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CONVERGENCE OF THE TIME-DOMAIN PERFECTLY MATCHED LAYER METHOD FOR ACOUSTIC SCATTERING PROBLEMS

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Abstract. In this paper we establish the stability and convergence of the time-domain perfectly matched layer (PML) method for solving the acoustic scattering problems. We first prove the well-posedness and the stability of the time-dependent acoustic scattering problem with the Dirichlet-to-Neumann boundary condition. Next we show the well-posedness of the unsplit-field PML method for the acoustic scattering problems. Then we prove the exponential convergence of the non-splitting PML method in terms of the thickness and medium property of the artificial PML layer. The proof depends on a stability result of the PML system for constant medium property and an exponential decay estimate of the modified Bessel functions.

Key Words. perfectly matched layer, acoustic scattering, exponential convergence, stability

1. Introduction

We consider the acoustic scattering problem with the sound-hard boundary condition on the obstacle

$$\frac{\partial u}{\partial t} = -\mathrm{div}\mathbf{p} + f(\mathbf{x},t), \qquad \frac{\partial \mathbf{p}}{\partial t} = -\nabla u \quad \text{in } [\mathbb{R}^2\backslash\bar{D}]\times(0,T), \tag{1.1}$$

$$\mathbf{p}\cdot\mathbf{n}_D = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{1.2}$$

$$\sqrt{r}(u-\mathbf{p}\cdot\hat{\mathbf{x}}) \to 0, \quad \text{as } r=|\mathbf{x}|\to\infty, \quad \text{a.e. } t\in(0,T), \tag{1.3}$$

$$u|_{t=0} = u_0, \quad \mathbf{p}|_{t=0} = \mathbf{p}_0. \tag{1.4}$$

Here u is the pressure and $\mathbf{p}$ is the velocity field of the wave. $D\subset\mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $\hat{\mathbf{x}}=\mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. $f, u_0, \mathbf{p}_0$ are assumed to be supported in the circle $B_R=\{\mathbf{x}\in\mathbb{R}^2 : |\mathbf{x}|<R\}$ for some $R>0$. (1.3) is the radiation condition which corresponds to the well-known Sommerfeld radiation condition in the frequency domain. We remark that the results in this paper can be easily extended to solve scattering problems with other boundary conditions such as the sound-soft or the impedance boundary condition on Γ_D.

One of the fundamental problems in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. The first objective of this paper is to prove the well-posedness and stability of the system (1.1)-(1.4) by imposing the

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Dirichlet-to-Neumann boundary condition on the $\Gamma_R = \partial B_R$. The proof depends on the abstract inversion theorem of the Laplace transform and the a priori estimate for the Helmholtz equation which seems to be new and is of independent interest. In Lax and Phillips [20], the scattering problem of the wave equation is studied by using the semigroup theory of operators in the absence of the source function f. We remark that the well-posedness of scattering problems in the frequency domain is well-known (cf. e.g. Colten and Kress [10]).

The non-local Dirichlet to Neumann boundary condition for (1.1)-(1.4) is the starting point of various approximate absorbing boundary conditions which have been proposed and studied in the literature, see the review papers Givoli [16], Tsynkov [25], Hagstrom [17] and the references therein. An interesting alternative to the method of absorbing boundary conditions is the method of perfectly matched layer (PML). Since the work of Bérenger [5] which proposed a PML technique for solving the time-dependent Maxwell equations in the Cartesian coordinates, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [27], Teixeira and Chew [24] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

There are two classes of time-domain PML methods for the wave scattering problems. The first class, called "split-field PML method" in the literature, includes the original Bérenger PML method. It is shown in Abarbanel and Gottlieb [2] that the Bérenger PML method is only weakly well-posed and thus may suffer instability in practical applications. The second class, the so-called "unsplit-field PML formulations" in the literature, is however, strongly well-posed. One such successful method is the uniaxial PML method developed in Sacks *et al* [23] and Gedney [15] for the Maxwell equations in the Cartesian coordinates. In the curvilinear coordinated, the split-field PML method is introduced in Collino and Monk [9] and the unsplit-field PML methods are introduced in Petropoulos [22] and [24] for Maxwell equations.

Although the tremendous attention and success in the application of PML methods in the engineering literature, there are few mathematical results on the convergence of the PML methods. For the Helmholtz equation in the frequency domain, it is proved in Lassas and Somersalo [19], Hohage *et al* [18] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. In Chen and Wu [8], Chen and Liu [7], an adaptive PML technique is proposed and studied in which a posteriori error estimate is used to determine the PML parameters. In particular, it is shown that the exponential convergence can be achieved for fixed thickness of the PML layer by enlarging PML medium properties. For the time-domain PML method, not much mathematical convergence analysis is known except the work in Hagstrom [17] in which the planar PML method in one space direction is considered for the wave equation. In de Hoop *et al* [12], Diaz and Joly [13], the PML system with point source is analyzed based on the Cagniard - de Hoop method.

The long time stability of the PML methods is also a much studied topic in the literature (see e.g. Bécache and Joly [3], Bécache *et al* [4], Appelö *et al* [1]). For a PML method to be practically useful, it must be stable in time, that is, the solution should not grow exponentially in time. We remark that the well-posedness of the

PML system which follows from the theory of symmetric hyperbolic systems allows the exponential growth of the solutions. In [3, 4, 1] the stability of the Cauchy problem of the PML systems is considered for the constant PML medium property by using the energy argument.

In this paper we will show the exponential convergence and stability of the unsplit-field PML method for the acoustic wave scattering problem in the polar coordinates. Our analysis starts with the well-posedness and stability of the scattering problem (1.1)-(1.4) in Section 2. We then introduce the upsplit-field PML method in the polar coordinates by following the procedure in [22] for Maxwell equations in Section 3. Also the well-posedness of the initial boundary value problem of the PML system is established. In Section 4 we prove the stability of the initial boundary value problem of the PML system for the constant medium property based on the method of the Laplace transform and the analysis in the frequency domain. In Section 5 we prove the exponential convergence of the PML method.

One of the key ingredients in our analysis is the following uniform exponential decay property of the modified Bessel function $K_n(z)$ (Lemma 5.1)

$$\frac{|K_n(s\rho_1+\tau)|}{|K_n(s\rho_2)|} \le e^{-\tau\left(1-\frac{\rho_2^2}{\rho_1^2}\right)},$$

for any $n \in \mathbb{Z}$, $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) > 0$, $\rho_1 > \rho_2 > 0$, and $\tau > 0$. The proof depends on the Macdonald formula for the integral representation of the product of modified Bessel functions and extends our earlier uniform estimate in [7] for the first Hankel function $H^1_\nu(z)$, $\nu \in \mathbb{R}$, in the upper-half complex plane.

2. The acoustic scattering problem

For any $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > 0$, we let $u_{\rm L} = \mathscr{L}(u)$ and $\mathbf{p}_{\rm L} = \mathscr{L}(\mathbf{p})$ be respectively the Laplace transform of u and $\mathbf{p}$ in time

$$u_{\rm L}(\mathbf{x},s) = \int_0^\infty e^{-st}u(\mathbf{x},t)dt, \qquad \mathbf{p}_{\rm L}(\mathbf{x},s) = \int_0^\infty e^{-st}\mathbf{p}(\mathbf{x},t)dt.$$

Since $\mathscr{L}(\partial_t u) = su_{\rm L} - u_0$ and $\mathscr{L}(\partial_t \mathbf{p}) = s\mathbf{p}_{\rm L} - \mathbf{p}_0$, by taking the Laplace transform of (1.1) we get

$$su_{\rm L} - u_0 = -\mathrm{div}\mathbf{p}_{\rm L} + f_{\rm L}, \qquad s\mathbf{p}_{\rm L} - \mathbf{p}_0 = -\nabla u_{\rm L} \quad \text{in } \mathbb{R}^2\backslash\bar{D}, \tag{2.1}$$

where $f_{\rm L} = \mathscr{L}(f)$. Because $f_{\rm L}$, u_0, $\mathbf{p}_0$ are supported inside B_R, we know that $u_{\rm L}$ satisfies the Helmholtz equation outside B_R

$$-\Delta u_{\rm L} + s^2 u_{\rm L} = 0.$$

Moreover, (1.3) implies that $u_{\rm L}$ satisfies the radiation condition

$$\sqrt{r}\left(\frac{\partial u_{\rm L}}{\partial r} + su_{\rm L}\right) \to 0, \qquad \text{as } r \to \infty.$$

We have the following series representation for $u_{\rm L}$ outside B_R [17]

$$u_{\rm L} = \sum_{n=-\infty}^{\infty} \frac{K_n(sr)}{K_n(sR)} u^n_{\rm L}(R,s)e^{\mathbf{i}n\theta}, \tag{2.2}$$

where $u^n_{\rm L}(R,s) = \frac{1}{2\pi}\int_0^{2\pi} u_{\rm L}(R,\theta,s)e^{-\mathbf{i}n\theta}d\theta$. Let $G : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ be the Dirichlet-to-Neumann operator for the Helmholtz equation

$$Gu_{\rm L}(R,\theta,s) = \frac{1}{s}\frac{\partial u_{\rm L}}{\partial r}\Big|_{\Gamma_R} = \sum_{n=-\infty}^{\infty} \frac{K_n'(sR)}{K_n(sR)} u^n_{\rm L}(R,s)e^{\mathbf{i}n\theta}. \tag{2.3}$$

Then, since $\mathbf{p}_0$ is supported in B_R, by (2.1),

$$\mathbf{p}_{\rm L}\cdot\hat{\mathbf{x}}+Gu_{\rm L}=0\quad\text{ on }\Gamma_R.$$

By taking the inverse Laplace transform we obtain the following Dirichlet-to-Neumann boundary condition for the acoustic scattering problem

$$\mathbf{p}\cdot\hat{\mathbf{x}}+\mathcal{T}(u)=0\quad\text{ on }\Gamma_R\times(0,T),\tag{2.4}$$

where $\mathcal{T}=\mathscr{L}^{-1}\circ G\circ\mathscr{L}$ and from (2.3) we know that

$$\mathcal{T}(u)(R,\theta,t)=\sum_{n=-\infty}^{\infty}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right)*u_n(R,t)\right]e^{\mathbf{i}n\theta},\tag{2.5}$$

where $u_n(R,t)=\mathscr{L}^{-1}(u_{\rm L}^n(R,s))=\frac{1}{2\pi}\int_0^{2\pi}u(R,\theta,t)e^{-\mathbf{i}n\theta}d\theta$.

Based on (2.4) we know that the original scattering problem (1.1)-(1.4) is reduced to the following problem on the bounded domain $\Omega_R\times(0,T)$, $\Omega_R=B_R\backslash\bar{D}$,

$$\frac{\partial u}{\partial t}=-\mathrm{div}\mathbf{p}+f,\qquad\frac{\partial\mathbf{p}}{\partial t}=-\nabla u\qquad\text{in }\Omega_R\times(0,T),\tag{2.6}$$

$$\mathbf{p}\cdot\mathbf{n}_D=0\quad\text{ on }\Gamma_D\times(0,T),\tag{2.7}$$

$$\mathbf{p}\cdot\hat{\mathbf{x}}+\mathcal{T}(u)=0\quad\text{ on }\Gamma_R\times(0,T),\tag{2.8}$$

$$u|_{t=0}=u_0,\qquad\mathbf{p}|_{t=0}=\mathbf{p}_0.\tag{2.9}$$

In this section we show that the reduced scattering problem on the bounded domain (2.6)-(2.9) is well-posed and stable. We first state the assumptions on the boundary and initial data:

(H1) $u_0\in H^2(\Omega_R)$ and $\mathrm{supp}(u_0)\subset B_R$;
(H2) $\mathbf{p}_0\in H(\mathrm{div};\Omega_R)$, $\mathrm{div}\mathbf{p}_0\in H^2(\Omega_R)$, and $\mathrm{supp}(\mathbf{p}_0)\subset B_R$;
(H3) $f\in H^1(0,T;L^2(\Omega_R))$, $f|_{t=0}=0$, and $\mathrm{supp}(f)\subset B_R\times(0,T)$;
(H4) Compatibility conditions: $\mathbf{p}_0\cdot\mathbf{n}_D=0$, $\nabla u_0\cdot\mathbf{n}_D=0$ on Γ_D.

