# Dynamic Properties of Computably Enumerable Sets

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November 28, 1995

#### Abstract

A set  $A \subseteq \omega$  is computably enumerable (c.e.), also called recursively enumerable, (r.e.), or simply enumerable, if there is a computable algorithm to list its members. Let  $\mathcal{E}$  denote the structure of the c.e. sets under inclusion. Starting with Post [1944] there has been much interest in relating the definable (especially  $\mathcal{E}$ -definable) properties of a c.e. set A to its "information content", namely its Turing degree, deg(A), under  $\leq_{\mathrm{T}}$ , the usual Turing reducibility. [Turing 1939]. Recently, Harrington and Soare answered a question arising from Post's program by constructing a nonemptly  $\mathcal{E}$ -definable property Q(A) which guarantees that A is incomplete  $(A <_{\mathrm{T}} K)$ . The property Q(A) is of the form  $(\exists C)[A \subset_{\mathrm{m}} C \& Q^{-}(A, C)]$ , where  $A \subset_{\mathrm{m}} C$  abbreviates that "Ais a major subset of C", and  $Q^{-}(A, C)$  contains the main ingredient for incompleteness.

A dynamic property P(A), such as prompt simplicity, is one which is defined by considering how fast elements elements enter A relative to some simultaneous enumeration of all c.e. sets. If some set in deg(A) is promptly simple then A is prompt and otherwise tardy. We introduce here two new tardiness notions, small-tardy(A, C) and Q-tardy(A, C). We begin by proving that small-tardy(A, C) holds iff A is small in C ( $A \subset_{s} C$ ) as defined by Lachlan [1968]. Our main result is that Q-tardy(A, C), which is more intuitive and easier to work with than the  $\mathcal{E}$ -definable counterpart,  $Q^{-}(A, C)$ , is exactly equivalent and captures the same incompleteness phenomenon.

<sup>\*</sup>The first author was supported by National Science Foundation Grant DMS 92-14048, and the second author by National Science Foundation Grant DMS 91-06714 and DMS 94-00825.

### 1 Introduction

Warning. From now on all sets and degrees will be c.e. unless specified otherwise. Post [16] initiated the study of the relationship between definable properties of a c.e. set A and its "information content" as measured by its Turing degree, deg(A), under the usual Turing reducibility  $\leq_{\rm T}$ . By the 1950's Myhill noticed that the c.e. sets form a lattice  $\mathcal{E}$  under inclusion and from then on most definable properties considered for c.e. sets were  $\mathcal{E}$ -definable. An exception is hyper-simplicity.

Friedberg and Muchnik solved Post's problem by constructing an incomplete and nonrecursive c.e. set, and invented the priority method to do it. The method was quickly developed into more sophisticated forms (infinite injury and the 0"'-method) and used to prove a number of theorems on c.e. sets and degrees. Sacks used the second method to construct an incomplete maximal set, Yates constructed a complete maximal set, and Martin [15] brought these results together and extended them in his beautiful theorem that the degrees of maximal sets are exactly  $H_1$ , the high degrees. Then Lachlan [8] and Shoenfield [17] proved that the degrees of the atomless sets (those with no maximal supersets) are  $\overline{L}_2$ , the complement of the low<sub>2</sub> degrees. Both properties of being maximal or atomless are  $\mathcal{E}$ -definable properties.

Meanwhile Soare [18] developed a new method for generating automorphisms of  $\mathcal{E}$ , and used it to show that maximal sets form an orbit. (The *orbit* of  $A \in \mathcal{E}$  is the set of all sets B which are automorphic to A, written  $A \simeq B$ .) The question stemming from Post's program remained open of whether there was an  $\mathcal{E}$ -definable property P(A) which guarantees that A is incomplete and nonrecursive. It seemed that automorphisms could be used to give a negative answer by showing that every nonrecursive set A has a complete set in its orbit. However, Harrington and Soare gave a negative answer to this question by proving the following.

**Theorem 1.1 (Harrington-Soare [3])** There is a nonempty  $\mathcal{E}$ -definable property Q(A) such that every c.e. set A satisfying Q(A) is noncomputable and Turing incomplete.

The property, which we shall describe fully in §4, is in two parts,

$$Q(A) \iff A \subset_{\mathbf{m}} B \& Q^{-}(A, C),$$

where  $A \subset_{\mathrm{m}} C$  abbreviates that "A is a major subset of C", and  $Q^{-}(A, C)$ , an  $\mathcal{E}$ -definable property with several quantifiers which contains the main ingredient for incompleteness. The property  $Q^{-}(A, C)$  succeeds but it is not very intuitive or easy to work with. The main achievement of the present paper is to produce a simpler and dynamic property, called Q-tardy(A, C), and to prove

(1) 
$$Q^{-}(A,C) \iff Q - tardy(A,C)$$

Hence, the dynamic property Q-tardy(A,C) is exactly equivalent to  $Q^{-}(A,C)$  (in the presence of  $A \subset_{\mathrm{m}} C$ ) and therefore captures the incompleteness phenomenon.

In §2 we discuss dynamic properties and particularly *promptness* properties, such as prompt simplicity, and their opposite, *i.e.*, *tardiness* properties. This will motivate our present tardiness property, Q-tardy(A, C).

