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THE RANKED PROBABILITY SCORE AND THE PROBABILITY SCORE: A COMPARISON [1, 2]

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ABSTRACT

In this paper, we compare the ranked probability score (RPS) and the probability score (PS) and examine the nature of the sensitivity of the RPS to distance. First, we briefly describe the nature of and the relationship between the frameworks within which the RPS and the PS were formulated. Second, we consider certain properties of the RPS and the PS including their range, their values for categorical and uniform forecasts, and their "proper" nature. Third, we describe the RPS and the PS in a manner that reveals the structure of and the relationship between these scoring rules. Fourth, we consider the RPS with reference to two definitions of distance and examine the nature of the sensitivity of the RPS to distance. The comparison of the RPS and the PS suggests that the RPS rather than the PS should be used to evaluate probability forecasts, at least in those situations in which the variable of concern is ordered.

1. INTRODUCTION

The probability score (PS) formulated by Brier (1950) is considered by most, if not all, meteorologists to be the "best" available scoring rule for evaluating probability forecasts. The status of the PS was recently further enhanced by the realization [3] that the PS was a proper scoring rule (Murphy and Epstein 1967*b*).[4] However, meteorologists have indicated for some time the need for a scoring rule that is sensitive to distance,[5] a need based upon considerations of distance [6] that are particularly relevant for forecasts of ordered variables such as temperature, precipitation amount, and ceiling height. Unfortunately, the PS does not, in general, satisfy this need. However, the ranked probability score (RPS) formulated by Epstein (1969) is sensitive to distance (Stael von Holstein 1970). In addition, the RPS is a proper scoring rule (Murphy 1969*d*). The author believes that, henceforth, the RPS rather than the PS should be used to evaluate probability forecasts of ordered variables.[7]

The purposes of this paper are (1) to compare the RPS and the PS and (2) to describe the nature of the sensitivity of the RPS to distance. Although the frameworks within which the RPS and the PS were formulated are not of particular concern in this paper, we briefly describe the nature of and the relationship between these frameworks in section 2. In section 3, we consider certain properties of the RPS and the PS. We describe the structure of the RPS and the PS and compare these structures in section 4. In section 5, we consider the RPS with reference to two definitions of distance and examine the nature of the sensitivity of the RPS to distance. Section 6 contains a brief summary and conclusion.

2. FORMULATION

In this section, we briefly consider the nature of the frameworks within which the RPS and the PS were formulated and the relationship between these frameworks.

Let the (row) vector $\mathbf{r}=(r_1, \ldots, r_K)$ denote a probability forecast of an ordered variable that has been divided into K mutually exclusive and collectively exhaustive classes or states. Then, r_k is the forecast probability of class k ($r_k \geq 0$, $\sum_k r_k=1$; $k=1, \ldots, K$).

RANKED PROBABILITY SCORE

The RPS was formulated within the framework of a K action–K state cost-loss ratio decision situation (Epstein 1969 and Stael von Holstein 1970).[8] The cost-loss matrix for this situation when $K=5$ is displayed in table 1. The set of actions $\{a_1, \ldots, a_5\}$ represents five decreasing levels of protection, and the set of states $\{s_1, \ldots, s_5\}$ represents five decreasing degrees (of severity) of weather. The quantity X represents the cost-loss ratio, the range of which is the unit interval [0, 1]. The matrix upon which the RPS is based is a utility matrix, the elements of which are linearly related to the sum of the corresponding elements in two matrices, the matrix in table 1 and its mirror image.[9] The RPS is the expected-utility measure (Murphy 1966, 1969*c*) that results from such a matrix when the cost-loss ratio X is assumed to have a uniform probability distribution.[10] The RPS when class j occurs, $\mathrm{RPS}_j(\mathbf{r})$,

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[3] In addition, Murphy (1966) has shown that, in certain special situations, the PS is a measure of the value of forecasts (refer to section 2 of the present paper). For a detailed description of the nature and properties of the PS, refer to Murphy (1969*a*).

[4] For a definition of proper scoring rules, refer to section 3 of the present paper.

[5] Whether or not a scoring rule is sensitive to distance depends upon the definition of *the term "distance" as well as the scoring rule itself. However, for the present, we shall* assume that a scoring rule is sensitive to distance if forecasts that concentrate their probability about the event that occurs receive better scores. We consider two definitions of distance and examine the sensitivity of the RPS to distance in section 5.

[6] In section 5, we indicate the nature of the concept of distance by considering (1) the differences between two forecasts and (2) two possible definitions of distance.

[7] If the variable of concern consists of only two events (e.g., rain and no rain), then "*all reasonable scoring rules are sensitive to distance*" (*Stael von Holstein 1970*). *From* another point of view, considerations of distance are not relevant in the two-event situation. Thus, scoring rules that are sensitive to distance are of particular interest when the number of events exceeds 2.

[8] The framework within which the RPS was formulated is an extension of the framework developed by Murphy (1966, 1969*c*).

[9] The mirror image of the matrix in table 1 is a matrix in which j and $6-j$ are interchanged ($j=1, 2$). A scoring rule based upon the sum of these two matrices is less dependent (with regard to its values) upon the class that occurs than a scoring rule based upon only one of these matrices. For example, if only the matrix in table 1 is considered, the maximum *(minimum) value will be a strictly increasing (an increasing) function of the class number;* and as a result, the range will not be symmetric about class 3 (p. 986 of Epstein 1969).

[10] The family of proper scoring rules that results if distributions other than the uniform distribution are considered is described by Stael von Holstein (1970).

TABLE 1.—*Cost-loss matrix for the decision situation within which the* RPS *was formulated when* $K=5$

Actions	States				
	s_1	s_2	s_3	s_4	s_5
a_1	X	X	X	X	X
a_2	$(3X+1)/4$	$3X/4$	$3X/4$	$3X/4$	$3X/4$
a_3	$(2X+2)/4$	$(2X+1)/4$	$X/2$	$X/2$	$X/2$
a_4	$(X+3)/4$	$(X+2)/4$	$(X+1)/4$	$X/4$	$X/4$
a_5	1	3/4	1/2	1/4	0

TABLE 2.—*Cost-loss matrix for the standard cost-loss ratio decision situation*

Actions	States	
	s_1 (weather)	s_2 (no weather)
a_1 (protect)	X	X
a_2 (do not protect)	1	0

can be expressed as

$$\mathrm{RPS}_j(\mathbf{r})=(3/2)-[1/2(K-1)]\sum_{i=1}^{K-1}\left[\left(\sum_{n=1}^{i} r_n\right)^2+\left(\sum_{n=i+1}^{K} r_n\right)^2\right]-[1/(K-1)]\sum_{i=1}^{K}|i-j|\, r_i \tag{1}$$

(p. 987 of Epstein 1969).