In the rest of this paper, we will always assume that f is extended so that $f\in H^1(0,+\infty;L^2(\Omega_R))$ and $\|f\|_{H^1(0,+\infty;L^2(\Omega_R))}\le C\|f\|_{H^1(0,T;L^2(\Omega_R))}$. We also remark that the assumption $f|_{t=0}=0$ in (H3) is not very restrictive in practical applications. If $f(x,0)\neq0$, let w be the solution of the equation $-\Delta w=f(x,0)$ in Ω_R with the boundary condition $\nabla w\cdot\mathbf{n}=0$ on $\partial\Omega_R$. Then $\mathbf{q}_0=-\nabla w$ satisfies $\mathrm{div}\,\mathbf{q}_0=f(x,0)$ and $(u,\mathbf{p}-\mathbf{q}_0)$ satisfies (1.1)-(1.4) with the source $f'=f-f(x,0)$ and the initial condition $\mathbf{p}_0'=\mathbf{p}_0-\mathbf{q}_0$.

The following theorem is the main result of this section.

Theorem 2.1. *Let the assumptions (H1)-(H4) be satisfied. Then the problem (2.6)-(2.9) has a unique solution* $u\in L^2(0,T;H^1(\Omega_R))\cap H^1(0,T;L^2(\Omega_R))$, $\mathbf{p}\in L^2(0,T;H(\mathrm{div},\Omega_R))\cap H^1(0,T;L^2(\Omega_R))$ *such that* $u|_{t=0}=u_0$, $\mathbf{p}|_{t=0}=\mathbf{p}_0$, *and for any* $v\in L^2(0,T;H^1(\Omega_R))$, $\mathbf{q}\in L^2(0,T;L^2(\Omega_R))$,

$$\int_0^T\left[\Big(\frac{\partial u}{\partial t},v\Big)-(\mathbf{p},\nabla v)-\langle\mathcal{T}(u),v\rangle_{\Gamma_R}\right]dt=\int_0^T(f,v)dt,\tag{2.10}$$

$$\int_0^T\left[\Big(\frac{\partial\mathbf{p}}{\partial t},\mathbf{q}\Big)+(\nabla u,\mathbf{q})\right]dt=0.\tag{2.11}$$

Here $\mathcal{T}(u)\in L^2(0,T;H^{-1/2}(\Gamma_R))$. *Moreover,* $(u,\mathbf{p})$ *satisfies the following stability estimate*

$$\begin{aligned}\max_{0\le t\le T}\big(\|\,\partial_t u\,\|_{L^2(\Omega_R)}+\|\,\nabla u\,\|_{L^2(\Omega_R)}+\|\,\partial_t\mathbf{p}\,\|_{L^2(\Omega_R)}+\|\,\mathrm{div}\mathbf{p}\,\|_{L^2(\Omega_R)}\big)\\ \le\ C\|(u_0,\mathbf{p}_0)\|_{\Omega_R}+C\|\partial_t f\|_{L^1(0,T;L^2(\Omega_R))},\end{aligned}\tag{2.12}$$

where $\|(u_0, \mathbf{p}_0)\|_{\Omega_R} = \| u_0 \|_{H^1(\Omega_R)} + \| \mathrm{div}\mathbf{p}_0 \|_{L^2(\Omega_R)}$.

We remark that the stability estimate (2.12) means the solution of the scattering problem does not grow in time. In the absence of the source f, however, it is proved in Lax and Phillips [20] by using the semigroup theory of operators that the solution of the scattering problem tends to zero in any bounder domain as time goes to infinity. For star-shaped obstacles, the exponential decay of scattering solutions for the wave equations is well-known, see, e.g. Morawetz [21].

The proof of Theorem 2.1 which depends on the abstract inversion theorem of the Laplace transform and the a priori estimate for the Helmholtz equation will be given in §2.2. In the following we first consider the properties of the modified Bessel functions to be used in the paper.

2.1. The modified Bessel function. For $\nu \in \mathbb{C}$, the modified Bessel functions $K_\nu(z)$, where $z \in \mathbb{C}$, is the solution of the ordinary differential equation

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} - (z^2 + \nu^2)y = 0, \tag{2.13}$$

which satisfies the following asymptotic behavior as $|z| \to \infty$

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}.$$

The importance of the function $K_\nu(z)$ in mathematical physics lies in the fact that it is a solution of (2.13) which tends to zero exponentially as $z \to \infty$ through positive values. We refer to the treatise Watson [28] for extensive studies on the functions $K_\nu(z)$.

The following lemma is proved in [28, P.439].

Lemma 2.2 (Macdonald formula)**.** *For any* $\nu \in \mathbb{C}$ *and* $z_1, z_2 \in \mathbb{C}$ *satisfying*

$$|\arg z_1| < \pi, \quad |\arg z_2| < \pi \quad \text{and } |\arg(z_1 + z_2)| < \frac{1}{4}\pi,$$

we have

$$K_\nu(z_1)K_\nu(z_2) = \frac{1}{2}\int_0^\infty e^{-\frac{v}{2} - \frac{z_1^2+z_2^2}{2v}} K_\nu\left(\frac{z_1 z_2}{v}\right) \frac{dv}{v}.$$

Lemma 2.3. *For any* $\nu \in \mathbb{R}$ *and* $z \in \mathbb{C}$ *such that* $\mathrm{Re}\,(z) > 0$*, we have*

$$|K_\nu(z)|^2 = \frac{1}{2}\int_0^\infty e^{-\frac{|z|^2}{2w} - \frac{z^2+\bar{z}^2}{2|z|^2}w} K_\nu(w)\frac{dw}{w}.$$

Proof. Since $K_\nu(\bar{z}) = \overline{K_\nu(z)}$ for real ν, we have

$$|K_\nu(z)|^2 = K_\nu(z)\overline{K_\nu(z)} = K_\nu(z)K_\nu(\bar{z}).$$

Since $\mathrm{Re}\,(z) > 0$, we have $|\arg(z + \bar{z})| = 0 < \frac{\pi}{4}$ and thus we can use Lemma 2.2 to obtain

$$|K_\nu(z)|^2 = \frac{1}{2}\int_0^\infty e^{-\frac{v}{2} - \frac{z^2+\bar{z}^2}{2v}} K_\nu\left(\frac{|z|^2}{v}\right) \frac{dv}{v}.$$

This proves the lemma after the change of variable $w = |z|^2/v$. □

An important consequence of this lemma is that for real ν, $K_\nu(z)$ has no zeros if $|\arg z| \le \frac{1}{2}\pi$ [28, P.511], which implies that $K_n(sR) \neq 0$ for any $n \in \mathbb{Z}, R > 0$, $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) > 0$. This justifies the writing of $K_n(sR)$ in the denominator in (2.3).

The following integral representation of $K_\nu(z)$ is useful in our analysis [28, P.181].

Lemma 2.4 (Schläfli integral representation)**.** *For any* $\nu \in \mathbb{R}$ *and* $z \in \mathbb{C}$ *such that* $|\arg(z)| < \frac{\pi}{2}$*, we have*

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)}\cosh(\nu t)dt.$$

Lemma 2.5. *For any* $\nu \in \mathbb{R}$ *and* $z \in \mathbb{C}$ *such that* $|\arg(z)| < \frac{\pi}{2}$*, we have*

$$K_\nu(|z|) \le |K_\nu(z)| \le K_\nu(\mathrm{Re}\,(z)).$$

Proof. First by Lemma 2.4 we have

$$\begin{aligned} |K_\nu(z)| &= \left|\int_0^\infty e^{-z\cosh(t)}\cosh(\nu t)dt\right| \\ &\le \int_0^\infty e^{-\mathrm{Re}\,(z)\cosh(t)}\cosh(\nu t)dt \\ &= K_\nu(\mathrm{Re}\,(z)). \end{aligned}$$

Next, if $z = z_1 + \mathbf{i}z_2$, $z_1, z_2 \in \mathbb{R}$, then

$$\frac{z^2+\bar{z}^2}{2|z|^2} = \frac{z_1^2 - z_2^2}{z_1^2+z_2^2} = 1 - \frac{2z_2^2}{z_1^2+z_2^2},$$

which, by Lemma 2.3, yields

$$\begin{aligned} |K_\nu(z)|^2 &= \frac{1}{2}\int_0^\infty e^{-\frac{|z|^2}{2w}-w}\cdot e^{\frac{2z_2^2}{z_1^2+z_2^2}w}K_\nu(w)\frac{dw}{w} \\ &\ge \frac{1}{2}\int_0^\infty e^{-\frac{|z|^2}{2w}-w}K_\nu(w)\frac{dw}{w} \\ &= K_\nu(|z|)^2. \end{aligned}$$

This completes the proof. □

Lemma 2.6. *For any* $\nu \in \mathbb{R}$ *and* $\rho_1 > \rho_2 > 0$*, we have*

$$K_\nu(\rho_1) \le e^{-(\rho_1-\rho_2)}K_\nu(\rho_2).$$

Proof. This is a direct consequence of the Schläfli integral representation in Lemma 2.3 and the fact that $\cosh(t) \ge 1$ for $t \ge 0$. □

2.2. The proof of Theorem 2.1. We start with the following lemma which can be proved by the standard energy argument.

Lemma 2.7. *Let the assumptions (H1)-(H4) be satisfied. Let* $U(\mathbf{x}, t)$ *be the solution of the following problem*

$$\begin{aligned} &\partial_{tt}U - \Delta U = 0 \quad \text{in } \Omega_R \times (0,T), \\ &\nabla U\cdot\mathbf{n}_D = 0 \ \text{ on } \Gamma_D, \qquad U = 0 \ \text{ on } \Gamma_R, \\ &U|_{t=0} = u_0, \ \ \partial_t U|_{t=0} = -\mathrm{div}\mathbf{p}_0 \ \text{ in } \Omega_R. \end{aligned}$$

Then

$$\begin{aligned} &\|\partial_t U\|^2_{L^2(\Omega_R)} + \|\nabla U\|^2_{L^2(\Omega_R)} = \|\nabla u_0\|^2_{L^2(\Omega_R)} + \|\mathrm{div}\mathbf{p}_0\|^2_{L^2(\Omega_R)}, \\ &\|\partial_{tt} U\|^2_{L^2(\Omega_R)} + \|\nabla(\partial_t U)\|^2_{L^2(\Omega_R)} = \|\Delta u_0\|^2_{L^2(\Omega_R)} + \|\mathrm{div}\mathbf{p}_0\|^2_{L^2(\Omega_R)}, \\ &\|\partial_{ttt} U\|^2_{L^2(\Omega_R)} + \|\nabla(\partial_{tt} U)\|^2_{L^2(\Omega_R)} = \|\Delta u_0\|^2_{L^2(\Omega_R)} + \|\Delta\mathrm{div}\mathbf{p}_0\|^2_{L^2(\Omega_R)}. \end{aligned}$$

Denote by $\mathbf{P} = \mathbf{p}_0 - \int_0^t \nabla U$. Let $u' = u - U$ and $\mathbf{p}' = \mathbf{p} - \mathbf{P}$. Then by (1.1) we know that

$$\frac{\partial u'}{\partial t} = -\mathrm{div}\mathbf{p}' + f, \qquad \frac{\partial \mathbf{p}'}{\partial t} = -\nabla u'. \tag{2.14}$$

By (H4), the boundary condition (1.2) becomes

$$\mathbf{p}' \cdot \mathbf{n}_D = 0 \quad \text{on } \Gamma_D. \tag{2.15}$$

By (2.4) we have

$$\mathbf{p}' \cdot \hat{\mathbf{x}} + \mathscr{T}(u') = -\mathbf{P} \cdot \hat{\mathbf{x}} \quad \text{on } \Gamma_R. \tag{2.16}$$

It is obvious that

$$u'|_{t=0} = 0, \qquad \mathbf{p}'|_{t=0} = 0. \tag{2.17}$$

Let $u'_{\mathrm{L}} = \mathscr{L}(u'), \mathbf{p}'_{\mathrm{L}} = \mathscr{L}(\mathbf{p}')$, and $\mathbf{P}_{\mathrm{L}} = \mathscr{L}(\mathbf{P})$. Then by taking the Laplace transform of (2.14)-(2.16) we obtain

$$su'_{\mathrm{L}} = -\mathrm{div}\mathbf{p}'_{\mathrm{L}} + f_{\mathrm{L}}, \quad s\mathbf{p}'_{\mathrm{L}} = -\nabla u'_{\mathrm{L}} \quad \text{in } \Omega_R, \tag{2.18}$$

$$\mathbf{p}'_{\mathrm{L}} \cdot \mathbf{n}_D = 0 \quad \text{on } \Gamma_D, \quad \mathbf{p}'_{\mathrm{L}} \cdot \hat{\mathbf{x}} = -Gu'_{\mathrm{L}} - \mathbf{P}_{\mathrm{L}} \cdot \hat{\mathbf{x}} \quad \text{on } \Gamma_R. \tag{2.19}$$

Notice that (2.18)-(2.19) is the standard scattering problem of the Helmholtz equation for u'_{L} whose well-posedness is guaranteed. Our strategy to show the well-posedness of (2.14)-(2.17) and thus (2.6)-(2.9) is to show the inverse Laplace transform of the solution $(u'_{\mathrm{L}}, \mathbf{p}'_{\mathrm{L}})$ of (2.18)-(2.19) is existent.

