Bernays-Gödel type-theory

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Dedicated to Saunders Mac Lane, on the occasion of his 90th birthday.

Abstract. There is a close relationship between category theory and logic. For example, elementary toposes have just enough properties to interpret intuitionistic higher-order logic, and we think of toposes as 'categories of sets'. In fact, a topos with a natural numbers object is an adequate universe in which to develop intuitionistic mathematics, and such a topos may be seen as a categorical analogue of a model of intuitionistic Zermelo-Fraenkel set-theory.

In this paper we implement the categorical analogue of Bernays-Gödel set-theory. We introduce the notion of *small structure* on a category, and if small structure satisfies certain axioms we can think of the underlying category as a category of classes. Our axioms imply the existence of a co-variant powerset monad on the underlying category of classes, which sends a class to the class of its small subclasses. *Simple* fixed points of this and related monads are shown to be models of intuitionistic Zermelo-Fraenkel set-theory (IZF).

There is a close relationship between category theory and logic. Categories with enough properties interpret various fragments of well-known logics. For example, Heyting categories have just enough properties to interpret many-sorted intuitionistic first-order logic; elementary toposes, in particular Grothendieck toposes, interpret intuitionistic higher-order logic; and regular categories, a weakening of the notion of Abelean categories, support the $\exists$-$\wedge$-fragment of (intuitionistic) first-order logic, nowadays called *regular* logic. Most categories we meet in practice support (only) intuitionistic logic, as opposed to classical logic satisfying the principle of *tertium non datur*, $p \vee \neg p$. This is one of the reasons why intuitionistic logic (and intuitionistic mathematics) can be of interest to mathematicians who base their reasoning on classical logic.

The slogan is that

categories behave like categories of sets,

as long as we do not use the axiom of choice, and as long as we reason *constructively*[1]. Thus we view a category as a kind of universe (not in the sense of Grothendieck universes), in which the logic supported by the category allows us to perform certain constructions. For example, in a topos all the basic operations on 'sets', like taking products, disjoint unions, 'power sets', and 'function sets' are available, and they behave as expected from intuitionistic higher-order logic. If there is in addition a natural numbers object, a 'set' satisfying Peano's postulates, we can develop a good deal of intuitionistic mathematics in such a category. In fact, the free topos generated by an object satisfying the Peano postulates is a good environment in which to do mathematics, that is, this topos *is* a good mathematical universe [10].

The foundations of mathematics accepted by most working mathematicians is however Zermelo-Fraenkel set-theory (ZF). The basic notion there is that of a *set*, the abstract model of a *collection*. There are other closely related systems, like for example that of Bernays and Gödel (BG) based on the distinction between sets and classes: every set is a class, but the only elements of classes are sets. The theory (BG) axiomatizes which classes behave well, that is, are sets. In fact, the philosophy of Bernays-Gödel set-theory is that *small* classes behave well.

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[1]There is more to be said. *Types* are not necessarily interpreted by *inhabited* objects, and objects in a category are rarely determined by the morphisms from the terminal object into them.

The aim of this paper is to develop an analogue of Bernays-Gödel set-theory (BG) in the framework of category theory. We change our slogan to

categories behave like categories of classes,

and accordingly we want to axiomatize what it means to be a *small* class, that is, a set.

We will start with a category $\mathcal{R}$, and think of it as a category of classes. Our basic assumption is that this category $\mathcal{R}$ of classes is *regular*, and this provides us with some basic operations on classes, notably products, equalizers, and images of maps between classes. *Small structure* on such a regular category consists of a family of subobjects $\mathcal{S}(X)$ of X, for each object X of $\mathcal{R}$. There are not enough subobjects to develop a decent theory, and so we axiomatize *families of small subobjects*. We consider reformulations of standard axioms of Bernays-Gödel set-theory, like (Replacement), (Pairing), (Union), (Powerset), (Separation), etc.

The most important axiom says that the small structure is *representable*, which means that for each class X of $\mathcal{R}$ the class $\mathrm{P}(X)$ of all small subclasses of X exists. It comes equipped with a membership relation $\varepsilon \hookrightarrow X \times \mathrm{P}(X)$. This allows to reformulate our axioms in terms of the internal logic of $\mathcal{R}$, and we almost recover the basic axioms of (BG).

Finally we show that if there is an object U which is isomorphic to $\mathrm{P}(U)$, then U can be equipped with a new membership relation $\epsilon \hookrightarrow U \times U$ such that (U, ϵ) is a model of intuitionistic Zermelo-Fraenkel set-theory (without (ϵ-Induction) and (Infinity), which can easily be added).

The program of this paper is not new. It was carried out in great detail in [7, 8] by André Joyal and Ieke Moerdijk, see also the recent paper by Alex Simpson [12]. The research presented here originates in our efforts to understand and apply the techniques and results of [8] to other set-theories, like for example Aczel's set-theory of non-well-founded sets [1], or Church's set-theory with a universal set [4].

Joyal and Moerdijk start with a Heyting pretopos, which to be considered a category of classes. Instead of axiomatizing families of subobjects they axiomatize maps $f\colon X \to Y$ between classes which have *small fibers*, which means that $f^{-1}(y)$ is a set for each $y \in Y$. Their axioms, though many, are very natural axioms on such maps, and include a representability axioms which states that there is a universal such map with small fibers (see the Appendix). From this data, Joyal and Moerdijk *construct* the co-variant powerset functor, which sends a class X to the class $\mathrm{P}(X)$ of its small subclasses. The book [8] contains the results that if $U \cong \mathrm{P}(U)$ then U is a model of (IZF), and also contains many existence theorems for such isomorphisms.

There are four main differences between our approach and that of [8].

(i) We favour families of small subobjects over maps with small fibers. In the presence of sufficiently strong axioms, both approaches are equivalent, that is, the classes of families of small subobjects, and of maps with small fibers determine each other uniquely.

We think that our approach is closer to the spirit of set-theory, and our ultimate goal is to construct models of such theories.

(ii) To construct models of various set-theories, one wants to have a fine control over the actual properties of the models to be constructed. In our approach these properties are controlled by closure properties of the class of small subobjects, in the approach taken by Joyal and Moerdijk these properties are controlled by closure properties of the class of maps with small fibers.

The approach used in [8] does not allow such a fine control; our approach does. In addition, it is not completely clear whether the axioms of [8] are too strong or not, that is, whether weaker axioms suffice to construct models of (IZF), or not.

However, there is something lost in our approach. For example, some of the axioms used in [8] stem from an axiomatization of the properties of open maps in the category of topological spaces. No such geometric intuition is present in our approach. In fact, our axioms are direct translations of the axioms of Bernays-Gödel set-theory.

(iii) Our representability axiom is weaker than that of [8]. Both axioms are natural in the corresponding frameworks, ours gives some basic properties immediately where Joyal and Moerdijk have to do some work.

(iv) We favour (Replacement) over (Collection). (Replacement) is more natural in our framework and was in fact the first axiom we isolated.

It turns out that the only differences between our axioms and those of [8] are their stronger representability axiom, and their collection axiom, which we replace by (Replacement). Our underlying category is a Heyting category with a subobject classifier and with finite stable and disjoint coproducts, which is only slightly weaker than the assumption of a Heyting pretopos.

A different approach was taken by Alex Simpson in [12]. He uses, as Joyal and Moerdijk, maps with small fibers, but he starts with only five axioms on a regular category, equivalent to our axioms (Union), (Separation), (Singleton), (Powerset), and (Representability), and proves results similar to ours. We advise the reader to consult that paper for further results, and for comments on the relationship between Heyting pretoposes and Heyting categories with stable and disjoint coproducts.

The reader will notice that most of our proofs are omitted. The reason is that most of them are absolutely elementary.

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1 Preliminaries

1.1 Regular categories. We recall that a category $\mathcal{R}$ is called *regular* if it has finite limits, if coequalizer of kernel pairs exist, and if regular epimorphisms are stable under pullbacks. In a regular category for each object X, the collection $\mathrm{Sub}(X)$ of subobjects of X is a meet semi-lattice (with largest element), and pullback along a map $f\colon X \to Y$ is a morphism of semi-lattices $f^{-1}\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$.

Every arrow $f\colon X \to Y$ in $\mathcal{R}$ can be factored as a regular epimorphism followed by a monomorphism, $X \twoheadrightarrow \exists_f X \hookrightarrow Y$, and this gives, for each arrow f, a monotone map $\exists_f\colon \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$, left-adjoint to f^{-1}, satisfying *Frobenius identity* $\exists_f(U \wedge f^{-1}(V)) = \exists_f(U) \wedge V$ for all U in $\mathrm{Sub}(X)$ and all V in $\mathrm{Sub}(Y)$. Moreover, for a pullback square

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

the *Beck-Chevalley conditions* $f^{-1}\exists_f = \exists_q g^{-1}$ and $p^{-1}\exists_f = \exists_g q^{-1}$ hold. We refer the reader to [2, 3, 5] for more details.

Regular categories have just enough structure to support *regular logic*, which is the $\exists$-$\wedge$-fragment of first-order logic, and we will make extensive use of this fact. For example, the *graph* of an arrow $f\colon X \to Y$ is the subobject $\mathrm{graph}(f) = \{(x, y) \mid f(x) = y\}$ of $X \times Y$.

We will also use *forcing* to determine truth of formulae. Recall that for a formula $\varphi(x)$ of the internal logic of $\mathcal{R}$ with $x\colon X$ a variable of type X, and for $a\colon A \to X$ a generalized element of X (at stage A), we say that A *forces* $\varphi(a)$, and write $A \Vdash \varphi(a)$, if a factors through $\{x \mid \varphi(x)\} \hookrightarrow X$. This forcing relation has the usual properties of forcing in categories, like *monotonicity* and *local character* with respect to regular epimorphisms. As well, $A \Vdash \varphi(a) \wedge \psi(a)$ if and only if $A \Vdash \varphi(a)$ and $A \Vdash \psi(a)$, and $A \Vdash \exists y \varphi(y, a)$ if and only if there is a regular epimorphism $p\colon B \twoheadrightarrow A$ and a generalized element $b\colon B \to Y$ such that $B \Vdash \varphi(b, a{\restriction}p)$. We refer the reader to [3] for detailed proofs.

