1065: On a Theorem of Brianchon and Poncelet

In a note of Margetson and Buckingham [4], they treat an interesting result of planar geometry. After we read it, we noticed that the following extension of their result holds.

Theorem 1. *If the vertices of $\triangle ABC$ lie on a given perpendicular hyperbola, then the orthocenter H also lies on the hyperbola, and the nine-point circle passes through the symmetric center of the hyperbola.*

According to [3], or [1], the *nine-point circle* was first found by Brianchon and Poncelet [2]. The circle passes through the nine important points of $\triangle ABC$; namely, the feets D, E, F of perpendiculars from A, B, C to the opposite edges; the midpoints L, M, N of the segment BC, CA, AB; and the midpoints of the segments AH, BH, CH.

Unfortunately, we found that Theorem 1 is not new. According to [3], it was already proved by Brianchon and Poncelet [2]. (However, we could not obtain it.) In the note, we consider a converse problem of the following type.

Theorem 2. *If a point Ω lies on the nine-point circle of a given $\triangle ABC$, where $\Omega \neq D, E, F$, then there is a perpendicular hyperbola which has Ω as the symmetric center and passes through the vertices A, B, C.*

To prove it, we use the following lemma.

Lemma. *Two points A and B of the xy-plane lie on the same perpendicular hyperbola $y = k/x$ if and only if the line AB meets both x- and y-axis at points, say, B' and A', respectively, and $\overrightarrow{A'A} = \overrightarrow{BB'}$.*

We think that the above lemma alone is sufficiently interesting. In the rest of the note, we give the proofs of Theorems 1 and 2. The proof of Theorem 1 is similar to that of [4] but quite different from that of [3].

Proof of Theorem 1. We denote by $y = k/x$ the perpendicular hyperbola, and by $\vec{a} = (a, k/a)$, $\vec{b} = (b, k/b)$, $\vec{c} = (c, k/c)$ the position vectors of A, B, C, respectively. We consider a point H' whose position vector $\vec{h} = (h, k/h)$, where $h = -k^2/abc$. We shall prove that H' coincides with the orthocenter H. Since we have $abch = -k^2$, we obtain that

$$\vec{a} \cdot \vec{h} + \vec{b} \cdot \vec{c} = (ah + bc)\left(1 + \frac{k^2}{abch}\right) = 0. \tag{1}$$

Similarly, we obtain that

$$\vec{a} \cdot \vec{h} + \vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{h} + \vec{c} \cdot \vec{a} = \vec{c} \cdot \vec{h} + \vec{a} \cdot \vec{b} = 0. \tag{2}$$

By using (2), we obtain that

$$(\vec{a} - \vec{b}) \cdot (\vec{c} - \vec{h}) = (\vec{a} - \vec{c}) \cdot (\vec{b} - \vec{h}) = 0, \tag{3}$$

or equivalently, $AB \perp CH'$ and $AC \perp BH'$. Thus the point H' coincides with the orthocenter H.

By the theorem of Euler line, the center S of the nine-point circle divides the segment HG internally in the ratio $3 : 1$, where G is the the center of gravity.

So the position vector of S is in the form $(\vec{a}+\vec{b}+\vec{c}+\vec{h})/4$. So, by using (2), we obtain that

$$\left|(\vec{a}+\vec{b}+\vec{c}+\vec{h})/4\right| = \left|(\vec{a}+\vec{b}-\vec{c}-\vec{h})/4\right|, \tag{4}$$

or equivalently, $\overline{S\Omega} = \overline{NS}$, where Ω is the origin. Thus the nine-point circle passes through Ω. □

Proof of Lemma. We denote by $(a, k_1/a)$ and $(b, k_2/b)$ the coordinates of A and B, respectively. By an easy calculation, the coordinates of A' is that

$$\left(0, \frac{-k_1 b + k_2 a}{a(a-b)} + \frac{k_2}{b}\right). \tag{5}$$

Thus the condition to satisfy $\overrightarrow{A'A} = \overrightarrow{BB'}$ is that

$$\frac{k_1}{a} - \left(\frac{-k_1 b + k_2 a}{a(a-b)} + \frac{k_2}{b}\right) = 0 - \frac{k_2}{b}. \tag{6}$$

It is reduced to $k_1 = k_2$. □

Proof of Theorem 2. We can assume without loss of generality that the orthocenter H is an inner point of $\triangle ABC$. Because, even if the point A, for example, is an inner point of $\triangle BCH$, we can exchange the labels of A and H. We can also assume that the point Ω lies on the open arc $\widehat{ND}$ (anticlockwise in the figure) of the nine-point circle. Because, even if the point Ω lies on the half-open arc $\widehat{FN}$ with the end N, for example, we can exchange the labels of A and B.