We first recall the following theorem in Treves [26, Theorem 43.1] which is the analog of the Paley-Wiener-Schwarz theorem for the Fourier transform of the distributions with compact support in the case of Laplace transform .

Lemma 2.8. *Let $\mathbf{h}(s)$ denote a holomorphic function in the half-plane $\mathrm{Re}\,(s) > \sigma_0$, valued in the Banach space E. The following conditions are equivalent:*
(i) *there is a distribution $T \in \mathcal{D}'_+(E)$ whose Laplace transform is equal to $\mathbf{h}(s)$;*
(ii) *there is a σ_1 real, $\sigma_0 \le \sigma_1 < \infty$, a constant $C > 0$, and an integer $k \ge 0$ such that, for all complex numbers s, $\mathrm{Re}\,(s) > \sigma_1$,*

$$\|\,\mathbf{h}(s)\,\|_E \le C(1 + |s|)^k.$$

Here $\mathcal{D}'_+$ is the space of distributions on the real line which vanish identically in the open negative half-line.

The following lemma on the Helmholtz scattering problems is of independent interest.

Lemma 2.9. *Let $s = s_1 + \mathbf{i}s_2$, $s_1 > 0, s_2 \in \mathbb{R}$. For any $g \in L^2(\Omega_R)$ and $\mu \in H^{-1/2}(\Gamma_R)$, let w be the weak solution of the following scattering problem*

$$-\Delta w + s^2 w = g \quad \text{in } \Omega_R, \tag{2.20}$$

$$\frac{\partial w}{\partial \mathbf{n}_D} = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial w}{\partial r} = sGw + \mu \quad \text{on } \Gamma_R. \tag{2.21}$$

Then there exists a constant C independent of s such that

$$\|\,\nabla w\,\|_{L^2(\Omega_R)} + \|\,sw\,\|_{L^2(\Omega_R)} \le \frac{C}{s_1}\Big(\|\,g\,\|_{L^2(\Omega_R)} + \|\,\mu\,\|_{H^{-1/2}(\Gamma_R)} + \|\,\bar{s}\mu\,\|_{H^{-1/2}(\Gamma_R)}\Big).$$

Proof. By testing (2.20) with any $v \in H^1(\Omega_R)$ and using the boundary conditions (2.21) we know that

$$\int_{\Omega_R} \Big(\nabla w \cdot \nabla \bar{v} + s^2 w \bar{v}\Big) dx - \langle sGw, v\rangle_{\Gamma_R} = (g, v) + \langle \mu, v\rangle_{\Gamma_R}, \tag{2.22}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega_R)$, and $\langle\cdot,\cdot\rangle_{\Gamma_R}$ is the duality pairing between $H^{-1/2}(\Gamma_R)$ and $H^{1/2}(\Gamma_R)$. Since $s^2 = s_1^2 - s_2^2 + \mathbf{i}(2s_1s_2)$, by choosing $v = w$ in (2.22) and taking respectively the real and imaginary part of the equation we get

$$\int_{\Omega_R} \Big[|\nabla w|^2 + (s_1^2 - s_2^2)|w|^2\Big] - \mathrm{Re}\,\langle sGw, w\rangle_{\Gamma_R} = \mathrm{Re}\Big[(g, w) + \langle \mu, w\rangle_{\Gamma_R}\Big], \tag{2.23}$$

$$2s_1s_2 \int_{\Omega_R} |w|^2 - \mathrm{Im}\,\langle sGw, w\rangle_{\Gamma_R} = \mathrm{Im}\Big[(g, w) + \langle \mu, w\rangle_{\Gamma_R}\Big]. \tag{2.24}$$

In the domain $\mathbb{R}^2\backslash\bar{B}_R$, we let

$$w(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{K_n(sr)}{K_n(sR)} w_n e^{\mathbf{i}n\theta}, \qquad w_n = \frac{1}{2\pi}\int_0^{2\pi} w(R,\theta) e^{-\mathbf{i}n\theta} d\theta. \tag{2.25}$$

It is clear that w satisfies the Helmholtz equation

$$-\Delta w + s^2 w = 0 \quad \text{in } \mathbb{R}^2\backslash\bar{B}_R, \tag{2.26}$$

and the Sommerfeld radiation condition

$$\sqrt{r}\left(\frac{\partial w}{\partial r} + sw\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{2.27}$$

Multiplying (2.26) by $\bar{w}$ and integrating over $B_\rho\backslash\bar{B}_R$, where $\rho > R$, we have

$$2s_1s_2 \int_{B_\rho\backslash\bar{B}_R} |w|^2 dx - \mathrm{Im}\int_{\Gamma_R\cup\Gamma_\rho} \frac{\partial w}{\partial \mathbf{n}} \bar{w} ds = 0.$$

Thus

$$2s_1s_2^2 \int_{B_\rho\backslash\bar{B}_R} |w|^2 - \mathrm{Im}\left(s_2\int_{\Gamma_\rho} \frac{\partial w}{\partial r}\bar{w}\right) + \mathrm{Im}\left(s_2\int_{\Gamma_R} \frac{\partial w}{\partial r}\bar{w}\right) = 0,$$

which yields, since $s_1 > 0$,

$$-\mathrm{Im}\left(s_2\int_{\Gamma_R} \frac{\partial w}{\partial r}\bar{w}\right) \geq -\mathrm{Im}\left(s_2\int_{\Gamma_\rho} \frac{\partial w}{\partial r}\bar{w}\right).$$

By Lemmas 2.5-2.6 we have, for $\rho > \frac{|s|}{s_1}R$,

$$\begin{aligned}\int_{\Gamma_\rho} |w|^2 = 2\pi \sum_{n=-\infty}^{\infty} \rho \frac{|K_n(s\rho)|^2}{|K_n(sR)|^2}|w_n|^2 &\leq 2\pi \sum_{n=-\infty}^{\infty} \rho \frac{|K_n(s_1\rho)|^2}{|K_n(|s|R)|^2}|w_n|^2 \\ &\leq \frac{\rho}{R} e^{-(s_1\rho - |s|R)} \| w \|_{L^2(\Gamma_R)}^2 \\ &\to 0, \quad \text{as } \rho \to \infty.\end{aligned}$$

On the other hand, by (2.27),

$$\left\| \frac{\partial w}{\partial r} + sw \right\|_{L^2(\Gamma_\rho)} \to 0, \quad \text{as } \rho \to \infty.$$

Thus we conclude that

$$-\mathrm{Im}\left(s_2\int_{\Gamma_R}\frac{\partial w}{\partial r}\bar{w}\right)\ge -\lim_{\rho\to\infty}\mathrm{Im}\left(s_2\int_{\Gamma_\rho}\frac{\partial w}{\partial r}\bar{w}\right)=0, \tag{2.28}$$

which, together with (2.24), implies

$$2s_1s_2^2\int_{\Omega_R}|w|^2dx\le s_2\mathrm{Im}\Big[(g,w)+\langle\mu,w\rangle_{\Gamma_R}\Big]. \tag{2.29}$$

Next, by Lemma 2.4,

$$|K_n(sr)|^2=\frac{1}{2}\int_0^\infty e^{-\frac{|s|^2r^2}{2\tau}-\frac{s^2+\bar{s}^2}{2|s|^2}\tau}K_n(\tau)\frac{d\tau}{\tau},$$

which is monotonely decreasing in r, thus $\frac{d}{dr}|K_n(sr)|^2\le 0$ and consequently

$$-\mathrm{Re}\left(\int_{\Gamma_R}\frac{\partial w}{\partial r}\bar{w}\right)=-\frac{1}{2}\int_{\Gamma_R}\frac{\partial}{\partial r}|w|^2=-\pi R\sum_{n=-\infty}^{\infty}\frac{d}{dr}\left[\frac{|K_n(sr)|^2}{|K_n(sR)|^2}\right]_{r=R}|w_n|^2\ge 0.$$

Therefore, by (2.23) and (2.29) we obtain

$$\begin{aligned}
&\int_{\Omega_R}\left[|\nabla w|^2+(s_1^2+s_2^2)|w|^2\right]dx\\
\le\ & 2s_2^2\int_{\Omega_R}|w|^2dx+\mathrm{Re}\Big[(g,w)+\langle\mu,w\rangle_{\Gamma_R}\Big]\\
\le\ & \frac{s_2}{s_1}\mathrm{Im}\Big[(g,w)+\langle\mu,w\rangle_{\Gamma_R}\Big]+\mathrm{Re}\Big[(g,w)+\langle\mu,w\rangle_{\Gamma_R}\Big]\\
=\ & \frac{1}{s_1}\mathrm{Re}\Big[(\bar{s}g,w)+\langle\bar{s}\mu,w\rangle_{\Gamma_R}\Big]\\
\le\ & \frac{C}{s_1}\|g\|_{L^2(\Omega_R)}\|sw\|_{L^2(\Omega_R)}\\
&+\frac{C}{s_1}\left(\|\bar{s}\mu\|_{H^{-1/2}(\Gamma_R)}\|\nabla w\|_{L^2(\Omega_R)}+\|\mu\|_{H^{-1/2}(\Gamma_R)}\|sw\|_{L^2(\Omega_R)}\right).
\end{aligned}$$

The lemma now follows from the Cauchy-Schwarz inequality. □

Lemma 2.10. *Let $s=s_1+\mathbf{i}s_2$ with $s_1>0, s_2\in\mathbb{R}$. We have*

$$-\mathrm{Re}\left(\frac{K_n'(sR)}{K_n(sR)}\right)\ge 0.$$

Proof. Since $w=K_n(sr)e^{\mathbf{i}n\theta}$ satisfies the Helmholtz equation (2.26) with the radiation condition. The argument to derive (2.28) implies that

$$-\mathrm{Im}\left(s_2s\frac{K_n'(sR)}{K_n(sR)}\right)\ge 0.$$

Moreover, since $\frac{d}{dr}|K_n(sr)|^2\le 0$ as noted in the proof of last lemma, we have $-\mathrm{Re}\left(s\frac{K_n'(sR)}{K_n(sR)}\right)\ge 0$. Now denote $s\frac{K_n'(sR)}{K_n(sR)}=\gamma_1+\mathbf{i}\gamma_2$ for $\gamma_1,\gamma_2\in\mathbb{R}$, then we know that $\gamma_1\le 0, s_2\gamma_2\le 0$. Therefore, since $s=s_1+\mathbf{i}s_2, s_1>0$, we obtain

$$\mathrm{Re}\left(\frac{K_n'(sR)}{K_n(sR)}\right)=\mathrm{Re}\left(\frac{\gamma_1+\mathbf{i}\gamma_2}{s}\right)=\frac{1}{|s|^2}\mathrm{Re}\left(\bar{s}\gamma\right)=\frac{1}{|s|^2}(s_1\gamma_1+s_2\gamma_2)\le 0.$$

This completes the proof. □

Lemma 2.11. *For any $\xi \in L^2(0,T;H^{1/2}(\Gamma_R))$, we have*

$$-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \mathcal{T}(\xi),\xi\rangle_{\Gamma_R} dt \ge 0.$$

Proof. Let

$$\xi(\theta,t)=\sum_{n=-\infty}^{\infty}\xi_n(t)e^{\mathbf{i}n\theta},\quad \xi_n(t)=\frac{1}{2\pi}\int_0^{2\pi}\xi(\theta,t)e^{-\mathbf{i}n\theta}dt.$$

By (2.5) we know that

$$\langle \mathcal{T}(\xi),\xi\rangle_{\Gamma_R}=2\pi R\sum_{n=-\infty}^{\infty}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right)*\xi_n\right]\bar{\xi}_n.$$

Denote $\tilde{\xi}_n=\xi_n\chi_{[0,T]}$, where $\chi_{[0,T]}$ is the characteristic function of the interval $(0,T)$. Then

$$\begin{aligned}
&-\int_0^T e^{-2s_1 t}\langle \mathcal{T}(\xi),\xi\rangle_{\Gamma_R}dt\\
=&\ -2\pi R\sum_{n=-\infty}^{\infty}\int_0^T e^{-2s_1 t}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right)*\xi_n\right]\bar{\xi}_n dt\\
=&\ -2\pi R\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2s_1 t}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right)*\tilde{\xi}_n\right]\bar{\tilde{\xi}}_n dt.
\end{aligned}$$

By the formula for the inverse Laplace transform $g(t)=\mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(g)(s_1+\mathbf{i}s_2))$, where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2, we know from the Plancherel identity that

$$-\int_0^T e^{-2s_1 t}\langle \mathcal{T}(\xi),\xi\rangle_{\Gamma_R}dt=-R\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{K_n'(sR)}{K_n(sR)}|\mathscr{L}(\tilde{\xi}_n)|^2 ds_2.$$

This completes the proof by Lemma 2.10. □

Now we are in the position to prove the main result of this section.