We will tacitly assume that for each object X of $\mathcal{R}$ the collection $\mathrm{Sub}(X)$ is a *set*, which is to say that $\mathcal{R}$ is *well-powered*. In this case the functor $\mathrm{Sub}(-)\colon \mathcal{R}^{\mathrm{op}} \to \mathsf{Set}$ (or $\mathrm{Sub}(-)\colon \mathcal{R}^{\mathrm{op}} \to \wedge\text{-}\mathsf{SLat}$ to the category of meet semi-lattices) is called the *subobject functor*. We note that, however, this is not a necessary assumption for our results.

1.2 The indexed subobject functor. Let R be a category with finite limits. An *I-indexed family of subobjects of X* is a subobject $S \rightarrowtail I \times X$, and we think of it as picking for each $i \in I$ the 'subobject' $\{x \mid (i,x) \in S\}$ of X. This gives the *indexed subobject functor*

$$\mathrm{Sub}_{(-)}(-)\colon \mathcal{R}^{\mathrm{op}} \times \mathcal{R}^{\mathrm{op}} \to \mathsf{Set}\,,$$

whose value on objects is $\mathrm{Sub}_I(X)$, the set of I-indexed families of subobjects of X, and whose value on arrows is given by pullback.

As the notation $\mathrm{Sub}_I(X)$ suggests the role of the indexing object I and that of X are quite different. In this paper we will *always* write the indexing object first, and we will often use the notation $S \hookrightarrow I \times X \to I$ to emphasize this distinction.

If the category $\mathcal{R}$ is regular there is also a *covariant* action in both variables. For example, a map $f\colon X \to Y$ sends the family of subobjects $S \rightarrowtail I \times X \to I$ to the family of subobjects $\{(i,y) \mid \exists x.f(x) = y \wedge (i,x) \in S\} \rightarrowtail I \times Y \to I$, which is the *image* of S under the map $I \times f$, also called the *fiberwise image* of f.

2 The basic axioms

In this section we introduce the central notion of this paper, that of a *small structure* on a regular category. We state the basic axioms, and discuss some derived 'axioms'.

2.1 Small structure. Let $\mathcal{R}$ be a regular category. As said before we are interested in axiomatizing properties of indexed families of *small* subobjects. The most basic axiom refers to an I-indexed family of subobjects $S \rightarrowtail I \times X \to I$ of X, and to an arbitrary map $\alpha\colon J \to I$. If we write S intuitively as $\sum_i S_i$ for $S_i = \{x \mid (i,x) \in S\}$ we get the re-indexing pullback square

$$\begin{tikzcd}
\alpha^{-1}(S) = \sum_j S_{\alpha(j)} \arrow[r] \arrow[d, hook] & \sum_i S_i = S \arrow[d, hook] \\
J \times X \arrow[r] \arrow[d] & I \times X \arrow[d] \\
J \arrow[r, "\alpha"'] & I\,.
\end{tikzcd}$$

Clearly, if $S \rightarrowtail I \times X \to I$ is an I-indexed family of *small* subobjects of X then $\alpha^{-1}(S) \rightarrowtail J \times X \to J$ should be a J-indexed family of small subobjects of X. The converse should be true provided that α is a 'surjection', which here means a regular epimorphism. This is exactly the content of the axiom (Independence)[2]:

(I) (Independence)
For every arrow $\alpha\colon J \to I$ and every I-indexed family of subobjects $S \rightarrowtail I \times X \to I$ of X, if S is an I-indexed family of *small* subobjects of X then $\alpha^{-1}(S)$ is a family of small subobjects too. If α is a regular epimorphism and $\alpha^{-1}(S)$ is a J-indexed family of *small* subobjects of X then S is a family of small subobjects.

Definition. *A* small structure *$\mathcal{S}$ on a regular category $\mathcal{R}$ is a class of indexed families of subobjects, referred to as* families of small subobjects, *such that axiom* (Independence) *is satisfied.*

[2] Thus the axiom (Independence) ensures that we axiomatize small subobjects of X, and smallness of families of subobjects of X does not depend on the indexing object, but is a property of the fibers.

We write $\mathcal{S}_I(X)$ for the set of I-indexed families of small subobjects of X. We say that a map $f\colon X \to Y$ has *small fibers* if its Graph[3] is a Y-indexed family of small subobjects of X, that is, if $\mathrm{Graph}(f) \rightarrowtail Y \times X \to Y$ is in $\mathcal{S}_Y(X)$. An object X is called *small* if $!_X\colon X \to 1$ has small fibers.

2.2 Induced small structure. For each object R of $\mathcal{R}$ there is an induced small structure $\mathcal{S}/R$ on the slice category $\mathcal{R}/R$ defined as follows: for objects $x\colon X \to R$ and $i\colon I \to R$ we say that $S \rightarrowtail I \times_R X \to I$ (over R) is an i-indexed family of small subobjects of x if $S \rightarrowtail I \times_R X \rightarrowtail I \times X \to I$ is in $\mathcal{S}_I(X)$.

Lemma. *For each object R of $\mathcal{R}$ the structure $\mathcal{S}/R$ is small structure on the slice category $\mathcal{R}/R$.*

Proof. For $\alpha\colon J \to I$ (Independence) follows from the diagram below, where all top squares are pullback squares.

$$\begin{tikzcd}
\alpha^{-1}(S) \arrow[r, hook] \arrow[d] & J \times_R X \arrow[r, hook] \arrow[d] & J \times X \arrow[r, hook] \arrow[d] & J \arrow[d, "\alpha"] \\
S \arrow[r, hook] \arrow[dr] & I \times_R X \arrow[r, hook] \arrow[d] & I \times X \arrow[r, hook] \arrow[dl] & I \arrow[dll] \\
& R & &
\end{tikzcd}$$

□

2.3 The basic axioms. For small structure on a regular category we consider the following additional axioms:

(B1) (Replacement)
For every arrow $f\colon X \to Y$ and every S in $\mathcal{S}_I(X)$, the object $\{(i,y) \mid \exists x.f(x) = y \wedge (i,x) \in S\}$ is in $\mathcal{S}_I(Y)$.

(B2) (Singleton)
For every X, the diagonal $\Delta_X\colon X \rightarrowtail X \times X \xrightarrow{\pi_1} X$ is in $\mathcal{S}_X(X)$.

(B3) (Union)
For every family $I \rightarrowtail J \times Y \to J$ of small subobjects of Y and every I-indexed family $A \rightarrowtail I \times X \to I$ of small subobjects of X, the J-indexed family of subobjects of X, $\{(j,x) \mid \exists y.(j,y) \in I \wedge (j,y,x) \in A\}$ is a family of small subobjects of X.

(B4) (Emptyset)
For every family $S \rightarrowtail J \times X \to J$ of small subobjects of X and every monomorphism $J \hookrightarrow I$, the I-indexed family $S \rightarrowtail J \times X \rightarrowtail I \times X$ is a family of small subobjects, that is, in $\mathcal{S}_I(X)$.

(B5) (Pairing)
For every object X, the partially ordered set $\mathcal{S}_I(X)$ is a join-semi lattice (not necessarily with a bottom element), and re-indexing preserves this structure.

(B6) (Separation)
For every X, the class of I-indexed families of subobjects is closed under subobjects, that is, for every X and every I, the class $\mathcal{S}_I(X)$ is an order-ideal in $\mathrm{Sub}(I \times X)$.

[3]We use $\mathrm{graph}(f) \rightarrowtail X \times Y$ (with lower case letter) for the graph of $f\colon X \to Y$, i.e., the object of pairs $\{(x,y) \mid f(x) = y\}$, and $\mathrm{Graph}(f)$ for the object $\{(y,x) \mid f(x) = y\}$.

(B7) (Powerset)
For every family of small subobjects $S \rightarrowtail I \times X \to I$, there exists an arrow (unique up to isomorphism in the slice category $\mathcal{R}/I$) $q_S \colon Q_S \to I$ which classifies small subfamilies of S. That is, for every $\alpha \colon J \to I$ there is a bijection between those small subobjects of $\alpha^*(X)$ contained in $\alpha^*(S)$, and arrows (in $\mathcal{R}/J$) $1 \to \alpha^*(q_S)$, natural in J.

We think of $\mathcal{R}$ as a category of classes, and families of *small* subclasses are families of *sets*.

Axiom (Replacement) says that the image of any family of small subobjects under a map between classes is again a family of small subobjects. (Singleton) requires for each $x \in X$ the subclass $\{x\}$ to be small. The main use of this axiom is to ensure that every $x \in X$ is contained in some small subclass. For (Union), the data are for each $j \in J$ a small subobject Y_j of Y, and for each j and each $y \in Y_j$ a small subobject $X_{j,y}$ of X. The axiom simply says that for each $j \in J$ the subobject $\bigcup_{y \in Y_j} X_{j,y} \hookrightarrow X$ is small. Axiom (Emptyset) just says that we can extend a given family of small subobjects by adding empty fibers, thereby making the empty subclass of X small. Roughly speaking this is the best we can do in regular logic. (Pairing) states that (external) binary unions of small sets exist. Modulo the other axioms, (Pairing) is equivalent to say that for $x_1, x_2 \in X$, the class $\{x_1, x_2\}$ exists and is small. The formulation of (Separation) is obvious, and for (Powerset) the idea is that $Q_S = \{(i, U) \mid U \subseteq S_i\}$, so that for each (small) $S_i \hookrightarrow X$ the class $\{U \mid U \subseteq S_i\}$ exists and is small.