The above result led us to a curious discovery Theorem 3.2 about the  $\mathcal{E}$ definable and new dynamic definitions of small subsets. Lachlan first defined the notion of A being a small subset of C, written  $A \subset_{s} C$ , in connection with his decision procedure for part of the elementary theory of  $\mathcal{E}$  as described in §3. This notion proved useful and other facts about small sets were added by Stob [20] (see [19, pp. 193-195]), and others. The property  $\hat{Q}(A) = (\exists C)[A \subset_{s} C]$ comes tantalizingly close to being a property like Q(A) which guarantees A incomplete, but not quite. We note that  $\hat{Q}(A)$  implies that A is not a promptly simple set by Corollary 3.3, but does not ensure that A is not of promptly simple degree.

The investigation of tardy properties with an eye toward incompleteness led naturally to a new tardiness property, small-tardy(A, C). Our other main result in the present paper is that,

(2) 
$$A \subset_{\mathrm{s}} C \iff \mathrm{small-tardy}(A, C).$$

This property small-tardy(A, C) gave new insight into the nature of small subsets, and led to a brand new and simpler  $\mathcal{E}$ -definable definition for the relation  $A \subset_{s} C$  which had been overlooked researchers for 25 years. The general point is that dynamic notions frequently are more intuitive and easier to work with than  $\mathcal{E}$ -definable ones. Each sheds light on the other, particularly when one can show equivalence of the two such notions.

We use the terms "computably enumerable (c.e.)" and "recursively enumerable (r.e.)" interchangably, and likewise "computable" and "recursive."

### 2 Dynamic Properties

Most properties of an r.e. set A are *static* properties in that they refer to A as a completed object without mention of the enumeration of A. Such include Post's properties of being simple or hh-simple, and Myhill's property of being maximal, all of which are also  $\mathcal{E}$ -definable properties. Another static property which is not  $\mathcal{E}$ -definable or even invariant under automorphisms is hyper-simplicity. A *dynamic* property on the other hand is one which is defined using an computable enumeration  $\{A_s\}_{s\in\omega}$  of A.

#### 2.1 The Extension Theorem and Automorphisms

The first essential use of a dynamic property was probably the covering hypothesis in the Extension Theorem of Soare's maximal set automorphism theorem [18]. Here there were several simultaneous enumerations of arrays of r.e. sets,  $\{U_n\}_{n\in\omega}$  and  $\{\hat{V}\}_{n\in\omega}$ , and it was important to measure for an element x which  $U_n$  sets it entered before entering certain  $\hat{V}_m$  sets.

#### 2.2 *d*-simple sets

In 1980 Lerman and Soare [11] attempted to capture part of the dynamic property of the Extension Theorem with an  $\mathcal{E}$ -definable property which is called *d*-simple, but they succeeded in capturing only a small part.

**Definition 2.1** A coinfinite set A is d-simple if for all X there exists  $Y \subseteq X$  such that

(3)  $X \bigcap \overline{A} = Y \bigcap \overline{A}, \text{ and }$ 

(4) 
$$(\forall Z)[(Z - X) \text{ infinite } \implies (Z - Y) \bigcap A \neq \emptyset].$$

The tension in constructing Y is that to meet (4) we wish to make Y as small as possible, but to meet (3) we must eventually put every element of X - A into Y. Every hh-simple is d-simple, and every d-simple set is simple. The degrees of d-simple sets include  $H_1$  and split  $L_1$ . Also a d-simple set cannot be small [11, p. 141]. (This old result takes on new significance in view of the present paper because d-simple sets behave like prompt sets and by the result here Theorem 3.2 on small sets, small sets must be tardy.) The major open question left over from Post's program is the following.

**Question 2.2** Find a necessary and sufficient condition on A for A to be automorphic to a complete set. In particular, is every d-simple set automorphic to a complete set?

The second question is not of great intrinsic interest itself, but it appears to be on the cutting edge of the symmetry between the methodologies for generating automorphisms and for producing invariant properties (such as Q(A)), and may therefore be useful in gaining insight into the completeness phenomenon and the first part of the question.

### 2.3 Promptly Simple Sets

The next significant advance came with the following definition of promptly simple sets by Maass [12].

**Definition 2.3** (i) A coinfinite r.e. set A is promptly simple if there is a computable function p and a computable enumeration  $\{A_s\}_{s\in\omega}$  of A such that for every e,

(5) 
$$W_e \text{ infinite} \Longrightarrow (\exists s) (\exists x) [x \in W_{e, \text{ at } s} \cap A_{p(s)}].$$

(ii) An r.e. set A is prompt if A has promptly simple degree namely,  $A \equiv_T B$  for some promptly simple set B, and an r.e. degree is prompt if it contains a prompt set.

(iii) An r.e. set or degree which is not prompt is tardy.

By the Promptly Simple Degree Theorem [19, Theorem XIII.1.7(iii)] a set A being prompt is equivalent to the following property which we may take as the definition. Let  $\{A_s\}_{s\in\omega}$  be any recursive enumeration of A. Then there is a recursive function p such that for all s,  $p(s) \geq s$ , and for all e,

(6)  $W_e \text{ infinite } \implies (\exists^{\infty} x) (\exists s) [x \in W_{e, \text{ at } s} \& A_s \upharpoonright x \neq A_{p(s)} \upharpoonright x],$ 

namely infinitely often A "promptly permits" on some element  $x \in W_e$ .

Promptly simply sets and degrees helped bring some dramatic advances in the subject. Maass [12] proved that any two promptly simple low sets are automorphic and discovered other properties of these sets [13]. Ambos-Spies, Jockusch, Shore, and Soare [1] used prompt degrees to unify and extend results about r.e. degrees, and promptness has been very influential ever since. (See [19, Chap. XIII].)