Proof of Theorem 2.1. Our starting point is the scattering problem of the Helmholtz equation (2.18)-(2.19) which has a unique solution $u_{\mathrm{L}}'\in H^1(\Omega_R)$ and $\mathbf{p}_{\mathrm{L}}'\in H(\mathrm{div};\Omega_R)$. It is obvious that u_{L}' satisfies

$$\begin{aligned}
&-\Delta u_{\mathrm{L}}'+s^2u_{\mathrm{L}}'=sf_{\mathrm{L}}\quad \text{in }\Omega_R,\\
&\frac{\partial u_{\mathrm{L}}'}{\partial \mathbf{n}_D}=0\quad\text{on }\Gamma_D,\qquad \frac{\partial u_{\mathrm{L}}'}{\partial r}=sGu_{\mathrm{L}}'+s\mathbf{P}_{\mathrm{L}}\cdot\hat{\mathbf{x}}\quad\text{on }\Gamma_R.
\end{aligned}$$

By Lemma 2.9, there exists a constant $C>0$ independent of s such that

$$\begin{aligned}
&\|\nabla u_{\mathrm{L}}'\|_{L^2(\Omega_R)}+\|su_{\mathrm{L}}'\|_{L^2(\Omega_R)}\\
\le&\ \frac{C}{s_1}\left(\|sf_{\mathrm{L}}\|_{L^2(\Omega_R)}+\|s\mathbf{P}_{\mathrm{L}}\cdot\hat{\mathbf{x}}\|_{H^{-1/2}(\Gamma_R)}+\||s|^2\mathbf{P}_{\mathrm{L}}\cdot\hat{\mathbf{x}}\|_{H^{-1/2}(\Gamma_R)}\right).
\end{aligned}$$

Moreover, by (2.18),

$$\begin{aligned}
&\|\mathrm{div}\mathbf{p}_{\mathrm{L}}'\|_{L^2(\Omega_R)}+\|s\mathbf{p}_{\mathrm{L}}'\|_{L^2(\Omega_R)}\\
\le&\ \frac{C}{s_1}\left(\|f_{\mathrm{L}}\|_{L^2(\Omega_R)}+\|sf_{\mathrm{L}}\|_{L^2(\Omega_R)}\right)\\
+&\ \frac{C}{s_1}\left(\|s\mathbf{P}_{\mathrm{L}}\cdot\hat{\mathbf{x}}\|_{H^{-1/2}(\Gamma_R)}+\||s|^2\mathbf{P}_{\mathrm{L}}\cdot\hat{\mathbf{x}}\|_{H^{-1/2}(\Gamma_R)}\right).
\end{aligned}\tag{2.30}$$

By [26, Lemma 44.1], $u'_{\rm L}, \mathbf{p}'_{\rm L}$ are holomorphic functions of s on the half plane $\mathrm{Re}\,(s) > \gamma > 0$, where γ is any positive constant. By Lemma 2.8 the inverse Laplace transform of $u'_{\rm L}, \mathbf{p}'_{\rm L}$ are existent and supported in $[0,\infty]$. Denote by $u' = \mathscr{L}^{-1}(u'_{\rm L})$, $\mathbf{p}' = \mathscr{L}^{-1}(\mathbf{p}'_{\rm L})$. Then, since $u'_{\rm L} = \mathscr{L}(u') = \mathscr{F}(e^{-s_1 t}u')$, where $\mathscr{F}$ is the Fourier transform in s_2, by the Parseval identity and (2.30), we have

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1 t}\left(\| \nabla u' \|^2_{L^2(\Omega_R)} + \| \partial_t u' \|^2_{L^2(\Omega_R)}\right) dt \\
= \;& 2\pi \int_{-\infty}^\infty \left(\| \nabla u'_{\rm L} \|^2_{L^2(\Omega_R)} + \| s u'_{\rm L} \|^2_{L^2(\Omega_R)}\right) ds_2 \\
\le \;& \frac{C}{s_1^2}\int_{-\infty}^\infty \| s f_{\rm L} \|^2_{L^2(\Omega_R)} ds_2 \\
+ \;& \frac{C}{s_1^2}\int_{-\infty}^\infty \left(\| s\mathbf{P}_{\rm L}\cdot\hat{\mathbf{x}} \|^2_{H^{-1/2}(\Gamma_R)} + \| |s|^2\mathbf{P}_{\rm L}\cdot\hat{\mathbf{x}} \|^2_{H^{-1/2}(\Gamma_R)}\right) ds_2.
\end{aligned}
$$

Since $f|_{t=0} = 0$ in Ω_R, $\mathbf{P}\cdot\hat{\mathbf{x}}|_{t=0} = \partial_t\mathbf{P}\cdot\hat{\mathbf{x}}|_{t=0} = 0$ on Γ_R, we have $\mathscr{L}(\partial_t f) = s f_{\rm L}$ in Ω_R and $\mathscr{L}(\partial_t\mathbf{P}\cdot\hat{\mathbf{x}}) = s\mathbf{P}_{\rm L}\cdot\hat{\mathbf{x}}$ on Γ_R. Moreover, notice that

$$
|s|^2\mathbf{P}_{\rm L}\cdot\hat{\mathbf{x}} = (2s_1 - s)s\mathbf{P}_{\rm L}\cdot\hat{\mathbf{x}} = 2s_1\mathscr{L}(\partial_t\mathbf{P}\cdot\hat{\mathbf{x}}) - \mathscr{L}(\partial_t^2\mathbf{P}\cdot\hat{\mathbf{x}}) \quad \text{on } \Gamma_R,
$$

we have

$$
\begin{aligned}
&\int_0^\infty e^{-2s_1 t}\left(\| \nabla u' \|^2_{L^2(\Omega_R)} + \| \partial_t u' \|^2_{L^2(\Omega_R)}\right) dt \\
\le \;& \frac{C}{s_1^2}\int_{-\infty}^\infty \| \mathscr{L}(\partial_t f) \|^2_{L^2(\Omega_R)} ds_2 \\
+ \;& C\left(1 + \frac{1}{s_1^2}\right)\int_{-\infty}^\infty \left(\| \mathscr{L}(\partial_t\mathbf{P}\cdot\hat{\mathbf{x}}) \|^2_{H^{-1/2}(\Gamma_R)} + \| \mathscr{L}(\partial_t^2\mathbf{P}\cdot\hat{\mathbf{x}}) \|^2_{H^{-1/2}(\Gamma_R)}\right) ds_2.
\end{aligned}
$$

Again by Parseval identity

$$
\begin{aligned}
(2.31)\quad &\int_0^\infty e^{-2s_1 t}\left(\| \nabla u' \|^2_{L^2(\Omega_R)} + \| \partial_t u' \|^2_{L^2(\Omega_R)}\right) dt \\
= \;& \frac{C}{s_1^2}\int_0^\infty e^{-2s_1 t}\| \partial_t f \|^2_{L^2(\Omega_R)} ds_2 \\
+ \;& C\left(1 + \frac{1}{s_1^2}\right)\int_0^\infty e^{-2s_1 t}\left(\| \partial_t\mathbf{P}\cdot\hat{\mathbf{x}} \|^2_{H^{-1/2}(\Gamma_R)} + \| \partial_t^2\mathbf{P}\cdot\hat{\mathbf{x}} \|^2_{H^{-1/2}(\Gamma_R)}\right) dt < \infty.
\end{aligned}
$$

This proves $u' \in L^2(0,T;H^1(\Omega_R)) \cap H^1(0,T;L^2(\Omega_R))$. Similarly, by (2.30), we have $\mathbf{p}' \in L^2(0,T;H(\mathrm{div};\Omega_R)) \cap H^1(0,T;L^2(\Omega_R))$. Moreover, by $Gu'_{\rm L} = -\mathbf{p}'_{\rm L}\cdot\hat{\mathbf{x}}$ on Γ_R and $\mathbf{p}' \in L^2(0,T;H(\mathrm{div};\Omega_R))$, we deduce that $\mathcal{T}(u') \in L^2(0,T;H^{-1/2}(\Gamma_R))$. By taking the inverse Laplace transform in (2.18)-(2.19) and using the definition of $u' = u - U$, $\mathbf{p}' = \mathbf{p} - \mathbf{P}$, one can easily show that $(u,\mathbf{p})$ satisfies (2.10)-(2.11).

It remains to prove the stability estimate (2.12). By (1.1) we know that u satisfies the wave equation

$$
\frac{\partial^2 u}{\partial t^2} - \Delta u = \frac{\partial f}{\partial t} \quad \text{in } \Omega_R \times (0,T).
$$

We multiply the equation by $\partial_t\bar{u}$, integrate over Ω_R, and use the boundary conditions (1.2), (2.4) to get

$$
\frac{1}{2}\frac{d}{dt}\left(\| \partial_t u \|^2_{L^2(\Omega_R)} + \| \nabla u \|^2_{L^2(\Omega_R)}\right) - \langle \partial_t(\mathcal{T}(u)), \partial_t u\rangle_{\Gamma_R} = (\partial_t f, \partial_t u).
$$

By (2.5) we know that

$$\langle \partial_t(\mathcal{T}(u)), \partial_t u\rangle_{\Gamma_R} = 2\pi R \sum_{n=-\infty}^{+\infty} \frac{d}{dt}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right) * u_n(R,t)\right] \cdot \frac{d\bar{u}_n}{dt}(R,t),$$

where $u_n(R,t) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta,t)e^{-\mathbf{i}n\theta}d\theta$. Since u_0 has compact support inside Ω_R, $u_n(R,0) = 0$, we have

$$\frac{d}{dt}\left[\mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right) * u_n(R,t)\right] = \mathscr{L}^{-1}\left(\frac{K_n'(sR)}{K_n(sR)}\right) * \frac{du_n(R,t)}{dt}.$$

Thus, by Lemma 2.11, we then get

$$-\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \partial_t(\mathcal{T}(u)), \partial_t u\rangle_{\Gamma_R} dt = -\mathrm{Re}\int_0^T e^{-2s_1 t}\langle \mathcal{T}(\partial_t u), \partial_t u\rangle_{\Gamma_R} dt \geq 0.$$

By (2.32) we then obtain after integration by parts

$$\begin{aligned} & e^{-2s_1 t}\left(\| \partial_t u \|^2_{L^2(\Omega_R)} + \| \nabla u \|^2_{L^2(\Omega_R)}\right) - \left(\| \mathrm{div}\mathbf{p}_0 \|^2_{L^2(\Omega_R)} + \| \nabla u_0 \|^2_{L^2(\Omega_R)}\right) \\ \leq\ & 2\| e^{-s_1 t}\partial_t f \|_{L^1(0,T;L^2(\Omega_R))} \| e^{-s_1 t}\partial_t u\|_{L^\infty(0,T;L^2(\Omega_R))}, \quad \forall t \in (0,T). \end{aligned}$$

Now the desired stability estimate follows by the standard argument and letting $s_1 \to 0$. □

3. The PML equation and the well-posedness

Now we turn to the introduction of the absorbing PML layer. We surround the domain Ω_R with a PML layer $\Omega^{\mathrm{PML}} = \{\mathbf{x} \in \mathbb{R}^2 : R < |\mathbf{x}| < \rho\}$. In the rest of this paper we assume $\rho \leq CR$ for some generic fixed constant $C > 0$.

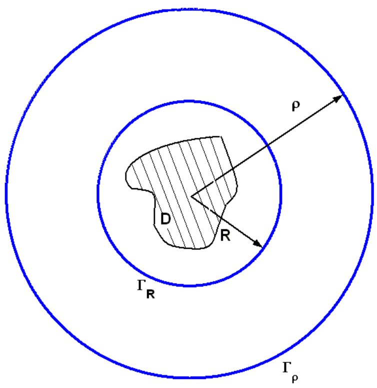

FIGURE 1. Setting of the scattering problem with the PML layer.