Lemma. *Each of the axioms is separately stable under slicing, that is, if the small structure $\mathcal{S}$ on $\mathcal{R}$ satisfies one of these axioms, so does the induced small structure $\mathcal{S}/R$ for each object R of $\mathcal{R}$.*$\square$

The reader might miss (Extensionality), (Foundations), and (Infinity). We get (Extensionality) from axiom (Representability) to be discussed in the next section. (Representability) states that for each X in $\mathcal{R}$, the class $\mathrm{P}(X)$ of all small subclasses of X exists. The assignment $X \mapsto \mathrm{P}(X)$ then extends to a co-variant functor, the *powerset functor*, which is part of a monad on $\mathcal{R}$. Later we will construct (in $\mathcal{R}$) models of intuitionistic set-theory as initial algebras of the powerset monad on $\mathcal{R}$, and the models constructed this way will satisfy (Foundations). For (Infinity) we will need the extra assumption that $\mathcal{R}$ has a natural numbers object N and that N is small.

Our axioms contain some redundancies. (Emptyset) is usually added to make models non-trivial, but in our case this is also a consequence of (Singleton), which has no set-theoretical pendant. The other dependencies are more surprising.

Proposition. *Let $\mathcal{S}$ be small structure on $\mathcal{R}$.*

(i) (Singleton) *and* (Union) *imply* (Replacement).

(ii) (Singleton), (Union), *and* (Separation) *imply* (Emptyset).

Later we will see that for *representable* small structures (Pairing) is also redundant.

Proof. To show the first part let $f \colon X \to Y$ be arbitrary. We first apply (Singleton) and pull $Y \rightarrowtail Y \times Y$ back along $f \times Y$ to deduce that $\{(x, f(x)) \mid x \in X\}$ is in $\mathcal{S}_X(Y)$.

If $S \in \mathcal{S}_I(X)$ is arbitrary we apply (Union) to this family and to the family $\{(i, x, f(x)) \mid (i, x) \in S\} \rightarrowtail S \times Y \to Y$, to deduce that

$$\{(i, y) \mid \exists x. (i, x) \in S \wedge y = f(x)\} \rightarrowtail I \times Y$$

is in $\mathcal{S}_I(Y)$.

For the second part we will only sketch a proof. (Union) implies that unions along maps with small fibers exists, so that (Emptyset) is a consequence of (Union) and the fact that monomor-

phisms have small fibers (a property called (Subsingleton) in the next section), applied to

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] & S = \sum_{j\in J} \bigcup_{i\in\alpha^{-1}(j)} S_i \arrow[d, hook] \\
I \times X \arrow[r] \arrow[d] & J \times X \arrow[d] \\
I \arrow[r, hook, "\alpha"'] & J\,.
\end{tikzcd}$$

The fact that monomorphisms do have small fibers is an easy consequence of (Singleton) and (Separation). □

2.4 Derived axioms. The following is a (non-exhaustive) list of other interesting axioms one might wish to consider, each of which can be derived from the axioms (B1)–(B7) above.

(C1) (Absoluteness)
If S is in $\mathcal{S}_I(X)$ and $E \hookrightarrow X$ is a monomorphism then $S \wedge (I \times E)$ is in $\mathcal{S}_I(E)$.

(C2) (Subsingleton)
Every monomorphism has small fibers.

(C3) (Strength)
For every S in $\mathcal{S}_I(X)$ and every map $\beta\colon I \to Y$, the I-indexed family $\{(i, x, \beta(i)) \mid (i, x) \in S\} \rightarrowtail I \times (X \times Y)$ is in $\mathcal{S}_I(X \times Y)$.

(C4) (Products)
If S is in $\mathcal{S}_I(X)$ and T is in $\mathcal{S}_J(Y)$ then $S \times T$ is a $(I \times J)$-indexed family of small subobjects of $X \times Y$, that is, in $\mathcal{S}_{I\times J}(X \times Y)$.

(C5) (Omega)
Every I-indexed family of subobjects of 1 is a family of small subobjects, that is, $\mathcal{S}_I(1) \cong \mathrm{Sub}(I \times 1)$ for all I.

(C6) (Meets)
Every $\mathcal{S}_I(X)$ is closed under binary meets.

(Absoluteness) should not be confused with (Separation), which has the same assumptions. Axiom (Subsingleton) is called 'separation' in [8], and indeed, together with (Union) and (Replacement) it implies (Separation).

In the next section we will discuss *representable* small structures, which are small structures such that for each object X the class $\mathrm{P}(X)$ of its small subobjects exists in the underlying category. (Replacement) makes P into an endo-functor on $\mathcal{R}$, and (Strength) is equivalent to a *tensorial strength* for this functor, see below. (Omega) makes $\mathrm{P}(1)$ into a subobject classifier.

For the rest of this section we mention some of the dependencies between our axioms, and show in particular that (C1)–(C6) are indeed consequences of our basic axioms.

Lemma. *(i)* (Emptyset) *and* (Singleton) *imply* (Subsingleton).

(ii) (Absoluteness) *and* (Singleton) *imply* (Subsingleton).

(iii) (Union), (Replacement), *and* (Subsingleton) *imply* (Separation).

(iv) (Union) *and* (Subsingleton) *imply* (Emptyset) *and* (Absoluteness).

(v) (Strength) *implies that* $S \in \mathcal{S}_I(X)$ *if and only if* S *is a small subobject of* $I^*(X)$ *in* $\mathcal{R}/I$. *The converse holds under the additional assumption* (Replacement).

(vi) (Singleton) *and* (Products) *imply* (Strength).

(vii) (Union) *and* (Strength) *imply* (Products).

(viii) (Replacement), (Singleton), *and* (Union) *imply* (Strength), *and thus also* (Products).

(ix) (Singleton) *and* (Separation) *imply* (Omega).

(x) (Separation) *implies* (Meet). □

Our axioms for small structure are closely modelled after the axioms of Bernays-Gödel set-theory. Instead, one might wish to start with axioms on a regular category which relate to the regular structure (terminal object, finite products, equalizers/monomorphisms, and regular epimorphisms). This suggests to start with the axioms

(i) $1 \rightarrowtail 1 \times 1 \to 1$ is small;

(ii) (Products);

(iii) (Absoluteness);

(iv) (Replacement);

or maybe a weakening of (Absoluteness) to regular monomorphisms. Except the first, all other three axioms are stable under slicing, and it is exactly (Singleton) which makes the first axiom stable.

Another desirable property is that S is in $\mathcal{S}_I(X)$ if and only if $S \hookrightarrow I^*X$ is a small subobject in $\mathcal{R}/I$ for the induced structure $\mathcal{S}/I$ on $\mathcal{R}$, the latter being equivalent to axiom (Strength). But this is a consequence of (Products) and (Singleton).

We might call a small structure on a regular category satisfying (Singleton), (Products), (Absoluteness), and (Replacement) a *regular* small structure. On top of this we have axioms of regular nature, like (Union), axioms of coherent nature, like (Emptyset) and (Pairing), and axioms of first-order or higher-order character, like (Omega) and (Powerset).

3 Representable small structure

In this section we introduce the axiom (Representability) and collect first consequences, the next section is then mainly devoted to (Representability) in connection with (Replacement), (Singleton), and (Union).

3.1 The axiom (Representability).

The last axiom we consider states that for each object X the collection $\mathrm{P}(X)$ of its small subobjects exists.

Definition. *Small structure* $\mathcal{S}$ *on a regular category* $\mathcal{R}$ *is said to be* representable *if for each object* X *of* $\mathcal{R}$, *the contravariant functor* $\mathcal{S}_{(-)}(X)$ *is representable. The representing object is denoted* $\mathrm{P}(X)$.

Explicitly this means that for each X there exists a universal $\mathrm{P}(X)$-indexed family of small subobjects of X, $\ni \rightarrowtail \mathrm{P}(X) \times X \to \mathrm{P}(X)$, called the *membership relation*. We write $x \,\varepsilon\, U$ instead of $(U, x) \in \ni$ for $x \in X$ and $U \in \mathrm{P}(X)$. A family of subobjects $S \rightarrowtail I \times X \to I$ is in $\mathcal{S}_I(X)$ if and only if there is a (unique) arrow $\chi_S \colon I \to \mathrm{P}(X)$ that fits into a pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & \ni \arrow[d, hook] \\ I \times X \arrow[r, "\chi_S \times X"'] & \mathrm{P}(X) \times X \,. \end{tikzcd}$$

We write $[x \mid (i,x) \in S]$ for $\chi_S(i)$. The brackets $[- \mid -]$ behave like ordinary set-forming brackets, since we have $x' \,\varepsilon\, [x \mid (i,x) \in S] \leftrightarrow (i,x') \in S$.

3.2 Extensionality. Every representable small structure satisfies extensionality.

Proposition. *Let S be a representable small structure. For generalized elements $U, U' \colon A \rightrightarrows \mathrm{P}(X)$ we have $U = U'$ if and only if $A \Vdash \forall x(x \,\varepsilon\, U \leftrightarrow x \in U')$. If the underlying regular category has implications between subobjects and universal quantification then this is equivalent to $\mathcal{R}$ satisfying $\forall U \forall U'(U = U' \leftrightarrow \forall x(x \,\varepsilon\, U \leftrightarrow x \,\varepsilon\, U'))$.* □

3.3 The order on $\mathrm{P}(X)$. Axiom (Meets) gives a meet operation on each $\mathrm{P}(X)$, which can be used to internalize the order on each $\mathcal{S}_I(X)$. Recall that (Meets) is a consequence of (Separation).