PROBABILITY SCORE

The PS when class j occurs, $\mathrm{PS}_j(\mathbf{r})$, can be expressed as

$$\mathrm{PS}_j(\mathbf{r})=(1-r_j)^2+\sum_{i=1}^{K} r_i^2,\quad (i\neq j). \tag{2}$$

Murphy (1966, 1969*c*) has shown that, whatever the original framework within which the PS [11] was formulated, $\mathrm{PS}_j(\mathbf{r})$ when $K=2$ is equivalent [12] to the expected-utility measure $\mathrm{EU}_j(\mathbf{r})$ formulated within the framework of the standard cost-loss ratio decision situation under the assumption that the cost-loss ratio is uniformly distributed. The matrix for the standard cost-loss ratio decision situation is displayed in table 2.[13] Specifically, when $K=2$

$$\mathrm{EU}_j(\mathbf{r})=1-(1/2)\mathrm{PS}_j(\mathbf{r}), \tag{3}$$

where

$$\mathrm{PS}_j(\mathbf{r})=2(1-r_j)^2,$$

or

$$\mathrm{PS}_j(\mathbf{r})=r_i^2+(1-r_j)^2,\quad (i\neq j).$$

Thus, $\mathrm{PS}_j(\mathbf{r})$ in eq (2), for K (>2) classes can be considered to represent a natural but neutral extension of $\mathrm{EU}_j(\mathbf{r})$ (refer to section 4 of the present paper).

RANKED PROBABILITY SCORE AND PROBABILITY SCORE

The framework within which the RPS was formulated can be considered to represent a generalization of the framework within which the measure $\mathrm{EU}_j(\mathbf{r})$ was formulated. Thus, the RPS represents, in one sense, a generalization of the PS.[14] When $K=2$, the RPS reduces to the PS. Specifically, when $K=2$,

$$\mathrm{RPS}_j(\mathbf{r})=1-(1/2)\mathrm{PS}_j(\mathbf{r}). \tag{4}$$

3. PROPERTIES

The properties of the PS, which have been described by Brier (1950), Hughes (1965), Murphy and Epstein (1967*a*, 1967*b*), and Sanders (1963, 1967) among others, have recently been summarized by Murphy (1969*a*). The properties of the RPS have been described by Epstein (1969), Murphy (1969*d*), and Stael von Holstein (1970). In this section, we briefly consider certain of these properties.

RANGE

The RPS, which has a positive orientation [15] and which assumes its values on the unit interval [0, 1], attains its maximum value of 1 when $r_j=1$ (i.e., for a categorical forecast of class j) and its minimum value of $1-[\max(j-1, K-j)/(K-1)]$ when $r_1(r_K)=1$ and $j\geq(\leq)\ (K+1)/2$ (i.e., for a categorical forecast of the class most distant from class j). Thus, while the maximum value of the RPS is 1 regardless of which class occurs, the minimum value of the RPS depends upon the class that occurs. In particular, the RPS assumes its absolute minimum value of 0 only when $r_1(r_K)=1$ and $j=K$ (1). For $j\neq1$ or K, the minimum value is greater than 0. The maximum and minimum values of the RPS are depicted in figure 1 for (a) K even and (b) K odd.[16]

The PS, which has a negative orientation and which assumes its values on the interval [0, 2], attains its maximum value of 2 when $r_i=1$ $(i\neq j)$ (i.e., for a categorical and incorrect forecast) and its minimum value of 0 when $r_j=1$ (i.e., for a categorical and correct forecast). Thus, the maximum and minimum values of the PS are not dependent upon which class occurs.

CATEGORICAL FORECASTS

For a categorical forecast, that is, when some $r_i=1$, $\mathrm{RPS}_j(\mathbf{r})$ in eq (1) reduces to

$$\mathrm{RPS}_j(\mathbf{r})=1-(|i-j|)/(K-1). \tag{5}$$

$\mathrm{RPS}_j(\mathbf{r})$ in eq (5) is displayed in table 3(a) for the situation in which $K=5$.

[11] The PS represents the sum of the squared deviations of certain indicator variables (the components of the observation vector) from their expected values (the components of the forecast vector, i.e., the probabilities) and as such is a rather natural measure of the degree of association between the forecast and the observation.

[12] Two scoring rules are equivalent if the scoring rules are linearly related (Winkler and Murphy 1968*b*).

[13] Murphy (1966, 1969*c*) considered only the matrix in table 2 and not its mirror image. Thus, the expected-utility measure E(U) (p. 867 of Murphy 1969*c*) is in essence equivalent but not identical to the expected-utility measure $\mathrm{EU}_j(\mathbf{r})$.

[14] We indicate in section 4, where the relationship between the RPS and the PS is examined in greater detail, that the RPS is, in another sense, a specialization of the PS.

[15] A scoring rule for which larger scores are better is said to have a positive orientation, and a scoring rule for which smaller scores are better is said to have a negative orientation (Winkler and Murphy 1968*b*). The RPS and the PS are examples of scoring rules with positive and negative orientations, respectively.

[16] The lines in figures 1(a) and 1(b) are intended to be schematic only. Since j is an integer, $\mathrm{RPS}_j(\mathbf{r})$ is of course a discrete function of j. Note that the maximum and minimum values are assumed only for categorical forecasts.

