Codable Sets and Orbits of Computably Enumerable Sets

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Abstract

A set X of nonnegative integers is *computably enumerable (c.e.)*, also called *recursively enumerable (r.e.)*, if there is a computable method to list its elements. Let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e \in \omega}, \subseteq)$. We previously exhibited a first order $\mathcal{E}$-definable property $Q(X)$ such that $Q(X)$ guarantees that X is not Turing complete (*i.e.*, does not code complete information about c.e. sets).

Here we show first that $Q(X)$ implies that X has a certain "slowness" property whereby the elements must enter X slowly (under a certain precise complexity measure of speed of computation) even though X may have high information content. Second we prove that every X with this slowness property is computable in some member of any nontrivial orbit, namely for any noncomputable $A \in \mathcal{E}$ there exists B in the orbit of A such that $X \leq_T B$ under relative Turing computability ($\leq_T$). We produce B using the Δ^0_3-automorphism method we introduced earlier.

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1 Introduction

Our fundamental goal is to study the relationship between *computability* properties and *definability* properties in the language $L(\subseteq)$ for the computably enumerable (c.e.) sets of natural numbers. Where possible we construct the desired $\mathcal{E}$ definable properties which say something about Turing information content of A; in other cases we prove nondefinability (and hence nonexistence of such a definable property) using automorphisms of $\mathcal{E}$; finally we relate definable properties to coding of information in orbits under $\mathrm{Aut}(\mathcal{E})$.

In the 1930's Gödel, Kleene, Turing and others introduced the notion of a *computable* (*i.e., recursive*) function. A *computably enumerable (c.e.)*, also called *recursively enumerable (r.e.)*, set of integers is one which is empty or the range of a computable function, namely one which can be generated by a computable process.

Let $\{W_e\}_{e\in\omega}$ be a standard indexing of the c.e. sets, and let $\mathcal{E}$ denote the structure of the computably enumerable sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subseteq)$. (Now $\mathcal{E}$ also forms a lattice but that does not play an important role in our results, and of course $\bigcup$ and $\bigcap$ are definable from $\subseteq$, so we consider $\mathcal{E}$ as a partially ordered set.) One of the basic tools of logic is the study of *definability* of various notions over certain structures. For sets the natural definability notion is that of subset $X \subseteq Y$ in the language $L(\subseteq)$, or equivalently a two sorted structure with the relation $x \in Y$, which is a structure clearly equidefinable with the first one. By *definability properties of the c.e. sets* we shall mean properties in $L(\subseteq)$ which are $\mathcal{E}$-definable.

Turing [20] defined *relative computability* denoted $X \leq_T Y$, for computing X given Y as an "oracle." Sets X and Y have the same *Turing degree*, written $X \equiv_T Y$, if $X \leq_T Y$ and $Y \leq_T X$, and may be thought of as coding the same information. We say that a c.e. set X is *high* if $X' \equiv_T \emptyset''$ and *low* if $X' \equiv_T \emptyset'$ where X' is the Turing jump of X. *Convention.* All sets and Turing degrees will be c.e. and $A, B, C, \ldots W, X, Y$ will range through c.e. sets unless specified otherwise, for example in §3 where we briefly mention n-r.e. sets.

Post's famous problem [16] was to produce a noncomputable c.e. set A which is not *complete*, namely such that $K \not\leq_T A$ where K is the complete c.e. set exhibited by Gödel. Most but not all of the properties suggested unsuccessfully by Post to guarantee incompleteness are $\mathcal{E}$-definable. The problem of producing an $\mathcal{E}$-definable incompleteness property was recently solved by Harrington and Soare.

Theorem 1.1 (Harrington-Soare [3]) *There is a nonempty $\mathcal{E}$-definable property $Q(A)$ such that every c.e. set A satisfying $Q(A)$ is noncomputable and Turing incomplete.*

Post was primarily concerned with incompleteness not with definable properties. However, we are concerned here with the deeper question of the *relationship between definability and information content (*i.e.*, Turing degree)* of $A \in \mathcal{E}$. in the sense of Martin's theorem [15] that the degrees of maximal sets are the high c.e. degrees. In this vein Harrington and Soare also exhibited [5] several other $\mathcal{E}$-definable properties $P(A)$ which imply something about the information content of A. For example, Harrington [19, p. 339] showed that there is an $\mathcal{E}$-definable property which holds exactly of the creative sets. Then Harrington and Soare then proved the following dual to Theorem 1.1 to produce a property $T(A)$ which implies that A must be *complete* (but noncreative).

Theorem 1.2 (Harrington-Soare [5]) *There is an $\mathcal{E}$-definable property $T(A)$ satisfied by a noncreative set (indeed a promptly simple set) A such that for all W, $T(W)$ implies that $K \leq_T W$.*

If we wish to prove that a proposed property is *not* $\mathcal{E}$-definable, we may use automorphims of $\mathcal{E}$. Indeed that is the *only* method we now know to prove nondefinability. Let $\mathrm{Aut}(\mathcal{E})$ denote the group of automorphisms of $\mathcal{E}$. Let $A \simeq B$ ($A \simeq_{\Delta^0_3} B$) denote that A is automorphic (Δ^0_3-automorphic) to B. The *orbit* of A, written orbit(A) or $[A]$, is $\{B : A \simeq B\}$. The infinite coinfinite computable sets constitute an orbit. An orbit $[A]$ is *nontrivial* if A is noncomputable (*i.e.*, $A >_T \emptyset$) and we will use *orbit* to mean *nontrivial orbit.*

Harrington and Soare recently introduced [4] a method for generating Δ^0_3-automorphisms of $\mathcal{E}$ and they have used it to prove a variety of nondefinability results such as Corollaries 1.7 and 1.6, which implies that every nontrivial orbit contains a set which is high and thus implies for example that Theorem 1.1 cannot be strengthened to assert that there is an $\mathcal{E}$-definable property $L(A)$ which guarantees that A is low. In this paper we use the Δ^0_3-automorphism method to prove the following much stronger version of these previous results.

Definition 1.3 (i) We say X can be *coded in the orbit of* A, denoted by $X \leq_T [A]$, if $X \leq_T B$ for some $B \in [A]$.

(ii) We say X is *codable* if X can be coded in every nontrivial orbit, namely if $X \leq_T [A]$ for every $A >_T \emptyset$.

Theorem 1.4 (Main Theorem) *If $Q(D)$ then D is codable.*

Corollary 1.5 *There is a single high c.e. set D which is codable.*

Proof. Fix D satisfying $Q(D)$. The first clause of the property $Q(D)$ asserts that there exist C such that $D \subset_m C$, namely D is a major subset of C. However, all major subsets are high [19, Exer. 1.19, p. 214]. ■

Corollary 1.6 (Harrington-Soare [4], Cholak [2]) *For every noncomputable c.e. set A there is a high c.e. set $B \in [A]$.* ■

Corollary 1.7 (Harrington) *[4, Theorem 9.1] Any nontrivial orbit can avoid any nontrivial downward cone, namely for all r.e. sets A and C such that $\emptyset <_T A$ and $C <_T K$ there is an c.e. set $B \simeq_{\Delta^0_3} A$ such that $B \not\leq_T C$.*

Proof. Given $C <_T K$ use the usual avoiding of downward cones method to choose $D \not\leq_T C$ such that $Q(D)$ and apply Theorem 1.4. ■

Corollary 1.8 *A set X is codable iff $X \leq_T D$ for some D satisfying $Q(D)$.*

Proof. If $X \leq_T D$ and $Q(D)$ then $X \leq_T D \leq_T [A]$ for every $A >_T \emptyset$ by Theorem 1.4. If $X \leq_T [A]$ for every $A >_T \emptyset$ then $X \leq_T C$ for some C and D such that $C \in [D]$ and $Q(D)$. Hence, $Q(C)$ because Q is $\mathcal{E}$-definable. ■

To prove the main Theorem 1.4 we first prove that for every D if $Q(D)$ then D is 2-tardy, where 2-tardy is a strengthening of the usual notion of tardy (which means being not of promptly simple degree, or equivalently having degree half of a minimal pair). We then prove that every 2-tardy set D can be coded into any nontrivial orbit. Thus, we see that $Q(D)$ implies that the elements of D are enumerated sufficiently slowly to allow D to be so coded, However, we know $Q(D)$ implies that D is high as above.

Corollary 1.9 *If S is a promptly simple set (or even of promptly simple degree) then S is not codable.* ■

Proof. If S is promptly simple (or even of prompt, *i.e.*, of promptly simple degree) and $S \leq_T C$ then C is also prompt [19, Corollary XIII.1.9, p. 287], thus not tardy, thus not 2-tardy, thus $\neg Q(C)$. Hence, by Corollary 1.8 S is not codable. ∎

Thus, codable sets can be high by Corollary 1.5 while noncodable sets can be low (choose a low promptly simple set in Corollary 1.9).

> *Therefore, one of our main conclusions is that the question of whether a set X can be coded into an arbitrary orbit $[A]$ depends more on the speed of enumeration of X (prompt or tardy) than on its information content (high or low).*

The fact that K is not codable has more to do with the fact that K is prompt (*i.e.*, of promptly simple degree) than that K has complete information content. For example, to show that $K \leq_T B$ for some $B \in [A]$ we must do very rapid coding, but if $Q(A)$ holds then $Q(B)$ holds for every $B \simeq A$ (because Q is $\mathcal{E}$-definable although "2-tardy" is not). Thus, B is 2-tardy, hence tardy, and hence incomplete. Thus, our present Theorem 1.4 improves our former Theorem 1.1.

To code D into $[A]$, the orbit of a noncomputable set A, we must construct $B \simeq A$ and a computable functional Ψ such that $D = \Psi(B)$. We must define the use function $\psi(n)$ to be a convenient element y not yet in B such that if n enters D then we can gradually move y into B. The property of D being 2-tardy will imply that there is a recursive function g (played by us) such that if n wants to enter D it must first declare that intention at some stage s and then wait until at least stage $g(s)$ before doing so. Since the automorphism machinery imposes considerable delay in putting $\psi(n)$ into B after first starting the process, we shall arrange that when n declares at stage s, we start $\psi(n)$ toward B immediately and make $g(s)$ so large that $\psi(x)$ has arrived in B by stage $g(s)$ before n has arrived in D.

We use the terms "computably enumerable (c.e.)" and "recursively enumerable (r.e.)" interchangably, and likewise "computable" and "recursive."

2 Dynamic Properties of C.E. Sets

2.1 Prompt and Tardy Sets and Degrees

Most properties of an r.e. set A are *static* properties in that they refer to A as a completed object without mention of the enumeration of A. Such include Post's properties of being simple or hh-simple, and Myhill's property of being maximal, all of which are also $\mathcal{E}$-definable properties. Another static property which is not $\mathcal{E}$-definable or even invariant under automorphisms is hyper-simplicity.

A *dynamic* property on the other hand is one which is defined using an computable enumeration $\{A_s\}_{s\in\omega}$ of A. The first essential use of a dynamic property was probably the covering hypothesis in the Extension Theorem of Soare's maximal set automorphism theorem [18]. Here there were several simultaneous enumerations of arrays of r.e. sets, $\{U_n\}_{n\in\omega}$ and $\{\widehat{V}\}_{n\in\omega}$, and it was important to measure for an element x which U_n sets it entered before entering certain $\widehat{V}_m$ sets. Lerman and Soare [11] attempted to capture part of this dynamic property with an $\mathcal{E}$-definable property called d-simple, but they succeeded in capturing only a very small part. The next significant advance came with the following definition of promptly simple sets by Maass [12].

Definition 2.1 (i) A coinfinite r.e. set A is *promptly simple* if there is a computable function p and a computable enumeration $\{ A_s \}_{s\in\omega}$ of A such that for every e,

$$(1) \qquad W_e \text{ infinite} \Longrightarrow (\exists s)\,(\exists x)\,[x \in W_{e,\,at\ s} \cap A_{p(s)}].$$

(ii) An r.e. set A is *prompt* if A has promptly simple degree namely, $A \equiv_T B$ for some promptly simple set B, and an r.e. degree is *prompt* if it contains a prompt set.

(iii) An r.e. set or degree which is not prompt is *tardy.*

By the Promptly Simple Degree Theorem [19, Theorem XIII.1.7(iii)] a set A being prompt is equivalent to the following property which we may take as the definition. Let $\{A_s\}_{s\in\omega}$ be any recursive enumeration of A. Then there is a recursive function p such that for all s, $p(s) \geq s$, and for all e,

$$(2) \qquad W_e \text{ infinite} \implies (\exists^\infty x)\,(\exists s)\,[x \in W_{e,\,at\ s} \ \ \&\ \ A_s\restriction x \neq A_{p(s)}\restriction x],$$

namely infinitely often A "promptly permits" on some element $x \in W_e$.

Promptly simply sets and degrees helped bring some dramatic advances in the subject. Maass [12] proved that any two promptly simple low sets are automorphic and discovered other properties of these sets [13]. Ambos-Spies, Jockusch, Shore, and Soare [1] used prompt degrees to unify and extend results about r.e. degrees, and promptness has been very influential ever since. (See [19, Chap. XIII].)

2.2 Almost Prompt Sets and Degrees

Harrington and Soare [4, Theorem 1.2] proved that every prompt set is automorphic to a complete set. They noticed that the same proof would work for a strictly larger dynamically defined class of sets called *almost prompt*, which are defined in terms of n-r.e. sets.

Definition 2.2 (i) A set $X \leq_T K$ is *n-r.e.* if $X = \lim_s X_s$ for some recursive sequence $\{ X_s \}_{s\in\omega}$ such that for all x, $X_0(x) = 0$ and

$$\text{card}\{ s : X_s(x) \neq X_{s+1}(x) \} \leq n.$$

(For example, the only 0-r.e. set is $\emptyset$, the 1-r.e. sets are the usual r.e. sets, and the 2-r.e. sets are the d.r.e. sets.)