Lemma. *For a representable small structure $\mathcal{S}$ on $\mathcal{R}$ axiom* (Meets) *is equivalent to the existence of maps $\wedge \colon \mathrm{P}(X) \times \mathrm{P}(X) \to \mathrm{P}(X)$ satisfying $x \,\varepsilon\, (U \wedge V) \leftrightarrow x \,\varepsilon\, U \wedge x \,\varepsilon\, V$.* □

It follows that the external order on each $\mathcal{S}_I(X)$ can be internalized, by $\leq$ being the equalizer

$$\leq \hookrightarrow \mathrm{P}(X) \times \mathrm{P}(X) \underset{\wedge}{\overset{\pi_1}{\rightrightarrows}} \mathrm{P}(X)\,.$$

Corollary. *For generalized elements $U, U' \colon A \rightrightarrows \mathrm{P}(X)$ the order on $\mathrm{P}(X)$ is characterized by $U \leq U'$ if and only if $A \Vdash \forall x(x \,\varepsilon\, U \to x \,\varepsilon\, U')$.* □

Of course, the order can also be derived from axiom (Pairing), which gives each $\mathrm{P}(X)$ the structure of a join-semi lattice.

Lemma. *Axiom* (Pairing) *is equivalent to the existence of natural maps $\vee \colon \mathrm{P}(X) \times \mathrm{P}(X) \to \mathrm{P}(X)$ making each $\mathrm{P}(X)$ into a join-semi lattice.* □

Using (Singleton) we get back the standard pairing map as the composite $X \times X \xrightarrow{\sigma \times \sigma} \mathrm{P}(X) \times \mathrm{P}(X) \xrightarrow{\vee} \mathrm{P}(X)$, sending (x,y) to $[x,y]$.

3.4 Absoluteness. Modulo the other axioms (Absoluteness) is equivalent to (Separation), and the following lemma may be seen as a regular formulation of (Separation).

Lemma. *A representable small structure satisfies* (Absoluteness) *if and only if for every monomorphism $E \hookrightarrow X$ there is a map $- \wedge E \colon \mathrm{P}(X) \to \mathrm{P}(E)$ satisfying $e \,\varepsilon\, (U \wedge E) \leftrightarrow e \,\varepsilon\, U$, where we view e as an element of X along the inclusion $E \hookrightarrow X$.* □

3.5 Stability under slicing. Our axioms are strong enough to ensure that each small structure $\mathcal{S}/R$ is representable. In this small section we will assume that each $\mathrm{P}(X)$ is ordered, a consequence of either (Meets), (Pairing), or (Separation).

Lemma. *Let $\mathcal{S}$ be a representable small structure on a regular category $\mathcal{R}$ satisfying* (Replacement), (Singleton), (Strength). *Then for each R in $\mathcal{R}$ the induced small structure $\mathcal{S}/R$ on $\mathcal{R}/R$ is representable, and the canonical regular functor $R^* \colon \mathcal{R} \to \mathcal{R}/R$ satisfies $\mathrm{P}_{\mathcal{S}/R}(R^*(X)) \cong R^*(\mathrm{P}(X))$ for all X in $\mathcal{R}$.*

Proof. It is quite obvious what the representable structure should be: Given $\alpha \colon X \to R$ in $\mathcal{R}/R$ the (total space of the object) $\mathrm{P}_{\mathcal{S}/R}(\alpha)$ should be $\{(r,U) \mid r \in R, U \in \mathrm{P}(X), x \,\varepsilon\, U \to \alpha(x) = r\}$, so we define (the total space of) $\mathrm{P}_{\mathcal{S}/R}(\alpha)$ as the object in the following pullback square

$$\begin{tikzcd}
\mathrm{P}_{\mathcal{S}/R}(\alpha) \arrow[r] \arrow[d, hook] & \geq \arrow[d, hook] \\
R \times \mathrm{P}(X) \arrow[r, "\sigma_R \times \mathrm{P}(\alpha)"'] & \mathrm{P}(R) \times \mathrm{P}(R)\,,
\end{tikzcd}$$

which makes the lemma straightforward. To prove that the pullback functor $R^*\colon \mathcal{R} \to \mathcal{R}/R$ preserves classes of small subobjects we consider $\pi_1\colon R \times Y \to R$ for Y arbitrary. Then

$$\mathrm{P}_{\mathcal{S}/R}(\pi_1) = \{(r, V) \mid r \in R, V \in \mathrm{P}(R \times Y), \pi_1(V) \leq [r]\},$$

which comes equipped with the canonical map $\mathrm{P}_{\mathcal{S}/R}(\pi_1) \hookrightarrow R \times \mathrm{P}(R \times Y) \xrightarrow{R \times \mathrm{P}(\pi_2)} R \times \mathrm{P}(Y)$. For the inverse we note that there is a map

$$R \times \mathrm{P}(Y) \to R \times \mathrm{P}(R \times Y), \qquad (r, U) \mapsto (r, [(r, y) \mid y \,\varepsilon\, U]),$$

which can be defined using the tensorial strength. Since $\pi_1[(r, y) \mid y \,\varepsilon\, U] \leq [r]$ this map factors through $\geq_{\mathrm{P}(R)}$, and induces a map $R \times \mathrm{P}(Y) \to \mathrm{P}_{\mathcal{S}/R}(\pi_1)$ which is inverse to $\mathrm{P}_{\mathcal{S}/R}(\pi_1) \to R \times \mathrm{P}(Y)$. □

4 Replacement, Singleton, and Union

In this section we focus on the three axioms (Replacement), (Singleton), and (Union) for a representable small structure $\mathcal{S}$ on a regular category $\mathcal{R}$. Axiom (Replacement) makes P into a contravariant functor $\mathrm{P}\colon \mathcal{R} \to \mathcal{R}$, which is the underlying functor of a monad on $\mathcal{R}$, a consequence of (Singleton) and (Union). In fact, the unit of this monad (at X) is the singleton map $\sigma_X\colon X \to \mathrm{P}(X)$, and multiplication is given by union $\bigcup^{(X)}\colon \mathrm{PP}(X) \to \mathrm{P}(X)$. It should be noted that σ_X and $\bigcup^{(X)}$ can be completely described using regular sequents of the internal logic of $\mathcal{R}$. The same is true for $\mathrm{P}(f)\colon \mathrm{P}(X) \to \mathrm{P}(Y)$ for $f\colon X \to Y$ an arrow in $\mathcal{R}$. Thus these axioms are of true regular nature (as opposed to, for example, (Pairing), which is a coherent axiom). These regular sequents do not capture all the information of (Replacement), (Singleton), and (Union); we have to add a stronger version of (Replacement).

4.1 Replacement. Axiom (Replacement) induces for each (fixed) $f\colon X \to Y$ and each index object I a map $\mathcal{R}(I, \mathrm{P}(X)) \to \mathcal{R}(I, \mathrm{P}(Y))$, natural in I, and hence a map $\mathrm{P}(f)\colon \mathrm{P}(X) \to \mathrm{P}(Y)$. By the definition of this map we have $\mathrm{P}(f)(U) = [y \mid \exists x.x \,\varepsilon\, U \wedge f(x) = y]$. The converse is also true: if there are maps $\mathrm{P}(f)$ satisfying the regular sequent above, then they internalize the direct image functor and the representable small structure satisfies (Replacement).

Lemma. *A representable small structure satisfies* (Replacement) *if and only if for each* $f\colon X \to Y$ *there is a map* $\mathrm{P}(f)\colon \mathrm{P}(X) \to \mathrm{P}(Y)$ *such that* $y \,\varepsilon\, \mathrm{P}(f)(U) \leftrightarrow \exists x.x \,\varepsilon\, U \wedge f(x) = y$. *Thus* P *becomes a covariant functor* $\mathrm{P}\colon \mathcal{R} \to \mathcal{R}$. □

4.2 Singleton.

Lemma. *A representable small structure satisfies* (Singleton) *if and only if there are maps* $\sigma_X\colon X \to \mathrm{P}(X)$ *such that* $x' \,\varepsilon\, \sigma_X(x) \leftrightarrow x' = x$. (Replacement) *makes* σ_X *natural in* X.

Proof. $\mathcal{R}$ satisfies the regular sequent $x' \,\varepsilon\, \sigma_X(x) \leftrightarrow x' = x$ if and only if the top square in

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook, "\Delta"'] & \ni \arrow[d, hook] \\
X \times X \arrow[r, "\sigma_X \times X"] \arrow[d, "\pi_1"'] & \mathrm{P}(X) \times X \arrow[d] \\
X \arrow[r, "\sigma_X"] & \mathrm{P}(X)
\end{tikzcd}$$

is a pullback square, from which the equivalence of $x' \,\varepsilon\, \sigma_X(x) \leftrightarrow x' = x$ and axiom (Singleton) follows using the universal property of $\ni$. □

The map σ_X is called the *singleton* map.

