\|\right)$ for some $j$ with $\|Tx_j\| > a$, so

$$\begin{array}{rcl} (k-1)^{-1} a & < & (k-1)^{-1}\|Tx_j\| \\ & = & \langle Tx_j, Tz\rangle \\ & = & \langle T^*Tx_j, z\rangle \\ & \leq & (k-1)^{-2}\|z\| \\ & < & (k-1)^{-2}Na \\ & < & (k-1)^{-1}a, \end{array}$$

a contradiction. Hence $\lambda_k = \lambda_{k-1}$ for all $k \geq N$. It follows that either $\lambda_k = 1$ for some $k$ or else $\lambda_k = 0$ for all $k$. In the former case, for each $k$ there exists $j \geq n_k$ such that $\|Tx_j\| > a$; in the latter, there exists $k$ such that $\|Tx_j\| < b$ for all $j \geq n_k$. Q.E.D.

**Lemma 5** *Let $H$ be a Hilbert space, let $T$ be a linear operator on $H$ such that $T^*$ exists and* $\mathrm{ran}(T)$ *is complete, and let $(x_n)$ be a sequence in $H$ such that $(T^*Tx_n)$ converges to* 0. *Then $(Tx_n)$ converges to* 0.

**Proof.** By Lemma 4, for each $\varepsilon > 0$ either $\|Tx_n\| > \varepsilon/2$ for infinitely many $n$ or else $\|Tx_n\| < \varepsilon$ for all sufficiently large $n$. In the former case, by passing to an appropriate subsequence, we may assume that $\|Tx_n\| > \varepsilon/2$ and $\|T^*Tx_n\| < n^{-2}$ for all $n$. Applying the Uniform Boundedness Theorem ([1], page 392, Problem 20; see also [15], page 61) to the bounded linear functionals $Tx \mapsto \langle Tx, nTx_n\rangle$ on $\mathrm{ran}(T)$, we can find $Tz \in \mathrm{ran}(T)$ such that the sequence $(|\langle Tz, nTx_n\rangle|)$ is unbounded. This contradicts the fact that

$$|\langle Tz, nTx_n\rangle| = |\langle z, nT^*Tx_n\rangle| \leq n^{-1}\|z\| \to 0 \text{ as } n \to \infty. * q.e.d. \qquad (1)$$

Before the next result we make a couple of observations:

- A sequentially continuous linear map between normed spaces maps Cauchy sequences to Cauchy sequences [8].
- If an operator on a Hilbert space has an adjoint, then the operator is sequentially continuous. (The proof involves a minor modification to that of Theorem 4 in [4].)

**Lemma 6** *Let $H$ be a Hilbert space, and $T$ an operator on $H$ such that $T^*$ exists and* $\mathrm{ran}(T)$ *is closed. Then* $\mathrm{ran}(T^*T)$ *is closed.*

**Proof.** Since $T^*$ is one-one on $\mathrm{ran}(T)$, there exists a linear mapping $S : \mathrm{ran}(T^*T) \to \mathrm{ran}(T)$ such that $ST^*Tx = Tx$ for all $x \in H$. Let $(x_n)$ be a sequence in $H$ such that $T^*Tx_n \to 0$. By Lemma 5, $S$ is sequentially continuous. Hence if $(T^*Tx_n)$ is a Cauchy sequence, then so is $(Tx_n)$. Since $\mathrm{ran}(T)$ is complete, there exists $x \in H$ such that $Tx_n \to Tx$. But $T^*$ is sequentially continuous, so $T^*Tx_n \to T^*Tx$. Thus $\mathrm{ran}(T^*T)$ is closed. Q.E.D.

We can now give the **Proof of Theorem 1.** Let $H$ be a Hilbert space, and $T$ a linear operator on $H$ such that $T^*$ exists and $\mathrm{ran}(T)$ is closed. By Lemmas 2 and 6, $\mathrm{ran}(T^*T)$ is both dense in $\mathrm{ran}(T^*)$ and complete. Hence $\mathrm{ran}(T^*)$ is complete. Moreover, for each $x \in H$ there exists $y \in H$ such that $T^*x = T^*Ty$; so

$$x = Ty + (x - Ty)\,,$$

where $Ty \in \mathrm{ran}(T)$ and

$$x - Ty \in \ker(T^*) = \mathrm{ran}(T)^{\perp}.$$

It follows from Lemma 5 of [4] that both $\mathrm{ran}(T)$ and $\ker(T^*)$ are located. Interchanging the roles of $T$ and $T^*$ in the foregoing, we now see that $\mathrm{ran}(T^*)$ and $\ker(T)$ are located. Q.E.D.

# 3 An Open Mapping Theorem

Recall that a linear mapping $T : X \to Y$ between normed spaces is

- **open** if there exists $r > 0$ such that for each $y \in \mathrm{ran}(T)$ with $\|y\| < r$ there exists $x \in X$ such that $\|x\| \leq 1$ and $Tx = y$;
- **sequentially open** if whenever $Tx_n \to 0$ there exists a sequence $(y_n)$ in $\ker(T)$ such that $x_n - y_n \to 0$.

Clearly, openness implies sequential openness. In classical mathematics openness and sequential openness are equivalent; this equivalence does not hold constructively ([6], Theorem 5).