Let $\alpha(r) = 1 + s^{-1}\sigma(r)$ be the artificial medium property, where $\sigma \geq 0$ for $r \in \mathbb{R}$ and $\sigma = 0$ for $r \leq R$. We remark that here we allow the function σ can be discontinuous. In particular, we will assume σ is a positive constant for $r > R$ in Sections 4 and 5. Denote $\tilde{r}$ the complex radius

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(\tau)d\tau = r\beta(r) & \text{if } r \geq R. \end{cases}$$

It is easy to see that for $r \geq R$, $\beta(r) = 1 + s^{-1}\hat{\sigma}(r)$, where

$$\hat{\sigma}(r) = \frac{1}{r}\int_0^r \sigma(\tau)d\tau.$$

We follow the development in [22] to introduce the PML equations. The starting point is the series representation of $u_{\rm L} = \mathscr{L}(u)$ outside B_R in (2.2)

$$u_{\rm L} = \sum_{n=-\infty}^{\infty} \frac{K_n(sr)}{K_n(sR)} u_{\rm L}^n(R,s)e^{{\bf i}n\theta}, \qquad u_{\rm L}^n(R,s) = \frac{1}{2\pi}\int_0^{2\pi} u_{\rm L}(R,\theta,s)e^{-{\bf i}n\theta}d\theta.$$

Based on the observation that $K_n(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$ as $|z| \to \infty$, we define the PML extension $\tilde{u}_{\rm L}$ as

$$\tilde{u}_{\rm L}(r,\theta,s) = \sum_{n=-\infty}^{\infty} \frac{K_n(s\tilde{r})}{K_n(sR)} u_{\rm L}^n(R,s)e^{{\bf i}n\theta}, \qquad \forall r > R. \tag{3.1}$$

Heuristically $\tilde{u}_{\rm L}(\tilde{r},\theta,s)$ will decay exponentially for large r. It is obvious that $\tilde{u}_{\rm L}$ satisfies $-\tilde{\Delta}\tilde{u}_{\rm L} + s^2\tilde{u}_{\rm L} = 0$ outside B_R, where $\tilde{\Delta} = \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial}{\partial\tilde{r}}\right) + \frac{1}{\tilde{r}^2}\frac{\partial^2}{\partial^2\theta}$ is the Laplace operator with respect to $(\tilde{r},\theta)$. Since $\tilde{r} = r\beta$ and $\dfrac{d\tilde{r}}{dr} = \alpha$, we know by using the chain rule that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\beta r}{\alpha}\frac{\partial\tilde{u}_{\rm L}}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\alpha}{\beta r}\frac{\partial\tilde{u}_{\rm L}}{\partial\theta}\right) + s^2\alpha\beta\tilde{u}_{\rm L} = 0. \tag{3.2}$$

The desired time-domain PML system will be obtained by taking the inverse Laplace transform of (3.2). To that purpose, we define

$$s\tilde{\mathbf{p}}_{{\rm L},r}^* = -\frac{\partial\tilde{u}_{\rm L}}{\partial r}, \quad s\tilde{\mathbf{p}}_{{\rm L},\theta}^* = -\frac{1}{r}\frac{\partial\tilde{u}_{\rm L}}{\partial\theta}, \quad s\tilde{u}_{\rm L}^* = \tilde{u}_{\rm L}, \tag{3.3}$$

and

$$\tilde{\mathbf{p}}_{{\rm L},r} = \frac{\beta}{\alpha}\tilde{\mathbf{p}}_{{\rm L},r}^*, \quad \tilde{\mathbf{p}}_{{\rm L},\theta} = \frac{\alpha}{\beta}\tilde{\mathbf{p}}_{{\rm L},\theta}^*, \tag{3.4}$$

Thus (3.2) becomes, since $s\alpha\beta = s + \sigma + \hat{\sigma} + s^{-1}\sigma\hat{\sigma}$,

$$s\tilde{u}_{\rm L} + (\sigma+\hat{\sigma})\tilde{u}_{\rm L} + \sigma\hat{\sigma}\tilde{u}_{\rm L}^* + {\rm div}\tilde{\mathbf{p}}_{\rm L} = 0. \tag{3.5}$$

Let, for $r > R$,

$$\tilde{u} = \mathscr{L}^{-1}(\tilde{u}_{\rm L}), \quad \tilde{\mathbf{p}} = \mathscr{L}^{-1}(\tilde{\mathbf{p}}_{\rm L}), \quad \tilde{u}^* = \mathscr{L}^{-1}(\tilde{u}_{\rm L}^*), \quad \tilde{\mathbf{p}}^* = \mathscr{L}^{-1}(\tilde{\mathbf{p}}_{\rm L}^*) \tag{3.6}$$

with $\tilde{u}|_{t=0} = 0, \tilde{\mathbf{p}}|_{t=0} = 0$, $\tilde{u}^*|_{t=0} = 0$, and $\tilde{\mathbf{p}}^*|_{t=0} = 0$.

Notice that $s\alpha = s + \sigma$, $s\beta = s + \hat{\sigma}$, by taking the inverse Laplace transform in (3.3)-(3.5), we get

$$\frac{\partial\tilde{\mathbf{p}}^*}{\partial t} = -\nabla\tilde{u}, \quad \frac{\partial\tilde{u}^*}{\partial t} = \tilde{u}, \tag{3.7}$$

$$\frac{\partial\tilde{\mathbf{p}}_r}{\partial t} + \sigma\tilde{\mathbf{p}}_r = \frac{\partial\tilde{\mathbf{p}}_r^*}{\partial t} + \hat{\sigma}\tilde{\mathbf{p}}_r^*, \quad \frac{\partial\tilde{\mathbf{p}}_\theta}{\partial t} + \hat{\sigma}\tilde{\mathbf{p}}_\theta = \frac{\partial\tilde{\mathbf{p}}_\theta^*}{\partial t} + \sigma\tilde{\mathbf{p}}_\theta^*, \tag{3.8}$$

$$\frac{\partial\tilde{u}}{\partial t} + (\sigma+\hat{\sigma})\tilde{u} + {\rm div}\tilde{\mathbf{p}} + \sigma\hat{\sigma}\tilde{u}^* = 0. \tag{3.9}$$

We rewrite (3.8) as

$$\frac{\partial\mathbf{p}}{\partial t} + \Lambda_1\tilde{\mathbf{p}} = \frac{\partial\mathbf{p}^*}{\partial t} + \Lambda_2\tilde{\mathbf{p}}^*, \tag{3.10}$$

where

$$\Lambda_1 = Q^T \begin{pmatrix} \sigma & 0 \\ 0 & \hat{\sigma} \end{pmatrix} Q, \quad \Lambda_2 = Q^T \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \sigma \end{pmatrix} Q, \quad Q = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The first order partial differential equations (3.7),(3.9),(3.10) for $(\tilde{u}, \tilde{\mathbf{p}}, \tilde{u}^*, \tilde{\mathbf{p}}^*)$ consist of the time-domain PML system for the acoustic scattering problem outside B_R. Since $\tilde{u} = u$, $\tilde{\mathbf{p}} = \mathbf{p}$ on Γ_R, $\tilde{u}, \tilde{\mathbf{p}}$ can be viewed as the extension of the solution $(u, \mathbf{p})$ of the problem (1.1)-(1.3). Moreover, since $\sigma = \hat{\sigma} = 0$ inside the circle B_R, if we set $\tilde{u} = \tilde{u}^* = u, \tilde{\mathbf{p}} = \tilde{\mathbf{p}}^* = \mathbf{p}$ in $(B_R \backslash \bar{D}) \times (0, T)$, then $(\tilde{u}, \tilde{\mathbf{p}}, \tilde{u}^*, \tilde{\mathbf{p}}^*)$ satisfies (3.7), (3.10), and, instead of (3.9),

$$\frac{\partial \tilde{u}}{\partial t} + \mathrm{div}\tilde{\mathbf{p}} + (\hat{\sigma} + \sigma)\tilde{u} + \sigma\hat{\sigma}\tilde{u}^* = f. \tag{3.11}$$

We summarize the above consideration in the following lemma.

Lemma 3.1. *Let* $(u, \mathbf{p})$ *be the solution of the problem* (2.6)-(2.9) *which is extended to be* $(\tilde{u}, \tilde{\mathbf{p}})$ *outside* B_R *according to* (3.6). *Let* $(\tilde{u}^*, \tilde{\mathbf{p}}^*)$ *be defined in* (3.6) *for* $r \geq R$ *and* $(\tilde{u}^*, \tilde{\mathbf{p}}^*) = (u, \mathbf{p})$ *for* $r < R$. *Then* $(\tilde{u}, \tilde{\mathbf{p}}, \tilde{u}^*, \tilde{\mathbf{p}}^*)$ *satisfies the PML system* (3.7),(3.10),(3.11) *outside* $\bar{D}$ *and satisfies the initial and boundary conditions* $\tilde{u}|_{t=0} = u_0, \tilde{\mathbf{p}}|_{t=0} = \mathbf{p}_0, \tilde{u}^*|_{t=0} = u_0, \tilde{\mathbf{p}}^*|_{t=0} = \mathbf{p}_0$ *in* $\mathbb{R}^2 \backslash \bar{D}$ *and* $\tilde{\mathbf{p}} \cdot \mathbf{n}_D = 0$ *on* $\Gamma_D \times (0, T)$.

We define the following initial-boundary value problem for $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ which is referred as the PML problem in the rest of this paper, where $\Omega_\rho = B_\rho \backslash \bar{D}$,

$$\frac{\partial \hat{u}}{\partial t} + \mathrm{div}\hat{\mathbf{p}} + (\sigma + \hat{\sigma})\hat{u} + \sigma\hat{\sigma}\hat{u}^* = f, \quad \frac{\partial \hat{u}^*}{\partial t} = \hat{u} \quad \text{in } \Omega_\rho \times (0, T), \tag{3.12}$$

$$\frac{\partial \hat{\mathbf{p}}^*}{\partial t} + \nabla \hat{u} = 0, \quad \frac{\partial \hat{\mathbf{p}}}{\partial t} + \Lambda_1 \hat{\mathbf{p}} = \frac{\partial \mathbf{p}^*}{\partial t} + \Lambda_2 \hat{\mathbf{p}}^* \quad \text{in } \Omega_\rho \times (0, T), \tag{3.13}$$

$$\hat{\mathbf{p}} \cdot \mathbf{n}_D = 0 \quad \text{on } \Gamma_D \times (0, T), \quad \hat{u} = 0 \quad \text{on } \Gamma_\rho \times (0, T), \tag{3.14}$$

$$\hat{u}|_{t=0} = u_0, \quad \hat{\mathbf{p}}|_{t=0} = \mathbf{p}_0, \quad \hat{u}^*|_{t=0} = u_0, \quad \hat{\mathbf{p}}^*|_{t=0} = \mathbf{p}_0 \quad \text{in } \Omega_\rho. \tag{3.15}$$

By the construction of the PML problem, $(\hat{u}, \hat{\mathbf{p}})$ is designed to approximate the solution $(u, \mathbf{p})$ of the original scattering problem in the domain $\Omega_R \times (0, T)$. The rigorous mathematical justification of this fact will be established in section 5.

Notice that (3.12)-(3.15) is a first order symmetric hyperbolic system whose well-posedness follows from the standard theory (see e.g. Chen [6]). Here we state the well-posedness of the PML problem (3.12)-(3.15) and omit the proof.

Theorem 3.2. *Let* $f \in L^\infty(0, T; L^2(\Omega_R))$. *Assume that* $u_0 \in H^1(\Omega_\rho)$, $\mathbf{p}_0 \in H(\mathrm{div}; \Omega_\rho)$ *are supported in* B_R *and satisfy the compatibility conditions* $\mathbf{p}_0 \cdot \mathbf{n}_D = 0$, $\nabla u_0 \cdot \mathbf{n}_D = 0$ *on* Γ_D. *Then the PML problem* (3.12)-(3.15) *has a unique strong solution* $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ *satisfying*

$$\hat{u} \in L^\infty(0, T; H^1(\Omega_\rho)) \cap W^{1,\infty}(0, T; L^2(\Omega_\rho)), \quad \hat{u}^* \in W^{1,\infty}(0, T; L^2(\Omega_\rho)),$$
$$\hat{\mathbf{p}} \in L^\infty(0, T; H(\mathrm{div}; \Omega_\rho)) \cap W^{1,\infty}(0, T; L^2(\Omega_\rho)), \quad \hat{\mathbf{p}}^* \in W^{1,\infty}(0, T; L^2(\Omega_\rho)).$$

4. The stability of the PML system

In this section we consider the stability of the initial boundary value problem of the PML system in the layer under the condition that the medium property σ is constant. In [3] and [4], the stability of the Cauchy problem of the PML system for the constant medium property is proved by using the energy estimate. Our argument is based on the method of the Laplace transform and the analysis in the frequency domain. In the rest of this paper we assume σ is a positive constant.