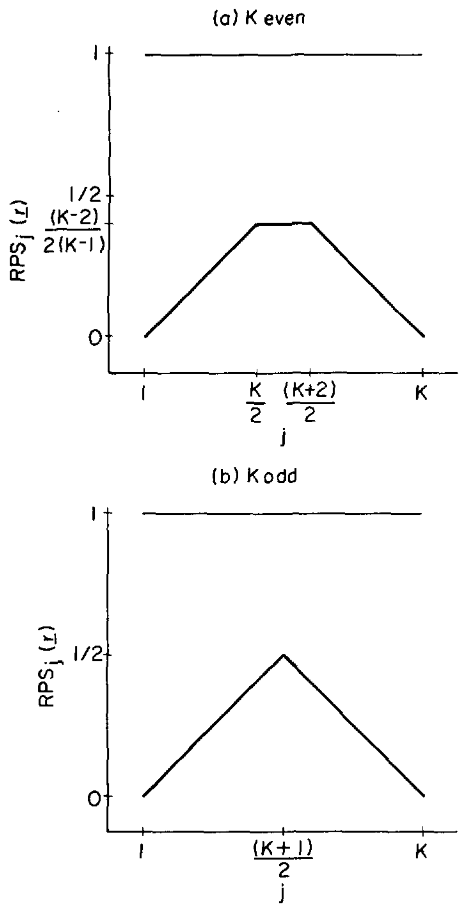

FIGURE 1.—Maximum and minimum values of the RPS for (a) K even and (b) K odd.

For a categorical forecast, $PS_j(\mathbf{r})$ in eq (2) reduces to

$$PS_j(\mathbf{r})=\begin{matrix}2 \text{ if } i\neq j \\ 0 \text{ if } i=j\end{matrix}. \qquad (6)$$

$PS_j(\mathbf{r})$ in eq (6) is displayed in table 3(b) for the situation in which $K=5$.

In table 3, note that, for $i \neq j$, the values of $PS_j(\mathbf{r})$ are constant, while the values of $RPS_j(\mathbf{r})$ depend upon the distance, in terms of classes, between i and j.

UNIFORM FORECASTS

As indicated by Epstein (p. 987 op. cit.), when the r_k are all equal to $1/K$, $RPS_j(\mathbf{r})$ in eq (1) reduces to

$$RPS_j(\mathbf{r})=(2/3)+(1/6K)+(j-1)(K-j)/K(K-1). \qquad (7)$$

$RPS_j(\mathbf{r})$ in eq (7) assumes its minimum value of $(4K+1)/6K$ when $j=1$ or K and its maximum value of either $(11K-1)/12K$ when $j=(K+1)/2$ and K is odd or $(11K^2-12K-2)/12K(K-1)$ when $j=K/2$ or $(K+2)/2$ and K is even. Note that, in the limit, that is, as K increases indefinitely, $RPS_j(\mathbf{r})$ in eq (7) is bounded between 2/3 and 11/12.

When the r_k are all equal to $1/K$, $PS_j(\mathbf{r})$ in eq (2) reduces to

$$PS_j(\mathbf{r})=(K-1)/K. \qquad (8)$$

Note that, in the limit, $PS_j(\mathbf{r})$ in eq (8) approaches 1.

TABLE 3.—*Scores assigned to categorical forecasts when $K=5$ and the scoring rule is* (a) *the* RPS *and* (b) *the* PS

(a) $RPS_j(\mathbf{r})$

		j: 1	2	3	4	5
i	1	1	3/4	1/2	1/4	0
	2	3/4	1	3/4	1/2	1/4
	3	1/2	3/4	1	3/4	1/2
	4	1/4	1/2	3/4	1	3/4
	5	0	1/4	1/2	3/4	1

(b) $PS_j(\mathbf{r})$

		j: 1	2	3	4	5
i	1	0	2	2	2	2
	2	2	0	2	2	2
	3	2	2	0	2	2
	4	2	2	2	0	2
	5	2	2	2	2	0

TABLE 4.—*Scores assigned to uniform forecasts when $K=5$ and the scoring rule is* (a) *the* RPS *and* (b) *the* PS

	j: 1	2	3	4	5
(a) $RPS_j(\mathbf{r})$	0.70	0.85	0.90	0.85	0.70
(b) $PS_j(\mathbf{r})$	0.80	0.80	0.80	0.80	0.80

The values of $RPS_j(\mathbf{r})$ in eq (7) and $PS_j(\mathbf{r})$ in eq (8) are displayed in table 4(a) and 4(b), respectively, when $K=5$. Note that $PS_j(\mathbf{r})$ is constant, while $RPS_j(\mathbf{r})$ is a maximum when $j=3$ and a minimum when $j=1$ or 5.

PROPER AND STRICTLY PROPER SCORING RULES

The concept of proper scoring rules was introduced into the meteorological literature by Murphy and Epstein (1967*b*). In that paper as well as in all the papers *published* through 1969 (Winkler and Murphy 1968*a*, 1968*b*; Murphy 1969*c*, 1969*d*, 1969*e*; Epstein 1969), the authors, in essence, identified two classes of scoring rules: (1) the class of proper scoring rules and (2) the class of improper scoring rules. In this paper and henceforth, we shall adopt the classification and terminology prescribed by Murphy (1969*b*) and Stael von Holstein (1970).

Let the (row) vector $\mathbf{p}=(p_1, \ldots, p_K)$ denote the forecaster's true belief (judgment). Then, p_k is the forecaster's subjective judgment that class K will occur ($p_k \geq 0$, $\sum_k p_k=1$; $k=1, \ldots, K$). Further, let $S_j(\mathbf{r})$ denote the score assigned by a scoring rule S to a forecaster's statement $\mathbf{r}$ when class j occurs, and let $S(\mathbf{r}, \mathbf{p})$ denote the forecaster's (subjective) expected score when his statement is $\mathbf{r}$ and his judgment is $\mathbf{p}$. Then, S is a proper scoring rule if

$$S(\mathbf{p}, \mathbf{p}) \geq S(\mathbf{r}, \mathbf{p}), \text{ for all } \mathbf{r}, \qquad (9)$$

and a strictly proper scoring rule if

$$S(\mathbf{p}, \mathbf{p}) > S(\mathbf{r}, \mathbf{p}), \text{ for all } \mathbf{r} \neq \mathbf{p}, \qquad (10)$$

where

$$S(\mathbf{r}, \mathbf{p})=\sum_{j=1}^{K} p_j S_j(\mathbf{r}).\text{[17]} \qquad (11)$$

[17] The scoring rule S is assumed to have a positive orientation.