(ii) Such a sequence $\{X_s\}_{s\in\omega}$ is called an *n-r.e. presentation* of X.

It is well-known and easy to show [19, Exercise III.3.8., p. 38] that for $n > 0$, X is n-r.e. iff

(3) $$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \dots \bigcup W_{e_{2k+1}}, \text{ or}$$

(4) $$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \dots \bigcup (W_{e_{2k+1}} - W_{e_{2k+2}}),$$

according as $n = 2k + 1$ is odd or $n = 2k + 2$ is even.

Definition 2.3 For $n = 0$ let $X^0_0 = \emptyset$. For $n > 0$ and $e = \langle e_1, e_2, \dots e_n\rangle$ define

(5) $$X^n_e = (W_{e_1} - W_{e_2}) \bigcup \dots,$$

as in (3) or (4) according as n is odd or even. We say that $\langle n, e\rangle$ is an *n-r.e. index* for X^n_e. Let

(6) $$X^n_{e,s} = (W_{e_1,s} - W_{e_2,s}) \bigcup \dots.$$

Definition 2.4 Let A be an r.e. set and let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A. We say A is *almost prompt*, abbreviated *a.p.*, if there is a nondecreasing recursive function $p(s)$ such that for all n and e,

$$X_e^n = \overline{A} \implies (\exists x)(\exists s)[x \in X_{e,s}^n \ \ \& \ \ x \in A_{p(s)}]. \tag{7}$$

Note that, as in the case of promptly simple, this definition is independent of the enumeration of A; if $p(s)$ works for the enumeration $\{A_s\}_{s\in\omega}$, and if $\{A'_s\}_{s\in\omega}$ is another enumeration of A, define $p'(s) = (\mu t)[A'_t \supseteq A_p(s)]$. We may think of Definition 2.4 as asserting that A will p-promptly hit every approximation $\{X_{e,s}^n\}_{s\in\omega}$ for every n-r.e. set $X_e^n = \overline{A}$ where the recursive approximation $X_{e,s}^n$ is determined by the *standard enumeration* $\{W_{e,s}\}_{e,s\in\omega}$ of the r.e. sets. In [4, Conversion Lemma 11.4] we prove that if we specify another collection of n-r.e. sets $\{\widehat{X}_e^n\}_{n,e\in\omega}$, by some recursive approximation $\{\widehat{X}_{e,s}^n\}_{n,e,s\in\omega}$, then there is a recursive function q such that A will q-promptly hit $\{\widehat{X}_{e,s}^n\}_{n,e,s\in\omega}$ if $\widehat{X}_e^n = \overline{A}$.

2.3 Very Tardy Sets

The negation of the property of almost prompt is called *very tardy*. An important special case of this is known as 2-tardy and is closely related to the property $Q(A)$ of Theorems 1.1 and 1.4 which is defined in §3.

Definition 2.5 Let A be an r.e. set and let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A.

(i) We say A is *very tardy* if A is not almost prompt, namely if for every nondecreasing recursive function $p(s)$,

$$(\exists n)(\exists e)[X_e^n = \overline{A} \ \ \& \ \ (\forall y)(\forall s)[y \in X_{e,s}^n \implies y \notin A_{p(s)}]]. \tag{8}$$

(ii) We say A is *n-tardy* if in (i) the fixed n works uniformly for *all* such functions p, namely for every nondecreasing recursive function $p(s)$,

$$(\exists e)[X_e^n = \overline{A} \ \ \& \ \ (\forall y)(\forall s)[y \in X_{e,s}^n \implies y \notin A_{p(s)}]]. \tag{9}$$

The main idea about a very tardy set A is that if $x \in X_{e,s}^n$ then x can later enter A eventually, but x must first undergo a delay until at least stage $p(s)+1$ before doing so. Since class of almost prompt sets is a strict extension of the class of prompt sets it follows that the class of very tardy sets is a strict subclass of the class of tardy sets, hence the name "very tardy." Note that A is 0-tardy iff $A = \omega$, and A is 1-tardy iff A is recursive. The 2-tardy sets play a special role in our work and have additional characterizations as follows.

Proposition 2.6 *For an r.e. set A the following are equivalent:*
(i) A is 2-tardy;
(ii) For every nondecreasing recursive function $p(s)$,

$$(\exists W_i \supseteq \overline{A})(\exists W_e \subseteq A)_{W_e \supseteq A \cap W_i}(\forall y)(\forall s)[y \in W_{i,s} - W_{e,s} \implies y \notin A_{p(s)}]].$$

(iii) For every nondecreasing recursive function $p(s)$,

$$(10) \quad (\exists W_i \supseteq \overline{A})(\exists W_e = A)(\forall y)(\forall s)[y \in W_{i,s} - W_{e,s} \implies y \notin A_{p(s)}]].$$

(iv) For every nondecreasing recursive function $p(s)$,

$$(11) \quad (\exists W_i \supseteq^* \overline{A})(\exists W_e = A)(\forall^\infty y)(\forall s)[y \in W_{i,s} - W_{e,s} \implies y \notin A_{p(s)}]].$$

where $(\forall^\infty y)$ denotes "for almost every y."

Proof. (i) $\iff$ (ii). For $k \in \omega$ let $k = \langle i, e\rangle$, so $X^2_k = W_i - W_e$ and $X^2_{k,s} = W_{i,s} - W_{e,s}$ by Definition 2.3. Now in (8) $X^2_k = \overline{A}$ iff $W_i - W_e = \overline{A}$ iff $W_i \supseteq \overline{A}$, $W_e \subseteq A$, and $W_e \supseteq W_i \cap A$. Thus, (ii) is seen to be merely a notational rewriting of (9) for the special case $n = 2$.

(ii) $\iff$ (iii). Fix $W_a = A$. Define $Y_{j,s} = W_{j,s} \cup W_{a,s}$. Let $\widehat{X}^2_k = Y_{i,s} - Y_{e,s}$ for $k = \langle i, e\rangle$. Hence, $X^2_k = \overline{A}$ iff $\widehat{X}^2_k = \overline{A}$ iff $Y_i = \omega$ and $Y_e = A$. By the Conversion Lemma 11.4 of [4] A is 2-tardy with respect to $\{X^2_k\}_{k\in\omega}$ iff A is 2-tardy with respect to $\{\widehat{X}^2_k\}_{k\in\omega}$.

(ii) $\iff$ (iv). Clearly, (iii) implies (iv). Assume (iv). Fix $W_i \supseteq^* \overline{A}$, $W_e = A$, and k such that for all $y > k$ and all s the matrix of (iv) holds, and $y \in \overline{A}$ implies $y \in W_i$. Let $S = A \cap [0, k]$, and $T = \overline{A} \cap [0, k]$. Using the Recursion Theorem and the Slowdown Lemma [19, Lemma XIII.1.5] there is an index j such that $W_j = (W_i - S) \cup T$ and $W_j \subseteq W_i \searrow W_j$. Hence, A satisfies (ii) via W_j and W_e. ∎

3 The Relation of $Q(A)$ to A Being 2-Tardy.

The purpose of this section is to prove that in the correct setting A satisfies $Q(A)$ iff A is 2-tardy. This enables us to work with the simpler and more intuitive property of 2-tardy in stead of the cumbersome property $Q(A)$ whose only advantage is that it is $\mathcal{E}$-definable while 2-tardy is not. We recall the definitions from [3].

Definition 3.1 (i) Let $A \subset_\infty C$ denote that $A \subset C$ and $C - A$ is infinite.

(ii) A subset A is a *major subset* of C if $A \subset_\infty C$ and for all e,

$$\overline{C} \subseteq W_e \implies \overline{A} \subseteq^* W_e.$$

(Note that if $A \subset_m C$ then both A and C are nonrecursive.)

(iii) $A \sqsubset B$ if there exists C such that $A \sqcup C = B$ (*i.e.*, $A \cup C = B$ and $A \cap C = \emptyset$).

(iv) If $\{X_s\}_{s\in\omega}$ and $\{Y_s\}_{s\in\omega}$ are recursive enumerations of r.e. sets X and Y define

$$X \setminus Y = \{z : (\exists s)[z \in X_s - Y_s]\},$$

the elements enumerated in X before Y and $X \searrow Y = (X \setminus Y) \cap Y$, the elements enumerated first in X and later in Y.

Definition 3.2 $\quad Q(A) : (\exists C)_{A \subset_m C}\ (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C}$ [

$$[B \cap (S - A) = D \cap (S - A)] \tag{12}$$

$$\implies (\exists T)[\overline{C} \subset T \ \&\ A \cap (S \cap T) = B \cap (S \cap T)]]. \tag{13}$$

The property $Q(A)$ (and the following proofs) should be visualized in the context of a two person game for r.e. sets in the sense of Lachlan [10] between the $\exists$-player (whom we call RED) who plays the r.e. sets A, C, D and T and the $\forall$-player (called BLUE) who plays the r.e. sets B and S. Intuitively, property $Q(A)$ asserts that if BLUE satisfies (12) (namely by making B copy D on $S - A$) then RED must construct $T \supset \overline{C}$ satisfying (13) (namely by making A copy B on $S \cap T$). Clearly $Q(A)$ implies $\emptyset <_T A$ because $A \subset_m C$.

3.1 $Q(A)$ implies A is 2-tardy

(The proof of the following theorem will be similar to the proof of Lemma 1 of [3] (which asserts that $Q(A)$ implies A is incomplete) since it proves a stronger result because 2-tardy sets are tardy and hence incomplete.)

Theorem 3.3 *If A satisfies property $Q(A)$ then A is 2-tardy.*

Proof. Fix A and $C \in \mathcal{E}$ such that A satisfies $Q(A)$ via C, and fix indices $W_a = A$ and $W_c = C$ such that $W_a \subseteq W_c \searrow W_a$, which we write,

$$A \subseteq C \searrow A, \tag{14}$$

because we define $A_s = W_{a,s}$ and $C_s = W_{c,s}$. To utilize the hypothesis $Q(A)$ BLUE will first split C into the disjoint union of uniformly r.e. sets $\{S_i\}_{i\in\omega}$, written $C = \sqcup_{i\in\omega} S_i$, and then on S_i BLUE will play B against $D = W_i$ to satisfy (12). Since $Q(A)$ holds, RED must reply with $T =$ some W_j to satisfy (13). Now BLUE will use a Π^0_2 guessing procedure (described in §3.1.2 below) to determine the correct values of i and j. We let $\alpha = \langle i, j\rangle$.

To better explain the basic α-module we shall assume in §3.1.1 two simplifying hypotheses (discharged later in §3.1.2), the first of which asserts that BLUE has fixed the correct i and j so that BLUE is playing single sets B and S and has the indices i and j (respectively) of single r.e. sets D and T such that if BLUE satisfies (12) then RED satisfies (13). Also all sets below except A, B, and C have subscript α which we drop for this subsection.

3.1.1 The basic α-module under simplifying assumptions

Now BLUE begins to satisfy (12) by first arranging that on $S - A$,

$$B \subseteq (D \searrow B). \tag{15}$$

Hence, RED must ensure that on $S \cap T$,

$$A \subseteq (B \searrow A), \tag{16}$$

because if $x \in (S \cap T \cap A) \setminus B$ then BLUE can restrain $x \in \overline{B}$ forever thereby refuting (13) while still maintaining (12) by ensuring (15) and (18) on $S - A$.

Now (15) and (16) together ensure that on $T \cap S$,

$$A \subseteq (D \searrow B \searrow A). \tag{17}$$

To achieve the rest of (12) for every x currently in $(D - B) \cap (S - A)$, after a finite number of stages of "restraint on x" BLUE will enumerate x in B. Thus, on $S - A$ BLUE will play

$$D - B = \emptyset. \tag{18}$$

This will force RED to ensure (13) by enumerating in A all x currently in $(B - A) \cap (S \cap T)$ so that on $S \cap T$,

$$B - A = \emptyset. \tag{19}$$

As a second simplifying assumption BLUE assumes in §3.1.1 that if (13) holds for T then (13) also holds with T replaced by a certain set $U \subseteq T$ which will be played by BLUE and which also satisfies

$$(U \cap C) \subseteq^* S. \tag{20}$$

(BLUE will discharge this assumption in §3.1.2.)

But $A \subset_m C$ and $\overline{C} \subseteq U$ (from (13)) imply

$$\overline{A} \subseteq^* U, \tag{21}$$

so from (20) and (21) we get

$$(C - A) \subseteq^* S. \tag{22}$$

3.1.2 Describing the α-module

We (BLUE) will define r.e. sets U_α, S_α, E_α, and B, whose indices we know in advance by the Recursion Theorem. Let $\{(D_i, T_j)\}_{i,j \in \omega}$ be an effective listing of all pairs of r.e. sets. Below BLUE will define r.e. sets $\{S_{i,j}\}_{i,j \in \omega}$ such that $C = \sqcup_{i,j \in \omega} S_{i,j}$. Now BLUE begins by playing for every i and j the set B on $S_{i,j}$ against D_i to satisfy (15) and (18) and therefore (12). Hence, (12) is also satisfied by the sets B, D_i, and $S_i = \sqcup_{j \in \omega} S_{i,j}$. Thus, for some j, T_j must satisfy (13) and hence (16) and (19) for B, D_i, and S_i, and therefore also for B, D_i, and $S_{i,j}$. Let $\alpha = \langle i, j \rangle$, and let D_α, S_α, and T_α denote $D_i = W_i$,

$S_{i,j}$, and $T_j = W_j$, respectively, and $D_{\alpha,s} = W_{i,s}$ and $T_{\alpha,s} = W_{j,s}$. For each α the conjunction of all the conditions in the matrices of (12) and (13) (with D, S, T replaced by D_α, S_α, and T_α respectively) is a Π^0_2 condition $F(\alpha)$. Hence, there is an r.e. sequence of r.e. sets $\{Z_\alpha\}_{\alpha\in\omega}$ such that for every α, $F(\alpha)$ holds iff $|Z_\alpha| = \infty$.