We consider the following initial boundary value problem of the PML system in the PML layer $\Omega^{\rm PML}=B_\rho\backslash\bar{B}_R$

$$\frac{\partial\phi}{\partial t}+{\rm div}\mathbf{\Phi}+(\sigma+\hat{\sigma})\phi+\sigma\hat{\sigma}\phi^*=0,\quad \frac{\partial\phi^*}{\partial t}=\phi\quad \text{in }\Omega^{\rm PML}\times(0,T), \tag{4.1}$$

$$\frac{\partial\mathbf{\Phi}^*}{\partial t}+\nabla\phi=0,\quad \frac{\partial\mathbf{\Phi}}{\partial t}+\Lambda_1\mathbf{\Phi}=\frac{\partial\mathbf{\Phi}^*}{\partial t}+\Lambda_2\mathbf{\Phi}^*\quad \text{in }\Omega^{\rm PML}\times(0,T), \tag{4.2}$$

$$\phi=0\quad \text{on }\Gamma_D\times(0,T),\qquad \phi=\xi\quad \text{on }\Gamma_\rho\times(0,T), \tag{4.3}$$

$$\phi|_{t=0}=0,\mathbf{\Phi}|_{t=0}=0,\phi^*|_{t=0}=0,\mathbf{\Phi}^*|_{t=0}=0\quad \text{in }\Omega^{\rm PML}. \tag{4.4}$$

Let $\phi_{\rm L}=\mathscr{L}(\phi),\mathbf{\Phi}_{\rm L}=\mathscr{L}(\mathbf{\Phi}),\phi^*_{\rm L}=\mathscr{L}(\phi^*)$, and $\mathbf{\Phi}^*_{\rm L}=\mathscr{L}(\mathbf{\Phi}^*)$, then we know that $\phi_{\rm L}$ satisfies the Helmholtz equation

$$-\nabla\cdot(A\nabla\phi_{\rm L})+s^2\alpha\beta\phi_{\rm L}=0\quad \text{in }\Omega^{\rm PML}, \tag{4.5}$$

where $A\nabla\phi_{\rm L}=\frac{\beta}{\alpha}\frac{\partial\phi_{\rm L}}{\partial r}\mathbf{e}_r+\frac{\alpha}{\beta r}\frac{\partial\phi_{\rm L}}{\partial\theta}\mathbf{e}_\theta$, $\mathbf{e}_r,\mathbf{e}_\theta$ are the unit vectors of the polar coordinates, and

$$s\mathbf{\Phi}_{\rm L}=-A\nabla\phi_{\rm L},\quad {\rm div}\mathbf{\Phi}_{\rm L}=-s\alpha\beta\phi_{\rm L}. \tag{4.6}$$

Let $a(\cdot,\cdot):H^1(\Omega^{\rm PML})\times H^1(\Omega^{\rm PML})\to\mathbb{C}$ be the sesqulinear form

$$a(\phi,\psi)=\int_R^\rho\int_0^{2\pi}\left(\frac{\beta r}{\alpha}\frac{\partial\phi}{\partial r}\frac{\partial\bar{\psi}}{\partial r}+\frac{\alpha}{\beta r}\frac{\partial\phi}{\partial\theta}\frac{\partial\bar{\psi}}{\partial\theta}+s^2\alpha\beta r\phi\bar{\psi}\right)drd\theta.$$

Lemma 4.1. *For any $\phi\in H^1(\Omega^{\rm PML})$ we have*

$$\mathrm{Re}\,[a(\phi,\phi)]+\frac{s_2}{s_1+\sigma}\mathrm{Im}\,[a(\phi,\phi)]\ge\frac{s_1^2}{(s_1+\sigma)^2}\|\phi\|^2_{*,\Omega^{\rm PML}}, \tag{4.7}$$

where

$$\|\phi\|^2_{*,\Omega^{\rm PML}}=\|A\nabla\phi\|^2_{L^2(\Omega^{\rm PML})}+\|\,s\alpha\beta\phi\,\|^2_{L^2(\Omega^{\rm PML})}.$$

Proof. Simple calculation shows that

$$\begin{aligned}\mathrm{Re}\,[a(\phi,\phi)] &= \int_R^\rho\int_0^{2\pi}\frac{(s_1+\hat{\sigma})(s_1+\sigma)+s_2^2}{|s_1+\sigma|^2}r\left|\frac{\partial\phi}{\partial r}\right|^2drd\theta\\ &+ \int_R^\rho\int_0^{2\pi}\frac{(s_1+\hat{\sigma})(s_1+\sigma)+s_2^2}{|s_1+\hat{\sigma}|^2}\frac{1}{r}\left|\frac{\partial\phi}{\partial\theta}\right|^2drd\theta\\ &+ \int_R^\rho\int_0^{2\pi}\left[(s_1+\sigma)(s_1+\hat{\sigma})-s_2^2\right]r|\phi|^2drd\theta,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}\,[a(\phi,\phi)] &= s_2\int_R^\rho\int_0^{2\pi}\frac{\sigma-\hat{\sigma}}{|s_1+\sigma|^2}r\left|\frac{\partial\phi}{\partial r}\right|^2drd\theta\\ &+ s_2\int_R^\rho\int_0^{2\pi}\frac{\hat{\sigma}-\sigma}{|s_1+\hat{\sigma}|^2}\frac{1}{r}\left|\frac{\partial\phi}{\partial\theta}\right|^2drd\theta\\ &+ s_2\int_R^\rho\int_0^{2\pi}(2s_1+\sigma+\hat{\sigma})r|\phi|^2drd\theta.\end{aligned}$$

Notice that $\sigma \geq \hat{\sigma}$, we have

$$
\begin{aligned}
& \operatorname{Re}[a(\phi,\phi)] + \frac{s_2}{s_1+\sigma}\operatorname{Im}[a(\phi,\phi)] \\
\geq\ & \int_R^\rho \int_0^{2\pi} \frac{(s_1+\hat{\sigma})(s_1+\sigma)+s_2^2}{|s_1+\sigma|^2} r \left|\frac{\partial \phi}{\partial r}\right|^2 drd\theta \\
+\ & \int_R^\rho \int_0^{2\pi} \frac{(s_1+\hat{\sigma})(s_1+\sigma)+s_1(s_1+\sigma)^{-1}s_2^2}{|s_1+\hat{\sigma}|^2} \frac{1}{r} \left|\frac{\partial \phi}{\partial \theta}\right|^2 drd\theta \\
+\ & \int_R^\rho \int_0^{2\pi} \left[(s_1+\sigma)(s_1+\hat{\sigma}) + \frac{s_1+\hat{\sigma}}{s_1+\sigma}s_2^2\right] r|\phi|^2 drd\theta
\end{aligned}
$$

The proof now follows easily by noticing that

$$
\|\phi\|^2_{*,\Omega^{\rm PML}} = \int_R^\rho \int_0^{2\pi} \left(\frac{|\beta|^2}{|\alpha|^2} r \left|\frac{\partial \phi}{\partial r}\right|^2 + \frac{|\alpha|^2}{|\beta|^2}\frac{1}{r}\left|\frac{\partial \phi}{\partial \theta}\right|^2 + |s\alpha\beta|^2 r|\phi|^2 \right) drd\theta.
$$

□

Lemma 4.2. *Let* $\xi \in H^1(0,T;H^{1/2}(\Gamma_\rho)) \cap H^2(0,T;H^{-1/2}(\Gamma_\rho))$. *Then there exists a function* $\zeta \in H^1(0,T;H^1(\Omega^{\rm PML})) \cap H^2(0,T;L^2(\Omega^{\rm PML}))$ *such that* $\zeta = 0$ *on* $\Gamma_R \times (0,T)$, $\zeta = \xi$ *on* $\Gamma_\rho \times (0,T)$, *and*

$$
\| \partial_t^2 \zeta \|_{L^2(0,T;L^2(\Omega^{\rm PML}))} \leq C\rho^{1/2} \| \partial_t^2 \xi \|_{L^2(0,T;H^{-1/2}(\Gamma_\rho))}, \tag{4.8}
$$

$$
\| \nabla \partial_t \zeta \|_{L^2(0,T;L^2(\Omega^{\rm PML}))} \leq C\rho^{-1/2} \| \partial_t \xi \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))}. \tag{4.9}
$$

Proof. Let

$$
\xi(\theta,t) = \sum_{n=-\infty}^{\infty} \xi_n(t) e^{\mathbf{i}n\theta}, \quad \xi_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \xi(\theta,t) e^{-\mathbf{i}n\theta} d\theta.
$$

Let $\chi_n \in C^\infty[R,\rho]$ such that $\chi_n(\rho) = 1$, $0 \leq \chi(r) \leq 1$, $|\chi_n'(r)| \leq C\delta_n^{-1}$ for $r \in [R,\rho]$, and $\mathrm{supp}(\chi_n) \subset (\rho-\delta_n,\rho)$, where $\delta_n = (\rho-R)/\sqrt{1+n^2}$, $n \in \mathbb{Z}$. Define

$$
\zeta(r,\theta,t) = \sum_{n=-\infty}^{\infty} \xi_n(t)\chi_n(r) e^{\mathbf{i}n\theta}.
$$

Then it is clear that $\zeta = 0$ on $\Gamma_R \times (0,T)$, $\zeta = \xi$ on $\Gamma_\rho \times (0,T)$. Next it is easy to see that

$$
\begin{aligned}
\| \partial_t^2 \zeta \|^2_{L^2(\Omega^{\rm PML})} &= 2\pi \sum_{-\infty}^{\infty} \int_R^\rho |\xi_n''(t)|^2 |\chi_n(r)|^2 rdr \\
&\leq 2\pi \sum_{-\infty}^{\infty} \int_{\rho-\delta_n}^\rho |\xi_n''(t)|^2 rdr \\
&\leq 2\pi\rho \sum_{-\infty}^{\infty} \delta_n |\xi_n''(t)|^2 \\
&\leq (\rho-R) \| \partial_t^2 \xi \|^2_{H^{-1/2}(\Gamma_\rho)}.
\end{aligned}
$$

This shows (4.8). Similarly we can prove (4.9). □

The following theorem is the main result of this section.

Theorem 4.3. *Let $\xi \in H^1(0,T;H^{1/2}(\Gamma_\rho)) \cap H^2(0,T;H^{-1/2}(\Gamma_\rho))$. Then the solution of the PML system* (4.1)-(4.4) *satisfies the following estimate*

$$\begin{aligned} & \| \partial_t \mathbf{\Phi} \|_{L^2(0,T;L^2(\Omega^{\rm PML}))} + \| {\rm div}\mathbf{\Phi} \|_{L^2(0,T;L^2(\Omega^{\rm PML}))} \\ \le \;& C(1+\sigma T)^2 \rho T(\| \partial_t^2 \xi \|_{L^2(0,T;H^{-1/2}(\Gamma_\rho))} + \rho^{-1} \| \partial_t \xi \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))}). \end{aligned}$$

Proof. Let ζ be the function defined in Lemma 4.2 which we extend to be a function in $H^1(0,+\infty;H^1(\Omega^{\rm PML})) \cap H^2(0,+\infty;L^2(\Omega^{\rm PML}))$ such that

$$\begin{aligned} \|\nabla \partial_t \zeta\|_{L^2(0,+\infty;L^2(\Omega^{\rm PML}))} &\le C \|\nabla \partial_t \zeta\|_{L^2(0,T;L^2(\Omega^{\rm PML}))}, \\ \|\partial_t^2 \zeta\|_{L^2(0,+\infty;L^2(\Omega^{\rm PML}))} &\le C \|\partial_t^2 \zeta\|_{L^2(0,T;L^2(\Omega^{\rm PML}))}. \end{aligned}$$

By testing (4.5) with $\phi_{\rm L} - \zeta_{\rm L} \in H_0^1(\Omega^{\rm PML})$, where $\zeta_{\rm L} = \mathscr{L}(\zeta)$, and integrating by parts we easily obtain

$$a(\phi_{\rm L}, \phi_{\rm L}) = a(\phi_{\rm L}, \zeta_{\rm L}).$$

By Lemma 4.1 we then have

$$\|\phi_{\rm L}\|^2_{*,\Omega^{\rm PML}} \le C \left(1+\frac{\sigma}{s_1}\right)^2 \frac{|s|}{s_1} |a(\phi_{\rm L}, \phi_{\rm L})| = C \left(1+\frac{\sigma}{s_1}\right)^2 \frac{|s|}{s_1} |a(\phi_{\rm L}, \zeta_{\rm L})|.$$

Since

$$|a(\phi_{\rm L}, \zeta_{\rm L})| \le \left(\| \nabla \zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})} + \| s\zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})}\right)^{1/2} \|\phi_{\rm L}\|_{*,\Omega^{\rm PML}},$$

we have

$$\|\phi_{\rm L}\|_{*,\Omega^{\rm PML}} \le C \left(1+\frac{\sigma}{s_1}\right)^2 \frac{|s|}{s_1} \left(\| \nabla \zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})} + \| s\zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})}\right)^{1/2}.$$

On the other hand, by (4.6) we know that

$$\|\phi_{\rm L}\|^2_{*,\Omega^{\rm PML}} = \| {\rm div}\mathbf{\Phi}_{\rm L} \|^2_{L^2(\Omega^{\rm PML})} + \| s\mathbf{\Phi}_{\rm L} \|^2_{L^2(\Omega^{\rm PML})}.$$