The classical Open Mapping Theorem states that a bounded linear mapping $T$ of a Banach space $X$ into a normed space $Y$ is open if and only if $\operatorname{ran}(T)$ is complete. A number of attempts have been made to prove this theorem, or something classically equivalent to it, constructively (see [17], [7], and [6]), using the full property of openness, but the theorems they have produced have had restrictive conditions on the locatedness of the image of a ball under $T$. A less restrictive result, using sequential openness rather than openness, appeared in [12], and provides one direction of implication in the following constructive **Sequential Open Mapping Theorem.**

**Theorem 7** *Let $T$ be a sequentially continuous linear mapping of a Banach space $X$ into a normed space $Y$, with a located kernel. Then $T$ is sequentially open if and only if* $\operatorname{ran}(T)$ *is complete.*

**Proof.** Assume first that $T$ is sequentially open, and let $(Tx_n)$ be a Cauchy sequence in $\operatorname{ran}(T)$. Then the linear mapping $T^{-1} : \operatorname{ran}(T) \to X/\ker(T)$ is sequentially continuous, so $(x_n)$ is a Cauchy sequence in $X/\ker(T)$. Since the latter space is complete, there exists $x \in X$ such that $\rho(x - x_n, \ker(T)) \to 0$. Choosing a sequence $(y_n)$ in $\ker(T)$ such that $\|x - x_n - y_n\| \to 0$, we see that $T(x - x_n - y_n) \to 0$ and therefore that $Tx_n \to Tx$. Hence $\operatorname{ran}(T)$ is complete.

Now assume, conversely, that $\operatorname{ran}(T)$ is complete. Then $T : X/\ker(T) \to \operatorname{ran}(T)$ is sequentially continuous, so, by Banach's Inverse Mapping Theorem [12], the inverse mapping from $\operatorname{ran}(T)$ onto $X/\ker(T)$ is sequentially continuous. This is equivalent to $T$ being sequentially open. Q.E.D.

For an operator with an adjoint on a Hilbert space, we can remove the hypothesis that the kernel be located. To do so, we note that in view of Lemma 2, we can adapt the proof of Theorem 2 of [4] to show that any sequentially open operator on $H$ with an adjoint has located range and kernel.

**Theorem 8** *Let $T$ be a linear operator on a Hilbert space $H$ such that $T^*$ exists. Then $T$ is sequentially open if and only if it has complete range.*

**Proof.** If $T$ is sequentially open, then $\ker(T)$ is located, as we noted above, and it follows from Theorem 7 that $\operatorname{ran}(T)$ is complete. Conversely, if $\operatorname{ran}(T)$ is complete, then $\ker(T)$ is located, by Theorem 1 above, and Theorem 7 shows that $T$ is sequentially open. Q.E.D.

# References


[1] Errett Bishop and Douglas Bridges, *Constructive Analysis*, Grundlehren der math. Wissenschaften Bd. **279**, Springer-Verlag, Heidelberg, 1985.

[2] Douglas Bridges, "A constructive look at positive linear functionals on $L(H)$", Pacific J. Math. **95**(1), 11-25, 1981.

[3] Douglas Bridges, "Constructive mathematics: a foundation for computable analysis", CDMTCS preprint, University of Auckland, 1997.

[4] Douglas Bridges and Hajime Ishihara, "Locating the range of an operator on a Hilbert space", Bull. London Math. Soc. **24**, 599-605, 1992.

[5] Douglas Bridges and Hajime Ishihara, "Absolute continuity and the uniqueness of the constructive functional calculus", Math. Logic Quarterly **40**(4), 519-527, 1995.

[6] Douglas Bridges and Hajime Ishihara, "A definitive constructive open mapping theorem", to appear in Math. Logic Quarterly.

[7] Douglas Bridges, William Julian, and Ray Mines, "A constructive treatment of open and unopen mapping theorems", Z. Math. Logik und Grundlagen der Math. Bd **35**, 29-43, 1989.

[8] Douglas Bridges and Ray Mines, "Sequentially continuous linear mappings in constructive analysis", to appear in J. Symbolic Logic.

[9] Douglas Bridges and Fred Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes **97**, Cambridge University Press, London, 1987.

[10] Douglas Bridges and Steve Reeves, "Constructive mathematics, in theory and programming practice", to appear in Philosophia Mathematica.

[11] Hajime Ishihara, "On the constructive Hahn–Banach theorem", Bull. London Math. Soc. **21**, 79–81, 1989.

[12] Hajime Ishihara, "A constructive version of Banach's Inverse Mapping Theorem", New Zealand J. Math. **23**, 71-75, 1995.

[13] Hajime Ishihara, "Sequential continuity of linear mappings in constructive mathematics", J. Universal Computer Science **3**, 1250–1254, 1997.

[14] Hajime Ishihara, "Locating subsets of a Hilbert space", to appear.

[15] Halsey L. Royden, "Aspects of constructive analysis", in *Errett Bishop: Reflections on Him and His Research*, Contemporary Mathematics **39**, Amer. Math. Soc., Providence RI, 1985.

[16] Walter Rudin, *Functional Analysis* (2nd Edn), McGraw-Hill, New York, 1991

[17] Gabriel Stolzenberg, "A critical analysis of Banach's Open Mapping Theorem", Northeastern University, Boston, Mass., 1971.

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