Defining U_α. Define r.e. set U_α by

$$(23) \qquad x \in U_{\alpha,s} \iff x \in U_{\alpha,s-1} \ \vee \ [\, x \in T_{\alpha,s} - C_s \quad \& \quad x \leq |Z_{\alpha,s}|\,].$$

By the Recursion Theorem with parameter α and the Slowdown Lemma [19, Lemma XIII.1.5] there is an index u_α (which we know in advance) such that

$$(24) \qquad U_\alpha = W_{u_\alpha} \quad \& \quad W_{u_\alpha} \subseteq (U_\alpha \setminus W_{u_\alpha}).$$

Defining S_α. If $x \in C_{s+1} - C_s$ choose the least α such that $x \in U_{\alpha,s}$, and enumerate x in $S_{\alpha,s+1}$. (If no such α exists enumerate x in $S_{x,s+1}$.) This defines an r.e. set S_α.

Defining E_α. Using the enumerations above for C, A, D_α, S_α, and $W_{e_\alpha} = U_\alpha$ we now define the r.e. set,

$$(25) \qquad E_\alpha = ((W_{u_\alpha} \cap S_\alpha) \searrow D_\alpha) \quad \cup \quad ((C \setminus W_{u_\alpha}) \searrow A).$$

This defines a recursive enumeration $\{E_{\alpha,s}\}_{s\in\omega}$ of the r.e. set E_α. Again by the Recursion Theorem with parameter α and the Slowdown Lemma there is an index e_α such that $W_{e_\alpha} = E_\alpha$ and $W_{e_\alpha} \subseteq (E_\alpha \setminus W_{e_\alpha})$.

Defining B. Fix a nondecreasing recursive function $p(s)$.

1. If $x \in (W_{u_\alpha,s} - W_{e_\alpha,s}) \cap W_{e_\alpha,s+1}$, then α-restrain x from B_t for all $t \leq p(s)$.

2. If $x \in (W_{u_\alpha,s} \cap S_{\alpha,s} \cap W_{e_\alpha,s+1}) - B_s$ and x is not α-restrained from B_{s+1} then enumerate x in B_{s+1}.

This defines a recursive enumeration $\{B_s\}_{s\in\omega}$ of the r.e. set B. Note that x can be α-restrained for only finitely many stages, starting when Step 1 first holds, and then never again after the α-restraint is dropped. Hence, there is no permanent restraint on x entering B so (18) holds. (Note that x can be α-restrained only if $x \in S_\alpha$ so x can never be α-restrained and also β-restrained for $\beta \neq \alpha$ because the S_α sets are disjoint. Thus, unlike the predecessor [3, Lemma 1], there is no injury and no conflict between α-strategies.)

Let α be the least β such that Z_β is infinite. Hence, D_α, S_α, T_α, and U_α satisfy the first two simplifying assumptions in §3.1.1 including (20), because by (23) Z_β and hence U_β and S_β are finite for every $\beta < \alpha$. Hence, (20), (21), and (22) hold. Define the finite set $\widehat{S}_\alpha = \cup_{\beta<\alpha} S_\beta$.

Lemma 3.4 *A is 2-tardy.*

Proof. Fix a nondecreasing recursive function $p(s)$. Apply the above construction to produce W_{u_α} and W_{e_α} for the least α satisfying $F(\alpha)$. Now

$$(W_{e_\alpha} \setminus C) \cap W_{u_\alpha} = \emptyset, \tag{26}$$

by (25). Next $W_{e_\alpha} = A$ and $W_{u_\alpha} = U_\alpha \supseteq \overline{C}$ by (23) because $T_\alpha \supseteq \overline{C}$. Hence, $U_\alpha \supseteq^* \overline{A}$ because $A \subset_m C$. We claim that (11) of Proposition 2.6 is satisfied by $W_i = W_{u_\alpha}$ and $W_e = W_{e_\alpha}$ so A is 2-tardy.

Suppose $x \notin \widehat{S}_\alpha$ and

$$x \in (W_{u_\alpha,s_1} - W_{e_\alpha,s_1}) \ \& \ x \in A_v. \tag{27}$$

Then

$$(\exists s_2 < s_1)[x \in C_{s_2}], \tag{28}$$

by (14). Then $x \in U_{\alpha,s}$. But $x \in U_\alpha \setminus C$ because $C \searrow U_\alpha = \emptyset$ by (23). Furthermore, when $x \in U_\alpha$ enters C, x enters S_α since $x \notin \widehat{S}_\alpha$. However, on $S_\alpha \cap U_\alpha$ we know $A \subseteq D_\alpha \searrow B \searrow A$ by (17). Hence, we may assume

$$(\exists s_3)_{s_1 \le s_3 \le s_1}[x \in W_{u_\alpha,s_3} \cap S_{\alpha,s_3} \cap \overline{W}_{e_\alpha,s_3} \cap W_{e_\alpha,s_3+1}] \tag{29}$$

(Namely, while in $W_{u_\alpha} \cap S_\alpha \cap \overline{W}_{e_\alpha}$, at stage s_3 element x "announces its intention" to eventually enter A by first entering $W_{e_\alpha,s+1}$.) By the action of the α-module, $x \notin B_t$ for all t, $s_3 + 1 \le t \le p(s_3)$. But then by (17), $x \notin A_t$, $s_3 + 1 \le t \le p(s_3)$. In (27) we must have $v > p(s_1)$ since p is nondecreasing. Hence, (11) of Proposition 2.6 is satisfied so A is 2-tardy. ∎

This completes the proof of Theorem 3.3. ∎

3.2 Small Subsets

Lachlan [9] introduced small sets in his program to construct canonical examples of certain diagrams and then rule out possible extensions so as to give a decision procedure for the $\overrightarrow{\forall}\,\overrightarrow{\exists}$-theory of the lattice of r.e. sets. The following definition is clearly equivalent to the standard definition given in [19, Definition X.4.10, p. 193].

Definition 3.5 A subset $A \subset C$ is a *small subset* of C (written $A \subset_s C$) if $A \subset_\infty C$ and for all X and Y, if

(i) $X \cap (C - A) \subseteq Y$, then

(ii) $(\exists Z)_{Z \subseteq X}\,[Z \supseteq (X - C)\ \ \&\ \ (Z \cap C) \subseteq Y]$.

If A is both a small subset and major subset of C we say it is a *small major subset* and write $A \subset_{sm} C$.

Note that (ii) is equivalent to the property

$$Y \cup (X - C) \text{ is r.e.} \tag{30}$$

For §3.3 we need the following consequence of small subsets.

Proposition 3.6 *If* $A \subset_s C$ *and if* $E = A$ *then there is a* $Z \supseteq \overline{A}$ *and* $\widehat{E} = E$ *such that*

$$(Z \searrow \widehat{E}) = (Z \cap C) \searrow \widehat{E}.$$

Proof. Let $S = E \searrow C$ and $\widehat{S} = C \setminus E$ so $C = S \sqcup \widehat{S}$. Now $C \supseteq A = E$ and $S \subseteq E = A$ so $\widehat{S} \supseteq (C - A)$. Apply Definition 3.5 with $X = \omega$, and $Y = \widehat{S}$ to obtain $Z \supseteq \overline{C}$, $Z \cap C \subseteq \widehat{S}$. Now $(Z \searrow E) \setminus C \subseteq S$ because $E = A \subseteq C$. But $Z \cap S = \emptyset$. Hence, $(Z \searrow E) \setminus C = \emptyset$. Thus, $Z \searrow E = (Z \cap C) \searrow E$. Now define $\widehat{E} = (Z \searrow E) \cup (C \searrow E)$ so $\widehat{E} = E$ and $(Z \searrow \widehat{E}) \searrow C = \emptyset$. ∎

It is interesting now to see that this important notion of small subset, just like the $Q(A)$ property, has a dynamic equivalent (iii) below which is proved in full in [7].

Theorem 3.7 (Harrington and Soare [7]) *Suppose* $A \subset_\infty C$. *Then the following are equivalent:*

(i) $A \subset_s C$*;*

(ii) $(\forall Y)[(C - A) \subseteq Y] \implies (\exists Z)[\overline{C} \subseteq Z \;\;\&\;\; Z \cap C \subseteq Y]$;

(iii) small-tardy(A, C)*:*

$$(31) \qquad (\forall f)(\exists T)[\overline{C} \subseteq T \;\;\&\;\; (\forall x)[x \in (T \cap C)_{at\ s} \implies x \notin A_{f(s)}]].$$

(In (iii) it is understood that f *ranges over recursive functions which are nondecreasing.)*

Note that (ii) is equivalent to the property,

$$(\forall Y \supseteq C - A)[Y \cup \overline{C} \text{ is r.e. }].$$

We refer to the property (iii) on $A \subset_\infty C$ as *small-tardy*(A, C) because it is a dynamic property. It is used in Corollary 3.10 to show that if $Q(A)$ holds via C then $A \subset_s C$.

Consider the property $\widehat{Q}(A) : (\exists C)[A \subset_s C]$. This resembles the property $Q(A)$ because $\widehat{Q}(A)$ implies that A is not a promptly simple *set* [7]. However, it does not guarantee that A is not of promptly simple *degree*, and therefore, unlike $Q(A)$ it does not ensure that the orbit of A contains only incomplete sets.

3.3 $Q(A)$ Holds For 2-tardy Small Major Subsets A

To complete the characterization of $Q(A)$ sets we now prove the following partial converse to Theorem 3.3. (The proof will be somewhat similar to the proof of Lemma 2 of [3] (which asserts that there is an A satisfying $Q(A)$) since given the existence of 2-tardy sets it proves a stronger result).

Theorem 3.8 *If* A *is 2-tardy and* $(\exists C)[A \subset_{sm} C]$ *then* $Q(A)$.

Proof. We let the opponent (BLUE) play one set B and we (RED) play one set D against B (rather than the infinitely many B_i and D_i as in Lemma 2 of [3]). Next we let $\{(S_j, \widehat{S}_j) : j \in \omega\}$ be an effective listing of all disjoint pairs of r.e. sets (*i.e.*, played by BLUE). RED must reply with a set $T_{\langle j,k \rangle}$

such that if B, D, and S_j satisfy (12) then $T_{\langle j,k\rangle}$ satisfies (13). Fix recursive enumerations A_s and C_s of A and C.

For each j define the nondecreasing partial recursive function $f_j(s)$ as follows. For each $x \leq s$ perform the following subroutine to obtain s'' depending on x:

1. If $x \in C_s$ define $s' = (\mu v \geq s)[x \in S_{j,v} \sqcup \widehat{S}_{j,v}]$.
2. If $x \in S_{j,s'} \cap D_{s'}$ let $s'' = (\mu v \geq s')[x \in B_v]$.

Define $f_j(s) = \max\{f_j(s-1),\ \max\{s''_x : x \leq s\}\}$.

If B, D, and S_j satisfy condition (12) of $Q(A)$, then $f_j(s)$ is total recursive. Now apply the hypothesis of 2-tardy to A and f_j, and apply Proposition 3.6, since $A \subset_s C$, and let $\alpha = \langle j,k\rangle$ to get a pair of sets I_α and E_α such that

$$\begin{aligned}(32)\qquad & I_\alpha \supseteq \overline{A} \ \ \&\ \ E_\alpha = A \ \ \&\ \ I_\alpha \cap (E_\alpha \setminus C) = \emptyset \\ & \&\ \ (\forall y)(\forall s)[y \in I_{\alpha,s} - E_{\alpha,s} \implies y \notin A_{f_j(s)}]].\end{aligned}$$

For $\alpha = \langle j,k\rangle$ let S_α, $\widehat{S}_\alpha$, T_α, and f_α denote S_j, $\widehat{S}_j$, $T_{\langle j,k\rangle}$, and f_j, respectively. We now use I_α and E_α to build T_α which satisfies (13). For each $\alpha = \langle j,k\rangle$ the conjunction of: (12) for (B, D, S_α); $S_\alpha \sqcup \widehat{S}_\alpha = C$; $B \subseteq C$; and the conditions in (32) is a Π^0_2 condition $F(\alpha)$. Let $\{Z_\alpha\}_{\alpha\in\omega}$ be an r.e. array of r.e. sets such that $F(\alpha)$ holds iff $|Z_\alpha| = \infty$.

Defining T_α. Define T_α by

$$(33)\qquad x \in T_{\alpha,s} \iff x \in T_{\alpha,s-1} \ \vee\ [\,x \in I_{\alpha,s} - C_s \quad \& \quad x \leq |Z_{\alpha,s}|\,].$$

Hence, $C \searrow T_\alpha = \emptyset$, $T_\alpha \subseteq I_\alpha$, and $T_\alpha \supseteq \overline{C}$ iff $|Z_\alpha| = \infty$.

Defining D. Suppose x enters C at some stage t. (By hypothesis $x \notin E_{\alpha,t}$.) Choose the least α such that $x \in T_{\alpha,t}$. For all $s \geq t$ let $x \in D_s$ iff $x \in E_{\alpha,s}$. (Namely, for the least such α let α define D in the sense that we let D copy E_α on $T_\alpha \searrow C$.)

Lemma 3.9 *$Q(A)$ holds.*

Proof. Suppose (12) holds for (B, D, S_j). Let $\alpha = \langle j, k\rangle$ be the least β such that Z_β is infinite. We must show that (13) holds for (B, S_α, T_α). Now $S_\alpha \sqcup \widehat{S}_\alpha = C$, and f_α is total.