The difference between proper and strictly proper scoring rules can be described briefly as: (1) a *proper* scoring rule is defined in such a way that the forecaster maximizes his expected score if he sets $\mathbf{r}$ equal to $\mathbf{p}$, but a forecast $\mathbf{r} \neq \mathbf{p}$ may also receive the same expected score, and (2) a *strictly proper* scoring rule is defined in such a way that the forecaster maximizes his expected score only if he sets $\mathbf{r}$ equal to $\mathbf{p}$. That is, a strictly proper scoring rule encourages complete honesty, while a proper scoring rule does not discourage complete honesty.[18] Clearly, strictly proper scoring rules would be preferred in general to proper scoring rules.

The RPS and the PS are both strictly proper scoring rules (Murphy 1969*b*). Thus if a forecaster wants to maximize (minimize) his expected RPS (PS), he should make his forecast $\mathbf{r}$ correspond exactly to his judgment $\mathbf{p}$. Note that any $\mathbf{r} \neq \mathbf{p}$ will receive a smaller (larger) expected score.

TABLE 5.—RPS_j ($\mathbf{r}$) *expressed in terms of the elements of the matrix* $\mathbf{r}'\mathbf{r}$ *for the situation in which* $K=5$

		$\mathbf{r}'\mathbf{r}$		
r_1^2	r_1r_2	r_1r_3	r_1r_4	r_1r_5
r_1r_2	r_2^2	r_2r_3	r_2r_4	r_2r_5
r_1r_3	r_2r_3	r_3^2	r_3r_4	r_3r_5
r_1r_4	r_2r_4	r_3r_4	r_4^2	r_4r_5
r_1r_5	r_2r_5	r_3r_5	r_4r_5	r_5^2
$j=1$: $\mathrm{RPS}_1(\mathbf{r})=1-(1/4)$ (sum of elements in $\mathbf{r}'\mathbf{r}$ multiplied by their respective coefficients below)				
0	0	0	0	0
0	1	1	1	1
0	1	2	2	2
0	1	2	3	3
0	1	2	3	4
$j=2$: $\mathrm{RPS}_2(\mathbf{r})=1-(1/4)$ (sum of elements in $\mathbf{r}'\mathbf{r}$ multiplied by their respective coefficients below)				
1	0	0	0	0
0	0	0	0	0
0	0	1	1	1
0	0	1	2	2
0	0	1	2	3
$j=3$: $\mathrm{RPS}_3(\mathbf{r})=1-(1/4)$ (sum of elements in $\mathbf{r}'\mathbf{r}$ multiplied by their respective coefficients below)				
2	1	0	0	0
1	1	0	0	0
0	0	0	0	0
0	0	0	1	1
0	0	0	1	2
$j=4$: $\mathrm{RPS}_4(\mathbf{r})=1-(1/4)$ (sum of elements in $\mathbf{r}'\mathbf{r}$ multiplied by their respective coefficients below)				
3	2	1	0	0
2	2	1	0	0
1	1	1	0	0
0	0	0	0	0
0	0	0	0	1
$j=5$: $\mathrm{RPS}_5(\mathbf{r})=1-(1/4)$ (sum of elements in $\mathbf{r}'\mathbf{r}$ multiplied by their respective coefficients below)				
4	3	2	1	0
3	3	2	1	0
2	2	2	1	0
1	1	1	1	0
0	0	0	0	0

4. STRUCTURE

In this section, we describe the RPS and the PS in a manner that we believe, reveals the structure of and the relationship between these scoring rules.

RANKED PROBABILITY SCORE

The $\mathrm{RPS}_j(\mathbf{r})$ in eq (1) can also be expressed as

$$\mathrm{RPS}_j(\mathbf{r})=1-[1/(K-1)]\left[\sum_{i=1}^{K}|i-j|r_i^2+2\sum_{i=1}^{j-2}\sum_{k=i+1}^{j-1}(j-k)r_ir_k + 2\sum_{i=j+1}^{K-1}\sum_{k=i+1}^{K}(i-j)r_ir_k\right], \quad (12)$$

an expression that reveals the structure of the RPS (see appendix). Note that the first term in brackets is a weighted sum of the squares of the r_k and the second and third terms are weighted sums of certain cross products of the r_k.

To facilitate this description of the structure of the RPS, we consider $\mathrm{RPS}_j(\mathbf{r})$ in eq (12) with reference to the symmetric matrix $\mathbf{r}'\mathbf{r}$ ($\mathbf{r}'$, a column vector, is the transpose of $\mathbf{r}$), a matrix whose elements consist of the squares and cross products of the r_k. In table 5, we display $\mathrm{RPS}_j(\mathbf{r})$ in terms of the elements of $\mathbf{r}'\mathbf{r}$ when $K=5$. Table 5 reveals that, for example,

$$\mathrm{RPS}_2\ (\mathbf{r})=1-(1/4)\ (r_1^2+r_3^2+2r_4^2+3r_5^2+2r_3r_4+2r_3r_5+4r_4r_5).$$

Consider the coefficient matrices in table 5. Note that the coefficients in the jth row and jth column are equal to 0. Further, note that the coefficients corresponding to the elements in $\mathbf{r}'\mathbf{r}$ that straddle j (for example, r_1r_3, r_1r_4, and r_1r_5 when $j=2$) are also equal to 0.[19] The coefficient of a square element, that is, an element on the (principal) diagonal of $\mathbf{r}'\mathbf{r}$, depends upon the absolute difference between the class number of concern (i, say) and the class number that occurs (j). The coefficient of a cross product, that is, an off-diagonal, element is equal to the coefficient assigned to the diagonal element in its row if it is above the diagonal and equal to the coefficient assigned to the diagonal element in its column if it is below the diagonal.

The symmetry of the RPS is evident upon examination of table 5. This symmetry facilitates the expression of the RPS for different values of K and j.