#### 2.4 Almost Prompt Sets and Degrees

The material from the next two subsections  $\S2.4$  and 2.5 is not strictly necessary for this paper but is helpful to understand other notions of promptness and tardiness.

Harrington and Soare [4, Theorem 1.2] proved that every prompt set is automorphic to a complete set. They noticed that the same proof would work for a strictly larger dynamically defined class of sets called *almost prompt*, which are defined in terms of n-r.e. sets.

**Definition 2.4** (i) A set  $X \leq_T K$  is *n*-*r.e.* if  $X = \lim_s X_s$  for some recursive sequence  $\{X_s\}_{s \in \omega}$  such that for all x,  $X_0(x) = 0$  and

$$\operatorname{card}\{s: X_s(x) \neq X_{s+1}(x)\} \le n.$$

(For example, the only 0-r.e. set is  $\emptyset$ , the 1-r.e. sets are the usual r.e. sets, and the 2-r.e. sets are the d.r.e. sets.)

(ii) Such a sequence  $\{X_s\}_{s \in \omega}$  is called an *n*-r.e. presentation of X.

It is well-known and easy to show [19, Exercise III.3.8., p. 38] that for n > 0, X is n-r.e. iff

(7) 
$$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \dots \bigcup W_{e_{2k+1}}, \text{ or }$$

(8) 
$$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \ldots \bigcup (W_{e_{2k+1}} - W_{e_{2k+2}}),$$

according as n = 2k + 1 is odd or n = 2k + 2 is even.

**Definition 2.5** For n = 0 let  $X_0^0 = \emptyset$ . For n > 0 and  $e = \langle e_1, e_2, \ldots e_n \rangle$  define

(9) 
$$X_e^n = (W_{e_1} - W_{e_2}) \bigcup \dots,$$

as in (7) or (8) according as n is odd or even. We say that  $\langle n, e \rangle$  is an *n*-r.e. index for  $X_e^n$ . Let

(10) 
$$X_{e,s}^n = (W_{e_1,s} - W_{e_2,s}) \bigcup \ldots$$

**Definition 2.6** Let A be an r.e. set and let  $\{A_s\}_{s\in\omega}$  be a recursive enumeration of A. We say A is almost prompt, abbreviated a.p., if there is a nondecreasing recursive function p(s) such that for all n and e,

(11) 
$$X_e^n = \overline{A} \implies (\exists x)(\exists s)[x \in X_{e,s}^n \& x \in A_{p(s)}].$$

Note that, as in the case of promptly simple, this definition is independent of the enumeration of A; if p(s) works for the enumeration  $\{A_s\}_{s\in\omega}$ , and if  $\{A'_s\}_{s\in\omega}$  is another enumeration of A, define  $p'(s) = (\mu t)[A'_t \supseteq A_p(s)]$ . We may think of Definition 2.6 as asserting that A will p-promptly hit every approximation  $\{X^n_{e,s}\}_{s\in\omega}$  for every n-r.e. set  $X^n_e = \overline{A}$  where the recursive approximation  $X^n_{e,s}$  is determined by the standard enumeration  $\{W_{e,s}\}_{e,s\in\omega}$ of the r.e. sets. In [4, Conversion Lemma 11.4] we prove that if we specify another collection of n-r.e. sets  $\{\widehat{X}^n_e\}_{n,e\in\omega}$ , by some recursive approximation  $\{\widehat{X}^n_{e,s}\}_{n,e,s\in\omega}$ , then there is a recursive function q such that A will q-promptly hit  $\{\widehat{X}^n_{e,s}\}_{n,e,s\in\omega}$  if  $\widehat{X}^n_e = \overline{A}$ .

#### 2.5 Very Tardy Sets

The negation of the property of almost prompt is called *very tardy*. An important special case of this is known as 2-tardy and is closely related to the property Q(A).

**Definition 2.7** Let A be an r.e. set and let  $\{A_s\}_{s \in \omega}$  be a recursive enumeration of A.

(i) We say A is very tardy if A is not almost prompt, namely if for every nondecreasing recursive function p(s),

(12) 
$$(\exists n)(\exists e)[X_e^n = \overline{A} \& (\forall y)(\forall s)[y \in X_{e,s}^n \implies y \notin A_{p(s)}]].$$

(ii) We say A is n-tardy if in (i) the fixed n works uniformly for all such functions p, namely for every nondecreasing recursive function p(s),

(13) 
$$(\exists e)[X_e^n = \overline{A} \& (\forall y)(\forall s)[y \in X_{e,s}^n \implies y \notin A_{p(s)}]].$$

The main idea about a very tardy set A is that if  $x \in X_{e,s}^n$  then x can later enter A eventually, but x must first undergo a delay until at least stage p(s)+1 before doing so. Since class of almost prompt sets is a strict extension of the class of prompt sets it follows that the class of very tardy sets is a strict subclass of the class of tardy sets, hence the name "very tardy." Note that A is 0-tardy iff  $A = \omega$ , and A is 1-tardy iff A is recursive. The 2-tardy sets play a special role in our work and have additional characterizations as follows, as we prove in [5].

**Proposition 2.8 (Harrington-Soare [5])** For an r.e. set A the following are equivalent:

(i) A is 2-tardy;

(ii) For every nondecreasing recursive function p(s),

(14)  $(\exists W_i \supseteq \overline{A})(\exists W_e = A)(\forall y)(\forall s)[y \in W_{i,s} - W_{e,s} \implies y \notin A_{p(s)}]].$ 

### 3 Small Subsets

Lachlan [9] introduced small sets in his program to construct canonical examples of certain diagrams and then rule out possible extensions so as to give a decision procedure for the  $\forall \exists$ -theory of the lattice of r.e. sets. The following definition is clearly equivalent to the standard definition as in [19, Definition 4.10, p. 193].