By the definition of $F(\alpha)$ the pair I_α and E_α witnesses that A is 2-tardy relative to f_α. Now $T_\alpha \subseteq I_\alpha$, and $T_\alpha \supset \overline{C}$ because $I_\alpha \supset \overline{C}$ and $|Z_\alpha| = \infty$. But the f_α delay ensures that on $S_\alpha \cap T_\alpha$ the sets obey the intended order of enumeration, namely $x \in A$ implies that $x \in D \searrow B \searrow A$, and hence $Q(A)$ holds. To verify this suppose $x \in T_\alpha \cap S_\alpha \cap A$. Then

$$x \in T_\alpha \searrow C \searrow S_\alpha \searrow A.$$

Hence,

$$x \in I_\alpha \searrow C \searrow S_\alpha \searrow A,$$

because $T \subseteq I_\alpha$, and $x \in T_\alpha$ implies $x \in I_\alpha \setminus C$. Hence,

$$x \in I_\alpha \searrow C \searrow E_\alpha \searrow A,$$

because $E_\alpha = A$ and $I_\alpha \cap (E_\alpha \setminus C) = \emptyset$ by (32). Hence,

$$x \in (I_\alpha \searrow C \searrow E_\alpha)_{\text{ at s}} \implies x \notin A_{f_\alpha(s)},$$

by the 2-tardy assumption. Hence,

$$x \in (I_\alpha \searrow C \searrow E_\alpha)_{\text{ at s}} \implies x \in B_{f_\alpha(s)},$$

by the definition of $f_\alpha(s)$. Therefore,

$$x \in (I_\alpha \searrow C \searrow E_\alpha)_{\text{ at s}} \implies x \in B \setminus A. \quad \blacksquare$$

This completes the proof of Theorem 3.8. ∎

Corollary 3.10 *(i)* $A \subset_{sm} C \implies [Q(A) \iff A$ *is 2-tardy*$]$.
(ii) $Q(A) \iff (\exists C)[A \subset_{sm} C$ *&* A *is 2-tardy*$]$.

Proof. (i) and (ii) ($\Longleftarrow$) follow immediately by Theorem 3.8 and Theorem 3.3. For (ii) ($\Longrightarrow$) assume $Q(A)$. Then $A \subset_m C$ by definition and A is 2-tardy by Theorem 3.8. But that proof actually establishes the following tardiness property, called *Q-tardy*(A, C), as we verify in [7],

(34)
$$A \subset_\infty C \;\;\&\;\; (\forall f)(\exists Z \supseteq \overline{C})(\exists E = A)_{(E \setminus C) \cap Z = \emptyset}$$
$$(\forall x)[x \in (Z \searrow C \searrow E)_{\text{at s}} \implies x \notin A_{f(s)}].$$

Now (34) obviously implies Theorem 3.7 (iii), namely small-tardy(A, C), so $A \subset_s C$ holds as well. ∎

3.4 Maximal 2-Tardy Sets

In this subsection we show that Theorem 3.8 cannot be improved to show that every 2-tardy set has property $Q(A)$. The other purpose is to give a better feeling for 2-tardy sets which will be the main topic of this paper from now on. The proof of the next theorem is similar to that of the existence of an incomplete maximal set by Sacks, or a maximal set whose degree is half of a minimal pair, because it generalizes both results since 2-tardy sets are tardy and hence incomplete. We assume familiarity with the maximal set construction and the tree method as in [19, Chap. X §3; Chap. XIV §1–3].

Theorem 3.11 *There is a maximal 2-tardy set A.*

Proof. To make A maximal it suffices to construct A coinfinite and satisfying for every e the requirement,

$$\mathcal{P}_e : \quad \overline{A} \subseteq^* \overline{W}_e \quad \vee \quad \overline{A} \subseteq^* W_e.$$

To construct A we have a sequence of markers $\{\Gamma_n\}_{n\in\omega}$, with a_n^s denoting the position of Γ_n at the end of stage s and such that

$$\overline{A}_s = \{a_0^s < a_1^s < \cdots\}.$$

Each marker Γ_e moves to maximize its e-state. To guarantee that A is 2-tardy we must meet for every e the requirement,

$$\mathcal{N}_e : \quad \varphi_e \text{ total and nondecreasing} \implies$$
$$(\exists W_i \supseteq \overline{A})(\exists W_j = A)(\forall x)[x \in W_{i,s} - W_{j,s} \implies x \notin W_{\varphi_e(s)}],$$

The strategy to meet $\mathcal{N}_e$ in the presence of $\mathcal{P}_n$, $n \geq e$, is the following. Let $F(e)$ be the Π_2^0 predicate asserting that φ_e is total and nondecreasing. Let $\{Z_e\}_{e\in\omega}$ be a recursive sequence such that $F(e)$ holds iff Z_e is infinite. Fix e. Construct I and J whose standard indices $W_i = I$ and $W_j = J$ satisfy $\mathcal{N}_e$.

1. Enumerate x in I_s if $x \notin (A_s \cup I_{s-1})$ and $x < |Z_{e,s}|$.
2. If $x \in I_s$ compute $t > s$ such that $x \in W_{i,t}$.
3. If $x \in W_{i,t}$ and some $\mathcal{P}_n$, $n \geq e$, wants to put x into A, then put x into J_{t+1} and compute let $v = (\mu r)[x \in W_{j,r+1}]$. Again $v > t$.

4. Attempt to compute $\varphi_e(v)$. Say $\varphi_e(v) = q$.

5. With priority $\mathcal{N}_e$ restrain x out of A_w for all $w \leq q$.

If φ_e is total then Step 3 always terminates, so every element x is restrained from A for at most finitely many stages. Hence, every lower priority $\mathcal{P}_n$, $n > e$, eventually succeeds in enumerating any element into A it chooses. If φ_e is not total then $\mathcal{N}_e$ may permanently restrain element x because Step 3 never terminates, but in this case Z_e is finite, so I_e is finite, and there are finitely many such x.

We must modify this strategy in the presence of $\mathcal{P}_n$, $n < e$. Consider $e = 1$. As in the tree method we now have two versions of $\mathcal{N}_1$, the Π^0_2 version, $\mathcal{N}^0_1$, which guesses that $W_0 \supseteq \overline{A}$, and the Σ^0_2 version, $\mathcal{N}^1_1$, which guesses that $W_0 \cap \overline{A} =^* \emptyset$. $\mathcal{N}^1_1$ acts as above but is injured and restarted whenever $W_0 \setminus A$ receives a new element. $\mathcal{N}^0_1$ acts as above except that in Step 1 we must have $x \in W_{0,s}$. Hence, $\mathcal{N}^0_1$ is never injured. These strategies are now put on a two branching tree in the usual manner. We leave the details to the reader. ∎

Corollary 3.12 *(i) The property A being 2-tardy does not guarantee that the orbit of A consists only of incomplete sets.*

(ii) The property of A being 2-tardy is not $\mathcal{E}$-definable.

Proof. (i) The maximal sets form an orbit by [19, Theorem XV.4.6], and there exist complete maximal sets. Thus, the 2-tardy set A of Theorem 3.11 is automorphic to a complete set.

(ii) If A being 2-tardy were $\mathcal{E}$-definable then by Theorem 3.11 every maximal set would be 2-tardy and hence incomplete. ∎

4 The Coding Theorem

Using the full automorphism machinery we proved in [4] a coding theorem which we may view here as a kind of "black box" without knowing anything of the internal workings of the automorphism machinery or the former proof. The present construction can thus be split into two parts performed *simultaneously*, first the basic automorphism construction with the full automorphism construction done by the automorphism "builder" (as specified in [4] and which we never see here), and the second done by the "coder" in the later sections of this paper. We first state the coding theorem, and then explain how it applies.

4.1 The Refined Coding Theorem

Theorem 4.1 (Refined Coding Theorem, Harrington-Soare [4]) *Let $A = U_0$ be a given nonrecursive r.e. set. Perform the basic coding construction (consisting of Steps 1–6, 11, $\hat{1}$–$\hat{5}$, $\hat{7}$, $\hat{8}$), and possibly with additional Steps $\hat{n}$, $9 \leq n < 11$, which we may specify later, and which may be performed at any stage during the construction, but which must satisfy condition ($\widehat{R}1$) below. Let T be the priority tree of the construction, f the true path through T, and $\{f_s\}_{s\in\omega}$ the recursive approximation to f so that $f = \liminf_s f_s$. Let $B = \widehat{U}_\rho$ where $\rho = f \upharpoonright 1$, and $B_s = \widehat{U}_{\rho,s}$. Let $g(\alpha, s)$ be a recursive function which we may define during the construction but such that for all $\alpha \in T$, $g(\alpha, s)$ is defined by the end of stage s. For $\alpha \in T$ and $i \in \omega$, define $t(\alpha, i)$ by*

$$t(\alpha,i) = \begin{cases} (\mu t)(\forall s \geq t)[i \leq g(\alpha,s)] & \text{if } t \text{ exists,} \\ \infty & \text{otherwise.} \end{cases}$$

*For every $\alpha \in T$, $\alpha \neq \lambda$ (the empty node on T), the construction will produce a d.r.e. set of α-*witnesses*, $\widehat{L}_\alpha = \lim_s \widehat{L}_{\alpha,s}$, and pairwise disjoint subsets $\widehat{L}'_{\alpha,i,s} \subseteq \widehat{L}_{\alpha,s}$ such that $|\widehat{L}'_{\alpha,i,s}| \leq 2$ and $\widehat{L}_{\alpha,s} = \bigcup \{\widehat{L}'_{\alpha,i,s} : i \leq g(\alpha,s)\}$, and from which $\hat{y}_{\alpha,i,s}$ and $\hat{y}'_{\alpha,i,s}$ are defined by*

$$\hat{y}_{\alpha,i,s} = (\mu\hat{x})[\hat{x} \in \widehat{L}'_{\alpha,i,s}], \text{ and } \hat{y}'_{\alpha,i,s} = (\mu\hat{x})[\hat{x} > \hat{y}_{\alpha,i,s} \ \& \ \hat{x} \in \widehat{L}'_{\alpha,i,s}],$$

if these elements exist. From $\widehat{L}_\alpha$ we may select a subset $\widehat{J}_\alpha = \lim_s \widehat{J}_{\alpha,s}$ of activated *α-witnesses using the additional Steps $\hat{n}$, $9 \leq n < 11$, providing that these steps satisfy the following property ($\widehat{R}1$).*

($\widehat{R}1$) If $\hat{x} \in \widehat{L}_{\alpha,s}$, then Step $\hat{n}$ may put $\hat{x}$ in $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$. Step $\hat{n}$ may not remove $\hat{x}$ from $\widehat{L}_\alpha$ or $\widehat{J}_\alpha$, or add $\hat{x}$ to $\widehat{L}_\alpha$. (It is understood that Step $\hat{n}$ may not perform any other action which would affect the automorphism machinery but Step $\hat{n}$ may perform additional external action, such as defining a use function $\psi^{B_s}(j)$.)

Assume $\alpha \subset f$, $\alpha \neq \lambda$. Choose v_α such that for all $s \geq v_\alpha$, α is not initialized and no $\beta <_L \alpha$ acts at stage s. Then for all $\hat{x}$ and s and all $i \geq 1$, *(i)* $(\forall\gamma \subseteq f)[lim\,inf_s\; g(\gamma, s) < \infty] \implies$ *A is* Δ^0_3*-automorphic to B;*

(ii) $\hat{x} \in \widehat{L}_{\alpha,s} - \widehat{J}_{\alpha,s} \implies \hat{x} \in \overline{B}_s$;

(iii) $[s \geq \; max\{v_\alpha, t(\alpha,i)\}$ & $\hat{y}_{\alpha,i,s} \in \widehat{J}_{\alpha,s}] \implies (\exists t > s)[\hat{y}_{\alpha,i,s} \in B_t]$;

(iv) $i \leq lim\,inf_s g(\alpha, s) \implies (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow]$;

(v) $[i \leq lim\,inf_s g(\alpha, s)$ & $(\exists^{<\infty} s)[\hat{y}_{\alpha,i,s} \in B_{s+1} - B_s]]$
$$\implies [lim_s \hat{y}_{\alpha,i,s} < \infty].$$

In addition if Steps $\hat{n}$, $9 \leq n < 11$, *satisfy the following condition* $(\widehat{R}2)$ *then conclusion (vi) holds for* α *and* i *as above.*

$(\widehat{R}2)$ *Step* $\hat{n}$ *may not put* $\hat{y}'_{\alpha,i,s}$ *into* $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$, *and may put* $\hat{y}_{\alpha,i,s}$ *into* $\widehat{J}_{\alpha,s+1} - \widehat{J}_{\alpha,s}$ *only if* $\hat{y}'_{\alpha,i,s}$ *is defined.*

(vi) $$i \leq lim\,inf_s g(\alpha, s) \implies [\,(a.e.\ s)[\hat{y}_{\alpha,i,s}\downarrow] \;\&\; (\exists^\infty s)[\hat{y}_{\alpha,i,s}\downarrow \;\;\&\;\; \hat{y}'_{\alpha,i,s}\downarrow]].$$

4.2 Explanation of the Refined Coding Theorem

In the proof of the Coding Theorem we used elements $x \in \omega$ on the A-side, and elements $\hat{x} \in \hat{\omega}$ on the B-side but in this paper we are only concerned with the B-side, so we shall drop all hats even though they are formally required. Suppose $\alpha \subset f$. To apply the coding theorem the "coder" specifies a recursive function $g(\alpha, s)$ for the number of coding witnesses for node α he desires at stage s.. Second the "builder" will produce a set L_α of α-witnesses labeled as $y_{\alpha,i,s}$ and $y'_{\alpha,i,s}$, $i \leq g(\alpha)$. (This corresponds roughly to properties (iv) and (v) of the Coding Theorem.) Such witnesses will remain in $\overline{B}$ until the coder takes some action. Third the coder may *activate* an α-witness x by enumerating it in his set J_α in which case (with finitely many exceptions) the "builder" will enumerate x into B eventually. Thus the coder can withhold an α-witness x from B by keeping x in L_α but withholding x from J_α (property (ii) below). The coder gives up direct control over enumerating elements into B but can indirectly enumerate into B an element $x \in L_\alpha$ by putting x into J_α (property (iii) below). Finally, the "builder" ensures that A is automorphic to B so long as the "coder" does nothing more to influence enumeration of elements into B.