4.3 Union and the monad structure on P.

Lemma. *If a representable small structure on a regular category $\mathcal{R}$ satisfies* (Union) *then there are maps $\bigcup^{(X)}\colon \mathrm{P}(\mathrm{P}(X)) \to \mathrm{P}(X)$ satisfying*

$$x \,\varepsilon\, \bigcup^{(X)} \mathfrak{A} \leftrightarrow \exists U\colon \mathrm{P}(X).U \,\varepsilon\, \mathfrak{A} \wedge x \,\varepsilon\, U$$

for all $\mathfrak{A}\colon \mathrm{PP}(X)$ and all $x\colon X$. (Replacement) *makes $\bigcup^{(X)}$ natural in X.*

Proof. For X, the map $\bigcup\colon \mathrm{P}(\mathrm{P}(X)) \to \mathrm{P}(X)$ is defined as follows. A map $i\colon J \to \mathrm{P}(\mathrm{P}(X))$ corresponds to a family $I \rightarrowtail J \times \mathrm{P}(X) \to J$ of small subobjects of $\mathrm{P}(X)$, and the composite $I \rightarrowtail J \times \mathrm{P}(X) \to \mathrm{P}(X)$ corresponds to a family $A \rightarrowtail I \times X \to I$ of small subobjects of X. By (Union) the J-indexed family $\{(j,x) \mid \exists A\colon \mathrm{P}(X).(j,U) \in I \wedge x \,\varepsilon\, U\}$ is small, and corresponds to a map $J \to \mathrm{P}(X)$. This construction is natural in J and defines $\bigcup$ with the property that $x \,\varepsilon\, \bigcup \mathfrak{A} \leftrightarrow \exists U\colon \mathrm{P}(X).U \,\varepsilon\, \mathfrak{A} \wedge x \,\varepsilon\, U$. Naturality of the union operations is straightforward. □

Proposition. *For a representable small structure on a regular category $\mathcal{R}$ satisfying* (Replacement), (Singleton), *and* (Union), *the triple* $(\mathrm{P}, \sigma_{(-)}, \bigcup^{(-)})$ *is a monad on $\mathcal{R}$.*

Proof. For the monad identities we argue in $\mathcal{R}$: The map $\mathrm{P}(\sigma_X)\colon \mathrm{P}(X) \to \mathrm{P}(\mathrm{P}(X))$ sends U to $[[x] \mid x \,\varepsilon\, U]$. By definition of $\bigcup^{(X)}$, $x' \,\varepsilon\, \bigcup^{(X)}[[x] \mid x \,\varepsilon\, U]$ if and only if there is some $x \,\varepsilon\, U$ so that $x' \,\varepsilon\, [x]$, i.e., if $x' = x$ for some $x \,\varepsilon\, U$, and $\mathrm{P}(\sigma_X)(U) = U$ for all $U \in \mathrm{P}(X)$.

Even simpler, $\sigma_{\mathrm{P}(X)}\colon \mathrm{P}(X) \to \mathrm{P}(\mathrm{P}(X))$ sends U to $[U]$, and $x' \,\varepsilon\, \bigcup^{(X)}[U]$ if and only if $x' \,\varepsilon\, U$ so that $\bigcup^{(X)} \sigma_{\mathrm{P}(X)}(U) = U$ for all $U \in \mathrm{P}(X)$.

Associativity of $\bigcup^{(X)}$ is proved in a similar way. □

Algebras for this monad are objects A equipped with a map $\bigvee\colon \mathrm{P}(A) \to A$ satisfying the equations

$$\bigvee [a] = a \qquad \text{and} \qquad \bigvee \bigcup \mathfrak{A} = \bigvee [\bigvee U \mid U \,\varepsilon\, \mathfrak{A}]$$

for all $a \in A$ and all $\mathfrak{A} \in \mathrm{P}(\mathrm{P}(A))$. We call such algebras *small-complete sup-lattices.* A morphism between such algebras $(A, \bigvee)$ and $(B, \bigvee)$ is simply a map $h\colon A \to B$ preserving small suprema in the sense that $h \bigvee^{(A)} U = \bigvee^{(B)} h(U)$ for all $U \in \mathrm{P}(A)$. Here $h(U)$ stands for $\mathrm{P}(h)(U) = [h(a) \mid a \,\varepsilon\, U]$, and the notation is justified since P internalizes the direct image.

For each object X of $\mathcal{R}$, $(\mathrm{P}(X), \bigcup^{(X)})$ is the *free* small-complete sup-lattice generated by X along the inclusion $\sigma_X\colon X \to \mathrm{P}(X)$. If $\alpha\colon X \to \mathrm{P}(Y)$ is an arbitrary map to the small-complete sup-lattice $\mathrm{P}(Y)$ then its extension $\bar{\alpha}\colon \mathrm{P}(X) \to \mathrm{P}(Y)$ is the unique map satisfying $y \,\varepsilon\, \bar{\alpha}(U) \leftrightarrow \exists x.x \,\varepsilon\, U \wedge \alpha(x) = y$.

4.4 Families of small subobjects versus maps with small fibers. Although we think that our axiomatization is closer to the spirit of set-theory nothing is really lost if one prefers maps with small fibers over families of small subobjects, as in [8].

Proposition. (Replacement), (Singleton), *and* (Union) *imply that $S \rightarrowtail I \times X \to I$ is in $\mathcal{S}_I(X)$ if and only if $S \to I$ is a map with small fibers, that is, if $S \rightarrowtail I \times S \to I$ is in $\mathcal{S}_I(X)$.*

Proof. If $S \to I$ has small fibers we use the diagram

$$\begin{tikzcd}
S \arrow[r, two heads, "\cong"] \arrow[d, hook] & S \arrow[d, hook] \\
I \times S \arrow[r, "I \times \pi_2"'] \arrow[d] & I \times X \arrow[d] \\
I \arrow[r] & I
\end{tikzcd}$$

and (Replacement) to show that $S \in \mathcal{S}_I(X)$. For the other direction we use (Singleton) and (Union) applied to the families $S \rightarrowtail S \times S \to S$ and $S \rightarrowtail I \times X \to I$ to deduce that the family $\{(i,s) \mid \exists x.(i,x) = s\} = \{(i,x) \mid i = \pi_1(s)\}$ is in $\mathcal{S}_I(S)$, that is, that $S \to I$ has small fibers. □

4.5 Local Replacement. We will need the following strengthening of (Replacement).

(B1*) (Local Replacement)
If $S \rightarrowtail I \times X \to I$ is a family of small subobjects and $g\colon S \to Y$ is arbitrary, then $\{(i,y) \mid \exists x.(i,x) \in S \wedge g(i,x) = y\}$ is in $\mathcal{S}_I(Y)$.

Lemma. (Replacement), (Singleton), *and* (Union) *imply* (Local Replacement).

Proof. Given $S \rightarrowtail I \times X \to I$ in $\mathcal{S}_I(X)$ we know that $S \rightarrowtail I \times S \to I$ is in $\mathcal{S}_I(S)$, and we can apply (Replacement). □

Lemma. *If* (Replacement) *holds and if* $S \in \mathcal{S}_I(X)$ *implies that* $S \to I$ *has small fibers, then* (Local Replacement) *holds. The converse is also true.*

Proof. If $S \rightarrowtail I \times X \to I$ and $f\colon S \to Y$ are given, then $S \rightarrowtail I \times S \to I$ is a family of small subobjects of S, and the result follows from (Replacement).

Conversely, for (Replacement), if $f\colon X \to Y$ is arbitrary and if S is in $\mathcal{S}_I(X)$, then we take $g\colon S \hookrightarrow I \times X \to X \to Y$ and apply (Local Replacement). If S is in $\mathcal{S}_I(X)$ then (Local Replacement) applied to $\mathrm{id}_S\colon S \to S$ shows that $\{(i,s) \mid \exists x.(i,s) \in S \wedge (i,x) = s\} \rightarrowtail I \times S \to I$ is in $\mathcal{S}_I(S)$, but this family is $S \rightarrowtail I \times S$. □

4.6 The complete characterization. We are now able to prove a complete characterization of (Replacement), (Singleton), and (Union) as promised at the beginning of this section.

Proposition. *A representable small structure satisfies* (Replacement), (Singleton), *and* (Union) *if and only if* $S \in \mathcal{S}_I(X)$ *implies that* $S \to I$ *has small fibers, and if there are maps* $\mathrm{P}(f)\colon \mathrm{P}(X) \to \mathrm{P}(Y)$ *for each* $f\colon X \to Y$ *in* $\mathcal{R}$, $\sigma_X\colon X \to \mathrm{P}(X)$, $\bigcup^{(X)}\colon \mathrm{PP}(X) \to \mathrm{P}(X)$ *for* $X \in \mathcal{R}$, *satisfying*

— $y \,\varepsilon\, \mathrm{P}(f)(U) \leftrightarrow \exists x.x \,\varepsilon\, U \wedge f(x) = y$;

— $x' \,\varepsilon\, \sigma_X(x) \leftrightarrow x' = x$;

— $x \,\varepsilon\, \bigcup^{(X)} \mathfrak{A} \leftrightarrow \exists U.U \,\varepsilon\, \mathfrak{A} \wedge x \,\varepsilon\, U$;

for all $x\colon X$, $U\colon \mathrm{P}(X)$, *and* $\mathfrak{A}\colon \mathrm{PP}(X)$.

Proof. We verified that (Replacement), (Singleton), and (Union) imply the regular sequents of the proposition, and (Singleton) and (Union) imply (Local Replacement), and thus if $S \in \mathcal{S}_I(X)$ then $S \to I$ has small fibers.

Conversely, the second and third regular sequent imply (Singleton) and (Replacement), so we have in addition (Local Replacement). To show (Union) it is enough to prove that the regular version of union and (Replacement) imply that we have unions along maps with small fibers: if then $S \rightarrowtail I \times X \to I$ and $I \rightarrowtail J \times Y \to J$ are given in $\mathcal{S}_I(X)$ and $\mathcal{S}_J(Y)$, then $I \to J$ has small fibers and we deduce that $\{(j,x) \mid \exists i.\pi_2(i) = j \wedge (i,x) \in S\} \rightarrowtail J \times X$ is a J-indexed family of small subobjects of X.

So assume the regular version of union and (Replacement), and take $S \in \mathcal{S}_I(X)$ arbitrary and $\alpha\colon I \to J$ a map with small fibers. The characteristic map of $\mathrm{Graph}(\alpha)$ is $a\colon J \to \mathrm{P}(I)$, $j \mapsto [i \mid \alpha(i) = j]$. If we write $\chi_S\colon I \to \mathrm{P}(X)$ for the characteristic map of S then the composite

$$J \xrightarrow{a} \mathrm{P}(I) \xrightarrow{\mathrm{P}(\chi_S)} \mathrm{PP}(X) \xrightarrow{\bigcup^{(X)}} \mathrm{P}(X)$$

sends j successively to $[i \mid \alpha(i) = j]$, $[S_i \mid \alpha(i) = j]$, and to $[x \mid \exists i.\alpha(i) = j \wedge (i,x) \in S]$. □

4.7 Strength. As we saw before the axioms (Replacement), (Singleton), and (Union) imply the axiom (Strength). Here we will give a regular version of this axiom.