**Definition 3.1** A subset  $A \subset C$  is a *small subset* of C (written  $A \subset_{s} C$ ) if  $A \subset_{\infty} C$  and for all X and Y, if

- (i)  $X \cap (C A) \subseteq Y$ , then
- (ii)  $(\exists Z)_{Z \subseteq X} [Z \supseteq (X C) \& (Z \cap C) \subseteq Y].$

If A is both a small subset and major subset of C we say it is a *small* major subset and write  $A \subset_{\text{sm}} C$ .

Note that the consequent of the implication in (ii) is equivalent to the property

(15)  $(\forall Y \supseteq C - A)[Y \cup \overline{C} \text{ is r.e. }].$ 

It is interesting now to see that this important notion of small subset, Theorem 3.2(i) below, just like the Q(A) property, has a dynamic equivalent, Theorem 3.2(iii), below which we now prove. It is particularly that the equivalent dynamic definition (iii) led to the discovery of another  $\mathcal{E}$ -definable definition (ii) below which is simpler than the original  $\mathcal{E}$ -definable one, but lay undiscovered for over 25 years.

**Theorem 3.2 (Harrington and Soare)** Suppose  $A \subset_{\infty} C$ . Then the following are equivalent:

(i) 
$$A \subset_{s} C$$
;  
(ii)  $(\forall Y)[[(C - A) \subseteq Y] \implies (\exists Z)[\overline{C} \subseteq Z \& Z \cap C \subseteq Y]];$   
(iii) small-tardy $(A, C)$ , namely:

(16) 
$$(\forall f)(\exists T)[\overline{C} \subseteq T \& (\forall x)[x \in (T \cap C)_{at s} \implies x \notin A_{f(s)}]].$$

(In (iii) it is understood that f ranges over recursive functions which are nondecreasing.)

Note that (ii) is equivalent to the property,

$$(\forall Y \supseteq C - A)[Y \cup \overline{C} \text{ is r.e. }].$$

We refer to the property (iii) on  $A \subset_{\infty} C$  as small-tardy(A, C) because it is a dynamic property.

*Proof.* (i)  $\implies$  (ii). Trivial. Let  $X = \omega$ .

(ii)  $\implies$  (iii). Fix a recursive function f as in (iii). We (BLUE) will build  $Y \supseteq (C - A)$ , so by (ii) the opponent (RED) must reply with  $Z = W_j$ for some j, satisfying (ii). Define  $W_{g(j)} = W_j \searrow C$ . If  $x \in (W_j \searrow C)_{\text{at s}}$ , then by the Recursion Theorem and Slowdown Lemma [19, Lemma XIII.1.5] we can compute  $t = (\mu v)[x \in W_{g(j),v}]$ , and know that t > s.

Namely, if  $x \in C_{s+1} - C_s$  take all j such that  $x \in W_{j,s}$  (necessarily  $j \leq s$ ). For each j compute  $t_{x,j} = (\mu v)[x \in W_{g(j),v}]$ . Let  $t = \max\{t_{x,j} : \text{all such } j\}$ . If  $x \notin A_{f(t)}$  then enumerate x in Y at stage f(t) + 1. Since every  $x \in C - A$  enters Y after some finite delay we have,

$$(17) C - A \subseteq Y.$$

However, no element once in A ever enters Y, so

(18) 
$$Y \cap A \subseteq Y \searrow A.$$

By (17) and (ii), RED must play some Z satisfying (ii). In (iii) we let  $T = Z \setminus C$ . Let  $W_j = T$ . Now  $\overline{C} \subseteq T$  because  $\overline{C} \subseteq Z$ . But  $T \cap C \subseteq Y$  by (ii) implies by (18) that  $T \cap C \subseteq Y \setminus A$ . Now  $A \searrow Y = \emptyset$ . Hence, for all x, if  $x \in (T \cap C)_{at s}$ , then

$$x \in (T \searrow C)_{\mathbf{at s}},$$

$$x \in W_{g(j),t}$$
 for some  $t > s$ ,

 $x \notin Y_{f(t)}$  by (18) and definition of Y,

 $x \notin A_{f(s)}$  since s < t and f is nondecreasing.

(iii)  $\implies$  (i). Fix  $A \subset_{\infty} C$  satisfying (iii). Given X and Y satisfying Definition 3.1 (i), we (RED) define Z satisfying Definition 3.1 (ii) as follows. Define

$$f(s) = (\mu t > s)(\forall x)[x \in X_s \cap C_s \implies x \in A_t \cup Y_t].$$

Such t exists by Definition 3.1 (i). Choose T satisfying (iii). Enumerate

(19) 
$$x \in Z_s \iff x \in Z_{s-1} \lor x \in (X_s \cap T_s) - C_s.$$

Now by (iii) for all x,

$$x \in (Z_s \cap C_s) \implies x \in (X_s \cap T_s), \text{ and}$$
$$x \in (T \cap C)_{at s} \implies x \notin A_{f(s)}, \text{ so}$$
$$x \in (Z \cap C)_{at s} \implies x \in Y_{f(s)}$$

by definition of f.