Finally, the coder must ensure that $\liminf_s g(\alpha, s) < \infty$, which implies that B is automorphic to A by Theorem 4.1 (i). This is a significant restriction. For example, one cannot code K into B by putting $y_{\alpha,i,s}$ into J_α exactly if $i \in K_s$ because for each $i \in \overline{K}$ one must keep $y_{\alpha,i,s} \in L_\alpha - J_\alpha$, which would cause $\liminf_s g(\alpha, s) = \infty$. (By the main result of Harrington and Soare [3] we know that we cannot always achieve $K \leq_T B$.) Nevertheless, the restriction $\liminf_s g(\alpha, s) < \infty$ still allows a lot of information to be coded into B as we shall see.)

5 The Codable Sets Theorem

The rest of this paper will be devoted to a proof of the following theorem which in light of Theorem 3.3 implies Theorem 1.4, and which asserts that if D is 2-tardy then D is codable.

Theorem 5.1 (Codable Sets Theorem) *If D is 2-tardy and A is nonrecursive then there exists B automorphic to A such that $D \leq_T B$.*

Proof. We begin with some preliminary definitions and remarks.

5.1 Defining the Delay Function h_β

Suppose that $\beta \subset f$, $y_\beta \in L_\beta$ and y_β is not removed from L_β before y_β enters B. (Such removal only occurs because $f_s <_L \beta$, which occurs finitely often, or because $g(\beta, s)$ decreases too far, which we can arrange not to happen since we play g.) By Theorem 4.1 (iii) we know that if we put y_β into J_β then eventually y_β enters B. However, in this paper we shall need the first conjunct in the conclusion of Theorem 4.1 (vi), so we must guarantee condition ($\widehat{R}2$) that $y_{\beta,s}$ enters J_β at stage $s+1$ only if $y'_{\beta,s}$ is defined. Hence, if we want to put y_β into B we first put y_β into a preliminary set J^-_β. If $y_{\beta,s} \in J^-_{\beta,s}$ and $y'_{\beta,s}$ is defined, then we put $y_{\beta,s}$ into $J_{\beta,s+1}$. If $\beta \subset f$ then by the second conclusion of Theorem 4.1 (vi) we know y'_β is eventually defined so any $y_\beta \in J^-_\beta$ must eventually enter J_β, and by (iii) y_β must later enter B. The next function measures the delay between the first and third events. Define

$$(35) \qquad h_\beta(s) = (\mu t > s)(\forall y)[y \in J^-_{\beta,s} \implies [y \in B_t \ \vee \ y \notin L_{\beta,t}]].$$

Note that h_β is a partial recursive function and may not be total if $\beta \not\subset f$, but if $\beta \subset f$ then h_β must be total. In particular, if $\alpha = \rho$ then h_ρ is defined as in (35). In addition, we define $E_\rho = D$ and $E_{\rho,s} = D_s$.

Now assume by induction on the length of β, $\rho \subseteq \beta$ and $\beta \subset f$, that h_β is total and $E_\beta = D$ (but not necessarily $E_{\beta,s} = D_s$). Now $E_\beta = D$ is 2-tardy so we can apply the 2-tardy characterization Proposition 2.6 (iii) with $p(s) = h_\beta(s+1)$ and with the previous 2-tardy set A of §2 replaced by the present 2-tardy set E_β to obtain the current version of (10),

$$(36) \quad (\exists W_i \supseteq \overline{D})(\exists W_e = D)(\forall y)(\forall s)[y \in W_{i,s} - W_{e,s} \implies y \notin E_{\beta,h_\beta(s+1)}].$$

5.2 Redefining the Priority Tree

In addition to the tasks that $\alpha \subset f$ must do for the automorphism construction, for this paper we need α to have the correct guess about W_i and W_e satisfying (36) for h_β where $\beta = \alpha^-$, the predecessor of α.

Definition 5.2 The definition of T is the same as in [4, Definition 2.11] except that there we used quadruples and put $\alpha = \beta\widehat{\ }\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha \rangle$ in T providing that $\beta \in T$ and conditions (i)–(viii) were met. Here we use quintuples and put $\alpha = \beta\widehat{\ }\langle \mathcal{M}_\alpha, \mathcal{R}_\alpha, \mathcal{B}_\alpha, k_\alpha, n_\alpha \rangle$ in T providing that $\beta \in T$, $n_\alpha \in \omega$, and the previous conditions (i)–(viii) of [4, Definition 2.11] are met.

Definition 5.3 (i) The *true path* $f \in [T]$ is defined by induction on n. Let $\beta = f \restriction n$ be consistent. Then $f \restriction (n+1)$ is the $<_L$-least $\alpha \in T$, $\alpha \supset \beta$, of length $m = n+1$ satisfying the previous conditions (i)–(v) of [4, Definition 2.12] and also,

$$(37) \quad n_\alpha = (\mu n)[n = \langle i, e \rangle \text{ and } W_i \text{ and } W_e \text{ satisfy (36) for } \beta = \alpha^-].$$

(ii) We also define i_α and e_α by $\langle i_\alpha, e_\alpha \rangle = n_\alpha$, and define $I_{\alpha,s} = W_{i_\alpha,s}$, $E_{\alpha,s} = W_{e_\alpha,s}$, $I_\alpha = W_{i_\alpha}$, and $E_\alpha = W_{e_\alpha}$. We may assume by Proposition 2.6 that these enumerations have been adjusted so that $E_{\alpha,s} \subseteq I_{\alpha,s+1}$. We arbitrarily define $I_\lambda = E_\lambda = \emptyset$.

Hence,

$$(38) \qquad (\forall \alpha \subset f)[E_\alpha = D \ \& \ h_\alpha \text{ is total }].$$

The first follows by induction on $\beta = \alpha^-$, (36), and (37). The second by (35) and the discussion following it.

Now by [4, Remark 2.13] this expansion of the priority tree T does not interfere with the automorphism construction of [4].

Assume $\alpha \in T$ and $\beta = \alpha^-$. Since α has the correct guess about h_β but not about its own h_α, α must use as traces elements from L_β not L_α. Thus, α will have a set of traces $y_\alpha \in L_\beta$, and when α wants to put y_α into B it puts y_α into a set K_α of *activated* α-traces. We define

$$J^-_{\beta,s} = \cup\{K_{\alpha,s} : \alpha^- = \beta\}, \tag{39}$$

so that if $\alpha \subset f$ then any $y_\alpha \in K_\alpha$ eventually enters B (or is cancelled). Thus, in dealing with α-traces we can deal only with α-level objects y_α and K_α and suppress explicit mention that they are really operating at level $\beta = \alpha^-$. (For ρ we have no $J^-_{\rho^-}$, because $\rho^- = \lambda$, so we allow $K_\rho \subset J_\rho$ itself which succeeds because $\rho \subset f$.)

For each $\alpha \in T$ we have a *pocket* P_α. We let $P_{\alpha,s}$ denote the set of elements in P_α at the end of stage s, and $P_{\alpha,\infty}$ be the set of permanent residents of P_α. Also define

$$P_{\supseteq\alpha,s} = \bigcup\{P_{\gamma,s} : \gamma \supseteq \alpha\} \text{ and } P_{<_L\alpha,s} = \bigcup\{P_{\delta,s} : \delta <_L \alpha\}.$$

Special roles will be played by the two nodes on T, $\lambda = f\restriction 0$ the empty node, and $\rho = f\restriction 1$, because $B = \widehat{U}_\rho$. Since $f_s <_L \rho$ for at most finitely many s and we will assume that the following coding construction takes place on nodes $\alpha \supseteq \rho$ and λ.

5.3 Attempting to Code D into B

Fix D and A. We must use the Refined Coding Theorem 4.1 to build B automorphic to A and simultaneusly build a recursive functional Ψ, with use function ψ, such that $D = \Psi^B$. (For notational convenience we often drop the superscript B or B_s from Ψ and ψ.)

We first put x in pocket P_α, for some α such that $|\alpha| = x$, when there are sufficiently many α-traces z_α, and we define $\psi(x) \geq z_\alpha$. Later x may move around among P_γ, $|\gamma| \leq x$. During each move of x from P_α some α-trace $z_\alpha \leq \psi(x)$ will enter B allowing $\psi(x)$ to become undefined and then redefined. If $x \in D = E_\rho$ then eventually $x \in P_{\rho,s}$ and some ρ-trace $z_\rho \leq \psi(x)$ enters B

allowing us to redefine $\psi(x)$, place x in P_λ, and redefine $\Psi(x) = 1$. If $x \notin D$ then x comes to rest in some P_α, $\alpha \neq \lambda$. Hence, $x \in D$ iff $x \in P_{\lambda,\infty}$. However, $\lim_s \psi_s(x)$ is recursive in B. Thus, B can compute the final position $P_{\alpha,\infty}$ of each x, so $D \leq_T B$. (We can suppress explicit mention of Ψ and concentrate on ψ. Namely, we can arrange that $\Psi_s(x)\downarrow$ iff $\psi_s(x)\downarrow$; and $\Psi_s(x) = 1$ iff $x \in P_{\lambda,s}$.)

We begin with some approximations which do not quite succeed but which motivate the construction which does succeed.

5.3.1 The First Approximation: Node ρ Alone.

Let $\rho =_{dfn} f\restriction 1$ as in Theorem 4.1 so $A = U_0$ and $B = \widehat{U}_\rho$. By Theorem 4.1 (iii) above, we know that if $y \in J_\rho$ then $y \in B$ eventually. Our first attempt to build ψ might be to choose from the potential ρ-trace set, L_ρ, an infinite disjoint set of ρ-traces $\{v_{\rho,x} : x \in \omega\}$, and to define $\psi_s(x) = v_{\rho,x}$. Later if $x \in D_t$, $t > s$, then we put $v_{\rho,x}$ into J_ρ from which it must eventually enter B at which time we can redefine $\psi(x)$ and put x in P_λ (setting $\Psi(x) = 1$). The difficulty is that if $\overline{D}$ is infinite (the only interesting case) then we are required to hold infinitely many different traces $\{v_{\rho,x}$: $x \in \overline{D}\}$ in $L_\rho - J_\rho$ forever, which can be done only by allowing $\lim_s g(\rho, s) = \infty$. This violates Theorem 4.1 (i) which requires that $\liminf_s g(\alpha, s) < \infty$ for all $\alpha \subset f$ in order to achieve A automorphic to B. (Namely, having these traces v_ρ stay forever at level ρ means that they ignore all other sets in the automorphism, U_n, V_n, $n > 0$, which obviously destroys the automorphism. The Δ_3-automorphism machinery of [4] allowed infinitely many such traces v_ρ to act at the ρ-level and perhaps even enter B, since lower nodes can guess at the infinite recursive set contributed to B by them, but it cannot tolerate infinitely many elements remaining permanently in $\overline{B}$ at level ρ, since otherwise we could code K into B.)

The next approximation is to still require every element x to have some ρ-trace $v_{\rho,x}$ as before but to have several different elements share the same trace. The obvious difficulty with sharing traces is that if $x_1 \in \overline{D}$ and x_1 shares a trace $v_{\rho,x}$ first with x_2, then with x_4, and so on where every x_{2n} enters D, then $lim_s \psi_s(x_1) = \infty$. The solution is to spread out the sharing of traces over the whole tree and to use the 2-tardy property to help arrange the change of traces for $\psi(x)$ as x advances up the tree toward node ρ.

5.3.2 The Second Approximation: Playing Only On Nodes $\alpha \subset f$.

In our second approximation we assume that $\alpha_n = f \upharpoonright n$ for all $n \in \omega$. Hence, $\alpha_1 = \rho$ and $\alpha_0 = f \upharpoonright 0 = \lambda$. For the moment we consider only the action along the α_n, $n \geq 1$, not along γ for any $\gamma \not\subset f$.

Case $\alpha = \rho = \alpha_1$. We have h_ρ defined as in (35). In addition we define $E_\rho = D$ and $E_{\rho,s} = D_s$.

Case $\alpha = \alpha_{n+1}$, $n \geq 1$. Let $\beta = \alpha^-$. By induction on n assume that $E_\beta = D$ and h_β are defined. Now $E_\beta = D$ is 2-tardy, so by (36), (37), and the fact that $\alpha \subset f$, we have

$$(40) \quad I_\alpha \supseteq \overline{D} \ \& \ E_\alpha = D \ \& \ (\forall y)(\forall s)[y \in I_{\alpha,s} - E_{\alpha,s} \implies y \notin E_{\beta,h_\beta(s+1)}].$$

By Definition 5.3 (ii) we may assume we are given enumerations $I_{\alpha,s} = W_{i_\alpha,s}$, $E_{\alpha,s} = W_{e_\alpha,s}$ such that $E_{\alpha,s} \subset I_{\alpha,s+1}$.