PROBABILITY SCORE

The $\mathrm{PS}_j(\mathbf{r})$ in eq (2) can also be expressed as

$$\mathrm{PS}_j(\mathbf{r})=\left(\sum_i r_i\right)^2+\sum_i r_i^2, \quad (i\neq j),$$

or

$$\mathrm{PS}_j(\mathbf{r})=2\sum_{i=1}^{K}r_i^2+2\sum_{i=1}^{K-1}\sum_{k=i+1}^{K}r_ir_k, \quad (i,k\neq j).^{20} \quad (13)$$

Note that the first term in eq (13) is twice the sum of the squares of the r_k (excluding r_j^2) while the second term is twice the sum of the cross products of the r_k (excluding r_ir_j for $i<j$ and r_jr_k for $k>j$).

[18] The concept of strictly proper (or proper) scoring rules can also be given another interpretation. From this point of view, the vector $\mathbf{p}$ consists of the sample climatological probabilities of the K classes (Brier 1950). The RPS (PS) is defined in such a manner that a forecast of the sample climatological probabilities maximizes (minimizes) the expected score for a constant forecast over the sample.

[19] Note that, when $j=1$ or K, no cross product elements straddle j.

[20] Note that $\mathrm{PS}_j(\mathbf{r})$ can be expressed simply as $2\sum_{i=1}^{K}\sum_{k=i}^{K} r_ir_k$, $(i,k\neq j)$.

TABLE 6.—$PS_j(\mathbf{r})$ *expressed in terms of the elements of the matrix* $\mathbf{r'r}$ *for the situation in which* $K=5$

$\mathbf{r'r}$

r_1^2	r_1r_2	r_1r_3	r_1r_4	r_1r_5
r_1r_2	r_2^2	r_2r_3	r_2r_4	r_2r_5
r_1r_3	r_2r_3	r_3^2	r_3r_4	r_3r_5
r_1r_4	r_2r_4	r_3r_4	r_4^2	r_4r_5
r_1r_5	r_2r_5	r_3r_5	r_4r_5	r_5^2

$j=1$: $PS_1(\mathbf{r})$ = sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below

0	0	0	0	0
0	2	1	1	1
0	1	2	1	1
0	1	1	2	1
0	1	1	1	2

$j=2$: $PS_2(\mathbf{r})$ = sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below

2	0	1	1	1
0	0	0	0	0
1	0	2	1	1
1	0	1	2	1
1	0	1	1	2

$j=3$: $PS_3(\mathbf{r})$ = sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below

2	1	0	1	1
1	2	0	1	1
0	0	0	0	0
1	1	0	2	1
1	1	0	1	2

$j=4$: $PS_4(\mathbf{r})$ = sum of elements $\mathbf{r'r}$ multiplied by their respective coefficients below

2	1	1	0	1
1	2	1	0	1
1	1	2	0	1
0	0	0	0	0
1	1	1	0	2

$j=5$: $PS_5(\mathbf{r})$ = sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below

2	1	1	1	0
1	2	1	1	0
1	1	2	1	0
1	1	1	2	0
0	0	0	0	0

TABLE 7.—*Difference between* $RPS_j(\mathbf{r})$ *and* $PS_j^*(\mathbf{r})$ *expressed in terms of the elements of the matrix* $\mathbf{r'r}$ *for the situation in which* $K=5$

$\mathbf{r'r}$

r_1^2	r_1r_2	r_1r_3	r_1r_4	r_1r_5
r_1r_2	r_2^2	r_2r_3	r_2r_4	r_2r_5
r_1r_3	r_2r_3	r_3^2	r_3r_4	r_3r_5
r_1r_4	r_2r_4	r_3r_4	r_4^2	r_4r_5
r_1r_5	r_2r_5	r_3r_5	r_4r_5	r_5^2

$j=1$: $RPS_1(\mathbf{r})-PS_1^*(\mathbf{r})=(1/4)$(sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below)

0	0	0	0	0
0	3	1	1	1
0	1	2	0	0
0	1	0	1	−1
0	1	0	−1	0

$j=2$: $RPS_2(\mathbf{r})-PS_2^*(\mathbf{r})=(1/4)$(sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below)

3	0	2	2	2
0	0	0	0	0
2	0	3	1	1
2	0	1	2	0
2	0	1	0	1

$j=3$: $RPS_3(\mathbf{r})-PS_3^*(\mathbf{r})=(1/4)$(sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below)

2	1	0	2	2
1	3	0	2	2
0	0	0	0	0
2	2	0	3	1
2	2	0	1	2

$j=4$: $RPS_4(\mathbf{r})-PS_4^*(\mathbf{r})=(1/4)$(sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below)

1	0	1	0	2
0	2	1	0	2
1	1	3	0	2
0	0	0	0	0
2	2	2	0	3

$j=5$: $RPS_5(\mathbf{r})-PS_5^*(\mathbf{r})=(1/4)$(sum of elements in $\mathbf{r'r}$ multiplied by their respective coefficients below)

0	−1	0	1	0
−1	1	0	1	0
0	0	2	1	0
1	1	1	3	0
0	0	0	0	0

$PS_j(\mathbf{r})$ in eq (13) is displayed in table 6 in terms of the elements of the matrix $\mathbf{r'r}$ when $K=5$. Note that, as in the case of $RPS_j(\mathbf{r})$, the coefficients in the jth row and jth column are equal to 0. However, note that, in the case of $PS_j(\mathbf{r})$, all the other coefficients are nonzero. Specifically, the square elements all have coefficients equal to 2, while the cross product elements all have coefficients equal to 1.

The symmetry of the PS is evident upon examination of table 6.

RANKED PROBABILITY SCORE AND PROBABILITY SCORE

The differences between the RPS and the PS, which are revealed by a comparison of tables 5 and 6, can be summarized briefly as follows: the RPS assigns to the elements in the symmetric matrix $\mathbf{r'r}$ weights equal to their distance, in class numbers, from class j, while the PS assigns to all the elements equal weights.

To facilitate the comparison of the RPS and the PS, we transform PS into PS*, a scoring rule with a positive orientation on the unit interval [0, 1] that is linearly related to PS. Specifically,

$$PS_j^*(\mathbf{r})=1-(1/2)PS_j(\mathbf{r}). \qquad (14)$$

$PS_j^*(\mathbf{r})$ and $RPS_j(\mathbf{r})$ possess the same orientation and range.