Lemma. *A representable small structure satisfies* (Strength) *if and only if there are maps* $t_{X,Y}\colon X \times \mathrm{P}(Y) \to \mathrm{P}(X \times Y)$ *satisfying* $(x,y) \varepsilon t_{X,Y}(x',V) \leftrightarrow x = x' \wedge y \varepsilon V$. (Replacement) *makes* $t_{X,Y}$ *natural in both variables.* □

The maps $t_{X,Y}$ provide the endo-functor P with a *tensorial strength* (see [9, 6]).

5 Separation and Powerset

5.1 The subobject classifier. Recall that (Singleton) and (Separation) imply (Omega).

Lemma. *For a representable small structure axiom* (Omega) *is equivalent to* $\ni\!\rightarrowtail \mathrm{P}(1) \times 1 \cong \mathrm{P}(1)$ *being a subobject classifier.* □

Note that this subobject classifier has not only the structure of a meet semi-lattice (with a largest element), but is a Brouwerian lattice, that is, there is an operation 'implication' uniquely determined by $p \wedge q \leq r$ if and only if $p \leq q \to r$. Indeed, we can construct the subobject

$$\{(p,q) \mid p \leq q\} \rightarrowtail \Omega \times \Omega$$

as the equalizer of the first projection and $\wedge\colon \Omega \times \Omega \to \Omega$. The classifying map for this subobject defines the implication operation on Ω.

Lemma. *Let $\mathcal{C}$ be a category with finite limits and a subobject classifier Ω. Then Ω (and every collection* $\mathrm{Sub}(X)$*, for X in $\mathcal{C}$) is a Brouwerian lattice, and pullback preserves implication between subobjects.* □

To appreciate the strength of axiom (Omega) we mention that not much more is needed to force a regular category with a subobject classifier to be a Heyting category[4].

Proposition. *Let $\mathcal{R}$ be a regular category with a subobject classifier Ω and suppose that universal quantification over Ω is defined. Then $\mathcal{R}$ is a Heyting category.*

[4] Freyd and Scedrov [5] call this a logos.

Proof. We already know that implication is defined. It follows that Ω (and thus every collection $\mathrm{Sub}(X)$ of subobjects) is a Heyting algebra with $\bot = \forall t\colon \Omega\ t$ and $p \vee q = \forall t\colon \Omega[(p \to t) \wedge (q \to t)] \to t]$. Moreover, universal quantification along projections is defined using existential quantification and universal quantification over Ω as $\forall x\colon X \varphi(x) = \forall t\colon \Omega[\exists x\colon X(\varphi(x) \leftrightarrow t) \to (t \leftrightarrow \top)]$. Finally, universal quantification along maps is defined using universal quantification along projections and implication. □

5.2 Separation. The axiom (Separation) is a strong axiom. As we saw above it implies (together with (Singleton)) the existence of a subobject classifier. Here we show that it also implies that universal quantification along *small* objects is defined.

Lemma. *Axiom* (Separation) *implies that for all small S and all I, $\mathcal{S}_I(S) = \mathrm{Sub}(I \times S)$.*

Proof. Since S is small the canonical map $!_S\colon S \to 1$ has small fibers, or equivalently, $S = \mathrm{Graph}(!_S) \hookrightarrow 1 \times S$ is in $\mathcal{S}_1(S)$. By pullback stability it holds that $I \times S \in \mathcal{S}_I(S)$ for all I, so that the lemma follows. □

Assuming in addition (Representability) we show that universal quantification along *small* objects is defined: if S is small and X is arbitrary, we define for $A \rightarrowtail X \times S$, the subobject $\forall_{\pi_1}(A)$ of X to be the equalizer of $\chi_A\colon X \to \mathrm{P}(S)$ and $\chi_{X \times S}\colon X \to \mathrm{P}(S)$. Note that even without (Separation) this defines a monotone map $\mathcal{S}_X(S) \to \mathrm{Sub}(X)$.

Proposition. *The map $\forall_{\pi_1}\colon \mathrm{Sub}(X \times S) \to \mathrm{Sub}(X)$ is right-adjoint to the pullback functor $\pi_1^{-1}\colon \mathrm{Sub}(X \times S) \to \mathrm{Sub}(X)$, and therefore the family of these defines universal quantification over the small object S.*

Proof. Let $S \hookrightarrow X$ be small with classifying map $\chi_S\colon 1 \to \mathrm{P}(S)$. If $A \hookrightarrow X \times S$ is arbitrary with classifying map $\chi_A\colon X \to \mathrm{P}(S)$ then $x \in \forall_{\pi_1}(A)$ if and only if $\chi_A(x) = [s \mid s \in S]$, that is, if $[s \mid (x, s) \in A] = [s \mid s \in S]$, from which adjointness follows. □

5.3 The powerset axiom. We remind the reader that each $\mathrm{P}(X)$ is ordered, provided that we assume (Meets), (Pairing), or (Separation).

Lemma. *Let $\mathcal{S}$ be small representable structure on $\mathcal{R}$ such that each $\mathrm{P}(X)$ is ordered. Then* (Powerset) *is equivalent to $\{(U, V) \mid U \geq V\} \hookrightarrow \mathrm{P}(X) \times \mathrm{P}(X) \xrightarrow{\pi_1} \mathrm{P}(X)$ being a family of small subobjects of $\mathrm{P}(X)$ for each object X of $\mathcal{R}$.*

Instead of this family we will use the induced map $\downarrow\colon \mathrm{P}(X) \to \mathrm{P}(\mathrm{P}(X))$, which is, in the presence of (Union), right-adjoint to $\bigcup^{(X)}$.

Proof. To define the map $\downarrow$ take $\chi_U\colon I \to \mathrm{P}(X)$ arbitrary. For simplicity we work in $\mathcal{R}/I$, thus $U\colon 1 \to \mathrm{P}(X)$. Clearly $\{V \mid U \geq V\} \to 1$ classifies small subobjects of X contained in U, so $\{V \mid U \geq V\}$ is small. (Replacement) implies that $\{V \mid U \geq V\} \hookrightarrow \mathrm{P}X$ is a small subobject of $\mathrm{P}(X)$, and it defines a global element $1 \to \mathrm{P}(\mathrm{P}(X))$. This construction is natural in I and defines $\downarrow\colon \mathrm{P}(X) \to \mathrm{P}(\mathrm{P}(X))$ with the property that $V \varepsilon \downarrow U$ if and only if $U \geq V$ for all $U, V \in \mathrm{P}(X)$.

Conversely, (working in $\mathcal{R}/I$), if $U \in \mathrm{P}(X)$ is given then $\{V \mid U \geq V\}$ is small and classifies those small subobjects of X contained in U, and (Powerset) is satisfied. □

Lemma. (Powerset) *and* (Singleton) *imply that* $\mathrm{P}(1)$ *is small.*

Proof. The class $\mathrm{P}(1)$ is ordered and has a maximal element $1 \to \mathrm{P}(1)$ corresponding to $1 \rightarrowtail 1 \times 1 \to 1$. Using (Replacement) and (Powerset) we deduce that $\{V \mid 1 \geq V\}$ is small, which is $\mathrm{P}(1)$ because 1 is maximal. □

5.4 A FIRST SUMMARY. Let us pause and summarize what we have proved.

THEOREM. *Let $\mathcal{S}$ be a representable small structure on a regular category $\mathcal{R}$ satisfying (B1)–(B7) except* (Pairing)*. Then $\mathcal{R}$ is a Heyting category with a subobject classifier $\ni\!\hookrightarrow \mathrm{P}(1)$. The same applies to every slice category $\mathcal{R}/R$. Moreover, $\mathcal{R}$ and each $\mathcal{R}/R$ have binary coproducts which are stable and disjoint.*

A different version of the theorem above was obtained by A. Simpson [12].

Proof. We have proved everything except the statement about coproducts. For every object X there is a canonical family of small subobjects, namely $\bot \rightarrowtail 1 \times X \to 1$, corresponding to $\emptyset\colon 1 \to \mathrm{P}(X)$. We can use the standard method from topos theory and define $X \amalg Y$ to be

$$\{\langle [x], \emptyset\rangle \mid x \in X\} \cup \{\langle \emptyset, [y]\rangle \mid y \in Y\} \rightarrowtail \mathrm{P}(X) \times \mathrm{P}(Y)\,,$$

which is $\{(U,V) \mid (\exists x.U = [x] \wedge V = \emptyset) \vee (U = \emptyset \wedge \exists y.V = [y])\}$. It is immediate from this definition that this defines the coproduct in $\mathcal{R}$ which is stable and disjoint. The same argument applies to each slice category. □

PROPOSITION. *The axioms* (Replacement)*,* (Emptyset)*, and* (Pairing) *are redundant for a representable small structure.*

This observation was independently made by Alex Simpson. In fact, in a different set-up he starts with these axioms and derives the others.