Notice the formula for the inverse Laplace transform

$$f(t) = \mathscr{F}^{-1}(e^{s_1 t} \mathscr{L}(f)(s_1 + \mathbf{i} s_2)),$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2, by the Parseval identity we have then

$$\begin{aligned} & \int_0^\infty e^{-2s_1 t} \left(\| \partial_t \mathbf{\Phi} \|^2_{L^2(\Omega^{\rm PML})} + \| {\rm div}\mathbf{\Phi} \|^2_{L^2(\Omega^{\rm PML})}\right) dt \\ = \;& 2\pi \int_{-\infty}^\infty \left(\| {\rm div}\mathbf{\Phi}_{\rm L} \|^2_{L^2(\Omega^{\rm PML})} + \| s\mathbf{\Phi}_{\rm L} \|^2_{L^2(\Omega^{\rm PML})}\right) ds_2 \\ \le \;& C \left(1+\frac{\sigma}{s_1}\right)^4 \frac{1}{s_1^2} \int_{-\infty}^\infty \left(\| s\nabla \zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})} + \| s^2 \zeta_{\rm L} \|^2_{L^2(\Omega^{\rm PML})}\right) ds_2 \\ = \;& C \left(1+\frac{\sigma}{s_1}\right)^4 \frac{1}{s_1^2} \int_0^\infty e^{-2s_1 t} \left(\| \nabla \partial_t \zeta \|^2_{L^2(\Omega^{\rm PML})} + \| \partial_t^2 \zeta \|^2_{L^2(\Omega^{\rm PML})}\right)^2 dt, \end{aligned}$$

where we have used the Parseval identity again in the last identity. This implies

$$\begin{aligned} & \int_0^T \left(\| \partial_t \mathbf{\Phi} \|^2_{L^2(\Omega^{\rm PML})} + \| {\rm div}\mathbf{\Phi} \|^2_{L^2(\Omega^{\rm PML})}\right) dt \\ \le \;& C \left(1+\frac{\sigma}{s_1}\right)^4 \frac{1}{s_1^2} e^{2s_1 T} \int_0^T \left(\| \nabla \partial_t \zeta \|^2_{L^2(\Omega^{\rm PML})} + \| \partial_t^2 \zeta \|^2_{L^2(\Omega^{\rm PML})}\right) dt. \end{aligned}$$

This completes the proof by taking $s_1 = T^{-1}$ and using Lemma 4.2. □

5. Convergence analysis

Let $(u, \mathbf{p})$ be the solution of the problem (2.6)-(2.9) and $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ be the solution of the PML problem (3.12)-(3.15). The purpose of this section is to prove that $(\hat{u}, \hat{\mathbf{p}})$ converges to $(u, \mathbf{p})$ exponentially in the domain $\Omega_R \times (0, T)$. We start with the following fundamental estimate for the modified Bessel function $K_\nu(z)$.

Lemma 5.1. *For any $\nu \in \mathbb{R}$, $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) > 0$, $\rho_1 > \rho_2 > 0$, and $\tau > 0$, we have*

$$\frac{|K_\nu(s\rho_1 + \tau)|}{|K_\nu(s\rho_2)|} \le e^{-\tau\left(1 - \frac{\rho_2^2}{\rho_1^2}\right)}. \tag{5.1}$$

Proof. Let $s = s_1 + \mathbf{i}s_2$ with $s_1 > 0, s_2 \in \mathbb{R}$. We consider two cases. First assume $s_1\rho_1 \ge |s|\rho_2$. Then by lemmas 2.5-2.6,

$$\frac{|K_\nu(s\rho_1 + \tau)|}{|K_\nu(s\rho_2)|} \le \frac{K_\nu(s_1\rho_1 + \tau)}{K_\nu(|s|\rho_2)} \le e^{-\tau} \le e^{-\tau\left(1 - \frac{\rho_2^2}{\rho_1^2}\right)},$$

which is exactly the estimate (5.1).

Next we assume $s_1\rho_1 \le |s|\rho_2$. Denote by $z_1 = s\rho_1 + \tau$ and $z_2 = s\rho_2$. Then by Lemma 2.4

$$\begin{aligned} &|K_\nu(z_1)|^2 \\ = &\frac{1}{2}\int_0^\infty e^{-\frac{|z_1|^2}{2w} - \frac{z_1^2 + \bar{z}_1^2}{2|z_1|^2}w} K_\nu(w)\frac{dw}{w} \\ = &\frac{1}{2}\int_0^\infty e^{-\frac{|z_1|^2 - |z_2|^2}{2w} - \left(\frac{z_1^2+\bar{z}_1^2}{2|z_1|^2} - \frac{z_2^2+\bar{z}_2^2}{2|z_2|^2}\right)w} \cdot e^{-\frac{|z_2|^2}{2w} - \frac{z_2^2+\bar{z}_2^2}{2|z_2|^2}w} K_\nu(w)\frac{dw}{w}. \end{aligned} \tag{5.2}$$

Now we are going to show that

$$e^{-\frac{|z_1|^2 - |z_2|^2}{2w} - \left(\frac{z_1^2+\bar{z}_1^2}{2|z_1|^2} - \frac{z_2^2+\bar{z}_2^2}{2|z_2|^2}\right)w} \le e^{-2\tau\left(1 - \frac{\rho_2^2}{\rho_1^2}\right)}. \tag{5.3}$$

The desired estimate (5.1) follows easily from (5.3) and Lemma 2.4.

To show (5.3), we first note that by simple calculation

$$\frac{z_1^2 + \bar{z}_1^2}{2|z_1|^2} - \frac{z_2^2 + \bar{z}_2^2}{2|z_2|^2} = \frac{2[(s_1\rho_1 + \tau)^2 s_2^2\rho_2^2 - s_1^2 s_2^2 \rho_1^2 \rho_2^2]}{|z_1|^2|z_2|^2} \ge \frac{2\tau^2 s_2^2 \rho_2^2}{|z_1|^2|z_2|^2}.$$

Since $|z_1|^2 \ge |s|^2\rho_1^2 = \frac{\rho_1^2}{\rho_2^2}|z_2|^2$, we obtain

$$\frac{|z_1|^2 - |z_2|^2}{|z_1|^2} = 1 - \frac{|z_2|^2}{|z_1|^2} \ge 1 - \frac{\rho_2^2}{\rho_1^2}.$$

By Cauchy-Schwarz inequality, for any $w > 0$,

$$\begin{aligned} e^{-\frac{|z_1|^2 - |z_2|^2}{2w} - \left(\frac{z_1^2+\bar{z}_1^2}{2|z_1|^2} - \frac{z_2^2+\bar{z}_2^2}{2|z_2|^2}\right)w} &\le e^{-2\left[\frac{(|z_1|^2 - |z_2|^2)}{2w}\cdot\frac{2\tau^2 s_2^2\rho_2^2}{|z_1|^2|z_2|^2}w\right]^{1/2}} \\ &\le e^{-2\tau\left(1 - \frac{\rho_2^2}{\rho_1^2}\right)^{1/2}\frac{s_2\rho_2}{|z_2|}}. \end{aligned}$$

Since $s_1\rho_1 \le |s|\rho_2$, we have

$$\frac{s_2^2\rho_2^2}{|z_2|^2} = \frac{s_2^2}{s_1^2 + s_2^2} \ge 1 - \frac{\rho_2^2}{\rho_1^2}.$$

This completes the proof. ☐

We also need the following estimate for the convolution which is widely used in the analysis of absorbing boundary conditions, e.g. in Hagstrom [17].

Lemma 5.2. *Let $f_1, f_2 \in L^2(0,T)$. For any $s_1 > 0$, we have*

$$\| f_1 * f_2 \|_{L^2(0,T)} \le e^{s_1 T} \left(\max_{-\infty < s_2 < \infty} |\mathscr{L}(f_1)(s_1 + \mathbf{i}s_2)| \right) \| f_2 \|_{L^2(0,T)}.$$

Proof. For the sake of completeness we give a proof here. We first note that by the formula for the inverse Laplace transform, we have

$$f(t) = \mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(f)(s_1 + \mathbf{i}s_2)),$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2. Let $\tilde{f}_2 = f_2\chi_{(0,T)}$, where $\chi_{(0,T)}$ is the characteristic function of the interval $(0,T)$. Then

$$\begin{aligned} \| f_1 * f_2 \|_{L^2(0,T)} &= \| f_1 * \tilde{f}_2 \|_{L^2(0,T)} \\ &= \| \mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(f_1 * \tilde{f}_2)(s_1 + \mathbf{i}s_2)) \|_{L^2(0,T)} \\ &\le e^{s_1 T} \| \mathscr{F}^{-1}(\mathscr{L}(f_1 * \tilde{f}_2)) \|_{L^2(-\infty,\infty)}, \end{aligned}$$

which by using the Parseval identity yields

$$\begin{aligned} \| f_1 * f_2 \|^2_{L^2(0,T)} &\le \frac{1}{2\pi} e^{2s_1 T} \| \mathscr{L}(f_1 * \tilde{f}_2) \|^2_{L^2(-\infty,\infty)} \\ &= \frac{1}{2\pi} e^{2s_1 T} \| \mathscr{L}(f_1) \cdot \mathscr{L}(\tilde{f}_2) \|^2_{L^2(-\infty,\infty)} \\ &\le \frac{1}{2\pi} e^{2s_1 T} \max_{-\infty < s_2 < \infty} |\mathscr{L}(f_1)(s_1 + \mathbf{i}s_2)| \cdot \| \mathscr{L}(\tilde{f}_2) \|^2_{L^2(-\infty,\infty)}. \end{aligned}$$

Again by using the Parseval identity

$$\begin{aligned} \| \mathscr{L}(\tilde{f}_2) \|^2_{L^2(-\infty,\infty)} = \| \mathscr{F}(e^{-s_1 t}\tilde{f}_2) \|^2_{L^2(-\infty,\infty)} &= 2\pi \| e^{-s_1 t}\tilde{f}_2 \|^2_{L^2(-\infty,\infty)} \\ &= 2\pi \| e^{-s_1 t} f_2 \|^2_{L^2(0,T)} \\ &\le 2\pi \| f_2 \|^2_{L^2(0,T)}. \end{aligned}$$

This completes the proof. □

Now for $r > R$, let $\tilde{\hat{u}}$ be the PML extension of $\hat{u}$ in the time domain

$$\tilde{\hat{u}}(r,\theta,t) = \sum_{n=-\infty}^{\infty} \left[\mathscr{L}^{-1}\left(\frac{K_n(s\tilde{r})}{K_n(sR)} \right) * \hat{u}_n(R,t) \right] e^{\mathbf{i}n\theta},$$

where $\hat{u}_n(R,t) = \frac{1}{2\pi}\int_0^{2\pi} \hat{u}(R,\theta,t)e^{-\mathbf{i}n\theta}d\theta$. Since $\hat{u}_n(R,0) = 0$ from the initial condition $\hat{u}|_{t=0} = u_0$ and u_0 is supported in B_R, we have

$$\partial_t \tilde{\hat{u}}(r,\theta,t) = \sum_{n=-\infty}^{\infty} \left[\mathscr{L}^{-1}\left(\frac{K_n(s\tilde{r})}{K_n(sR)} \right) * (\partial_t \hat{u}_n)(R,t) \right] e^{\mathbf{i}n\theta}.$$

Since $s\tilde{\rho} = s\rho + \rho\hat{\sigma}(\rho)$, by Lemmas 5.2 and 5.1 we know that, for any $s_1 > 0$,

$$\begin{aligned}
&\| \partial_t \tilde{\hat{u}} \|^2_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \\
= \;& 2\pi\rho \sum_{n=-\infty}^{\infty} (1+n^2)^{1/2} \left\| \mathscr{L}^{-1}\left(\frac{K_n(s\tilde{\rho})}{K_n(sR)}\right) * (\partial_t \hat{u}_n)(R,t) \right\|^2_{L^2(0,T)} \\
\le \;& 2\pi\rho e^{2s_1 T} \sum_{n=-\infty}^{\infty} (1+n^2)^{1/2} \max_{-\infty<s_2<\infty} \left| \frac{K_n(s\tilde{\rho})}{K_n(sR)} \right|^2 \| \partial_t \hat{u}_n(R,t) \|^2_{L^2(0,T)} \\
\le \;& \frac{\rho}{R} e^{2s_1 T} \max_{-\infty<n<\infty} \max_{-\infty<s_2<\infty} \left| \frac{K_n(s\tilde{\rho})}{K_n(sR)} \right|^2 \| \partial_t \hat{u} \|^2_{L^2(0,T;H^{1/2}(\Gamma_R))} \\
\le \;& \frac{\rho}{R} e^{2s_1 T} e^{-2\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2}\right)} \| \partial_t \hat{u} \|^2_{L^2(0,T;H^{1/2}(\Gamma_R))},
\end{aligned}$$

which implies, since the estimate is valid for any $s_1 > 0$,

$$\| \partial_t \tilde{\hat{u}} \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \le C e^{-\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2}\right)} \| \partial_t \hat{u} \|_{L^2(0,T;H^{1/2}(\Gamma_R))}. \tag{5.4}$$

Similarly, we can show that

$$\| \partial_t^2 \tilde{\hat{u}} \|_{L^2(0,T;H^{-1/2}(\Gamma_\rho))} \le C e^{-\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2}\right)} \| \partial_t^2 \hat{u} \|_{L^2(0,T;H^{-1/2}(\Gamma_R))}. \tag{5.5}$$

Now we are in the position to prove the convergence of the time-domain PML method.