In table 7, we display the difference between $RPS_j(\mathbf{r})$ and $PS_j^*(\mathbf{r})$ in terms of the elements of the matrix $\mathbf{r'r}$ when $K=5$. The coefficients in the jth row and jth column are, of course, equal to 0. Note that the coefficients of the square elements are positive and decrease in both directions away from class j. Further, note that the coefficients of the cross product elements are all positive, except those for $r_4r_5(r_1r_2)$ when $j=1$ (5). Thus, when $K=5$, $RPS_j(\mathbf{r})$ is greater than $PS_j^*(\mathbf{r})$ for all $\mathbf{r}$ when $j=2$, 3, or 4 and for most, but not all, $\mathbf{r}$ when $j=1$ or 5.[21]

[21] The difference between $RPS_j(\mathbf{r})$ and $PS_j^*(\mathbf{r})$ is positive for all $\mathbf{r}$ when $K=3$ and for most, but not all, $\mathbf{r}$ when $K=4$.

In summary, the PS is, in essence, an unweighted or neutral scoring rule while the RPS is a weighted scoring rule. Thus, in a sense, the RPS is a specialization of the PS, although equal weights themselves are, in reality, special weights.

5. SENSITIVITY TO DISTANCE

In section 1, we stated that the RPS was sensitive to distance. To indicate the relevance of the concept of distance, at least in an intuitive manner, consider the forecasts $\mathbf{r}=(0.0, 0.1, 0.3, 0.4, 0.2)$ and $\mathbf{r}'=(0.0, 0.3, 0.1, 0.4, 0.2)$ on an occasion when class 4 occurs. Note that $\mathbf{r}$ and $\mathbf{r}'$ consist of the same probabilities and that $r_4=r_4'=0.4$; the difference between $\mathbf{r}$ and $\mathbf{r}'$ is simply that $r_2=0.1$ and $r_3=0.3$, while $r_2'=0.3$ and $r_3'=0.1$. The PS would assign $\mathbf{r}$ and $\mathbf{r}'$ the same score (0.50).[22] However, if the variable of concern is ordered, many meteorologists and others would consider, no doubt, $\mathbf{r}$ to be a better forecast than $\mathbf{r}'$. The basis for this conclusion would be, in general, that $\mathbf{r}$ is closer than $\mathbf{r}'$ to class 4. Thus, the concept of distance appears to be of relevance in evaluating forecasts, at least if the forecasts relate to ordered variables. The RPS, which assigns weights to the elements of the matrix $\mathbf{r}'\mathbf{r}$ according to their distance from the class which occurs, takes distance into account (see below). Specifically, the RPS would assign $\mathbf{r}$ and $\mathbf{r}'$ the scores 0.9475 and 0.9275, respectively. In this section, we consider the RPS with reference to two definitions of distance and examine the nature of the sensitivity of the RPS to distance.

TAILS SUMS

Stael von Holstein (1970) has formulated one such definition according to which a forecast $\mathbf{r}'$ is more distant than a forecast $\mathbf{r}$ from the class that occurs (j) if $\mathbf{r}'\neq\mathbf{r}$ and

$$R_i'\geq R_i,\quad (i=1,\ldots,j-1), \tag{15}$$

and

$$R_i'\leq R_i,\quad (i=j,\ldots,K-1) \tag{16}$$

where

$$R_i=\sum_{k=1}^{i} r_k.$$

Then, a scoring rule S (say) is sensitive to distance according to inequalities (15) and (16) if

$$S_j(\mathbf{r})>S_j(\mathbf{r}')$$

whenever $\mathbf{r}'$ is more distant than $\mathbf{r}$ from class j (refer to footnote 17). This definition of distance treats the tails[23] of a forecast separately and implies that any portion of either tail of $\mathbf{r}'$ contains at least as much mass, that is, probability, as the same portion of that tail of $\mathbf{r}$. In particular, this definition does not permit the transfer of mass from one tail to the other.[24] Stael von Holstein (1970) shows that the RPS is sensitive to distance according to this definition of distance.

SYMMETRIC SUMS

Another definition of distance has been formulated by Murphy (1970)[25] according to which $\mathbf{r}'$ is more distant than $\mathbf{r}$ from class j if $\mathbf{r}'\neq\mathbf{r}$ and

$$C_i'\leq C_i,\ [i=0, 1, \ldots, \max(j-1, K-j)] \tag{17}$$

where

$$C_i=\sum_{k=j-i}^{j+i} r_k.$$

Then, the scoring rule S is sensitive to distance according to inequality (17) if

$$S_j(\mathbf{r})>S_j(\mathbf{r}')$$

whenever $\mathbf{r}'$ is more distant than $\mathbf{r}$ from class j. This definition of distance, which treats symmetric portions of the tails about class j simultaneously, is based upon a less restrictive assumption concerning the nature of the differences between $\mathbf{r}'$ and $\mathbf{r}$ than that upon which Stael von Holstein's definition is based. In particular, this definition permits the transfer of mass from one tail to the other. However, we can show rather easily that the RPS is not sensitive to distance according to this definition of distance.

RPS SENSITIVITY TO DISTANCE

To indicate the nature of the sensitivity of the RPS to distance, four forecasts for a situation in which $K=5$ and $j=3$ are displayed in table 8. An original forecast $\mathbf{r}$ has been modified in three different ways to yield, in turn, the forecasts $\mathbf{r}^*$, $\mathbf{r}'$, and $\mathbf{r}''$: (1) for $\mathbf{r}^*$, 0.10 is transferred from class 1 to class 2 [$\mathbf{r}^*$ is less distant than $\mathbf{r}$ according to inequalities (15), (16), and (17), and the mass remains in the same tail], (2) for $\mathbf{r}'$, 0.02 is transferred from class 1 to class 4 [$\mathbf{r}'$ is less distant than $\mathbf{r}$ according to (17) but not according to inequalities (15) and (16), and the mass moves to the other tail], and (3) for $\mathbf{r}''$, 0.10 is transferred from class 1 to class 4 [$\mathbf{r}''$ is less distant than $\mathbf{r}$ according to inequality (17) but not according to (15) and (16), and the mass moves to the other tail]. Note that $\mathrm{RPS}_3(\mathbf{r}'')<\mathrm{RPS}_3(\mathbf{r})<\mathrm{RPS}_3(\mathbf{r}')<\mathrm{RPS}_3(\mathbf{r}^*)$. The fact that the RPS is sensitive to distance according to inequalities (15) and (16) ensures that $\mathrm{RPS}_3(\mathbf{r}^*)>\mathrm{RPS}_3(\mathbf{r})$, while the fact that $\mathrm{RPS}_3(\mathbf{r}'')<\mathrm{RPS}_3(\mathbf{r})$ indicates that the RPS is not sensitive to distance according to (17). However, the fact that $\mathrm{RPS}_3(\mathbf{r}')>\mathrm{RPS}_3(\mathbf{r})$ implies that the RPS is sensitive to distance for certain transfers of mass from one tail to the other.