For each $n \in \omega$ we have at every stage two α_n-traces v_{α_n} and y_{α_n}, whose approximations at the end of stage s are $v_{\alpha_n,s}$ and $y_{\alpha_n,s}$. We let z_α be a variable representing an α-trace v_α or y_α and later w_α. To begin with for each $x \leq s$ such that $\psi_s(x)\uparrow$ we put x in P_{α_x}, and define $\psi_{s+1}(x) = v_{\alpha_x}$. Later if $x \in P_{\alpha,s}$, and $x \in I_{\alpha,s} - E_{\alpha,s}$, but $x \in E_{\alpha,s+1}$ (namely $x \in E_{\alpha,\text{ at } s+1}$), and $\psi_s(x) = z_\alpha$ (where $z_\alpha = v_\alpha$ or y_α) then we put trace z_α into $K_{\alpha,\text{ at } s+1}$, and hence into $J^-_{\beta,\text{ at } s+1}$ by (39). Now $J^-_\beta \subset B$ because $\beta \subset f$ so z_α enters B at some stage $t+1 > s+1$. At stage $t+1$ we move x to P_β, and define $\psi_{t+1}(x) = y_{\beta,t}$ if $\beta \neq \lambda$; and $= 0$ if $\beta = \lambda$. If $\alpha = \rho$ then $\beta = \lambda$ and the process ends because $E_\lambda = \emptyset$.

This algorithm succeeds because if $x \notin D$ then $x \in P_{\alpha_x,\infty}$ and $\psi(x) = v_{\alpha_x}$. If $x \in D$ then $x \in E_{\alpha_n}$ for all $n \geq 1$, so it follows by induction on $x - n$ that x enters and later leaves every P_{α_n}, $x \leq n \leq 1$, until x reaches P_λ. Hence, $D \leq_T B$.

5.3.3 Extracting Information From the Second Approximation

The method above works well and indeed our full construction will work much like this along $\alpha \subset f$. However, determining whether $\alpha \subset f$ is a $\emptyset''$

question so we shall have to add some extra features to handle those $\gamma \not\subset f$. Let us first extract some information from the above procedure. Suppose

(41) $$[x \in P_{\alpha,s} \ \& \ x \in E_{\alpha, \text{ at } s+1} \ \& \ z_{\alpha,s} \in J^-_{\beta, \text{ at } s+1}], \text{ and}$$

(42) $$\psi_s(x) = z_{\alpha,s} \ \& \ z_{\alpha,s} \in B_{\text{ at } t+1}$$

Then $s+1 < t+1 \leq h_\beta(s+1)$ by (36). But

(43) $$x \in E_{\alpha, \text{ at } s+1} \implies x \in I_{\alpha,s} - E_{\alpha,s}.$$

Hence, by (40) or (36),

(44) $$x \notin E_{\beta,h_\beta(s+1)}, \text{ and thus } x \notin E_{\beta,t+1}.$$

Hence,

(45) $$(\forall n)(\forall y)[[\, y \in E_{\alpha_{n+1}, \text{ at } s} \ \& \ y \in P_{\alpha_n, \text{ at } t} \ \& \ y \in E_{\alpha_n, \text{ at } u}\,] \implies u > t > h_{\alpha_n}(s)\,].$$

Definition 5.4 Let $\alpha \in T$, $|\alpha| > 0$, and $\beta = \alpha^-$, the predecessor of α.

(i) x has *discharged* α *at stage* t if $(\exists s < t)[x \in E_{\alpha, \text{ at } s} \ \& \ h_\beta(s)\downarrow = t]$.

(ii) x has *discharged* α *by stage* v if x has discharged α at some stage $t \leq v$.

Thus, the algorithm works as follows in two phases at each α. During phase one x waits in P_α until $x \in E_\alpha$ say at $s+1$. Then at stage $s+1$ we put the α-trace z_α associated with $\psi(x)$ into K_α (and hence in J^-_β) and begin phase two during which x waits in P_α until $z_\alpha \in B$ at some stage $t+1 > s+1$ whereupon x moves to P_β and we redefine $\psi(x) = y_\beta$. At stage $t+1$ phase two ends and x *discharges* α.

The significance of (44) and (45) is that that we have enforced a *tower of delay.* Namely, when x arrives in P_β at stage $t+1$ we know that $x \notin E_{\beta,t+1}$ and also that $\psi(x)$ is free to be redefined $\geq y_\beta$. Thus, each E_{α_n} is just another enumeration of D except that after x enters $E_{\alpha_{n+1}}$ it must undergo a very long delay before entering E_{α_n} thus giving us time to adjust traces and redefine $\psi(x)$. *This is the essence* of how the 2-tardy hypothesis is used to prove $D \leq_T B$.

For $\gamma \not\subset f$ either phase may terminate in an infinite wait. For $\alpha \subset f$ the message " $x \in E_\alpha$" gives us ever stronger assurance that $x \in D$ as we move up the tree toward ρ. Notice that ρ has only two traces at any time, $v_{\rho,s}$ and $y_{\rho,s}$, and only the later is assigned to the infinitely many elements x arriving from nodes $\alpha \supset \rho$. Thus, at any given time many $x \in P_{\rho,s}$ may "share" $y_{\rho,s}$ in the sense that each has $\psi_s(x) = y_{\rho,s}$. If one such element x_1 is not yet in E_ρ when another x_2 enters E_ρ, causes $y_{\rho,s}$ to enter B, say at $t+1$, and change value to $y_{\rho,t+1}$ as x_2 enters P_λ, then x_1 simply stays in P_ρ and we redefine $\psi_{t+1}(x_1) = y_{\rho,t+1}$. Normally, this is dangerous because it threatens $\lim_s \psi_s(x_1) = \infty$. However, if $\alpha \subset f$ then $E_\alpha = D$ by (40) so if x in P_ρ from $\alpha \subset f$ then $x \in E_\alpha$ so $x \in E_\rho$ eventually. This sharing of traces requires at most a finite number of traces for each α and thus apply the Theorem 4.1 to build the automorphism.

5.3.4 The Third Approximation: Expanding to Nodes $\gamma \not\subset f$.

Our final construction must work on all nodes $\beta \subset f$ so we must add extra β-traces $w_{\beta,\gamma}$ to handle nodes γ such that $\beta \subset f <_L \gamma$, and $u_{\beta,\delta}$ for nodes $\delta \supset \beta$ such that $\delta <_L f$. Thus, node β will need β-traces: v_β, and $u_{\beta,\alpha}$, $y_{\beta,\alpha}$, $w_{\beta,\alpha}$ for every α such that $\alpha^- = \beta$ and $\alpha \leq_L f_s$. These traces will be ordered

$$(46) \qquad \begin{aligned} & v_\beta < \dots u_{\beta,\delta} < y_{\beta,\delta} < w_{\beta,\delta} < \dots u_{\beta,\alpha} < y_{\beta,\alpha} < w_{\beta,\alpha} < \dots \\ & \quad u_{\beta,\gamma} < y_{\beta,\gamma} < w_{\beta,\gamma}, < \dots, \text{ where } \delta <_L \alpha <_L \gamma \leq_L f_s. \end{aligned}$$

We refer to all these as *β-traces* and in addition we refer to the three traces, $u_{\beta,\alpha}$, $y_{\beta,\alpha}$, $w_{\beta,\alpha}$ as *(β,α)-traces.*

The β-traces v_β and $y_{\beta,\alpha}$ behave roughly as before except that every element x arriving in P_β from P_α after discharging α is assigned $y_{\beta,\alpha}$ as trace so $\psi(x) = y_{\beta,\alpha}$.

The trace $w_{\beta,\alpha}$ works as follows. Suppose that $\beta = \alpha^- = \gamma^-$, $\alpha <_L \gamma$, $\alpha \subset f$. Suppose that for some $\xi \supseteq \gamma$, $x \in P_{\xi,r}$, and $\psi_r(x)\downarrow = z_\xi$, where z_ξ represents some ξ-trace. Suppose that later $\alpha \subset f_s <_L \gamma$ say for $s > r$. Then we move x to $P_{\beta,s}$, put $w_{\beta,\alpha}$ into K_β, and the automorphism machinery automatically cancels all θ-traces for $\theta >_L f_s$. Our order of appointment of traces will guarantee that $w_{\beta,\alpha} < v_\gamma < z_\xi = \psi_r(x)$. If $\beta \subset f$ then $w_{\beta,\alpha}$ eventually enters B at which time $\psi(x)$ becomes undefined and starts all over. (If later $f_t <_L \beta$ before $w_{\beta,\alpha}$ enters B then we repeat the above argument with a new trace $w_{\beta',\alpha'}$ for some $\beta' \subset \alpha' \subset f_t$.)

Definition 5.5 A node δ is *incorrect* by stage t if

(47) $$(\exists s \leq t)(\exists \alpha \subseteq \delta)(\exists \beta \subset \alpha)[x \in E_{\beta,s} - E_{\alpha,s}].$$

By (40) if α is incorrect by stage s then $\alpha \not\subset f$. Hence, we may assume that our recursive approximation $\{f_s\}_{s\in\omega}$ and construction have been arranged to ensure that

(48) $$(\forall s)[\alpha \subset f_s \implies \alpha \text{ is correct by } s], \text{ and}$$

(49) $$\alpha \text{ incorrect by } s \implies [P_{\supseteq\,\alpha,\,s} = \emptyset \text{ and no } \alpha - \text{traces are defined at } s].$$

The trace $u_{\beta,\alpha}$ works as follows. Suppose $x \in P_{\alpha',s} - E_{\beta,s}$ and $x \in E_{\beta,s+1} - E_{\alpha,s+1}$, for $\beta = \alpha^-$ and some $\alpha' \supseteq \alpha$. (Hence, α and α' are incorrect at stage $s+1$.) Move x to $P_{\beta,s+1}$ and put $u_{\beta,\alpha}$ into K_β, from which it will later enter B if $\beta \subset f$, and will cause $\psi(x)$ to become undefined. By (49) this occurs, and therefore $u_{\beta,\alpha}$ enters K_β, at most once after which α will forever be seen to be incorrect, and $P_{\supseteq\alpha,\,s} = \emptyset$.

5.3.5 Counting the Number of Traces Needed

Define $k(m,s) = \text{ card}\{\gamma : |\gamma| = m \;\;\&\;\; \gamma <_L f_s\}$. Let $|\alpha| = n+1$ and $\beta = \alpha^-$. Now α needs traces v_α and $y_{\alpha,\gamma}$, $u_{\alpha,\gamma}$, $w_{\alpha,\gamma}$ for all $\gamma \leq_L f_s$, γ an immediate extension of α. Hence, α needs $1 + 3 \times k(n+2,s)$ many traces. But each $\alpha \leq_L f_s$ of length $n+1$ needs this many, and there are $k(n+1,s)$ many such α. Thus, for $|\beta| > 1$ our request in defining $g(\beta,s)$, which will determine L_β where these traces will be found, must be,

(50) $$g(\beta,s) = m(n,s) =_{dfn} k(n+1,s) \times (1 + 3 \times k(n+2,s)).$$

For $\beta = \rho$ we define $g(\rho,s) = m(1,s) + m(0,s)$ because $L_\lambda = \emptyset$ and $J_\lambda = \emptyset$ so ρ must use its own L_ρ not L_λ for its $m(0,s)$ many ρ-traces and in addition provide $m(1,s)$ many ρ-traces for those α, $|\alpha| = 2$. Hence, $K_\rho \subset J_\rho^-$ not J_λ^-, and the obvious minor adjustments must be made in the text for the case of $\alpha = \rho$ which we leave to the reader.

If $g(\beta,s) = k$ the procedure of [4] automatically attempts to produce $2k$ candidates in L_β so that each z_α will have a backup trace z'_α. As these $2k$ many elements begin to appear in L_β we assign them first to α_1, the first immediate successor of β, up to $2 \times [1 + 3 \times k(n+2,s)]$ many traces, a primary trace for each category described above and a backup trace. Next we assign

$2 \times [1 + 3 \times k(n+2, s)]$ many traces to α_2, the next immediate successor of β and so on. This means that every trace assigned to α_1 is smaller than any trace assigned to α_2, and so on. From the traces assigned to α_1 we assign the least two as trace v_α and its backup v'_α, the next least four to $y_{\alpha,\gamma}$, $w_{\alpha,\gamma}$ and their backups, $y'_{\alpha,\gamma}$, $w'_{\alpha,\gamma}$, for γ the immediate successor of α_1 which is $<_L$-minimal, and so on until all traces for α_1 are appointed together with backup traces. This procedure automatically guarantees that when defined the β traces satisfy the inequality (46).

Second if β, δ, α, and γ as in (46), so $\delta <_L \alpha <_L \gamma$, and $\alpha \subset f$ then we shall prove,

(51) $$\lim_s v_{\beta,s} < \infty,$$

(52) $$(\forall z_{\beta,\delta})\,[\lim_s z_{\beta,\delta,s}\downarrow],$$

(53) $$\lim_s u_{\beta,\alpha,s} < \infty \;\;\&\;\; \lim_s y_{\beta,\alpha,s} = \infty \;\;\&\;\; \lim_s w_{\beta,\alpha,s} = \infty, \text{ and}$$

(54) $$(\forall z_\gamma)\,[\lim_s z_{\gamma,s}\uparrow],$$

where from now on $z_{\beta,\alpha}$ represents an arbitrary (β,α)-trace, $u_{\beta,\alpha}$, $y_{\beta,\alpha}$, or $w_{\beta,\alpha}$, and z_β represents an arbitrary β-trace, v_β, $u_{\beta,\alpha}$, $y_{\beta,\alpha}$, or $w_{\beta,\alpha}$.