Proof. It remains to verify that (Pairing) can be derived. The underlying category has binary coproducts which are stable, so that (Union) and (Singleton) (and (Replacement), a consequence of these two axioms) imply that $\mathrm{P}(X) \times \mathrm{P}(Y) \cong \mathrm{P}(X \amalg Y)$, the isomorphism sending the small $U \hookrightarrow X$ and $V \hookrightarrow Y$ to $U \amalg V \hookrightarrow X \times Y$. To prove (Pairing) we apply (Replacement) to the codiagonal $X \amalg X \to X$ and get the pairing map as the composite $\mathrm{P}(X) \times \mathrm{P}(X) \cong \mathrm{P}(X \amalg X) \to \mathrm{P}(X)$. □

THEOREM. *Let $\mathcal{S}$ be a representable small structure on a regular category. Then the axioms (B1)–(B7) and* (Extensionality) *are satisfied if and only if the following hold for x, x' of type X, U, U', V of type $\mathrm{P}(X)$, $\mathfrak{A}$ of type $\mathrm{P}(\mathrm{P}(X))$, and w of type W:*

(C1) $\forall U[\forall x\, \varepsilon\, U \exists^! w \phi(x,w) \to \exists V \forall x\, \varepsilon\, V \phi(x,w)]$,
for every formula ϕ of the internal language of $\mathcal{R}$;

(C2) $\forall x \exists U.x\, \varepsilon\, U$,

(C3) $\forall \mathfrak{A} \forall x.x\, \varepsilon \bigcup \mathfrak{A} \leftrightarrow \exists U(x\, \varepsilon\, U \wedge U\, \varepsilon\, \mathfrak{A})$;

(C4) $\exists U.U = U$;

(C5) $\forall x \forall x' \exists U.x\, \varepsilon\, U \wedge x'\, \varepsilon\, U$

(C6) $\forall U \exists U' \forall x.x\, \varepsilon\, U' \leftrightarrow x\, \varepsilon\, U \wedge \phi$,
for every formula ϕ of the internal language of $\mathcal{R}$;

(C7) $\forall U \exists \mathfrak{A} \forall V. V \varepsilon \mathfrak{A} \leftrightarrow V \leq U$,
where $V \leq V$ *abbreviates* $\forall x. x \varepsilon V \to x \varepsilon U$;

(C8) $\forall U \forall V. U = V \leftrightarrow \forall x (x \varepsilon U \leftrightarrow x \varepsilon V)$.

Proof. Assume (B1)–(B7) and (Extensionality). Then $\mathcal{R}$ is a Heyting category by the previous theorem. (C8) is (Extensionality), (C7) is mainly a consequence of (Powerset), (C5) follows from (Pairing). For (C4) we saw that $\perp \rightarrowtail 1 \times X \to 1$ is small, and $\chi_\perp = \emptyset : 1 \to \mathrm{P}(X)$ is a global section of $\mathrm{P}(X)$. (C3) follows from (Union), (C2) from (Singleton).

Statement (C6) is an immdiate consequence of (Separation), so that it remains to prove (C1). Let Y be given, and $U : A \to \mathrm{P}(Y)$ a generalized element of $\mathrm{P}(Y)$ at stage A so that $A \Vdash \forall y \exists^! w \phi(y, w)$, that is, in $\mathcal{R}/A$ we have that $U : 1 \to \mathrm{P}(Y)$ is a global element of $\mathrm{P}(Y)$ and $\mathcal{R}/A \models \forall y \exists^! w \phi(y, w)$. Since $\mathcal{R}/A$ is regular there is a unique arrow $f : Y \to W$ such that $\mathcal{R}/A \models \forall y \phi(y, f(y))$. Let V be $f(U) \in \mathrm{P}(W)$, which exists by (Replacement). Then $\mathcal{R}/A \models \exists V \forall y \varepsilon U \exists w \varepsilon V. \phi(y, w)$, and (C1) follows since Y and $U : A \to \mathrm{P}(Y)$ were arbitrary.

For the converse assume that $\mathcal{R}$ is Heyting and satisfies (C1)–(C8). We will argue informally in $\mathcal{R}$. If $S \in \mathrm{P}(X)$ and $f : X \to Y$ are arbitrary we consider $\phi(x, y) \equiv f(x) = y$. Then $\forall x \exists^! y \phi(x, y)$ and certainly $\forall x \varepsilon S \exists^! y \phi(x, y)$. By (C1) there is $V \in \mathrm{P}(Y)$ so that $\forall x \varepsilon S \exists y \varepsilon V f(x) = y$, that is, $\forall x \varepsilon S f(x) \varepsilon V$. Use (C6) to form $V' = \{y \mid y \varepsilon V \wedge \exists x. f(x) = y\}$. Then $V' \in \mathrm{P}(Y)$ and $V' = f(S)$. This proves (Replacement). Note that (C1) implies in fact (Local Replacement).

For (Singleton) we use again (C6), this time together with (C2), and (Union) follows from (C3) and (C1). (Emptyset) is a consequence of (C4) and (C6), (Pairing) follows from (C5) and (Union), and (C6) implies (Separation).

Finally, if $S \rightarrowtail I \times X \to I$ is a family of small subobjects of X then we define $Q_S = \{(i, U) \mid i \in I, U \in \mathrm{P}(X), U \varepsilon \downarrow \chi_S(i)\}$, for $\chi_S : I \to \mathrm{P}(X)$ the characteristic function of S. The map $\pi_1 : Q_S \to I$ classifies small subfamilies of S, and we proved (Powerset). □

6 Intuitionistic Zermelo-Fraenkel set-theory

By intuitionistic Zermelo-Fraenkel set-theory (IZF) we mean the following one-sorted set-theory formulated in intuitionistic first-order logic (we favour (Replacement) over (Collection)):

(Z1) (Extensionality)
$x = x' \leftrightarrow \forall y. y \epsilon x \leftrightarrow y \epsilon x')$;

(Z2) (Pairing)
$\exists z. x \epsilon z \wedge x' \epsilon z$;

(Z3) (Union)
$\exists y \forall z. z \epsilon y \leftrightarrow \exists w (z \epsilon w \wedge w \epsilon x)$;

(Z4) (Separation)
$\exists y \forall z. z \epsilon y \leftrightarrow z \epsilon x \wedge \varphi$,
for every formula φ;

(Z5) (Powerset)
$\exists y \forall z. z \epsilon y \leftrightarrow z \subseteq x)$,
where $z \subseteq x$ abbreviates $\forall w. w \epsilon z \to w \epsilon x$;

(Z6) (Replacement)
$\forall y \epsilon x \exists^! w \varphi \to \exists z \forall y \epsilon x \exists w \epsilon z \varphi$,
for every formula φ;

(Z7) (ϵ-Induction)
$\forall x(\forall y \,\epsilon\, x\varphi(y) \to \varphi(x)) \to \forall x\varphi(x)$,
for every formula φ;

(Z8) (Infinity)
$\exists x(\emptyset \,\epsilon\, x \wedge \forall y \,\epsilon\, x.y \cup \{y\} \,\epsilon\, x)$,
where $\emptyset$ stands for the emptyset, and $\cup$ for (binary) union.

We will not treat the case of atoms, but refer the reader to [8].

6.1 Simple algebras as models. Let $\mathcal{S}$ be representable small structure on a regular category $\mathcal{R}$. Let U be an object of $\mathcal{R}$ such that $d\colon U \xrightarrow{\cong} \mathrm{P}(U)$ with inverse c. Intuitively, c codes small subclasses as elements of U.

Example. Let $\mathcal{R}$ is Set, the category of sets, and let κ be an inaccessible cardinal. If small means having cardinality less than κ, then families of small subobjects satisfy all our axioms including (Infinity). The powerset functor $\mathrm{P}_{\leq\kappa}$ is the functor sending a set X to the set of its subsets of cardinality strictly less than κ, and a function $f\colon X \to Y$ is send to

$$\mathrm{P}_{\leq\kappa}(f)\colon \mathrm{P}_{\leq\kappa}(X) \to \mathrm{P}_{\leq\kappa}(Y)\,, \qquad S \mapsto f(S)\,.$$

If U is the set V_κ, the set at the kth stage of the van Neumann hierarchy, then there are isomorphisms $\mathrm{P}_{\leq\kappa}(V_\kappa) \cong V_\kappa$.

If $\kappa = \omega$, then the same construction will give a representable small structure satisfying (B1)–(B7), but *not* (Infinity).

Define a binary relation ϵ on U, called membership, by

$$x \,\epsilon\, y \qquad \text{if and only if} \qquad x \,\varepsilon\, d(y)\,,$$

where $\varepsilon \rightarrowtail U \times \mathrm{P}(U)$ is the membership relation obtained from small structure.

Proposition. *If the small structure satisfies (B1)–(B8) then (U, ϵ) is a model of (Z1)–(Z6).*

Proof. As an example we treat (Z1), i.e., (Extensionality). We know that (U, ε) satisfies $\forall U \forall U'(U = U' \leftrightarrow \forall y.y \,\varepsilon\, U \leftrightarrow y \,\varepsilon\, U')$. Since $y \,\varepsilon\, U$ if and only if $y \,\epsilon\, c(U)$ we can write $x = c(U)$, $x' = c(U')$, and (U, ϵ) satisfies $\forall x \forall x'(x = x' \leftrightarrow \forall y.y \,\epsilon\, x \leftrightarrow y \,\epsilon\, x')$. The other cases are similar. □

By transfer of structure, U is a small-complete sup-lattice, with supremum given by

$$\bigvee\colon \mathrm{P}(U) \to U\,, \qquad S \mapsto d\bigcup[c(s) \mid s \in S]\,,$$

and c and d are isomorphisms of small-complete sup-lattices. We call the algebra $(U, \bigvee)$ *simple* (or, to be more precise, $c\sigma$-simple), if U has no subalgebra V such that $c\sigma|V\colon V \to U$ factors through V.

We note that many simple algebras arise as initial albras of monads. If $\mathcal{R}$ is locally presentable and if P is accessible then small-complete sup-lattices $(A, \bigvee\colon \mathrm{P}(A) \to A)$ together with a monotone (inflationary, etc.) endomap $A \to A$ are algebras for another monad P'. Initial algebras for these monads are simple in our sense. For the existence of such simple algebras we refer the reader to Chapters 3 and 4 of [8].