Since the circle , whose diameter is AB intersects the nine-point circle at two points D and E, the open arc $\widehat{ED}$ (anticlockwise in the figure) is included in the interior of , . Since the point Ω lies on the open arc $\widehat{ND}$, which is included in the interiror of , , we obtain $\angle A\Omega B > 90°$. (The condition $\Omega \neq D$ is used here!) We take two points A' and B' on the line AB so that $\overline{N\Omega} = \overline{NA'} = \overline{NB'}$. Since the point Ω lies on the circle whose diameter is $A'B'$, we obtain $\angle A'\Omega B' = 90°$. So we can choose the lines $\Omega B'$ and $\Omega A'$ as the new x- and y-axis. Then by the lemma, the points A and B lie on the same perpendicular hyperbola $y = k/x$.

We shall show that the point C also lies on the hyperbola when the origin Ω lies on the nine-point circle. Since we have $\overline{\Omega S} = \overline{NS}$, or equivalently, (4), we obtain that

$$(\vec{a}+\vec{b})\cdot(\vec{c}+\vec{h}) = 0. \tag{7}$$

Since H is the orthocenter of $\triangle ABC$, we obtain $AB \perp CH$ and $AC \perp BH$, or equivalently, (3). By combining (3) and (7), we obtain (2). By putting $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, $\vec{c} = (c_1, c_2)$, $\vec{h} = (h_1, h_2)$ to (2), we obtain that

$$\begin{pmatrix} a_1 & a_2 & b_1c_1 + b_2c_2 \\ b_1 & b_2 & c_1a_1 + c_2a_2 \\ c_1 & c_2 & a_1b_1 + a_2b_2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{8}$$

The condition of (8) to have a solution (h_1, h_2) is that

$$(b_1b_2 - c_1c_2)(a_2^2 - a_1^2) + (c_1c_2 - a_1a_2)(b_2^2 - b_1^2) + (a_1a_2 - b_1b_2)(c_2^2 - c_1^2) = 0. \tag{9}$$

Since we have $a_1a_2 = b_1b_2 = k$, we obtain that

$$(c_1c_2 - k)(a_1^2 - b_1^2)\left(1 + \frac{k^2}{a_1^2b_1^2}\right) = 0. \tag{10}$$

Since we have $A \neq B$ and $N \neq \Omega$, we obtain $c_1c_2 = k$. □

By Theorem 2, we can deduce that a curve which enjoy the second conclusion of Theorem 1 must be a perpendicular hyperbola. However, we have no answers to the same problem for the first conclusion.

Problem. Suppose whenever the vertices of $\triangle ABC$ lie on a given curve γ, so does the orthocenter H. Then must the curve γ be a perpendicular hyperbola?

References

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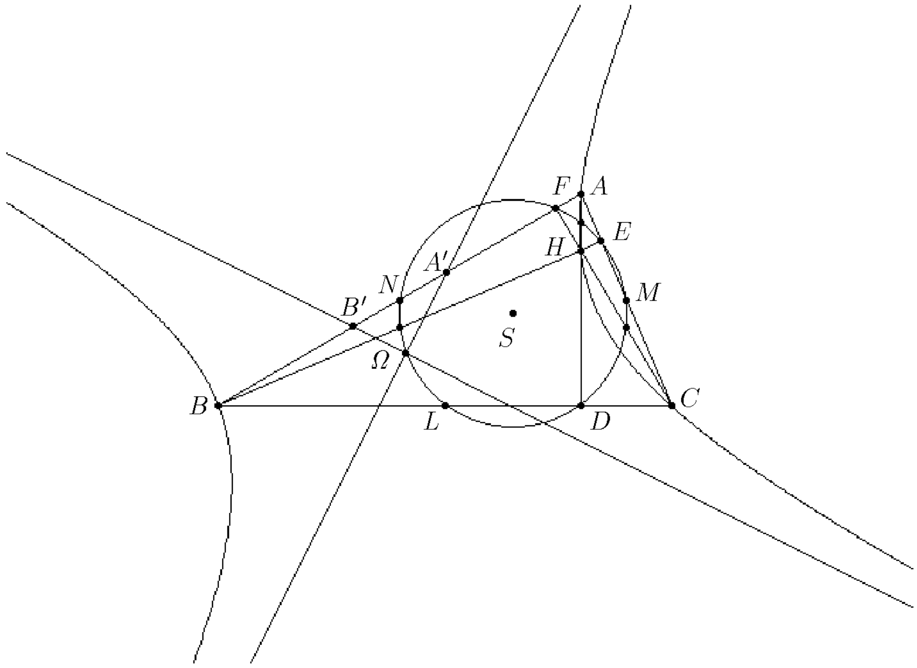
Kenzi ODANI: Department of Mathematics, Aichi University of Education, Kariya-shi, Aichi 448-8542, Japan; kodani@auecc.aichi-edu.ac.jp
Shihomi TAKASE: 20 Okuseko, Horikiri, Atsumi-cho, Aichi 441-3627, Japan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