Consider the property  $\widehat{Q}(A) : (\exists C)[A \subset C]$ . This resembles the property Q(A) because  $\widehat{Q}(A)$  implies that A is not a promptly simple set. However, it does not guarantee that A is not of promptly simple degree, and therefore, unlike Q(A) it does not ensure that the orbit of A contains only incomplete sets.

**Corollary 3.3** If  $A \subset_{s} C$  then A is not a promptly simple set.

*Proof.* Let  $A \subset_{s} C$ . Let p(s) be any nondecreasing total recursive function. By Theorem 3.2 (iii) there exists  $T \supseteq \overline{C}$  such that

$$(\forall x)[x \in (T \cap C)_{\mathbf{at s}} \implies x \notin A_{f(s)}]].$$

Hence,  $W_j = T \cap C$  witnesses that A fails to satisfy (5).

## 4 Q(A) And Tardy Properties

In the following definition we separate the first property of Q(A) into two parts: the first part  $Q^{-}(A, C)$  which is equivalent to a purely dynamic property and is the key to satisfying tardiness and hence incompleteness; and the second part asserting  $A \subset_{\mathbf{m}} C$ , which is entirely standard.

**Definition 4.1** (i)  $Q^{-}(A, C) : A \subset_{\infty} C \& (\forall B \subseteq C) (\exists D \subseteq C) (\forall S)_{S \sqsubset C}$  [

 $(20) \qquad \qquad [B \cap (S - A) = D \cap (S - A)]$ 

(21) 
$$\implies (\exists T)[\overline{C} \subset T \And A \cap (S \cap T) = B \cap (S \cap T)]].$$
  
(*ii*)  $Q(A) : (\exists C)[A \subset_{\mathbf{m}} C \And Q^{-}(A, C)].$ 

**Definition 4.2** If  $A \subset C$  then Q-tardy (A, C) holds if

(22) 
$$A \subset_{\infty} C \& (\forall f) (\exists I \supseteq \overline{C}) (\exists E = A)_{(E \setminus C) \cap I = \emptyset} (\forall x) [x \in (I \searrow C \searrow E)_{at s} \implies x \notin A_{f(s)}].$$

The main result of the present paper is the following.

**Theorem 4.3 (Main Theorem)**  $Q^{-}(A, C) \iff Q$ -tardy(A, C).

We prove this in the next two theorems. The first resembles Lemma 1 of [3], and the second Lemma 2 there.

**Theorem 4.4**  $Q^{-}(A, C) \implies Q\text{-}tardy(A, C).$ 

*Proof.* Fix A and  $C \in \mathcal{E}$  such that A satisfies Q(A) via C, and fix indices  $W_a = A$  and  $W_c = C$  such that  $W_a \subseteq W_c \searrow W_a$ , which we write,

because we define  $A_s = W_{a,s}$  and  $C_s = W_{c,s}$ . To utilize the hypothesis Q(A)BLUE will first split C into the disjoint union of uniformly r.e. sets  $\{S_i\}_{i \in \omega}$ , written  $C = \bigsqcup_{i \in \omega} S_i$ , and then on  $S_i$  BLUE will play B against  $D = W_i$  to satisfy (20). Since Q(A) holds, RED must reply with T = some  $W_j$  to satisfy (21). Now BLUE will use a  $\Pi_2^0$  guessing procedure (described in §4.2 below) to determine the correct values of i and j. We let  $\alpha = \langle i, j \rangle$ .

To better explain the basic  $\alpha$ -module we will assume in §4.1 two simplifying hypotheses (discharged later in §4.2), the first of which asserts that BLUE has fixed the correct *i* and *j* so that BLUE is playing single sets *B* and *S* and has the indices *i* and *j* (respectively) of single r.e. sets *D* and *T* such that if BLUE satisfies (20) then RED satisfies (21). Also all sets below except *A*, *B*, and *C* have subscript  $\alpha$  which we drop for this subsection.

### 4.1 The basic $\alpha$ -module under simplifying assumptions

Now BLUE begins to satisfy (20) by first arranging that on S - A,

$$(24) B \subseteq (D \searrow B)$$

Hence, RED must ensure that on  $S \cap T$ ,

because if  $x \in (S \cap T \cap A) \setminus B$  then BLUE can restrain  $x \in \overline{B}$  forever thereby refuting (21) while still maintaining (20) by ensuring (24) and (27) on S - A.

Now (24) and (25) together ensure that on  $T \cap S$ ,

To achieve the rest of (20) for every x currently in  $(D - B) \cap (S - A)$ , after a finite number of stages of "restraint on x" BLUE will enumerate x in B. Thus, on S - A BLUE will play

$$(27) D - B = \emptyset$$

This will force RED to ensure (21) by enumerating in A all x currently in  $(B-A) \cap (S \cap T)$  so that on  $S \cap T$ ,

$$B - A = \emptyset.$$

As a second simplifying assumption BLUE assumes in §4.1 that if (21) holds for T then (21) also holds with T replaced by a certain set  $U \subseteq T$  which will be played by BLUE and which also satisfies

$$(29) (U \cap C) \subseteq^* S$$

(BLUE will discharge this assumption in §4.2.) But  $A \subset_{\mathrm{m}} C$  and  $\overline{C} \subseteq U$  (from (21)) imply