Consider two forecasts $\mathbf{r}$ and $\mathbf{r}'$ where $r_m'=r_m-\epsilon$, $r_n'=r_n+\epsilon$, and $r_k'=r_k$ for all $k\neq m, n$ [$k, m, n=1, \ldots, K$; $m<n$; $0<\epsilon\leq\min(r_m, 1-r_n)$]. Thus, $\mathbf{r}$ is transformed into $\mathbf{r}'$ by transferring a mass ϵ from class m to class n. Then,

$$\mathrm{RPS}_j(\mathbf{r}')>(<)\mathrm{RPS}_j(\mathbf{r})$$

[22] The PS is not, of course, the only scoring rule that would assign $\mathbf{r}$ and $\mathbf{r}'$ the same score. In fact, prior to the formulation of the RPS, all known strictly proper scoring rules would have assigned $\mathbf{r}$ and $\mathbf{r}'$ the same score.

[23] The tails of a forecast are defined with respect to the class that occurs (j). The left-hand tail consists of the r_k for which $k<j$, and the right-hand tail consists of the r_k for which $k>j$.

[24] Mass, that is, probability, can of course be transferred to or from class j and either tail.

[25] Refer also to Murphy and Epstein (1967a, pp. 753–754) and Stael von Holstein (1970).

TABLE 8.—*Four forecasts that indicate the nature of the sensitivity of the* RPS *to distance*

Forecasts	$RPS_3(\cdot)$
$\mathbf{r} = (0.10, 0.10, 0.60, 0.10, 0.10)$	0.9750
$\mathbf{r}^* = (0.00, 0.20, 0.60, 0.10, 0.10)$	0.9775
$\mathbf{r}' = (0.08, 0.10, 0.60, 0.12, 0.10)$	0.9757
$\mathbf{r}'' = (0.00, 0.10, 0.60, 0.20, 0.10)$	0.9725

TABLE 9.—*Terms in inequality (18) for the forecast pairs* (a) $(\mathbf{r}, \mathbf{r}^*)$, (b) $(\mathbf{r}, \mathbf{r}')$, *and* (c) $(\mathbf{r}, \mathbf{r}'')$ *(refer also to table 8)*. [LHS (RHS) *refers to left (right)-hand side of inequality (18).*]

Forecast pair	m	n	ϵ	j	k_0	$\bar{k}_n^R$	$\bar{k}_m^L$	LHS	RHS
(a) $(\mathbf{r}, \mathbf{r}^*)$	1	2	0.10	3	3	1.10	0	1.10	0.05
(b) $(\mathbf{r}, \mathbf{r}')$	1	4	0.02	3	3	0.10	0	0.10	0.03
(c) $(\mathbf{r}, \mathbf{r}'')$	1	4	0.10	3	3	0.10	0	0.10	0.15

if

$$(\bar{k}_n^R - \bar{k}_m^L) - (\bar{k}_0 - j) > (<) (n-m)\epsilon/2 \tag{18}$$

where

$$\bar{k}_n^R = \sum_{k=n+1}^{K} (k-n) r_k,$$

$$\bar{k}_m^L = \sum_{k=1}^{m-1} (m-k) r_k,$$

and

$$\bar{k}_0 = \sum_{k=1}^{K} k r_k.$$

However, since the RPS is sensitive to distance according to inequalities (15) and (16) and as a result $RPS_j(\mathbf{r}') > RPS_j(\mathbf{r})$ if $n \le j$ and $RPS_j(\mathbf{r}') < RPS_j(\mathbf{r})$ if $m \ge j$, the situations of particular interest are those in which $m < j < n$, that is, in which the mass (ϵ) is transferred from the left to the right tail.[26] In such situations, $\bar{k}_n^R$, the right-hand portion of the first moment of k about n, is in the right tail; and $\bar{k}_m^L$, the left-hand portion of the first moment of k about m, is in the left tail. We shall refer to $\bar{k}_m^R$ and $\bar{k}_m^L$ as the right and left partial tail moments, respectively. Then, the first term in inequality (18) is the difference between the right and left partial tail moments, while the second term in (18) is the difference between the first moment of k about zero ($\bar{k}_0$), that is, the mean class number, and j, the number of the class that occurs. Thus, inequality (18) indicates that the sensitivity of the RPS to the transfer of mass depends upon the relative magnitudes of the difference between the partial tail moments and the difference between the mean class number and j (for particular values of m, n, and ϵ). In table 9, we display the terms in inequality (18) for three pairs of forecasts: (a) $(\mathbf{r}, \mathbf{r}^*)$, (b) $(\mathbf{r}, \mathbf{r}')$, and (c) $(\mathbf{r}, \mathbf{r}'')$ (refer also to table 8). An examination of table 9 and similar tables for other forecasts and transfers of mass will reveal something of the nature of the sensitivity of the RPS to transfers of mass between the tails of forecasts.[27]

The nature of the sensitivity of the RPS to distance does not, of course, ensure that we as evaluators will always be completely satisfied with the scores themselves. For example, the RPS assigns the same score (0.925) to the forecast $\mathbf{r} = (0.5, 0.3, 0.1, 0.1, 0.0)$ whether class 1 or class 2 occurs. However, with a scoring rule that is in some sense sensitive to distance, such problems are inevitable. While the RPS is, in the author's opinion, the best scoring rule available to evaluate probability forecasts of ordered variables, we must examine the sensitivity of other expected-utility measures to distance.