Now by (51) and (53) we can require that when we define any α-trace $z_{\alpha,s}$ we have,

(55) $$(\forall\beta \subset \alpha)(\forall\xi \subset \alpha)_{\xi^- = \beta}(\forall\gamma <_L \alpha)_{\gamma^- = \beta}\,[u_{\beta,\xi,s}\downarrow < z_{\alpha,s} \;\;\&\;\; w_{\beta,\gamma,s}\downarrow < z_{\alpha,s}].$$

Also whenever such a β-trace $w_{\beta,\gamma}$ or $u_{\beta,\xi}$ enters B we cancel all α-traces under Step 5 and later reappoint them. Thus, we ensure that (55) always holds.

Next by (52) and the same cancellation policy we can ensure that,

(56) $$(\forall\delta <_L \alpha)(\forall\delta - \text{trace } z_\delta)(\forall\alpha - \text{trace } z_\alpha)(\forall s)[z_{\delta,s}\downarrow \implies z_{\delta,s} < z_{\alpha,s}].$$

Finally, we may assume

(57) $$(\forall\alpha)(\forall s)[\alpha - \text{trace } z_{\alpha,s}\downarrow \implies z_{\alpha,s} > 0].$$

5.4 The Full Construction

For every $\alpha \subseteq \rho$ let the α-traces be defined as in §5.3.5. For $\theta \in T$ and s define the predicate $R(\theta, s)$ for "θ is ready at s,"

$$(58)\qquad R(\theta, s) \equiv (\forall \beta \subseteq \theta)[v_{\beta,s}\downarrow \\ \&\ (\forall \alpha \subseteq \theta)[\alpha^- = \beta \implies y_{\beta,\alpha,s}\downarrow] \\ \&\ (\forall \gamma)(\forall \alpha)[\beta \subset \alpha \subseteq \theta\ \&\ \gamma <_L \alpha \implies w_{\beta,\gamma,s}\downarrow]].$$

Notice that if $\alpha \subset f$ then these α-traces are defined for almost all s by Theorem 4.1 (vi). Define the chip functions,

$$(59)\qquad c_1(x, \theta, s) = \ \mathrm{card}\{t < s : R(\theta, t)\ \&\ \theta \subseteq f_t\ \&\ |\theta| = x\}, \\ c_2(x, \theta, s) = \ \mathrm{card}\{s : \psi_s(x)\uparrow\ \&\ \psi_{s+1}(x)\downarrow = \theta\}, \\ c(x, \theta, s) = c_1(x, \theta, s) - c_2(x, \theta, s).$$

Whenever the defining clause for $c_1(x, \theta, t)$ holds we can think of θ as acquiring another "chip" for x, and spending a chip whenever the defining clause for $c_2(x, \theta, t)$ holds (necessarily because Step 1 applies). The function $c(x, \theta, s)$ represents the balance of chips available. Note that if $\theta = f\upharpoonright x$ then $\lim_s c_1(x, \theta, s) = \infty$.

In addition to the use function $\psi(x)$ we shall need an auxiliary function $\widehat{\psi}(x)$ such that $\widehat{\psi}(x) \leq \psi(x)$ and if $x \in P_\alpha$ then $\widehat{\psi}(x) = z_\alpha$ some α-trace such that if z_α enters B then we shall undefine $\psi(x)$. The point is that $\psi(x)$ once defined does not change in value until $\widehat{\psi}(x)$ enters B so $\psi \leq_T B$, while $\widehat{\psi}$ may change in value several times before $\widehat{\psi}(x)$ enters B so $\widehat{\psi} \not\leq_T B$.

Stage $s = 0$. Every trace and function $\psi(x)$, $\widehat{\psi}(x)$, $x \in \omega$ is undefined, and $P_\alpha = \emptyset$ for all $\alpha \in T$.

Stage $s+1$. At stage $s+1$ for each $x \leq s$ perform the first of these steps on x which applies. (We often drop the subscript s or $s+1$ from ψ, $\widehat{\psi}$, P_α, and α-traces if the context makes the meaning clear.) If a γ-trace z_γ is cancelled at stage $s+1$ then we put $z_{\gamma,s}$ into $K_{\gamma,s+1}$.

Step 1. Correcting $\gamma >_L f_s$. Suppose $\alpha^- = \beta = \gamma^-$, $\alpha <_L \gamma$, and $\alpha \subset f_{s+1}$.

(a) Put $w_{\beta,\alpha,s}$ into $K_{\beta,s+1}$. Cancel all γ-traces and all (β,γ)-traces. (The automorphism machinery [4, Step $\hat{7}$ Case 2 and Lemma 7.2] immediately removes them from L_β by our definition of g in (50).)

(b) In addition if $x \in P_{\xi,s}$ for some $\xi \supseteq \gamma$, then move x into $P_{\beta,s+1}$, and define $\widehat{\psi}_{s+1}(x) = w_{\beta,\alpha,s}$.

Step 2. Defining $\psi(x)$. Suppose $\psi_s(x)\uparrow$. Define

$$\pi(x,s) = (<_L -\text{least } \theta)[\ |\theta| = x \ \ \& \ \ x < c(x,\theta,s)]$$

if such θ exists and let $\pi(x,s)$ be undefined otherwise. If $\pi(x,s)\downarrow$ then find $\alpha \subseteq \pi$ of longest length such that no $\beta \subseteq \alpha$ has been discharged by x by stage s. (Since λ never discharges any x such α will exist.) Put x in P_α.

(a) If $\alpha = \pi$ define $\psi_{s+1}(x) = \widehat{\psi}_{s+1}(x) = v_{\alpha,s}$.

(b) If $\pi \supset \alpha \supseteq \rho$ define $\psi_{s+1}(x) = \widehat{\psi}_{s+1}(x) = y_{\alpha,\gamma,s}$ where $\alpha \subset \gamma \subseteq \pi$ and $\gamma^- = \alpha$. If $\alpha = \lambda$ define $\psi_{s+1}(x) = 0$.

Step 3. Undefining $\psi(x)$. Suppose $x \in P_{\xi,s}$ and $z \in B_{s+1} - B_s$, where

$$z = \widehat{\psi}_s(x) \ \vee \ [z < \widehat{\psi}_s(x) \ \ \& \ \ [z = u_{\beta,\alpha,s} \ \vee \ z = w_{\beta,\delta,s}]], \tag{60}$$

for some $\beta \subset \alpha \subset \xi$, $\alpha^- = \beta$, $\delta^- = \beta$, $\delta <_L \alpha$. Then let $\psi_{s+1}(x)$ and $\widehat{\psi}_{s+1}(x)$ be undefined, and remove x from P_ξ.

Step 4. Entering E_α. (a) If $x \in P_{\alpha,s} - E_{\alpha,s}$, $\widehat{\psi}_s(x) = z_\alpha$, and $x \in E_{\alpha,s+1}$, then put z_α in $K_{\alpha,s+1}$.

(b) Suppose $x \in P_{\xi,s} \cap E_{\beta,s+1}$, for $\beta \subset \alpha \subseteq \xi$, $\alpha^- = \beta$. (Hence, α and ξ become incorrect at $s+1$.) Put $u_{\beta,\alpha,s}$ in $K_{\beta,s+1}$, move x to $P_{\beta,s+1}$, and define $\widehat{\psi}_{s+1}(x) = u_{\beta,\alpha,s}$.

Step 5. Correcting for $\beta \subset \alpha$. (a) If $\beta \subset \alpha$ and $u_{\beta,\alpha,s} \in B_{s+1} - B_s$ then cancel all α'-traces for all $\alpha' \supseteq \alpha$.

(b) If $\beta \subset \alpha$, $\beta \subset \delta$, $\delta <_L \alpha$, and $w_{\beta,\delta,s} \in B_{s+1} - B_s$ then cancel all α'-traces for all $\alpha' \supseteq \alpha$.

5.5 The Verification

Lemma 5.6 *(i)* $(\forall \delta <_L f)[\bigcup_s P_{\supseteq\delta,\,s} =^* \emptyset]$.

(ii) $(\forall \delta <_L f)(\forall x)[(\exists^\infty s)[x \in \bigcup_s P_{\supseteq\delta,\,s}] \implies (\exists \theta \supseteq \delta)[x \in P_{\supseteq\theta,\,\infty}]$.

Proof. (i) Fix $\delta <_L f$. Choose s_0 such that for all $s \geq s_0$, $f_s \not\leq_L \delta$, and for all $\theta \supseteq \delta$, $\lim_s c_1(x,\theta,s) = c_1(x,\theta,s_0)$, and $\lim_s c(x,\theta,s) = c(x,\theta,s_0)$. Let $k = \text{card}\{t : \delta \supseteq f_t\}$. Only $x < k$ can ever enter $P_{\supseteq\delta}$ under Step 2 because of the definition of π there. No x can enter $P_{\supseteq\delta}$ under Step 1 at any stage $s \geq s_0$. These are the only two steps which will bring x into $P_{\supseteq\delta}$ because Step 1 removes x from P_α and Step 4 only moves x from P_α to P_β for $\beta \supset \alpha$. Hence, any x will reenter $P_{\supseteq\delta}$ at most finitely often.

(ii) Suppose $x \in \bigcup_s P_{\supseteq\delta,s}$ for infinitely many s. After stage s_0 x cannot enter or reenter $P_{\supseteq\delta}$ so $x \in P_{\supseteq\delta,s}$ for all $s \geq s_1$ for some $s_1 \geq s_0$. Step 3 cannot apply to x else x is removed from its present P_ξ, $\xi \geq \delta$, and not returned to another P_α until Step 2 later applies. Step 4 can apply to x at most finitely often while x remains in $P_{\supseteq\delta}$ because each such application moves x from some P_α to P_β for $\beta \subset \alpha$. Hence, after a finite number of applications of Step 4 x settles in $P_{\xi,\infty}$ for some $\xi \supseteq \delta$. ∎

Lemma 5.7 *(i)* $(\forall\beta)(\forall\alpha)_{\alpha^-=\beta}\,(\exists^{\leq 1} s)[u_{\beta,\alpha,s} \in K_{\beta,s+1} - K_{\beta,s}$ *by Step 4b*$]$.

(ii) $(\forall\delta <_L f)[\beta = \delta^- \implies (\exists^{<\infty} s)[z_{\beta,\delta,s+1} \neq z_{\beta,\delta,s}]]$,
where $z_{\beta,\delta,s}$ *denotes an arbitrary* (β,δ)*-trace* $u_{\beta,\delta,s}$, $y_{\beta,\delta,s}$, *or* $w_{\beta,\delta,s}$.

(iii) $(\forall\alpha \subset f)[\beta = \alpha^- \implies lim_s u_{\beta,\alpha,s}\downarrow]$.

Proof. (i) Suppose $u_{\beta,\alpha,s} \in K_{\beta,s+1} - K_{\beta,s}$ because Step 4 (b) applied to x at stage $s+1$. Then all $\alpha' \supseteq \alpha$ are incorrect at all stages $t \geq s+1$. Hence, $\alpha' \not\subseteq f_t$ and $P_{\alpha',t} = \emptyset$ for all $t \geq s+1$, by our convention that incorrect nodes γ are removed from T and $P_{\gamma,s} = \emptyset$. Thus, Step 4 can never apply to α' again so $u_{\beta,\alpha,t}$ never enters K_β.

(ii) Fix δ. Assume (ii) for all $\delta' <_L \delta$. By (i) choose s_0 such that: for all $s \geq s_0$, $f_s \not\leq_L \delta$; $u_{\theta,\xi,s} \notin K_{\theta,s+1} - K_{\theta,s}$ by Step 4 (b) for $\theta \subset \xi \subseteq \delta$; all δ'-traces have settled in value for $\delta' <_L \delta$; and by Lemma 5.6 that $P_{\supseteq\delta,s} = P_{\supseteq\delta,s_0}$ for all $s \geq s_0$.

Now after stage s_0 Step 1 cannot apply to δ so $w_{\beta,\delta}$ cannot change in value. No $x \in P_{\supseteq\delta,s}$, $s \geq s_0$, ever leaves and in particular does not enter P_β so $y_{\beta,\gamma}$ and $u_{\beta,\gamma}$ cannot change value at stage $s \geq s_0$.

(iii) Fix $\alpha \subset f$ and $\beta = \alpha^-$. By (i) choose s_0 such that for all $s \geq_0$, $f_s \geq_L \alpha$; all δ-traces, $\delta <_L \beta$, have settled by s_0; assume by induction for all $\theta \subset \xi \subset \beta$, $\xi^- = \theta$, that $u_{\theta,\xi}$ does not change in value at $x \geq s_0$; and by (i) that $u_{\beta,\alpha}$ does not enter K_β after s_0. Thus, Steps 3 and 5 do not cause any α-trace to be cancelled after stage s_0. Thus, $u_{\beta,\alpha}$ never changes in value after s_0. ∎

Lemma 5.8 *(i)* $(\forall x)(\exists\theta)[x \in P_{\theta,\infty}]$.
(ii) $(\forall\alpha \subset f)[lim_s v_{\alpha,s} < \infty]$.

Proof. (i) Fix x. Let $\alpha = f \upharpoonright x$. Choose s_0 such that $f_s \not<_L \alpha$ for any $s \geq s_0$, and for all $\theta <_L \alpha$ if $|\theta| = x$ and $\theta \subset f_s$ for some $s \leq s_0$, then $\lim_s c_1(x,\theta,s) = c_1(x,\theta,s_0)$, $\lim_s c(x,\theta,s) = c(x,\theta,s_0)$, and for all $\beta \subseteq \theta$ or $\beta \subseteq \alpha$, $x \in E_\beta$ iff $x \in E_{\beta,s_0}$, and $x \in E_{\beta,\text{ at } t}$, $t \leq s_0$, implies $h_\beta(t) < s_0$.