Proposition. *If U is simple then (U, ϵ) satisfies (Z1)–(Z7).*

Proof. It remains to veryfy (ϵ-Induction). As above we write ϵ for the membership relation on U, that is, $x \epsilon y$ if and only if $x \varepsilon d(y)$. Let $\varphi(y)$ be any formula. For simplicity we assume that $\varphi(y)$ has no other free variables. Otherwise we will work in an appropriate slice category of $\mathcal{R}$. Suppose that $U \models \forall x(\forall y \epsilon x\ \varphi(y) \to \varphi(x))$ and let

$$V = \{x \in U \mid \varphi(x) \wedge \forall y \epsilon x.\varphi(y))\}.$$

We will show that V is a subalgebra of U, and that $c\sigma \restriction V$ factors though V. Since U is simple we deduce that $V = U$, and in particular $\forall x \varphi(x)$.

To show that V is a subalgebra of U we have to show that $\mathrm{P}(V) \to \mathrm{P}(U) \to U$ factors though $V \hookrightarrow V$. Take $S \in \mathrm{P}(V)$ arbitrary and let $u = \bigvee^{(U)} S = c \bigcup \mathrm{P}(d)S$. We will first verify $\forall y \epsilon u.\varphi(y)$. If $y \epsilon u$ then $y \varepsilon du = \bigcup \mathrm{P}(d)S$, so there is some $s \varepsilon S$ such that $y \varepsilon d(s)$, or $y \epsilon s$. Because $S \subseteq V$ we know that $\varphi(s)$ and since $y \epsilon s$ we also know that $\varphi(y)$, which proves the claim.

To check that $\varphi(u)$ holds it is enough to verify that $\forall y \epsilon u.\varphi(y)$, which we just did. Thus V is a subalgebra of U.

The map $c\sigma$ sends $v \in V$ to $c[v]$ in U. To show that $c\sigma(v)$ is an element of V it is enough to show that $\forall y \epsilon c\sigma(v).\varphi(y)$. But if $y \epsilon c\sigma(v)$ then $y \varepsilon \sigma(v)$, so that $y = v$ and $\varphi(y)$ clearly holds. □

6.2 The axiom (Infinity). The most natural way to state the axiom of infinity is that there is a natural numbers object N and N is small. Recall that a *natural numbers object* in a category $\mathcal{C}$ with finite products is an object N together with maps $0\colon 1 \to N$ (zero), and $s\colon N \to N$ (successor) such that for arrows $f\colon I \to X$ and $g\colon I \times X \to X$ in $\mathcal{C}$ there is a unique arrow $h\colon I \times N \to X$ for which

$$\begin{tikzcd}
I \times 1 \arrow[r, "I \times 0"] \arrow[d, "\cong"'] & I \times N \arrow[r, "I \times s"] \arrow[d, "(\pi_2, h)"] & I \times N \arrow[d, "h"] \\
I \arrow[r, "f"'] & I \times X \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes. Without parameters this means that for all $z\colon 1 \to X$ and all maps $g\colon X \to X$ there is a unique map h such that

$$h(0) = z \qquad \text{and} \qquad f(s(n)) = g(h(x))$$

for all $n \in N$.

If $\mathcal{C}$ has finite limits so has the slice category $\mathcal{C}/C$ for each $C \in \mathcal{C}$, and N is a natural numbers object if and only if for each object C, $(C^*(N), C \times 0, C \times s)$ satisfies the 'parameter free' version of the axiom.

Let S be small structure on a regular category $\mathcal{R}$ with a natural numbers object N. The axiom of infinity reads as follows:

(B9) (Infinity)
The natural numbers object N is small.

Lemma. *Let $\mathcal{S}$ be a representable small structure on a regular category satisfying* (Replacement) *and* (Infinity). *If $1 \xrightarrow{x_0} X \xrightarrow{g} X$ are arbitrary arrows then $\mathcal{R}$ satisfies*

$$\exists V\colon \mathrm{P}(X)(\forall x(x \varepsilon V \leftrightarrow x = x_0 \vee \exists y.y \varepsilon V \wedge g(y) = x)).$$

The lemma also holds for maps $f\colon I \to X$ and $g\colon I \times X \to X$.

Proof. By the recursion principle there is a map $h\colon N \to X$ such that

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x_0"'] & N \arrow[r, "s"] \arrow[d, "h"] & N \arrow[d, "h"] \\ & X \arrow[r, "g"'] & X \end{tikzcd}$$

commutes. Since N is small there is the global element $N\colon 1 \to \mathrm{P}(N)$, and we let $V = \mathrm{P}(h)(N)\colon 1 \to \mathrm{P}(X)$. Then U satisfies

$$\begin{aligned} x \,\varepsilon\, V \quad &\leftrightarrow \quad \exists m.h(m) = x \\ &\leftrightarrow \quad x = h(0) \vee \exists n.h(s(n)) = x\,. \end{aligned}$$

But $h(0) = x_0$ and $h(s(n)) = g(h(n))$, from which the lemma follows. □

Let again d be an isomorphism $U \to \mathrm{P}(U)$ with inverse c, and define the membership relation $x \,\epsilon\, y$ by $x \,\varepsilon\, d(y)$.

THEOREM. *Let S be a representable structure on a regular category $\mathcal{R}$ satisfying (B1)–(B9). Then (U, ϵ) is a model of (Z1)–(Z6) and (Z8). So if U is simple, then (U, ϵ) is a model of intuitionistic Zermelo-Fraenkel set-theory.*

Proof. It remains to verify (Z8). We first note that the empty set is given by

$$\emptyset\colon 1 \xrightarrow{\perp} \mathrm{P}(X) \xrightarrow{c} X\,,$$

and that there is the map $-\cup\{-\}\colon X \to X$ defined by $X \to X \times X \xrightarrow{d\times\sigma} \mathrm{P}(X)\times\mathrm{P}(X) \xrightarrow{\vee} \mathrm{P}(X) \xrightarrow{c} X$. This map has the usual property, namely

$$\begin{aligned} y \,\epsilon\, x \cup \{x\} \quad &\leftrightarrow \quad y \,\varepsilon\, d(x \cup \{x\}) \\ &\leftrightarrow \quad y \,\varepsilon\, d(x) \cup \sigma(x) \\ &\leftrightarrow \quad y \,\varepsilon\, d(x) \vee y \,\varepsilon\, \sigma(x) \\ &\leftrightarrow \quad y \,\epsilon\, x \vee y \,\epsilon\, c\sigma(x) \\ &\leftrightarrow \quad y \,\epsilon\, x \vee y \,\epsilon\, \{x\}\,, \end{aligned}$$

where $\{-\} = c\sigma\colon X \to \mathrm{P}(X) \to X$ is the singleton function of X satisfying

$$\begin{aligned} y \,\epsilon\, \{x\} \quad &\leftrightarrow \quad y \,\varepsilon\, d\{x\} \\ &\leftrightarrow \quad y \,\varepsilon\, \sigma(x) \\ &\leftrightarrow \quad y = x\,. \end{aligned}$$

We apply the previous lemma to these two maps, and find that U satisfies the first-order sentence

$$\exists z\colon \mathrm{P}(X)(\forall x.x \,\varepsilon\, z \leftrightarrow x = \emptyset \vee \exists y.y \,\varepsilon\, z \wedge x = y \cup \{y\})\,.$$

If we write $\bar{z}$ for cz then this translates into $\exists \bar{z}(\forall x.x \,\epsilon\, \bar{z} \leftrightarrow x = \emptyset \vee \exists y.y \,\epsilon\, \bar{z} \wedge x = y \cup \{y\})$, in particular (Z8). □

7 Appendix: The axioms for small maps from Joyal and Moerdijk

The book [8] uses the following axioms for a class S of maps with small fibers on a Heyting pretopos (a Heyting category with finite coproducts that are disjoint and stable):

(A1) Any isomorphism belongs to $\mathcal{S}$, and $\mathcal{S}$ is closed under composition.

(A2) (Stability)
In a pullback square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"'] & X \end{tikzcd}$$

if f belongs to $\mathcal{S}$ then so does g.

(A3) (Descent)
In any pullback square (1), if g belongs to $\mathcal{S}$ and p is an epimorphism then f belongs to $\mathcal{S}$.

(A4) The maps $0 \to 1$ and $1 + 1 \to 1$ belong to $\mathcal{S}$.

(A5) (Sums)
If $Y \to X$ and $Y' \to X'$ belong to $\mathcal{S}$ then so does their sum $Y + Y' \to X + X'$.

(A6) (Quotients)
In any commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X, & \end{tikzcd}$$

if p is an epimorphism and g belongs to $\mathcal{S}$ then so does f.

(A7) (Collection)
For any two arrows $p\colon Y \twoheadrightarrow X$ and $f\colon X \to A$ where p is an epimorphism and f belongs to S, there exists a quasi-pullback square of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, two heads, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, two heads, "h"'] & & A \end{tikzcd}$$

where h is an epimorphism and g belongs to $\mathcal{S}$.

Here such a square is called a quasi-pullback square if the evident arrow $Z \to B \times_A X$ is an epimorphism.

(S1) (Exponentiability axiom)
Every map in $\mathcal{S}$ is exponentiable.

(S2) (Representability axiom)
There exists a map $\pi\colon E \to U$ in $\mathcal{S}$ which is universal in the following sense: for any map $f\colon Y \to X$ in $\mathcal{S}$ there exists a diagram

$$\begin{tikzcd} Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[r] \arrow[d, "f'"] & E \arrow[d, "\pi"] \\ X & X' \arrow[l, two heads, "p"] \arrow[r] & U \end{tikzcd}$$

in which p is an epimorphism and both squares are pullbacks.

In addition, Joyal and Moerdijk consider sometimes the axioms below. We note that it is a consequence of the axioms above that the associated class of small subobjects is representable by a functor P. The axiom (S5) below requires of course the further assumption that the underlying Heyting pretopos has an (indexed) natural numbers object N.

(S3) (Powerset)
If $X \to B$ belongs to $\mathcal{S}$ then so does $P_{\mathcal{S}}(X \to B)$.

(S4) (Separation)
Every monomorphism belongs to $\mathcal{S}$.

(S5) (Infinity)
For the natural number object N of the pretopos the map $N \to 1$ is small.

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