(30) 
$$\overline{A} \subseteq^* U,$$

so from (29) and (30) we get

$$(31) (C-A) \subseteq^* S$$

#### 4.2 Describing the $\alpha$ -module

We (BLUE) will define r.e. sets  $U_{\alpha}$ ,  $S_{\alpha}$ ,  $E_{\alpha}$ , and B, whose indices we know in advance by the Recursion Theorem. Let  $\{(D_i, T_j)\}_{i,j\in\omega}$  be an effective listing of all pairs of r.e. sets. Below BLUE will define r.e. sets  $\{S_{i,j}\}_{i,j\in\omega}$  such that  $C = \bigsqcup_{i,j\in\omega}S_{i,j}$ . Now BLUE begins by playing for every i and j the set Bon  $S_{i,j}$  against  $D_i$  to satisfy (24) and (27) and therefore (20). Hence, (20) is also satisfied by the sets B,  $D_i$ , and  $S_i = \bigsqcup_{j\in\omega}S_{i,j}$ . Thus, for some j,  $T_j$  must satisfy (21) and hence (25) and (28) for B,  $D_i$ , and  $S_i$ , and therefore also for B,  $D_i$ , and  $S_{i,j}$ . Let  $\alpha = \langle i, j \rangle$ , and let  $D_{\alpha}$ ,  $S_{\alpha}$ , and  $T_{\alpha}$  denote  $D_i = W_i$ ,  $S_{i,j}$ , and  $T_j = W_j$ , respectively, and  $D_{\alpha,s} = W_{i,s}$  and  $T_{\alpha,s} = W_{j,s}$ . For each  $\alpha$  the conjunction of all the conditions in the matrices of (20) and (21) (with D, S, T replaced by  $D_{\alpha}$ ,  $S_{\alpha}$ , and  $T_{\alpha}$  respectively) is a  $\Pi_2^0$  condition  $F(\alpha)$ . Hence, there is an r.e. sequence of r.e. sets  $\{Z_{\alpha}\}_{\alpha\in\omega}$  such that for every  $\alpha$ ,  $F(\alpha)$  holds iff  $|Z_{\alpha}| = \infty$ .

**Defining**  $U_{\alpha}$ . Define r.e. set  $U_{\alpha}$  by

(32) 
$$x \in U_{\alpha,s} \iff x \in U_{\alpha,s-1} \lor [x \in T_{\alpha,s} - C_s \& x \le |Z_{\alpha,s}|].$$

By the Recursion Theorem with parameter  $\alpha$  and the Slowdown Lemma [19, Lemma XIII.1.5] there is an index  $u_{\alpha}$  (which we know in advance) such that

(33) 
$$U_{\alpha} = W_{u_{\alpha}} \quad \& \quad W_{u_{\alpha}} \subseteq (U_{\alpha} \setminus W_{u_{\alpha}}).$$

Defining  $S_{\alpha}$ . If  $x \in C_{s+1} - C_s$  choose the least  $\alpha$  such that  $x \in U_{\alpha,s}$ , and enumerate x in  $S_{\alpha,s+1}$ . (If no such  $\alpha$  exists enumerate x in  $S_{x,s+1}$ .) This defines an r.e. set  $S_{\alpha}$ .

**Defining**  $E_{\alpha}$ . Using the enumerations above for C, A,  $D_{\alpha}$ ,  $S_{\alpha}$ , and  $W_{e_{\alpha}} = U_{\alpha}$  we now define the r.e. set,

(34) 
$$E_{\alpha} = ((W_{u_{\alpha}} \cap S_{\alpha}) \searrow D_{\alpha}) \cup ((C \setminus W_{u_{\alpha}}) \searrow A).$$

This defines a recursive enumeration  $\{E_{\alpha,s}\}_{s\in\omega}$  of the r.e. set  $E_{\alpha}$ . Again by the Recursion Theorem with parameter  $\alpha$  and the Slowdown Lemma there is an index  $e_{\alpha}$  such that  $W_{e_{\alpha}} = E_{\alpha}$  and  $W_{e_{\alpha}} \subseteq (E_{\alpha} \setminus W_{e_{\alpha}})$ .

**Defining** B. Fix a nondecreasing recursive function p(s).

- 1. If  $x \in (W_{u_{\alpha},s} W_{e_{\alpha},s}) \cap W_{e_{\alpha},s+1}$ , then  $\alpha$ -restrain x from  $B_t$  for all  $t \leq p(s)$ .
- 2. If  $x \in (W_{u_{\alpha},s} \cap S_{\alpha,s} \cap W_{e_{\alpha},s+1}) B_s$  and x is not  $\alpha$ -restrained from  $B_{s+1}$  then enumerate x in  $B_{s+1}$ .

This defines a recursive enumeration  $\{B_s\}_{s\in\omega}$  of the r.e. set B. Note that x can be  $\alpha$ -restrained for only finitely many stages, starting when 1. first holds, and then never again after the  $\alpha$ -restraint is dropped. Hence, there is no permanent restraint on x entering B so (27) holds. (Note that x can be  $\alpha$ -restrained only if  $x \in S_{\alpha}$  so x can never be  $\alpha$ -restrained and also  $\beta$ -restrained for  $\beta \neq \alpha$  because the  $S_{\alpha}$  sets are disjoint. Thus, unlike the predecessor [3, Lemma 1], there is no injury and no conflict between  $\alpha$ -strategies.)

Let  $\alpha$  be the least  $\beta$  such that  $Z_{\beta}$  is infinite. Hence,  $D_{\alpha}$ ,  $S_{\alpha}$ ,  $T_{\alpha}$ , and  $U_{\alpha}$ satisfy the first two simplifying assumptions in §4.1 including (29), because by (32)  $Z_{\beta}$  and hence  $U_{\beta}$  and  $S_{\beta}$  are finite for every  $\beta < \alpha$ . Hence, (29), (30), and (31) hold. Define the finite set  $\hat{S}_{\alpha} = \bigcup_{\beta < \alpha} S_{\beta}$ .