Theorem 5.3. *Let* $(u, \mathbf{p})$ *be the solution of the problem* (2.6)-(2.9) *and* $(\hat{u}, \hat{\mathbf{p}}, \hat{u}^*, \hat{\mathbf{p}}^*)$ *be the solution of the PML problem* (3.12)-(3.15). *Then there exists a constant* $C > 0$ *depending only on* ρ/R *but independent of* $\sigma, R, \rho,$ *and* T *such that*

$$\begin{aligned}
&\max_{0\le t\le T} \left(\| u-\hat{u} \|_{L^2(\Omega_R)} + \| \mathbf{p}-\hat{\mathbf{p}} \|_{L^2(\Omega_R)} \right) \\
\le \;& C(1+\sigma T)^2 \rho T^{3/2} e^{-\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2}\right)} \| \partial_t^2 \hat{u} \|_{L^2(0,T;H^{-1/2}(\Gamma_R))} \\
+ \;& C(1+\sigma T)^2 T^{3/2} e^{-\rho\hat{\sigma}(\rho)\left(1-\frac{R^2}{\rho^2}\right)} \| \partial_t \hat{u} \|_{L^2(0,T;H^{1/2}(\Gamma_R))}.
\end{aligned}$$

Proof. Since $\sigma = 0$ for $r \le R$ we know that $\hat{\mathbf{p}}^* = \hat{\mathbf{p}}$. From (2.6) and (3.12)-(3.13) we know that

$$\frac{\partial(u-\hat{u})}{\partial t} + \mathrm{div}(\mathbf{p}-\hat{\mathbf{p}}) = 0 \quad \text{in } \Omega_R \times (0,T), \tag{5.6}$$

$$\frac{\partial(\mathbf{p}-\hat{\mathbf{p}})}{\partial t} + \nabla(u-\hat{u}) = 0 \quad \text{in } \Omega_R \times (0,T). \tag{5.7}$$

By testing (5.6) with $v \in H^1(\Omega_R)$ and using (2.5) we know that

$$\left(\frac{\partial(u-\hat{u})}{\partial t}, v\right) - (\mathbf{p}-\hat{\mathbf{p}}, \nabla v) - \langle \mathcal{T}(u-\hat{u}), v \rangle_{\Gamma_R} = \langle \hat{\mathbf{p}}\cdot\hat{\mathbf{x}} + \mathcal{T}(\hat{u}), v \rangle_{\Gamma_R}. \tag{5.8}$$

Let $w = u-\hat{u}$ and $w^* = \int_0^t (u-\hat{u})dt$. From (5.7) we have $\mathbf{p}-\hat{\mathbf{p}} = -\nabla w^*$. Thus by taking $v = w$ in (5.8) we have

$$\frac{1}{2}\frac{d}{dt}\left(\| w \|^2_{L^2(\Omega_R)} + \| \nabla w^* \|^2_{L^2(\Omega_R)} \right) - \langle \mathcal{T}(w), w \rangle_{\Gamma_R} = \langle \hat{\mathbf{p}}\cdot\hat{\mathbf{x}} + \mathcal{T}(\hat{u}), w \rangle_{\Gamma_R}. \tag{5.9}$$

Denote

$$X(0,T;\Omega_R) = \{ v \in L^\infty(0,T;L^2(\Omega_R)), v^* = \int_0^t v dt \in L^\infty(0,T;H^1(\Omega_R)) \}.$$

It is clear $X(0,T;\Omega_R)$ is Banach space with the norm

$$\|v\|_{X(0,T;\Omega)} = \sup_{0\le t\le T}\left(\| v \|^2_{L^2(\Omega_R)} + \| \nabla v^* \|^2_{L^2(\Omega_R)}\right)^{1/2}.$$

Define

$$Y(0,T;\Gamma_R) = \{\varphi : \int_0^T \langle\varphi, v\rangle_{\Gamma_R} dt < \infty, \quad \forall v \in X(0,T;\Omega_R)\}.$$

It is easy to see that $Y(0,T;\Gamma_R)$ is also a Banach space with the norm

$$\|\varphi\|_{Y(0,T;\Gamma_R)} = \sup_{v\in X(0,T;\Omega_R)} \frac{\left|\int_0^T\langle\varphi, v\rangle_{\Gamma_R} dt\right|}{\|v\|_{X(0,T;\Omega_R)}}.$$

Since $w|_{t=0} = 0, w^*|_{t=0} = 0$, from (5.9) and Lemma 2.11 we obtain

$$\|e^{-s_1 t}w\|^2_{X(0,T;\Omega_R)} \le C\|e^{-s_1 t}(\hat{\mathbf{p}}\cdot\hat{\mathbf{x}} + \mathcal{T}(\hat{u}))\|_{Y(0,T;\Gamma_R)}\|e^{-s_1 t}w\|_{X(0,T;\Omega_R)}.$$

Therefore by letting $s_1 \to 0$,

$$\sup_{0\le t\le T}\left(\|w\|_{L^2(\Omega_R)} + \|\nabla w^*\|_{L^2(\Omega_R)}\right) \le C\|\hat{\mathbf{p}}\cdot\hat{\mathbf{x}} + \mathcal{T}(\hat{u})\|_{Y(0,T;\Gamma_R)}. \tag{5.10}$$

For $\hat{u}_{\rm L} = \mathscr{L}(\hat{u})$ we define its PML extension as in (3.1)

$$\tilde{\hat{u}}_{\rm L} = \sum_{n=-\infty}^{\infty}\frac{K_n(s\tilde{r})}{K_n(sR)}\hat{u}^n_{\rm L}(R,s)e^{\mathbf{i}n\theta}, \quad \forall r > R,$$

where $\hat{u}^n_{\rm L} = \frac{1}{2\pi}\int_0^{2\pi}\hat{u}_{\rm L}(R,\theta,s)e^{-\mathbf{i}n\theta}d\theta$. Similar to (3.3)-(3.4) we introduce

$$s\tilde{\hat{\mathbf{p}}}^*_{\rm L} = -\nabla\tilde{\hat{u}}_{\rm L}, s\tilde{\hat{u}}^*_{\rm L} = \tilde{\hat{u}}_{\rm L}, \tilde{\hat{\mathbf{p}}}_{\rm L} = -{\rm diag}(\beta/\alpha, \alpha/\beta)\nabla\tilde{\hat{u}}_{\rm L},$$

where ${\rm diag}(\beta/\alpha, \alpha/\beta)$ is the diagonal matrix with the principal elements $\beta/\alpha, \alpha/\beta$. Define

$$\tilde{\hat{u}} = \mathscr{L}^{-1}(\tilde{\hat{u}}_{\rm L}), \tilde{\hat{\mathbf{p}}} = \mathscr{L}^{-1}(\tilde{\hat{\mathbf{p}}}_{\rm L}), \tilde{\hat{u}}^* = \mathscr{L}^{-1}(\tilde{\hat{u}}^*_{\rm L}), \tilde{\hat{\mathbf{p}}}^* = \mathscr{L}^{-1}(\tilde{\hat{\mathbf{p}}}^*_{\rm L}).$$

Then we know that $(\tilde{\hat{u}}, \tilde{\hat{\mathbf{p}}}, \tilde{\hat{u}}^*, \tilde{\hat{\mathbf{p}}}^*)$ satisfies (3.7),(3.9) and (3.10) in $\Omega^{\rm PML}\times(0,T)$ and $\mathcal{T}(\hat{u}) = -\tilde{\hat{\mathbf{p}}}\cdot\hat{\mathbf{x}}$ on $\Gamma_R\times(0,T)$.

To estimate $\|(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})\cdot\hat{\mathbf{x}}\|_{Y(0,T;\Gamma_R)}$ we notice that any function $v \in X(0,T;\Omega_R)$ can be extended to $\Omega^{\rm PML}\times(0,T)$ (still denoted by v) such that $v = 0$ on $\Gamma_\rho\times(0,T)$ and $\|v\|_{X(0,T;\Omega^{\rm PML})} \le C\|v\|_{X(0,T;\Omega_R)}$. Thus

$$\|(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})\cdot\hat{\mathbf{x}}\|_{Y(0,T;\Gamma_R)} \le C\sup_{v\in X(0,T;\Omega^{\rm PML})}\frac{\left|\int_0^T\langle(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})\cdot\hat{\mathbf{x}}, v\rangle_{\Gamma_R}dt\right|}{\|v\|_{X(0,T;\Omega^{\rm PML})}}.$$

On the other hand, since $v = 0$ on Γ_ρ, we have

$$\int_0^T\langle(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})\cdot\hat{\mathbf{x}}, v\rangle_{\Gamma_R}dt = \int_0^T({\rm div}(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}), v)_{\Omega^{\rm PML}} + (\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}, \nabla v)_{\Omega^{\rm PML}}.$$

Integrating by parts we obtain

$$\begin{aligned}
&\int_0^T(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}, \nabla v)_{\Omega^{\rm PML}}dt \\
&= (\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}(\cdot,T), \nabla v^*(\cdot,T))_{\Omega^{\rm PML}} - \int_0^T(\partial_t(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}), \nabla v^*)_{\Omega^{\rm PML}}dt \\
&\le \max_{0\le t\le T}\|\nabla v^*\|_{L^2(\Omega^{\rm PML})}\cdot\int_0^T\|\partial_t(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})\|_{L^2(\Omega^{\rm PML})}dt,
\end{aligned}$$

where we have used the fact that $(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}})|_{t=0} = 0$. Therefore,

$$\|(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}) \cdot \hat{\mathbf{x}}\|_{Y(0,T;\Gamma_R)} \leq C \int_0^T \left(\| \operatorname{div}(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}) \|_{L^2(\Omega^{\rm PML})} + \| \partial_t(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}) \|_{L^2(\Omega^{\rm PML})} \right),$$

and consequently, by (5.10),

$$\begin{aligned} &\sup_{0\leq t\leq T} \left(\|w\|_{L^2(\Omega_R)} + \|\nabla w^*\|_{L^2(\Omega_R)} \right) \\ \leq \quad & C \int_0^T \left(\| \operatorname{div}(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}) \|_{L^2(\Omega^{\rm PML})} + \| \partial_t(\hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}) \|_{L^2(\Omega^{\rm PML})} \right) dt. \end{aligned}$$

Let $\phi = \hat{u} - \tilde{\hat{u}}, \mathbf{\Phi} = \hat{\mathbf{p}} - \tilde{\hat{\mathbf{p}}}, \phi^* = \hat{u}^* - \tilde{\hat{u}}^*, \mathbf{\Phi}^* = \hat{\mathbf{p}}^* - \tilde{\hat{\mathbf{p}}}^*$. Then $(\phi, \mathbf{\Phi}, \phi^*, \mathbf{\Phi}^*)$ satisfies (4.1)-(4.4) with $\xi = -\tilde{\hat{u}}|_{\Gamma_\rho}$. By Theorem 4.3 we have then

$$\begin{aligned} &\sup_{0\leq t\leq T} \left(\|w\|_{L^2(\Omega_R)} + \|\nabla w^*\|_{L^2(\Omega_R)} \right) \\ \leq \quad & C(1+\sigma T)^2 \rho T^{3/2} \left(\| \partial_t^2 \tilde{\hat{u}} \|_{L^2(0,T;H^{-1/2}(\Gamma_\rho))} + \rho^{-1} \| \partial_t \tilde{\hat{u}} \|_{L^2(0,T;H^{1/2}(\Gamma_\rho))} \right). \end{aligned}$$

This completes the proof by using (5.4)-(5.5). □

Since σ is constant in (R, ρ), $\rho\hat{\sigma} = \sigma\rho(1 - R/\rho)$. Theorem 5.3 implies that exponential convergence of the PML method can be achieved for any fixed thickness of the layer by enlarging the PML absorbing parameter σ which increases as $\ln T$ for large T.

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