6. CONCLUSION

The relationship between the frameworks within which the PS and the RPS were formulated was briefly described in section 2. Certain properties of the RPS and the PS are described in section 3, including their range (the minimum value of the RPS depends upon which class occurs), their values for categorical and uniform forecasts, and their strictly proper nature. In section 4, we expressed the RPS and the PS in terms of the matrix $\mathbf{r}'\mathbf{r}$, the elements of which are the squares and cross products of the r_k (the components of the forecast vector $\mathbf{r}$). The comparison of the RPS and the PS reveals that (1) in the RPS the elements are weighted by their distance, in terms of the differences between class numbers, from the class that occurs (j) and (2) in the PS the elements are all weighted equally. In the two-class situation, in which the concept of distance is not particularly meaningful, the RPS and the PS are equivalent, that is, linearly related. In section 5, we examined the sensitivity of the RPS to distance (the PS is not sensitive to distance). We found that the RPS is sensitive to distance according to one definition of distance but not according to another less restrictive definition of distance. The examination of the RPS has revealed something of the nature of the sensitivity of the RPS to distance.

As indicated in the introduction, meteorologists concerned with the evaluation of probability forecasts have indicated the need for a scoring rule that possessed the desirable properties of the PS and that in addition was in some sense sensitive to distance. The RPS is such a scoring rule. Thus, the RPS would appear to be a particularly appropriate scoring rule for the evaluation of probability forecasts when sensitivity to distance is of concern, that is, when the variable of concern is ordered and the number of states of concern exceeds 2. The author would like to encourage meteorologists to use the RPS, at least as a supplemental measure, in such situations so that we as evaluators can accumulate information relative to its absolute and relative performance.

[26] The situation in which a mass (ϵ) is transferred from class n to class m would, of course, yield similar results.

[27] A full treatment of the transfer of mass problem would require the consideration of a vector $\epsilon = (\epsilon_1, \ldots, \epsilon_K)$ where $r_k + \epsilon_k \ge 0$, $\sum_k r_k = 1$, and $\sum_k \epsilon_k = 0$ $(k = 1, \ldots, K)$.

APPENDIX

In this appendix, we demonstrate that eq (1) and (12) are equivalent. Equation (1) can be rewritten as

$$\mathrm{RPS}_j(\mathbf{r})=(3/2)-[1/2(K-1)]\left[\sum_{i=1}^{K-1}\left(\sum_{n=1}^{i}r_n\right)^2+\sum_{i=1}^{K-1}\left(\sum_{n=i+1}^{K}r_n\right)^2\right]-[1/(K-1)]\sum_{i=1}^{K}|i-j|r_i,$$

or

$$\mathrm{RPS}_j(\mathbf{r})=(3/2)-[1/2(K-1)]\left[\sum_{i=1}^{j-1}\left(\sum_{n=1}^{i}r_n\right)^2+\sum_{i=j}^{K-1}\left(\sum_{n=1}^{i}r_n\right)^2+\sum_{i=1}^{j-1}\left(\sum_{n=i+1}^{K}r_n\right)^2+\sum_{i=j}^{K-1}\left(\sum_{n=i+1}^{K}r_n\right)^2\right]-[1/(K-1)]\sum_{i=1}^{K}|i-j|r_i,$$

or

$$\mathrm{RPS}_j(\mathbf{r})=(3/2)-[1/2(K-1)]\sum_{i=1}^{j-1}\left[\left(\sum_{n=1}^{i}r_n\right)^2+\left(1-\sum_{n=1}^{i}r_n\right)^2\right]-[1/2(K-1)]\sum_{i=j}^{K-1}\left[\left(1-\sum_{n=i+1}^{K}r_n\right)^2+\left(\sum_{n=i+1}^{K}r_n\right)^2\right]-[1/(K-1)]\sum_{i=1}^{K}|i-j|r_i,$$

or

$$\mathrm{RPS}_j(\mathbf{r})=1+[1/(K-1)]\sum_{i=1}^{j-1}\left[\sum_{n=1}^{i}r_n-\left(\sum_{n=1}^{i}r_n\right)^2\right]+[1/(K-1)]\sum_{i=j}^{K-1}\left[\sum_{n=i+1}^{K}r_n-\left(\sum_{n=i+1}^{K}r_n\right)^2\right]-[1/(K-1)]\sum_{i=1}^{j-1}(j-i)r_i-[1/(K-1)]\sum_{i=j+1}^{K}(i-j)\,r_i,$$

or, since

$$\sum_{i=1}^{j-1}\sum_{n=1}^{i}r_n=\sum_{i=1}^{j-1}(j-i)r_i$$

and

$$\sum_{i=j}^{K-1}\sum_{n=i+1}^{K}r_n=\sum_{i=j+1}^{K}(i-j)r_i,$$

$$\mathrm{RPS}_j(\mathbf{r})=1-[1/(K-1)]\sum_{i=1}^{j-1}\left(\sum_{n=1}^{i}r_n\right)^2-[1/(K-1)]\sum_{i=j}^{K-1}\left(\sum_{n=i+1}^{K}r_n\right)^2,$$

or

$$\mathrm{RPS}_j(\mathbf{r})=1-[1/(K-1)]\left[\sum_{i=1}^{j-1}(j-i)r_i^2+2\sum_{i=1}^{j-2}\sum_{k=i+1}^{j-1}(j-k)r_ir_k\right]-[1/(K-1)]\left[\sum_{i=j+1}^{K}(i-j)r_i^2+2\sum_{i=j+1}^{K-1}\sum_{k=i+1}^{K}(i-j)r_ir_k\right],$$

or

$$\mathrm{RPS}_j(\mathbf{r})=1-[1/(K-1)]\left[\sum_{i=1}^{K}|i-j|r_i^2+2\sum_{i=1}^{j-2}\sum_{k=i+1}^{j-1}(j-k)r_ir_k+2\sum_{i=j+1}^{K-1}\sum_{k=i+1}^{K}(i-j)r_ir_k\right],$$

which is identical to eq (12).

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