**Lemma 4.5** Q-tardy(A, C) holds.

*Proof.* Fix a nondecreasing recursive function p(s). Apply the above construction to produce  $W_{u_{\alpha}}$  and  $W_{e_{\alpha}}$  for the least  $\alpha$  satisfying  $F(\alpha)$ . Now

(35) 
$$(W_{e_{\alpha}} \setminus C) \cap W_{u_{\alpha}} = \emptyset,$$

by (34). Next  $W_{e_{\alpha}} = A$  and  $W_{u_{\alpha}} = U_{\alpha} \supseteq \overline{C}$  by (32) because  $T_{\alpha} \supseteq \overline{C}$ . Define  $k = \max(\widehat{S}_{\alpha})$ , and  $G = [0, k] \cap A$ . Let  $I_{\alpha} = W_{u_{\alpha}} - G$ . We claim that  $I_{\alpha}$  and  $E_{\alpha} = W_{e_{\alpha}}$  satisfy (22) so Q-tardy(A, C) holds. Suppose  $x \notin \widehat{S}_{\alpha}$  and

(36) 
$$x \in (W_{u_{\alpha},s_1} - W_{e_{\alpha},s_1}) \& x \in A_t.$$

Then

$$(\exists s_2 < t) [x \in C_{s_2}],$$

by (23). Then  $x \in U_{\alpha,s}$ . But  $x \in U_{\alpha} \setminus C$  because  $C \searrow U_{\alpha} = \emptyset$  by (32). Furthermore, when  $x \in U_{\alpha}$  enters C, x enters  $S_{\alpha}$  since  $x \notin \widehat{S}_{\alpha}$ . However, on  $S_{\alpha} \cap U_{\alpha}$  we know  $A \subseteq D_{\alpha} \searrow B \searrow A$  by (26). Hence, we may assume

$$(38) \qquad (\exists s_3)_{s_1 \leq s_3 \leq t} [x \in W_{u_\alpha, s_3} \cap S_{\alpha, s_3} \cap \overline{W}_{e_\alpha, s_3} \cap W_{e_\alpha, s_3+1}]$$

(Namely, while in  $W_{u_{\alpha}} \cap S_{\alpha} \cap \overline{W}_{e_{\alpha}}$ , at stage  $s_3$  element x "announces its intention" to eventually enter A by first entering  $W_{e_{\alpha},s+1}$ .) By the action of the  $\alpha$ -module,  $x \notin B_t$  for all  $t, s_3 + 1 \le t \le p(s_3)$ . But then by (26),  $x \notin A_t$ ,  $s_3 + 1 \le t \le p(s_3)$ . In (36) we must have  $v > p(s_1)$  since p is nondecreasing. Hence, (22) so Q-tardy(A, C) holds.

This completes the proof of Theorem 4.4.

**Theorem 4.6** Q-tardy $(A, C) \implies Q^{-}(A, C)$ .

*Proof.* We let the opponent (BLUE) play one set B and we (RED) play one set D against B (rather than the infinitely many  $B_i$  and  $D_i$  as in Lemma 2 of [3]). Next we let  $\{(S_j, \hat{S}_j) : j \in \omega\}$  be an effective listing of all disjoint pairs of r.e. sets (*i.e.*, played by BLUE). RED must reply with a set  $T_{\langle j,k \rangle}$ such that if B, D, and  $S_j$  satisfy (20) then  $T_{\langle j,k \rangle}$  satisfies (21). Fix recursive enumerations  $A_s$  and  $C_s$  of A and C.

For each j define the nondecreasing partial recursive function  $f_j(s)$  as follows. For each  $x \leq s$  perform the following subroutine to obtain s'' depending on x:

If x ∈ C<sub>s</sub> define s' = (µv ≥ s)[x ∈ S<sub>j,v</sub> ⊔ Ŝ<sub>j,v</sub>].
 If x ∈ S<sub>i,s'</sub> ∩ D<sub>s'</sub> let s'' = (µv ≥ s')[x ∈ B<sub>v</sub> ∪ A<sub>v</sub>].

Define  $f_j(s) = \max\{f_j(s-1), \max\{s''_x : x \le s\}\}.$ 

If B, D, and  $S_j$  satisfy condition (20) of Q(A), then  $f_j(s)$  is total recursive. Now applying the hypothesis of Q-tardy(A, C) to C, A, and  $f_j$ , and letting  $\alpha = \langle j, k \rangle$ , we get a pair of sets  $I_{\alpha}$  and  $E_{\alpha}$  such that

(39) 
$$I_{\alpha} \supseteq \overline{A} \& E_{\alpha} = A \& I_{\alpha} \cap (E_{\alpha} \setminus C) = \emptyset \\ \& (\forall y)(\forall s)[y \in I_{\alpha,s} - E_{\alpha,s} \implies y \notin A_{f_{j}(s)}]].$$

For  $\alpha = \langle j, k \rangle$  let  $S_{\alpha}$ ,  $\hat{S}_{\alpha}$ ,  $T_{\alpha}$ , and  $f_{\alpha}$  denote  $S_j$ ,  $\hat{S}_j$ ,  $T_{\langle j,k \rangle}$ , and  $f_j$ , respectively. We now use  $I_{\alpha}$  and  $E_{\alpha}$  to build  $T_{\alpha}$  which satisfies (21). For each  $\alpha = \langle j, k \rangle$  the conjunction of: (20) for  $(B, D, S_{\alpha})$ ;  $S_{\alpha} \sqcup \hat{S}_{\alpha} = C$ ;  $B \subseteq C$ ; and the conditions in (39) is a  $\Pi_2^0$  condition  $F(\alpha)$ . Let  $\{Z_{\alpha}\}_{\alpha \in \omega}$  be an r.e. array of r.e. sets such that  $F(\alpha)$  holds iff  $|Z_{\alpha}| = \infty$ .