By Lemmas 5.6 and 5.7, Choose $s_1 \geq s_0$ such that

$$(\forall s \geq s_1)(\forall\delta <_L \alpha)[P_{\supseteq\delta,s} = P_{\supseteq\delta,\infty}, \text{ and} \tag{61}$$

$$(\forall s \geq s_1)(\forall\beta)(\forall\xi)(\forall\delta <_L \alpha)(\forall\theta = \delta^-)[\beta \subset \xi \subseteq \alpha \implies [u_{\beta,\xi,s} = u_{\beta,\xi,s_1} \;\&\; w_{\theta,\delta,s} = w_{\theta,\delta,s_1}]]. \tag{62}$$

At $s \geq s_1$ Step 4 cannot apply to $x \in P_{\theta,s}$, $\theta \subseteq \alpha$, and Steps 1 and 5 cannot apply to v_α. Hence, either x comes to rest in $P_{\delta,\infty}$ for some $\delta <_L \alpha$ or perhaps $\delta \subset \alpha$ including $\delta = \lambda$, or else Step 3 undefines $\psi(x)$ at some stage $x \geq s_1$.

In the latter case Step 2 must redefine $\psi(x)$ at some stage $s+1 > s_0$ with $\alpha \leq_L \pi(x,s)$. If $\alpha <_L \pi(x,s) = \gamma$ then at some stage $t > s$ we have $\beta \subset \alpha \subset f_t$, $\beta \subset \gamma$, and $\gamma^- = \beta$, for some $\beta \subset \alpha$. Hence, by Step 4 we know x enters $P_{\beta,t+1}$, and $w_{\beta,\alpha,t}$ enters $K_{\beta,t+1}$, and later enters B at some stage $u+1 \geq t+1$ because $\beta \subset f$. When $\psi(x)$ was last defined at stage $s+1$ and x placed in some P_θ, $\theta \supseteq \gamma$, we defined $\psi_{s+1}(x) = v_\theta$ for some θ-trace z_θ. By (46), (55), and (56), we have the following inequalities at s,

$$w_{\beta,\alpha} < w_{\beta,\gamma} < \widehat{\psi}(x) = \psi(x) = z_\theta, \tag{63}$$

and thus at all stages r, $s \leq r \leq u$, we have

$$w_{\beta,\alpha,r} = \widehat{\psi}_r(x) \leq \psi_r(x) = z_{\theta,s},$$

by induction on s and the action of Steps 4 and 5 the only ones which can alter the value of $\widehat{\psi}(x)$ while $\psi(x)$ remains defined. Thus, at stage $u+1$ Step 3 causes $\psi(x)$ and $\widehat{\psi}(x)$ to become undefined.

While $\pi(x,s) >_L \alpha$ during the above process $c_2(x,\alpha,s)$ is not increasing but $c_1(x,\alpha,s)$ is going to infinity. Hence, by the definition of $c(x,\alpha,s)$ we must have $\pi(x,s) = \alpha$ and Step 2 applies to redefine $\psi(x)$ at $s+1$ for some $s \geq s_0$.

Case 1. $x \notin E_{\alpha,s}$. Then by $\alpha \subset f$ and the choice of s_0, for all $\beta \subseteq \alpha$, $E_\alpha = E_\beta$ and $x \notin E_{\beta,s}$. Hence, $x \notin E_\rho = D$. None of Steps 1–5 will apply to x again. Thus, for all $r \geq s+1$, $x \in P_{\alpha,r}$ and $\psi_r(x) = \widehat{\psi}_r(x) = v_{\alpha,s} = v_{\alpha,r}$. Note that $v_{\alpha,s} = v_{\alpha,r}$ bedause no $x' = neqx$ is ever assigned trace $v_{\alpha,s}$ at stage $r > s$ because Step 2(i) would require $x' = x$.

Case 2. $x \in E_{\alpha,s}$. Then $x \in E_{\beta,s}$ for all $\rho \subseteq \beta \subseteq \alpha$ by choice of s_0. Hence, by Step 2 Case 1 we place x in E_λ where it remains forever.

(ii) In either case we see that $\lim_s v_{\alpha,s} < \infty$ because: x will never cause v_α to change in value again; no other $x' \neq x$ can every move v_α for any reason; and v_α never changes in value after s_1 due to a change in higher priority w or u markers under Step 5 because of (62). ■

Lemma 5.9 $(\forall \gamma)(\forall s)[z_{\gamma,s}\downarrow \Longrightarrow$

$$(64) \qquad (\forall \beta \subset \gamma)(\forall \alpha <_L \gamma)_{\alpha^- = \beta}\,[w_{\beta,\alpha,s}\downarrow < z_{\gamma,s}]$$

$$(65) \qquad \&\ (\forall \beta \subset \gamma)(\forall \alpha \subset \gamma)_{\alpha^- = \beta}\,[u_{\beta,\alpha,s}\downarrow < z_{\gamma,s}]].$$

Proof. Whenver $z_{\gamma,s+1}$ is newly defined then (64) and (65) must hold by our convention (55) for appointing traces. Suppose by induction on s that $z_{\gamma,s}\downarrow$ and that (64) and (65) hold for some s. Then they clearly hold for $s+1$ unless either $u_{\beta,\alpha}$ or $w_{\beta,\alpha}$ changes value at stage $s+1$. This can only happen under Step 1 or Step 5 in which case z_γ is cancelled also. ■

Lemma 5.10

(*i*) $(\forall x)(\forall s)[[\widehat{\psi}_s(x)\downarrow\ \&\ \widehat{\psi}_{x+1}(x)\downarrow] \Longrightarrow [\widehat{\psi}_{s+1}(x) \leq \widehat{\psi}_s(x) \leq \psi_s(x)]]$.

(*ii*) $(\forall x)(\forall s)[x \in P_{\alpha,s} \Longrightarrow (\exists \alpha\text{-}trace\ z_{\alpha,s})[\widehat{\psi}_s(x) = z_{\alpha,s}]]$.

Proof. (i) The proof is by induction on s. Assume that $\psi_{s+1}(x)$ is defined or redefined at stage $s+1$ under Step 2 with $\pi(x,s) = \theta$. Then either x

enters $P_{\theta,s+1}$ and we define $\psi_{s+1}(x) = \widehat{\psi}_{s+1}(x) = v_{\theta,s}$ or else x enters $P_{\beta,s+1}$ for some $\beta \subset \theta$ and we define $\psi_{s+1}(x) = \widehat{\psi}_{s+1}(x) = y_{\beta,\xi,s}$ for $\beta \subset \xi \subseteq \theta$, and $\xi^- = \beta$. Fix $t \geq s+1$ and assume by induction that (i) and (ii) hold for t. Assume $x \in P_{\alpha,t}$ and $\widehat{\psi}_t(x) = z_{\alpha,t}$ an α-trace. Assume that $\widehat{\psi}_{t+1}(x) \neq \widehat{\psi}_t(x)$. If Step 3 applied then $\widehat{\psi}_{t+1}(x)$ and $\psi_{t+1}(x)$ are both undefined. Since we are assuming that $\widehat{\psi}_{t+1}(x)\downarrow$ we must have that Step 1 or Step 4 applied to x at stage $t+1$. If Step 1 applied then $x \in P_{\delta,t+1}$ and $\widehat{\psi}_{t+1}(x) = w_{\delta,t} = w_{\delta,t+1}$, and we have $w_{\delta,t} < z_{\alpha,t}$ by Lemma 5.9. If Step 4 applied then $x \in P_{\beta,t+1}$, $\beta \subset \alpha$, and $\widehat{\psi}_{t+1}(x) = u_{\beta,\xi,t} = u_{\beta,\xi,t+1}$, for $\beta \subset \xi \subset \alpha$, and we have $u_{\beta,\xi,t} < z_{\alpha,t}$ by Lemma 5.9. Hence, (i) and (ii) hold at stage $t+1$. ■

Lemma 5.11 *(i)* $\psi(x) = lim_s \psi_s(x) \; < \infty$.
(ii) $\psi \leq_T B$.

Proof. (i) Fix x. By Lemma 5.8 (i) fix α and s_0 such that for all $s \geq s_0$, $x \in P_{\alpha,s}$. If $\alpha = \lambda$ then $\psi_s(x) = 0$ for all $s \geq s_0$. If $\alpha \neq \lambda$ then $\psi_{s_0}(x) = z_{\alpha,s_0}$ for some α-trace z_α. Suppose that $\widehat{\psi}_t(x) = z_{\alpha,t} = z_{\alpha,s_0}$ for some $t \geq s_0$. At stage $t+1$, Step 1 cannot apply to x else x moves to P_β for some $\beta \subset \alpha$; Step 2 cannot apply because $x \in P_{\alpha,t}$ already; Steps 3 and 4 cannot apply to x else x is removed from P_α at $t+1$; and Step 5 applies only to traces and never to elements $x \in P_\alpha$. Hence, no part of the construction ever changes the value of $\psi_t(x)$ and thus $\psi_t(x) = \psi_{s_0}(x)$ for all $t \geq s_0$.

(ii) By (i) we know that $\psi(x)$ exists. From Lemma 5.10 we know,

$$(\forall x)(\forall s)[[\widehat{\psi}_s(x)\downarrow \;\;\&\;\; \widehat{\psi}_{x+1}(x)\downarrow] \implies [\widehat{\psi}_{s+1}(x) \leq \widehat{\psi}_s(x) \leq \psi_s(x)]].$$

From the construction Steps 3 and 5 it follows that,

$$\widehat{\psi}_s(x)\downarrow \;\;\&\;\; \widehat{\psi}_{s+1}(x)\uparrow \implies (\exists z \leq \widehat{\psi}_s(x))[x \in B_{s+1} - B_s].$$

Thus, $\psi \leq_T B$ because given x choose s such that $B_s \upharpoonright y = B \upharpoonright y$ where $y = 1 + \psi_s(x)$. Hence, $\psi(x) = \psi_t(x)$ for all $t \geq s$. ■

Lemma 5.12 *(i)* $x \in D \implies [x \in P_{\lambda,\infty} \;\;\&\;\; \psi(x) = 0]$.
(ii) $x \in \overline{D} \implies (\exists \alpha \neq \lambda)[x \in P_{\alpha,\infty} \;\;\&\;\; \psi(x) > 0]$.
(iii) $D \leq_T B$.

Proof. (i) Suppose $x \in D$. Suppose toward a contradiction that for some $\alpha \neq \lambda$ and for all $s \geq s_0$, $x \in P_{\alpha,s}$ and $\psi_s(x) = \psi(x)$. Choose the longest β, $\rho \subseteq \beta \subseteq \alpha$, such that $\beta \subset f$. Now $E_\beta = D$ because $\beta \subset f$. Hence, there exists $s_1 \geq s_0$ such that $x \in E_{\beta,s_1}$. At stage s_1 trace u_{β,ξ,s_1} is put into K_β and we define $\widehat{\psi}_{s_1}(x) = u_{\beta,\xi,s_1}$, for $\beta \subset \xi \subseteq \alpha$, $\xi^- = \beta$. If $\beta = \alpha$ then this trace is z_{α,s_1}. In either case since $\beta \subset f$ the trace later enters B, at some stage $t > s_1$, the function $\psi_t(x)$ becomes undefined by Step 3 and x moves to another P_γ, $\gamma \neq \alpha$, contrary to choice of s_0.

(ii) Fix $x \in \overline{D} = \overline{E}_\rho$. Hence, Step 2 never applies to x. However, this is the only way an element can enter P_λ or have $\psi(x) = 0$. Hence, x never enters P_λ or has $\psi(x) = 0$. Thus, by Lemmas 5.8 and 5.11 there exists some some s_0, some $\alpha \neq \lambda$, and some α-trace z_α, such that for all $s \geq s_0$, $x \in P_{\alpha,s}$ and $\psi_s(x) = z_\alpha$. Finally, $z_\alpha > 0$ because of our convention that all traces are greater than 0.

(iii) Now $D \leq_T B$ follows immediately by (i) and (ii) and Lemma 5.11. ■

The Refined Coding Theorem 4.1 (i) and Lemma 5.12 (iii) together establish the conclusion of Theorem 5.1. ■

Remark 5.13 The crucial use of the hypothesis "D is 2-tardy" is the following. Suppose $x \in P_{\alpha,s} - E_{\alpha,s}$ and

$$v_\beta < v_\alpha < y_{\alpha,\gamma} = \psi(x) < y_{\beta,\alpha},$$

which is the likely configuration since $y_{\beta,\alpha}$ has increased in response to other x' discharging β. Now without "D 2-tardy" and its consequences (44) and (45), we may have $x \in E_\beta \setminus E_\alpha$, at which point we must put a β-trace into K_β to redefine $\psi(x)$. The only such β-traces z_β available below $\psi(x)$ are v_β or $u_{\beta,\alpha}$, since the others are going to infinity and may already exceed $\psi(x)$. However, repeating this for a fixed z_β over infinitely many x will cause $\lim_s z_{\beta,s} = \infty$, thus preventing z_β from carrying out this role, and negating the effect of sharing traces. With "D 2-tardy" and its consequences (44) and (45), we may expect that $x \in E_\alpha$ first at which time the α-trace $y_{\alpha,\gamma}$ enters K_α and later B before $x \in E_\beta$. Thus, $\psi(x)$ is redefined to $y_{\beta,\alpha}$ before x enters P_β or E_β. Therefore, when x finally enters E_β we can assign to $\psi(x)$ a relatively cheap β-trace, $y_{\beta,\alpha}$ which is going to infinity anyway, rather than an expensive β-trace, v_β or $u_{\beta,\alpha}$, which we want to converge.

6 Open Questions

Question 1 *A 3-tardy $\implies$ A is codable?*

Or equivalently,

Question 2 *A 3-tardy $\implies$ $(\exists B$ 2-tardy $)[A \leq_T B]\Gamma$*

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