Define  $T_{\alpha}$  by

(40) 
$$x \in T_{\alpha,s} \iff x \in T_{\alpha,s-1} \lor [x \in I_{\alpha,s} - C_s \& x \le |Z_{\alpha,s}|].$$

Hence,  $C \searrow T_{\alpha} = \emptyset$ ,  $T_{\alpha} \subseteq I_{\alpha}$ , and  $T_{\alpha} \supseteq \overline{C}$  iff  $|Z_{\alpha}| = \infty$ .

Suppose x enters C at some stage t. (By hypothesis  $x \notin E_{\alpha,t}$ .) Choose the least  $\alpha$  such that  $x \in T_{\alpha,t}$ . For all  $s \ge t$  let  $x \in D_s$  iff  $x \in E_{\alpha,s}$ . (Namely, for the least such  $\alpha$  let  $\alpha$  define D in the sense that we let D copy  $E_{\alpha}$  on  $T_{\alpha} \searrow C$ .)

Lemma 4.7  $Q^-(A, C)$  holds.

*Proof.* Suppose (20) holds for  $(B, D, S_j)$ . Let  $\alpha = \langle j, k \rangle$  be the least  $\beta$  such that  $Z_\beta$  is infinite. We must show that (21) holds for  $(B, S_\alpha, T_\alpha)$ . Now  $S_\alpha \sqcup \widehat{S}_\alpha = C$ , and  $f_\alpha$  is total.

By the definition of  $F(\alpha)$  the pair  $I_{\alpha}$  and  $E_{\alpha}$  witnesses that A is 2-tardy relative to  $f_{\alpha}$ . Now  $T_{\alpha} \subseteq I_{\alpha}$ , and  $T_{\alpha} \supset \overline{C}$  because  $I_{\alpha} \supset \overline{C}$  and  $|Z_{\alpha}| = \infty$ . But the  $f_{\alpha}$  delay ensures that on  $S_{\alpha} \cap T_{\alpha}$  the sets obey the intended order of enumeration, namely  $x \in A$  implies that  $x \in D \searrow B \searrow A$ , and hence Q(A)holds. To verify this suppose  $x \in T_{\alpha} \cap S_{\alpha} \cap A$ . Then

$$x \in T_{\alpha} \searrow C \searrow S_{\alpha} \searrow A.$$

Hence,

$$c \in I_{\alpha} \searrow C \searrow S_{\alpha} \searrow A,$$

because  $T \subseteq I_{\alpha}$ , and  $x \in T_{\alpha}$  implies  $x \in I_{\alpha} \setminus C$ . Hence,

1

$$x \in I_{\alpha} \searrow C \searrow E_{\alpha} \searrow A$$

because  $E_{\alpha} = A$  and  $I_{\alpha} \cap (E_{\alpha} \setminus C) = \emptyset$  by (39). Hence,

$$x \in (I_{\alpha} \searrow C \searrow E_{\alpha})$$
 at  $s \implies x \notin A_{f_{\alpha}(s)},$ 

by the 2-tardy assumption. Hence,

 $x \in (I_{\alpha} \searrow C \searrow E_{\alpha})$  at  $s \implies x \in B_{f_{\alpha}(s)}$ ,

by the definition of  $f_{\alpha}(s)$ . Therefore,

$$x \in (I_{\alpha} \searrow C \searrow E_{\alpha}) \text{ at } s \implies x \in B \setminus A.$$

This completes the proof of Theorem 4.6.

### 5 Relation of *Q*-tardy to 2-tardy

Note that Q-tardy(A, C) implies small-tardy(A, C). In [5, Theorems 3.3 and 3.8] we prove the following results.

**Theorem 5.1** (i)  $A \subset_{sm} C \implies [Q(A) \iff A \text{ is } 2\text{-tardy}].$ (ii)  $Q(A) \iff (\exists C)[A \subset_{sm} C \& A \text{ is } 2\text{-tardy}].$ 

Thus, it is not true that Q(A) holds iff A is 2-tardy, but this does hold if  $A \subset_{\text{sm}} C$  for some C.

What is the relation between Q-tardy(A, C) and and 2-tardy(A)? If Q-tardy(A, C) and  $A \subset_{\mathbf{x}} C$  where x denotes either major subset  $\subset_{\mathbf{m}}$  or weak major subset  $\subset_{\mathbf{wm}}$ , a slightly weaker condition, then 2-tardy(A) holds. Also if 2-tardy(A) and  $A \subset_{\mathbf{s}} C$  then Q-tardy(A, C) holds. These are all fairly easy to prove, and they establish the relationship between 2-tardy(A) and Q-tardy(A, C).

In [5, Theorem 3.11] we prove that there is a maximal 2-tardy set and hence: (i) the property of A being 2-tardy does not guarantee that the orbit of A consists only of incomplete sets; and (ii) the property of A being 2-tardy is not  $\mathcal{E}$ -definable.

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